The strong CP problem, cluster decomposition, and the index theorem

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CP^3-Origins

The aim:

Challenge the conventional view of the strong CP problem by showing that a careful **infinite 4d volume** limit implies that **QCD does not violate CP** regardless of the value of the **θ angle**

The plan:

The strong CP problem from the bottom up

Constraints from chiral symmetries

Fermion correlators from cluster decomposition and the index theorem

Reduction to finite volume

The strong CP problem from the bottom up

CP violation from the neutron dipole moment

Spin operator $S^i = \frac{i}{8}\epsilon^{ijk}[\gamma^j, \gamma^k]$

Electric field $E_i = F_{0i}$

$$\mathcal{L}_{\text{eff}} \supset \frac{i}{4} f(q^2)\bar{N}\gamma^\mu\gamma^\nu\gamma_5 F_{\mu\nu} N \propto \frac{i}{8}\bar{N}[\gamma^\mu, \gamma^\nu]\gamma_5 F_{\mu\nu} N = \frac{i}{8}\bar{N}[\gamma^\mu, \gamma^\nu]\tilde{F}_{\mu\nu} N \supset \bar{N}(\vec{S}\cdot\vec{E})N$$

CP-odd!

Neutron dipole moment $d_n = f(0)$

Chiral Lagrangian

Goldstones from $U(3)_L \times U(3)_R \to U(3)_V$ $U = \langle U \rangle e^{\mathrm{i}\frac{\Pi^a \sigma^a}{\sqrt{2} f_\pi}} \sim \bar{\psi} P_R \psi$

Neutron-proton doublet $N = \begin{pmatrix} p \\ n \end{pmatrix}$

Quark masses $M = \begin{pmatrix} m_u e^{i\alpha_u} & & \\ & m_d e^{i\alpha_d} & \\ & & m_s e^{i\alpha_s} \end{pmatrix}$

CP-odd phases

Lagrangian

$$\mathcal{L}_{\pi,\mathrm{p,n}} \supset \frac{1}{4} f_\pi^2 \mathrm{Tr} D_\mu U D^\mu U^\dagger + (a f_\pi^3 \mathrm{Tr} M U + |b| e^{-i\xi} f_\pi^4 \mathrm{det} U + \mathrm{h.c.})$$

$$+ i \bar{N} \not{D} N - (m_N \bar{N} \tilde{U} P_L N + ic\, \bar{N} \gamma^\mu \tilde{U}^\dagger D_\mu \tilde{U} P_L N + d\, \bar{N} \tilde{M}^\dagger P_L N + e\, \bar{N} \tilde{U} \tilde{M} \tilde{U} P_L N + \mathrm{h.c.})$$

($\tilde{U}$: projection into u,d flavours)

CP-odd terms in the neutron interactions

Writing

$$\langle U \rangle = \begin{pmatrix} e^{i\varphi_u} & & \\ & e^{i\varphi_d} & \\ & & e^{i\varphi_s} \end{pmatrix}$$

Minimizing $\mathcal{L}_{\rm pion}[U = \langle U \rangle]$ w.r.t. angles:

$$m_i \varphi_i = \tilde{m}(m_u, m_d, m_s)(\xi + \alpha_u + \alpha_d + \alpha_s)$$

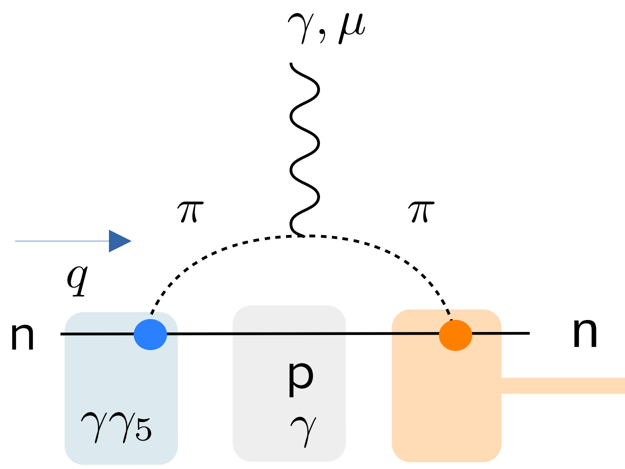
Substituting φ_i in $\mathcal{L}_{\rm neutron}$ and after appropriate field redefinition $N \to \mathcal{N}(N, U)$

$$\mathcal{L}_{\rm neutron} \supset -\frac{2c+1}{f_\pi}\partial_\mu \pi^a \bar{\mathcal{N}} T^a \gamma^\mu \gamma_5 \mathcal{N} + \frac{2(d+e)\tilde{m}}{f_\pi}(\xi + \alpha_u + \alpha_d + \alpha_s)\bar{\mathcal{N}}\, \pi^a T^a\, \mathcal{N}$$

CP-even

CP-odd

Neutron dipole moment



$$-i\mathcal{M} \propto (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_5 q_\nu$$

$$\mathcal{L}_{\rm eff} \supset \frac{i}{4}(\xi + \alpha_u + \alpha_d + \alpha_s) f(q^2)\bar{N}\gamma^\mu\gamma^\nu\gamma_5 F_{\mu\nu}N \supset (\xi + \alpha_u + \alpha_d + \alpha_s) f(q^2)\bar{N}(\vec{S}\cdot\vec{E})N$$

- CP-odd term $\propto (\xi + \alpha_u + \alpha_d + \alpha_s)$ gives contribution to neutron dipole moment

Summary: d_n from chiral Lagrangian

ChPT result $d_n \propto (\xi + \alpha_u + \alpha_d + \alpha_s)$

Experimental bound $|d_n| < 1.8 \times 10^{-26} e \cdot cm$ [nEDM collaboration 2020]

What is the value of ξ in terms of fundamental parameters?

