Comments on Arxiv:2001.07152

Kiwoon Choi, Oct. 12, 2022

It is possible that some (even many) of my comments may not be correct, although at the moment they look correct to me.

It is also quite probable that some of my comments are irrelevant and due to my misunderstanding of the viewpoint taken in arXiv:2001.07152.

The discussions in arXiv:2001.07152 involve two parameters both of which are part of the parameters describing the domain of the path integral over YM gauge fields:

"VT=Euclidean spacetime volume"
as the range of the spacetime coordinate of field variables

"N= maximal winding number"
as the range of the possible value of ν $=\int FF^*$

As N corresponds to the maximum of the spacetime integral of a gauge-invariant local operator, it is proportional to VT, and roughly given by

$$N = \max(\nu) \sim VT\Lambda^4 ,$$

where Λ denotes the UV cutoff scale of the theory and we used simple dimensional analysis implying that the allowed maximal value of the YM gauge field strength is $\max(F) \sim \Lambda^2$.

As a specific example, one can consider field configurations on the Euclidean spacetime which is fully packed by small instantons with size $\rho \sim \Lambda^{-1}$, for which

$$N \sim VT/\rho^4 .$$

More generic configurations involving n instantons and n* anti-instantons with $\max(n+n^*) \sim VT/\rho^4$ leads to a similar result: $N = \max(n-n^*) = O(VT/\rho^4)$.

The winding # $\nu = \int FF^*$ can also be written as a surface integral of the CS 3-form over the 3-dim boundary. Then one may think it is weird that the maximal value of ν expressed as a 3-dim surface integral is proportional to the volume of the 4-dimensional bulk spacetime.

The point is that unlike FF^*, the local value of CS 3-form is not gauge-invariant, so can not be identified as a local physical quantity on the boundary surface.

Only the integer-valued integral of the CS 3-form over the entire boundary (=3-sphere) is a gauge-invariant quantity, and it represents the total # of twist (or windings) made by FF^* over the bulk spacetime inside the boundary.

We don't need to first take

$$VT \rightarrow \text{infinity (or } r = \text{ radius of the Euclidean spacetime} \rightarrow \text{infinity)}$$

to ensure that the possible gauge field configurations can be classified by the integer-valued winding #. Before taking r → infinity, one can impose a boundary condition limiting the asymptotic behavior of the gauge field strength as

$$F \sim 1/r^3 \quad \text{(vector potential} = A \sim \text{pure-gauge} + O(1/r^2)),$$

which would ensure that ν = integer + $O(1/r)$.

Then all possible gauge field configurations can be classified by the integer-part of ν for a finite but large enough value of "r".

Of course, one needs to take r → infinity at some stage, presumably at the last stage.
It has been argued by S. Coleman in his Erice lecture that once r is taken to infinity, the only relic of the above boundary condition is the quantized winding #, and any gauge-invariant quantity other than the winding # is not affected by this boundary condition when r → infinity.

It has been argued in arXiv:2001.07152 that within a certain approximation, the partition function Z_ν of the topological sector with an winding number ν can be approximated by the modified Bessel function:

$$Z_\nu (X) \sim I_\nu(X) \quad \text{(with a ν-independent coefficient),}$$

where the dimensionless variable X is given by

$$X = \beta VT,$$

and β can be interpreted as an effective lagrangian or effective potential describing the amplitudes for the tunneling process between different n-vacua.

To examine the behavior at VT → infinity of the total partition function

$$Z = \sum Z_\nu (X) \quad (-N \leq \nu \leq N)$$

and also the behavior of the associated physical amplitudes, one should compare X with N using

$$N \sim VT\Lambda^4 \ \ (\text{or} \sim VT/\rho^4) \quad \& \quad X = \beta VT$$

Then for any value of VT, combining dimensional analysis with semiclassical computation, one finds

$$X/N \sim \beta/\Lambda^4 \text{ (or } \sim \beta\rho^4) \sim \exp(-8\pi^2/g^2_{QCD}(\rho)) \times (m\rho)^{n1} \times (\psi_L\psi_R\, \rho^3)^{n2}$$

where ψ denotes the light quarks, n1 and n2 are model-dependent non-negative integers, and ρ can be interpreted either as "the size of the corresponding instantons" or "the length scale of UV physics generating the effective lagrangian β".

The above result shows that X/N rapidly vanishes in the limit

$$g^2_{QCD}(\rho) \to 0 \ (\rho \to 0) \quad \text{or} \quad m\rho \to 0.$$

This implies that it is more sensible to take N → infinity before X → infinity than the opposite order, which would result in the strong CP violation by nonzero θ-bar.

There are several ways to introduce potentially CP-violating angles in QCD, all of which lead to the same physical consequence:

1) Choice of the physical Hilbert space defined by the behavior under a large gauge transformation,
2) Choice of the functional derivative representation of the conjugate momenta of the gluon vector potential in the Schrodinger picture
3) Simple addition of the θ-term in the lagrangian,
4) Choice of the phase of Det(M_q)

As they all lead to the same physical consequence, there exists only one relevant angle combination θ-bar describing the physical consequence of any of the above 4 choices.

At any rate, the above 4 choices are a part of the definition of the theory.
Therefore θ-bar is one of the defining parameters of the theory, and physical observables can depend on the value of θ-bar, which can be confirmed by explicit computation.

As stressed by Coleman and Weinberg, the cluster decomposition relation for Z_ν is a good approximation only when both Ω_1 & Ω_2 are large enough, so that the surface effect from the boundary between Ω_1 & Ω_2 can be ignored.

On the other hand, the authors used the relation even when one of Ω_1 & Ω_2 becomes arbitrarily small, and then argued that Z_ν is given by the modified Bessel function $I_\nu(X)$ of $X = \beta VT$ (β = appropriate effective lagrangian).

As the authors consider the limit that Ω_1 or Ω_2 becomes arbitrarily small, while keeping the cluster decomposition relation, this may correspond to making an implicit assumption that the winding #s of Ω_1 & Ω_2 are provided by point-like objects each of which carries integer-valued winding #. This looks qualitatively similar to the dilute instanton gas approximation, which may explain why this approach yields same form of Z_ν as the dilute instanton gas approximation.