Response to Kiwoon Choi's comments on arXiv:2001.0715

In regards to the maximum topological charge

As the topological charge counts the number of times that the gauge group can be wrapped around a sphere at infinity, there is no proportionality between the volume and the maximum topological charge. As an example considered by Coleman in his Erice lecture "The uses of instantons", section 3.2, consider a 2D gauge theory in a finite disk with a compact U(1) gauge group, and demand pure gauge configurations at the circle at the boundary. Then one can wrap the circle associated with the compact U(1) group an arbitrary number of times around the circle at the boundary of spacetime. This is so regardless of the size of the disk, and Coleman at no point mentions an upper bound on the topological charge/winding number. Similarly, lattice theorists working with a finite 4D torus still talk about arbitrary topological sectors, without capping them in terms of the volume. See e.g. the classic lattice reference hep-lat/9207009 by Lüscher and Narayanan, whose formulae include sums over topological charge reaching up to infinity.

Even when the topological charge is unbounded regardless of the volume, one may still attempt to talk about a probability distribution for the topological charge. This can be done for example in a dilute instanton picture by interpreting the vacuum energy density at each topological sector as a classical probability distribution. If one proceeds in this way, one finds in fact that the expected fluctuations of the topological charge grow with the square root of the volume, rather than the volume as argued by Kiwoon. While the average number of instantons and anti-instantons do grow as VT, their difference, which gives the topological charge, goes as $\sqrt{VT}$. This means that the most probable topological charge grows much slower than the spacetime volume as the latter is increased.

In any case, we emphasize that this applies to an estimate of the most probable topological charge, rather than an absolute cutoff. Moreover, the interpretation of the instanton density as a classical probability can be questioned as it ignores potential quantum interference effects.

The relevant question in our point of view is when one can actually justify the use of integer topological charges. As emphasized in our paper and talks, the topological charge is only *required* to be an integer for an infinite spacetime volume, which justifies our order of limits.

In regards to the different ways to introduce CP-violating angles in QCD

We agree on the different ways that CP violating angles can be introduced in QCD. We also agree that the conclusions about the physical CP violation should not depend on how the angles are introduced. Our work shows that going through all the different ways, one can conclude that there is no CP violation and that observables do not depend on the parameter $\bar{\theta}$. While our article 2001.07152 focuses on the CP violating angles from the explicit θ term and $\arg\det M_q$ (ways 3 and 4 in Kiwoon's comments), the talk by Björn explicitly addressed the phases in canonical quantization (ways 1 and 2).

In regards to the use of arbitrarily small volumes

We use zero space-time volumes $\Omega = 0$ simply as an intermediate step in mathematical calculations because, under the assumption of analyticity in Ω, one can recover a function for arbitrary Ω in terms of its derivatives at zero. Our solutions for the partition functions solve exactly the infinite set of clustering relations for arbitrary values of Ω, and when we extract physical conclusions we always do so for large/infinite volumes, where clustering holds as mentioned by Kiwoon. Moreover, note that for $\Omega = 0$ the partition functions become zero for all topological sectors except the trivial one, which shows explicitly that Kiwoon's concern of shrinking nontrivial topological configurations to a point does not apply, as for a point-like spacetime we get no nontrivial topology. Hence our construction makes no assumption about point-like distributions of topological charge.

We emphasize that traditional results in the literature assume/comply with analyticity of the partition function in the spacetime volume. Hence, if our assumption of analyticity is to be questioned, the traditional results leading to CP violation should also be questioned.

As a general remark, we would like to emphasize one more time our main objections concerning the standard derivations of CP violation in the strong interactions: (1) There appears to be no valid reason why integer topological sectors should follow in finite volumes (in particular, field eigenstates on surfaces at finite time do not correspond to physical states) and (2) the usual expression for the θ vacua does not give a properly normalizable state. This remains an issue even after smearing the field eigenstate configurations of the prevacua, and the remedy appears when appreciating that large gauge transformation lead to redundant descriptions of the same physical state. In the presentation by Björn it is shown that the Hilbert space in which matrix elements satisfy a consistent Heisenberg time evolution leads to CP-even observables.