The UV perspective: QCD θ angle

$$S_{\text{QCD}} = \int \mathrm{d}^4x \left[-\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{g^2\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} + \sum_{i=1}^{N_f} \overline{\psi}_i \left(i\gamma^\mu D_\mu - m_i e^{i\alpha_i \gamma_5} \right) \psi_i \right] .$$

$\not{CP}$ $\not{CP}$

θ-term is a total derivative and thus corresponds to a boundary term

→ it can never contribute in perturbation theory:

effects of θ are nonperturbative

S_θ is CP-odd! Expected to contribute to the neutron dipole moment

The θ angle and the Chiral Lagrangian

From before

$$\mathcal{L}_{\rm pion} = \frac{1}{4} f_\pi^2 {\rm Tr} D_\mu U D^\mu U^\dagger + (a f_\pi^3 {\rm Tr} M U + |b| e^{-i\xi} f_\pi^4 {\rm det} U + {\rm h.c.})$$

$$|d_n| \propto (\xi + \alpha_u + \alpha_d + \alpha_s)$$

It is generally thought that $\xi = \theta$ [Baluni, Crewther et al]

For generic θ misaligned with α_i one would expect CP violation, $d_n \neq 0$

Strong CP problem, thought to imply $\bar{\theta} \equiv \theta + \sum_i \alpha_i < 10^{-10}$

How to fix ξ?

- Using symmetry arguments related to anomalous chiral symmetries

  $\xi = \theta$ thought to be the unique possibility (→ CP violation)

- Matching correlators between QCD and effective Lagrangians

  Only real computation that we know of is `t Hooft's, using dilute instanton gas and yielding $\xi = \theta$ (→ CP violation)

Our work

- We have noted an ambiguity in the choices of ξ compatible with chiral symmetries

  This talk

- We have recomputed Green's functions in the dilute instanton gas, in Euclidean and Minkowski spacetime, and found $\xi = -\sum_i \alpha_i$ (→ no CP violation)

  B. Garbrecht's talk

- We also have a computation of fermion correlators which does not rely on instantons, yielding the same conclusion

  This talk

Constraints from chiral symmetries

Spurious chiral symmetry

The partition function changes under **chiral field redefinitions** due to **masses** and **anomaly**

$$\psi \to e^{i\beta\gamma_5}\psi \qquad \bar{\psi} \to \bar{\psi} e^{i\beta\gamma_5}$$

$$Z(\theta, \alpha_j) \to Z(\theta - 2N_f\beta, \alpha_j + 2\beta)$$

fermion mass phases

Spurion symmetry: Z invariant under chiral transformations plus "spurion" transf:

$$\psi \to e^{i\beta\gamma_5}\psi \qquad \bar{\psi} \to \bar{\psi} e^{i\beta\gamma_5}$$

$$\theta \to \theta + 2N_f\beta, \quad \mathfrak{m}_j = m_j e^{\mathrm{i}\alpha_j} \to e^{-2i\beta}\mathfrak{m}_j$$

Effective Lagrangians for QCD should **respect spurion symmetry**

Spurious symmetry in the chiral Lagrangian

$$\psi \to e^{i\beta\gamma_5}\psi \qquad \bar\psi \to \bar\psi e^{i\beta\gamma_5} \quad \Rightarrow \quad U \to e^{2i\beta}U \qquad N \to e^{i\beta\gamma_5}N \qquad \bar N \to \bar N e^{i\beta\gamma_5}$$

$$\mathfrak{m}_j = m_j e^{\mathrm{i}\alpha_j} \to e^{-2i\beta}\mathfrak{m}_j \quad \Rightarrow \quad M \to e^{-2i\beta}M$$

+2 -2 -2 +2 -2N_f +2N_f

$$\mathcal{L}_{\pi,\mathrm{p,n}} \supset \frac{1}{4} f_\pi^2 \mathrm{Tr} D_\mu U D^\mu U^\dagger + (a f_\pi^3 \mathrm{Tr} M U + |b| e^{-i\xi} f_\pi^4 \det U + \mathrm{h.c.})$$

∓1 ±1 -1 +2 -1 +1 -2 +2 -1 -1 +2 -1 -1+2-2+2 -1

$$+i\bar N \not{D} N - (m_N \bar N \tilde U P_L N + ic\,\bar N \gamma^\mu \tilde U^\dagger D_\mu \tilde U P_L N + d\,\bar N \tilde M^\dagger P_L N + e\,\bar N \tilde U \tilde M \tilde U P_L N + \mathrm{h.c.})$$

Spurious symmetry in the chiral Lagrangian

Spurious chiral symmetry requires

$$\xi \to \xi + 2N_f\beta$$

More than one possibility in terms of fundamental parameters θ, α_i !

