Hadronic Light-by-Light contribution to a_μ using $N_f = 2 + 1 + 1$ twisted mass ensembles

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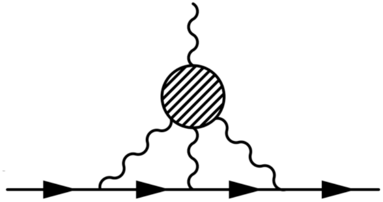
27-29 August 2025

Computational setup

Our lattice calculation of a_μ^{HLbL}:

- $N_f = 2 + 1 + 1$ dynamical flavors: $u = d = \ell$ (isospin symmetry) $+\ s$ and c quarks
- Quark masses tuned to the physical point
- Twisted mass fermions at maximal twist $\implies O(a)$-improvement on lattice artifacts
- Continuum extrapolation: $a[\mathrm{fm}] = 0.04892 - 0.07948$
- Volumes: $L[\mathrm{fm}] = 5.09 - 5.46$



Ensemble	a^{iso} [fm]	L [fm]
B64	0.07948(11)	5.09
C80	0.06819(14)	5.46
D96	0.056850(90)	5.46
E112	0.04892(11)	5.48

Summary of the results

Strange contribution

	$\bar{\mathcal{L}}^{(3)}$	$\bar{\mathcal{L}}^{(\Lambda=0.4)}$
$a_\mu^{\text{HLbL, conn}}(s)$	2.37(45)	2.57(39)

Charm contribution

	$\bar{\mathcal{L}}^{(3)}$	$\bar{\mathcal{L}}^{(\Lambda=0.4)}$
$a_\mu^{\text{HLbL, conn}}(c)$	1.75(17)	1.70(18)

Light $(u+d)$ contribution, B64 ensemble, $\bar{\mathcal{L}}^{(\Lambda=0.4)}$

$$a_\mu^{\text{HLbL, conn}}(\ell) = 124(12)\,, \quad a_\mu^{\text{HLbL,(2+2)}}(\ell) = -50.9(7.9)$$

Master formula

The HLbL contribution is the convolution of a QED kernel with the 4-point polarization function [1]:

$$\boxed{a_\mu^{\text{HLbL}} = \frac{m_\mu e^6}{3} \int d^4x \int d^4y \, \bar{\mathcal{L}}_{[\rho,\sigma]\mu\nu\lambda}(x,y) \, i\Pi_{\rho\mu\nu\lambda\sigma}(x,y) \,,}$$

where:

$$i\Pi_{\rho\mu\nu\lambda\sigma}(x,y) = \int d^4z \, (-z_\rho) \, \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle$$

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Features of the integrand

- $\mathcal{L} \cdot \Pi$ is a Lorentz scalar. We can fix a $\hat{y}$ direction and integrate over $|y|$.
- The QED kernel is known. From the lattice we extract $i\Pi$.

$|y|$ integrand

Fixing the frame of reference

When integrating over x and y, what matters are $|x|$, $|y|$ and $c_\beta = \frac{x \cdot y}{|x||y|}$.

$\implies$ Direction $\hat{y}$ fixed, angular integration over Ω_y done exactly.

$$a_\mu^{\mathrm{HLbL}} = \boxed{\int_0^\infty d|y|\, f(|y|)} = \frac{m_\mu e^6}{3} 2\pi^2 |y|^3 \int_x \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) \left[\int_z (-z_\rho) i\Pi_{\mu\nu\sigma\lambda}(x,y,z) \right]$$

Sum over topologies

$$f(|y|) = \sum_{\mathrm{Topology}} f^{(\mathrm{Topology})}(|y|)$$

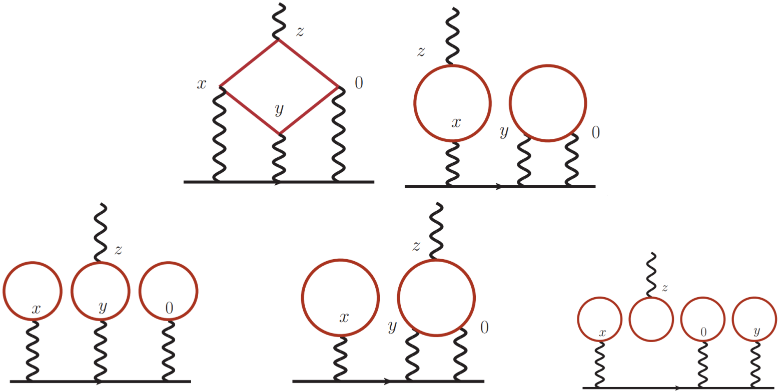
Each topology is classified by the type of Wick contractions.