$\xi = \theta$ Usual option, assumed by [Baluni, Crewther et al] → CP violation

$$d_n \propto (\xi + \sum_i \alpha_i) = (\theta + \sum_i \alpha_i) \equiv \bar{\theta}$$

$\xi = -\sum_i \alpha_i$ Alternative option → CP conservation

$$d_n \propto (\xi + \sum_i \alpha_i) = 0$$

Fermion correlators from cluster decomposition and the index theorem

- In order to resolve the ambiguity, we must **match effective $\det U$ term in** the chiral Lagrangian with **results for correlators in QCD, paying special attention to complex phases**

- We will derive an **effective Lagrangian** capturing this correlators and match to

$$\mathcal{L}_{\text{eff}}^{\text{QCD}} \leftrightarrow \mathcal{L}_{ChPT} \propto e^{-\mathrm{i}\xi}\det U + e^{\mathrm{i}\xi}\det U^{\dagger} \sim e^{-\mathrm{i}\xi}\prod_{i=1}^{N_f}\bar{\psi}_i P_R \psi_i + e^{\mathrm{i}\xi}\prod_{i=1}^{N_f}\bar{\psi}_i P_L \psi_i$$

Read ξ from phases in effective vertices derived from QCD

- Next we proceed to calculate the **phase of QCD correlators** starting from the **path integral** and using **clustering** and the **index theorem**.

Towards correlators: vacuum path integral

$$\int_{\phi_i,\phi_f,T}\left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S_T} = \langle\phi_f|e^{-\mathrm{i}HT}|\phi_i\rangle = \sum_n e^{-\mathrm{i}E_nT}\langle\phi_f|n\rangle\langle n|\phi_i\rangle$$

To get a **vacuum transition amplitude** we can take the **infinite T limit,**

$$Z = \lim_{T\to\infty e^{-i0+}}\int_T\left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S_T} \sim \lim_{T\to\infty e^{-i0+}}\langle 0|e^{-\mathrm{i}HT}|0\rangle$$

To recover the vacuum amplitude for **finite T**, one would **need to know the wave functional of the vacuum**

$$\langle 0|e^{-iHT}|0\rangle = \int[\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\langle 0|\phi_f\rangle\langle\phi_f|e^{-iHT}|\phi_i\rangle\langle\phi_i|0\rangle$$

$$= \int[\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\langle 0|\phi_f\rangle\langle\phi_i|0\rangle\int_{\phi_i,\phi_f,T}\left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S}$$

Wrap-up: the importance of boundary conditions

Infinite T method

$$Z = \lim_{T\to\infty e^{-i0_+}} \int_T \left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S_T} \sim \lim_{T\to\infty e^{-i0_+}} \langle 0|e^{-\mathrm{i}HT}|0\rangle$$

Boundary conditions remain arbitrary!

Wave functional method

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\langle 0|\phi_f\rangle\langle\phi_i|0\rangle \int_{\phi_i,\phi_f,T} \left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S}$$

Boundary conditions are fixed by unknown wave functional, need additional reweighting

Wrap-up: the importance of boundary conditions

▶ To ensure projection into vacuum, we use the Euclidean path integral for infinite V T, without the need to enforce particular b.c.s

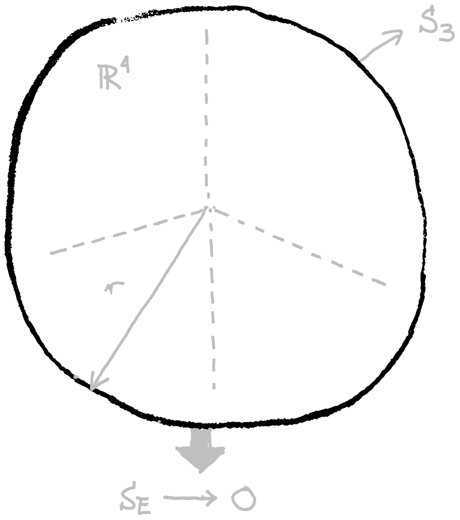
This is in contrast with lattice simulations at finite volume with periodic b.c.s

This requires to subtract contamination from excited states!

Finite action constraints and topology

Euclidean path integral can be formulated in terms of a sum of integrations over **steepest descent flows** that start from **finite action saddles** [Witten]

In **infinite spacetime, gauge fields at saddles** must be **pure gauge transf. at ∞**



$$A_m(r,\theta,\varphi,s) = \frac{i}{g} U_r(\theta,\varphi,s)\,\partial_m U_r^\dagger(\theta,\varphi,s), \quad U_r(\theta,\varphi,s) \in SU(3)$$

Finite action constraints and topology

This leads to maps S_3 ⟶ SU(3) that fall into equivalence or **homotopy classes**

"wrappings" of SU(3) over S_3 that cannot be connected by continuous deformations

The **steepest descent flows** are **continuous**

▶ the full flow from a saddle point falls into the same homotopy class

Homotopy classes characterized by **integer topological charge** Δn

▶ In an **infinite spacetime** $Z = \sum_{\Delta n} Z_{\Delta n}$

Topological charge and the index theorem

Atiyah-Singer's index theorem relates the **topological charge** to the **eigenspectrum** of the **Euclidean Dirac operator**

$$\not{D}\varphi^\lambda = \lambda\varphi^\lambda$$

Zero modes:

$$\not{D}\varphi^0 = 0$$

Index theorem

$$\Delta n = \#(\text{Right-handed zero modes of } \not{D}) - \#(\text{Left-handed zero modes of } \not{D})$$

$$\not{D}\psi_R = 0 \qquad \not{D}\psi_L = 0$$

The θ term and the topological charge

The θ **term** turns out to be **proportional to the topological charge**

$$-S_\theta^E = i\theta \int d^4x \frac{g^2}{64\pi^2} \epsilon_{mnrs} F^a_{mn} F^a_{rs} = i\theta \Delta n$$

In an **infinite spacetime** $Z = \sum_{\Delta n} e^{i\Delta n \theta} \tilde{Z}_{\Delta n}$

Remember: Integer topological charge only enforced for infinite volume

Strategy to compute correlators

The aim is to constrain the functional dependence of the partition functions $Z_{\Delta n}$ on $VT \equiv \Omega,\ \Delta n,\ \mathfrak{m}_j = m_j e^{i\alpha_j}$

Fermion masses can be understood as sources for the integrated fermion correlators [Leutweyler & Smilga]

$$\mathcal{L} \supset \sum_j \left(\bar{\psi}_j(\mathfrak{m}_j^* P_L + \mathfrak{m}_j P_R)\psi_j\right)$$

These correlators should be sensitive to global CP-violating phases

$$\frac{\partial}{\partial \mathfrak{m}_i} Z_{\Delta n} = -\int d^4x\, \langle \bar{\psi}_i P_R \psi_i \rangle_{\Delta n}, \qquad \frac{\partial}{\partial \mathfrak{m}_i^*} Z_{\Delta n} = -\int d^4x\, \langle \bar{\psi}_i P_L \psi_i \rangle_{\Delta n}.$$

Cluster decomposition

$$Z(\Omega) = \sum_{\Delta n=-\infty}^{\infty} \int_{\Delta n} \mathcal{D}\phi e^{-S_\Omega[\phi]+i\Delta n\theta} \equiv \sum_{\Delta n=-\infty}^{\infty} e^{i\Delta n\theta}\tilde{Z}_{\Delta n}(\Omega)$$

4D volume

Factorizing path integral a la [Weinberg]

$$\tilde{Z}_{\Delta n}(\Omega = \Omega_1 + \Omega_2) = \int_{\Delta n} \mathcal{D}\phi e^{-S_{\Omega_1+\Omega_2}[\phi]} = \sum_{\Delta n_1} \int_{\Delta n_1} \mathcal{D}\phi e^{-S_{\Omega_1}[\phi]} \int_{\Delta n - \Delta n_1} \mathcal{D}\phi e^{-S_{\Omega_2}[\phi]}$$

$$\tilde{Z}_{\Delta n}(\Omega = \Omega_1 + \Omega_2) = \sum_{\Delta n_1=-\infty}^{\infty} \tilde{Z}_{\Delta n_1}(\Omega_1)\tilde{Z}_{\Delta n-\Delta n_1}(\Omega_2)$$

Cluster decomposition

Want to solve this infinite number of relations by following these steps:

- Factorize all complex phases and obtain a set of relations for real functions
- Find a suitable Ansatz
- Assuming analiticity in Ω, we can construct the full function by computing all the derivatives at $\Omega = 0$.

Factorizing out all the complex phases

With θ factored out, additional complex phases can only come from α_i, i.e. from the integration over fermion fields

Fermionic path integrals give determinants of massive Euclidean Dirac operator

$$\int \mathcal{D}\psi\mathcal{D}\bar{\psi}e^{-\bar{\psi}(\not{D}+\mathfrak{m}P_R+\mathfrak{m}^*P_L)\psi} \propto \det(-\not{D}-\mathfrak{m}P_R-\mathfrak{m}^*P_L)$$

- The eigenvalues of $\det(-\not{D}-\mathfrak{m}P_R-\mathfrak{m}^*P_L)$ can be constructed in terms of those of $\not{D}$

Factorizing out all the complex phases

Nonzero eigenvalues of $\not{D}$ come in pairs that differ in sign

$$\not{D}\varphi^\lambda = \lambda\varphi^\lambda \to \not{D}(\gamma_5\varphi^\lambda) = -\lambda(\gamma_5\varphi^\lambda)$$

This leads to pairs of eigenvalues of $\det(-\not{D} - \mathfrak{m}P_R - \mathfrak{m}^* P_L)$

$$\mathfrak{m} \equiv m_{\mathrm{R}} + \mathrm{i}\gamma^5 m_{\mathrm{I}}$$

$$\begin{pmatrix} \not{D} + m_{\mathrm{R}} + \mathrm{i}\gamma^5 m_{\mathrm{I}} & 0 \\ 0 & \not{D} + m_{\mathrm{R}} + \mathrm{i}\gamma^5 m_{\mathrm{I}} \end{pmatrix} \begin{pmatrix} \varphi^\lambda \\ \gamma^5\varphi^\lambda \end{pmatrix} = \begin{pmatrix} \lambda + m_{\mathrm{R}} & \mathrm{i}m_{\mathrm{I}} \\ \mathrm{i}m_{\mathrm{I}} & -\lambda + m_{\mathrm{R}} \end{pmatrix} \begin{pmatrix} \varphi^\lambda \\ \gamma^5\varphi^\lambda \end{pmatrix}.$$

The matrix has eigenvalues $\xi(\lambda), \xi^*(\lambda)$ with $\xi(\lambda)\xi^*(\lambda) = |\mathfrak{m}|^2 + |\lambda|^2$

- Modes associated with $\lambda \neq 0$ give a real contribution to $\det(-\not{D} - \mathfrak{m}P_R - \mathfrak{m}^* P_L)$

Factorizing out complex phases

Zero modes of $\not{D}$ contribute phases to $\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L)$

$$\not{D}\varphi^0 = 0, \quad \varphi^0 = P_{R/L}\varphi^0 \Rightarrow (\not{D} + \mathfrak{m} P_R + \mathfrak{m}^* P_L)\varphi^0 = |\mathfrak{m}| e^{\pm i\alpha} P_{R/L}\varphi^0$$

Total phase of $\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L)$ follows from index theorem

$$\begin{aligned}
&\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L) \\
&= (-\mathrm{e}^{\mathrm{i}\alpha})^{\#(\text{r.h. zero modes}) - \#(\text{l.h. zero modes})} |\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L)| \\
&= (-e^{\mathrm{i}\alpha})^{\Delta n} |\det(-\not{D} - \mathfrak{m} P_R - \mathfrak{m}^* P_L)|
\end{aligned}$$

Factorizing out complex phases

Finally considering all fermion flavours, and defining $\bar{\alpha} \equiv \sum_i \alpha_i$

$$\tilde{Z}_{\Delta n}(\Omega) = e^{i\Delta n \bar{\alpha}} g_{\Delta n}(\Omega) \Rightarrow g_{\Delta n}(\Omega_1 + \Omega_2) = \sum_{\Delta n_1 = -\infty}^{\infty} g_{\Delta n_1}(\Omega_1) g_{\Delta n - \Delta n_1}(\Omega_2)$$

Real

Parity properties

$$\Delta n = \int d^4x \frac{g^2}{64\pi^2} \epsilon_{mnrs} F^a_{mn} F^a_{rs}$$ changes sign under parity

the real functions $g_{\Delta n}(\Omega)$ are insensitive to CP-odd phases from fermion masses

$$g_{-\Delta n}(\Omega) = g_{\Delta n}(\Omega)$$

Towards an Ansatz

$$g_{\Delta n}(\Omega_1 + \Omega_2) = \sum_{\Delta n_1 = -\infty}^{\infty} g_{\Delta n_1}(\Omega_1) g_{\Delta n - \Delta n_1}(\Omega_2)$$

$\Omega_2, = 0$ → $$g_{\Delta n}(\Omega_1) = \sum_{\Delta n_1 = -\infty}^{\infty} g_{\Delta n_1}(\Omega_1) g_{\Delta n - \Delta n_1}(0)$$

→ $$g_{\Delta n}(0) = \delta_{\Delta n, 0}$$

This and the previous parity considerations motivate **Ansatz**

$$g_{\Delta n}(\Omega) = g_{|\Delta n|}(\Omega) = \Omega^{|\Delta n|} f_{|\Delta n|}(\Omega^2), \quad f_{|\Delta n|}(0) \neq 0, \quad f(0) = 1$$

Taking derivatives

$$g_{\Delta n}(\Omega) = \Omega^{|\Delta n|} f_{|\Delta n|}(\Omega^2)$$

$$g'_{\Delta n}(\Omega) = |\Delta n|\, \Omega^{|\Delta n|-1} f_{|\Delta n|}(\Omega^2) + 2\,\Omega^{|\Delta n|+1} f'_{|\Delta n|}(\Omega^2)$$ → $$g'_{\Delta n}(0) = \delta_{|\Delta n| 1} f_1(0).$$

Taking derivatives of cluster relations w.r.t. Ω_1

$$g'_{\Delta n}(\Omega_1 + \Omega_2) = \sum_{\Delta n_1 = -\infty}^{\infty} g'_{\Delta n_1}(\Omega_1) g_{\Delta n - \Delta n_1}(\Omega_2).$$

$\Omega_1 = 0$ → $$g'_{\Delta n}(\Omega_2) = \sum_{\Delta n_1 = -\infty}^{\infty} g'_{\Delta n_1}(0) g_{\Delta n - \Delta n_1}(\Omega_2).$$

$$g'_{\Delta n}(\Omega_2) = f_1(0)\big(g_{\Delta n+1}(\Omega_2) + g_{\Delta n-1}(\Omega_2)\big).$$

Recursive solutions for derivatives

$$g'_{\Delta n}(\Omega_2) = f_1(0)(\, g_{\Delta n+1}(\Omega_2) + g_{\Delta n-1}(\Omega_2)).$$

Taking derivatives of the above expression w.r.t. Ω_2:

$$\frac{d^m g_{\Delta n}}{d\Omega^m} \text{ related to } \frac{d^{m-1} g_{\Delta m}}{d\Omega^{m-1}}$$

Proceeding recursively, we can relate $\frac{d^n g_{\Delta n}}{d\Omega^n}$ to $g_{\Delta n}(\Omega)$

$$\frac{d^m}{d\Omega^m} g_{\Delta n}(\Omega) = (f_1(0))^m \sum_{k=0}^{m} \binom{m}{k} g_{\Delta n-m+2k}(\Omega).$$

Series expansion around the origin

Setting $\Omega_2 = 0$ and remembering $g_{\Delta n}(0) = \delta_{\Delta n,0}$

$$\frac{d^m}{d\Omega^m} g_{\Delta n}(0) = \begin{cases} (f_1(0))^m \begin{pmatrix} m \\ \frac{m-\Delta n}{2} \end{pmatrix}, & \text{if} \quad m - \Delta n = 0, 2, 4, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Taylor expansion

$$g_{\Delta n}(\Omega) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{d\Omega^m} g_{\Delta n}(0)\, \Omega^m = \sum_{k=0}^{\infty} \frac{1}{(|\Delta n| + 2k)!} (f_1(0))^{|\Delta n|+2k} \begin{pmatrix} |\Delta n| + 2k \\ k \end{pmatrix} \Omega^{|\Delta n|+2k}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!(|\Delta n| + k)!} \left(\frac{2 f_1(0)\, \Omega}{2} \right)^{|\Delta n|+2k} = I_{\Delta n}(2 f_1(0) \Omega).$$

Final result of partition function

There is a unique solution with a single real parameter $f_1(0) \equiv \beta$

$$f_{\Delta n}(\Omega) = I_{\Delta n}(2\beta\Omega)$$

$$Z_{\Delta n} = \mathrm{e}^{\mathrm{i}\Delta n(\theta+\bar{\alpha})} I_{\Delta n}(2\beta\Omega)$$

Matches results of [Leutweyler & Smilga] achieved in a different way!

Making dependence on complex masses explicit:

$$Z_{\Delta n}(\Omega) = e^{\mathrm{i}\Delta n(\theta - \mathrm{i}/2\sum_j \log(\mathfrak{m}_j/\mathfrak{m}_j^*))} I_{\Delta n}(2\beta(\mathfrak{m}_k\mathfrak{m}_k^*)\,\Omega)$$

Mass dependence and correlators

Taking derivatives with respect to $\mathfrak{m}$, $\mathfrak{m}^*$ gives averaged integrated correlators

Spurion chiral charge -2

+2Nf -2N_f -2

$$\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L \psi_i\rangle_{\Delta n} = -\,e^{\mathrm{i}\Delta n(\theta+\bar{\alpha})}\left(\frac{\beta}{2\mathfrak{m}_i^*}\left(I_{\Delta n+1}(2\beta\Omega) - I_{\Delta n-1}(2\beta\Omega)\right)\right.$$
$$\left. + \mathfrak{m}_i\left(I_{\Delta n+1}(2\beta\Omega) + I_{\Delta n-1}(2\beta\Omega)\right)\frac{\partial}{\partial(\mathfrak{m}_i\mathfrak{m}_i^*)}\,\beta(\mathfrak{m}_k\mathfrak{m}_k^*)\right)$$

-2

Summing over topological sectors

-2 -2

$$\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L \psi_i\rangle = \lim_{N\to\infty}\lim_{VT\to\infty}\frac{\sum_{\Delta n=-N}^{N}\frac{1}{VT}\int d^4x\,\langle\bar{\psi}_i P_L\psi_i\rangle_{\Delta n}}{\sum_{\Delta m=-N}^{N} Z_{\Delta m}} = 2\mathfrak{m}_i\,\partial_{\mathfrak{m}_i\mathfrak{m}_i^*}\beta(\mathfrak{m}_k\mathfrak{m}_k^*),$$

Topological classification only enforced in infinite volume, which fixes ordering

Result due to all Bessel functions having a common asymptotic behaviour

$$I_{\Delta n}(2\beta\Omega) = I_0(2\beta\Omega)(1+\mathcal{O}((\beta\Omega)^{-1}))$$

General correlators

Taking **higher-order derivatives** w.r.t. $\mathfrak{m}_i, \mathfrak{m}_i^*$, yields general integrated correlators

-2N$_f$ -2N$_f$

$$\langle \left(\prod_{j=1}^{N_f} \int d^4 x_j (\bar\psi_j P_L \psi_j) \right) \rangle = e^{\mathrm{i} \sum_j \alpha_j} f(\mathfrak{m}_k^* \mathfrak{m}_k)$$

Reproduced by the following **effective interaction** (after factoring out ordinary props/)

-2N$_f$ +2N$_f$

$$\mathcal{L}_{\rm eff} \supset e^{\mathrm{i} \sum_j \alpha_j} \Gamma \prod_j \bar\psi_j P_R \psi_j$$

To be **matched** to **chiral Lagrangian** with $U \sim \bar\psi P_R \psi$

$$\xi = -\sum_j \alpha_j$$

$$\mathcal{L}_{\rm pion} \supset |b| e^{-i\xi} f_\pi^4 \det U$$

Consequences for d_n and CP violation

$$d_n \propto \xi + \alpha_u + \alpha_d + \alpha_s = 0$$

- All phases of all fermion correlators are fixed by the α_i:
- θ disappears
- All phases can be eliminated with chiral field redefinitions

No CP violation in fermion correlators

Where we did depart from the usual results?

- Only in the ordering of limits!
- Opposite order of limits yields traditional picture of CP-violation

Reduction to finite volume

Finite volumes in an infinite spacetime

- To project into the vacuum for finite Ω requires knowing vacuum wave functional

- We aim to derive an effective finite-volume description starting from an infinite-volume path integral guaranteed to capture the vacuum state

- The finite volume description can can help make contact with lattice computations

Finite volumes in an infinite spacetime

Assume local operator $\mathcal{O}_1$ with support in finite spacetime volume Ω_1

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n=-\infty}^{\infty} e^{i\Delta n\theta} \int_{\Delta n} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_\Omega[\phi]}}{\sum_{\Delta n=-\infty}^{\infty} e^{i\Delta n\theta} \int_{\Delta n} \mathcal{D}\phi\, \mathrm{e}^{-S_\Omega[\phi]}}$$

$$= \frac{\sum_{\Delta n=-\infty}^{\infty} \sum_{\Delta n_1=-\infty}^{\infty} e^{i\Delta n\theta} \int_{\Delta n_1} \mathcal{D}\phi\, \mathcal{O}_1\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2=\Delta n-\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n=-\infty}^{\infty} \sum_{\Delta n_1=-\infty}^{\infty} e^{i\Delta n\theta} \int_{\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2=\Delta n-\Delta n_1} \mathcal{D}\phi\, \mathrm{e}^{-S_{\Omega_2}[\phi]}} .$$

[Note: Integer Δn_1 is only an approximation, carried out in a surface kept finite, with reduced impact in full path integral.]

Finite volumes in an infinite spacetime

Path integrations over Ω_2 give just the partition functions we calculated before

In the infinite volume limit the Bessel functions tend to common value and dependence on Δn factorizes out and cancels:

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, (-1)^{-N_f \Delta n_1} \mathrm{e}^{-\mathrm{i}\,\alpha \Delta n_1} \mathcal{O}_1 \, \mathrm{e}^{-S_{\Omega_1}[\phi]}}{\sum_{\Delta n_1=-\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, (-1)^{-N_f \Delta n_1} \mathrm{e}^{-\mathrm{i}\,\alpha \Delta n_1} \mathrm{e}^{-S_{\Omega_1}[\phi]}}\,.$$

We recover a path integration over a finite volume, without θ dependence

Extra phases precisely cancel those from fermion determinants in Ω_1

No interferences between different topological sectors: CP is conserved

Conclusions

QCD with an arbitrary θ **does not predict CP violation**, as long as the sum over topological sectors is performed at **infinite volume**

This **ordering of limits** is the correct one because the topological classification is only enforced for an infinite volume

For **local observables** one can recover CP-conserving expectation values from **path integrals in a finite subvolume without θ dependence**

In B. Garbrecht’s talk:

- Same conclusions using instantons
- Further arguments supporting CP conservation based on canonical quantization

Thank you!

Additional material

Baluni's CP-violating effective Lagrangian

Baluni's CP-violating Lagrangian (used by [Crewther et al]) is based on searching for field redefinitions that minimize the QCD mass term

$$\mathcal{L}_M(U_{R,L}) = \bar{\psi} U_R^\dagger M U_L \psi_L + \text{h.c.}, \quad U_{R,L} \in SU_{R,L}(3)$$

$$\langle 0|\delta\mathcal{L}|0\rangle = \min_{U_{R,L}} \langle 0|\mathcal{L}_M(U_{R,L})|0\rangle$$

However, there is an extra assumption: that the phase of the fermion condensate is aligned with θ

$$\langle \bar{\psi}_R \psi_L \rangle = \Delta e^{\mathrm{i}c\theta} \mathbb{I}$$

This assumption does not hold for the chiral Lagrangian with $\xi = -\alpha$

The η' mass

Chiral Lagrangian with alignment in the phases of mass terms and anomalous terms still predicts a **nonzero value of the η' mass**

$$\mathcal{L} = f_\pi^2 \mathrm{Tr}\partial_\mu U \partial^\mu U^\dagger + a f_\pi^3 \mathrm{Tr} M U + |b| e^{i \arg \det M} f_\pi^4 \det U + \mathrm{h.c.}$$

$$m_{\eta'}^2 = 8|b| f_\pi^2$$

Can be seen to be **proportional** to the **topological susceptibility** over **finite volumes** of the **pure gauge theory**, in line with [Witten, Di Vecchia & Veneziano]

Classic arguments linking topological susceptibility to CP violation ([Shifman et al]) rely on analytic expansions in θ which **don't apply** with our limiting procedure

***Z* becomes non-analytic in θ.** This possibility has been mentioned by [Witten]

[Witten, Nucl. Phys. B 156 (1979)]

the physics is of order e^{-N}, contrary to the basic assumptions of this paper, or else the physics is non-analytic as a function of θ, In the latter case, which is quite plausible, the singularities would probably be at $\theta = \pm\pi$, as Coleman found for the massive Schwinger model [10]. It is also quite plausible that θ is not really an angular variable.)

To write a formal expression for $d^2E/d\theta^2$, let us think of the path integral formulation of the theory:

$$Z = \int dA_\mu \exp i \int \mathrm{Tr}\left[-\tfrac{1}{4}F_{\mu\nu} + \frac{g^2\theta}{16\pi^2 N} F_{\mu\nu}\tilde{F}_{\mu\nu}\right]. \tag{5}$$

Partition function and analiticity

Usual partition function is analytic in θ

$$Z_{\text{usual}} = \lim_{VT\to\infty} \lim_{\substack{N\to\infty \\ N\in N}} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = \mathrm{e}^{2\mathrm{i}\kappa_{N_f} VT\cos(\bar{\alpha}+\theta+N_f\pi)}$$

θ-dependence of observables (giving CP violation) usually relies on θ expansion. e.g.

$$\frac{\langle \Delta n \rangle}{\Omega} = \mathrm{i}\,(\theta-\theta_0)\left.\frac{\langle \Delta n^2 \rangle}{\Omega}\right|_{\theta_0} + \mathcal{O}(\theta-\theta_0)^2$$

topological susceptibility

[Shifman et al]

In our limiting procedure the former is not valid, as Z becomes nonanalytic in θ

$$Z = \lim_{\substack{N\to\infty \\ N\in N}} \lim_{VT\to\infty} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = I_0(2\mathrm{i}\kappa_{N_f} VT) \lim_{\substack{N\to\infty \\ N\in N}} \sum_{|\Delta n|\leq N} \mathrm{e}^{\mathrm{i}\Delta n(\bar{\alpha}+\theta+N_f\pi)}$$

θ drops out from observables, there is no CP violation

The QCD angle from the vacuum state

Hamiltonian is zero for pure gauge transformations, with integer n_{CS}: Expect degenerate classical pre-vacua $|n_{\rm CS}\rangle \equiv |n\rangle$

If the true vacuum $|\omega\rangle$ were to be a linear combination of the classical prevacua

$$|\omega\rangle = \sum_n f(n)|n\rangle$$

Demanding invariance up to a phase under gauge transformations in the Δn class

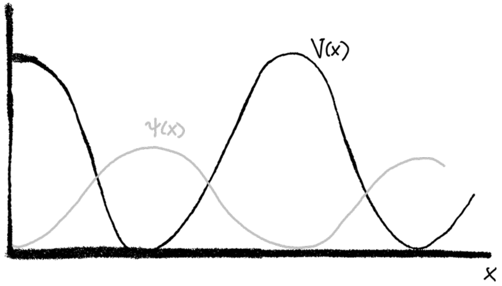
$$U_{\Delta n}|\omega\rangle = \sum_n f(n)|n+\Delta n\rangle = e^{i\Delta n\theta}|\omega\rangle \Rightarrow f(n) = e^{-in\theta}$$

$$Z(\theta) = \langle\omega|e^{-HT}|\omega\rangle = \sum_m\sum_n \langle m|e^{-HT}e^{i\theta(m-n)}|n\rangle = \mathcal{N}\sum_{\Delta n}\langle n+\Delta n|e^{-HT}e^{i\theta\Delta n}|n\rangle$$

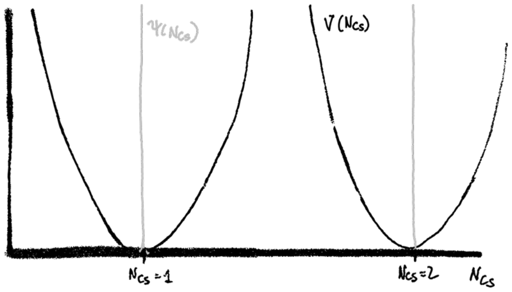
$$= \mathcal{N}\sum_{\Delta n}\int_{\Delta n}\mathcal{D}\phi\, e^{-S_\theta+\dots}$$

Can one use the “θ vacuum” at finite volume?

Bloch wave function in QM:



vs θ vacuum having support only on classical vacua



Too naive! Have to use path integral in infinite 4D volume to project into vacuum

Dvali's footnote

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[2] The 3-form language of [14] clarifies the claim of [24] that by changing the order of limits in ordinary instanton calculation, one ends up with $\vartheta = 0$. In this approach one performs calculation in the finite volume and then takes it to infinity. In 3-form language the meaning of this is rather transparent. The finite volume is equivalent of introducing an infrared cutoff in form of a shift of the massless pole in (28) away from zero. This effectively gives a small mass to the 3-form. For any non-zero value of the cutoff, the unique vacuum is $E_0 = 0$ which is equivalent to $\vartheta = 0$. Other states $E \neq 0$ (corresponding to $\vartheta \neq 0$) have finite lifetimes which tend to infinity when cutoff is taken to zero. In this way the $\vartheta \neq 0$ vacua are of course present but one is constrained to $\vartheta = 0$ by the prescription of the calculation. Thus, changing the order of limits by no means eliminates the ϑ-vacua. As usual, when taking the limit properly, one must keep track of states that become stable in that limit. These are the states with $\vartheta \neq 0$ ($E \neq 0$), which become the valid vacua in the infinite volume limit. The effect is in certain sense equivalent to introducing an auxiliary axion and then decoupling it.

Dvali's 3-form formalism

[Dvali] has the following line of reasoning from which he concludes that QCD violates CP

Nonzero topological susceptibility at zero momentum / full volume → Massless pole in CS current-current correlators → EFT for massless 3-form with CP violating vacua

With **our ordering of limits**, we have that the **topological susceptibility** is:

zero at zero momentum/full volume

nonzero at finite volume/nonzero momentum

▶ **Dvali's first premise is violated and his argument does not apply**

Dvali's criticism

[Dvali] argues that in a calculation at finite volume which is then sent to infinity,
CP violation can't be captured because the infrared regulation gives a mass to the 3 form.

We make the following observations:

[`t Hooft]'s original calculations (at finite volume, taken to ∞ in the end) lead to CP violation for arbitrary ϑ, in conflict with Dvali's argument

If finite volume is problematic, more reason to take the infinite volume limit as soon as possible, as we do, leading to no CP violation for arbitrary ϑ

Dvali's formalism has no explicit/direct link to UV ϑ parameter

Dvali's critique of finite volumes can be turned against his own construction, as it is based on assuming nonzero topological susceptibility, while the only nonperturbative evidence for it comes from lattice results at finite volume