Topologies

Lattice calculation: Π is found in terms of quark fields:

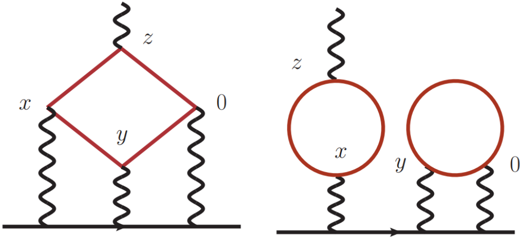
$$j_\mu(x) = \frac{2}{3}(\overline{u}\gamma_\mu u)(x) - \frac{1}{3}(\overline{d}\gamma_\mu d)(x) - \frac{1}{3}(\overline{s}\gamma_\mu s)(x) + \frac{2}{3}(\overline{c}\gamma_\mu c)(x) \tag{1}$$

The 4-point function results in 5 topologies of Wick contractions [2]:



Topologies hierarchy

We restrict to the leading contributions: **fully-connected** and $(\mathbf{2}+\mathbf{2})$.



Argument (large N_c limit)

- At the $\mathrm{SU}_f(3)$ symmetric point, the last 3 topologies do not contribute.
- In the large N_c limit they are suppressed [2] + numerical evidence from the lattice [3] (RBC/UKQCD 2016)

Wick contractions

On the lattice we compute the average over the gauge configurations of:

$$\langle j_\mu(x)\, j_\nu(y)\, j_\sigma(z)\, j_\lambda(0)\rangle = -2\,\mathrm{Re}\left\{ I^{(1)}_{\mu\nu\sigma\lambda}(x,y,z,0) + I^{(2)}_{\mu\nu\sigma\lambda}(x,y,z,0) + I^{(3)}_{\mu\nu\sigma\lambda}(x,y,z,0) \right\}\,, \tag{2}$$

where:

$$I^{(1)}_{\mu\nu\sigma\lambda}(x,y,z,w) = \mathrm{Tr}\left\{\gamma_\mu S(x,y)\gamma_\nu S(y,z)\gamma_\sigma S(z,w)\gamma_\lambda S(w,x)\right\} \tag{3}$$

$$I^{(2)}_{\mu\nu\sigma\lambda}(x,y,z,w) = I^{(1)}_{\mu\nu\lambda\sigma}(x,y,w,z) \tag{4}$$

$$I^{(3)}_{\mu\nu\sigma\lambda}(x,y,z,w) = I^{(1)}_{\lambda\nu\sigma\mu}(w,y,z,x) \tag{5}$$

Calculation trick

We use these symmetries, by changing the kernel $\rightarrow \sim (1/3)$ of the Dirac inversions.

Numerical inversion

$$D\psi = \eta \;\rightarrow\; \psi = S\eta$$

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Choice of the kernel

Kernel freedom

At $L \to \infty$ we can change the QED kernel by some irrelevant terms.

$$\bar{\mathcal{L}}^{(\Lambda)}_{[\rho,\sigma];\mu\nu\lambda}(x,y) = \bar{\mathcal{L}}_{[\rho,\sigma];\mu\nu\lambda}(x,y) - \frac{\partial}{\partial x_\mu}(x_\alpha e^{-\Lambda m^2 x^2/2})\bar{\mathcal{L}}_{[\rho,\sigma];\alpha\nu\lambda}(0,y) - \frac{\partial}{\partial y_\nu}(y_\alpha e^{-\Lambda m^2 y^2/2})\bar{\mathcal{L}}_{[\rho,\sigma];\mu\alpha\lambda}(x,0)$$

Advantage for lattice calculation

- Lattice data becomes noisier for increasing $|y|$
- At coarse lattice spacing we have a few points at short distances
- We choose a kernel that is not too long-ranged to improve the quality of the integral

Our choice of $\hat{y}$ and of the kernel

Checking: Discretization Effects

We check the consistency of the results computed along the directions:

- $y/a \propto (1,1,1,1)$: hyper-diagonal $\implies |y|/a = k \cdot 2$
- $y/a \propto (0,1,1,1)$: spatial diagonal $\implies |y|/a = k \cdot \sqrt{3}$

Checking: Finite Volume Effects

If Finite Volume Effects are negligible, we expect consistency between different kernels. We consider [1]:

- $\bar{\mathcal{L}}^{(\Lambda=0.4)}$
- $\bar{\mathcal{L}}^{(3)}$

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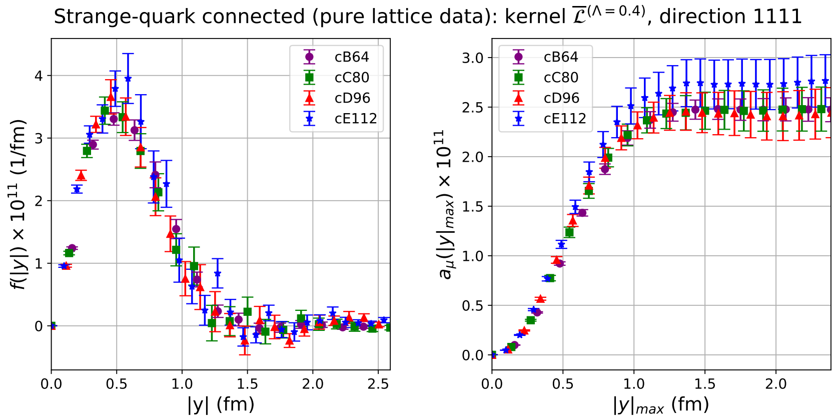
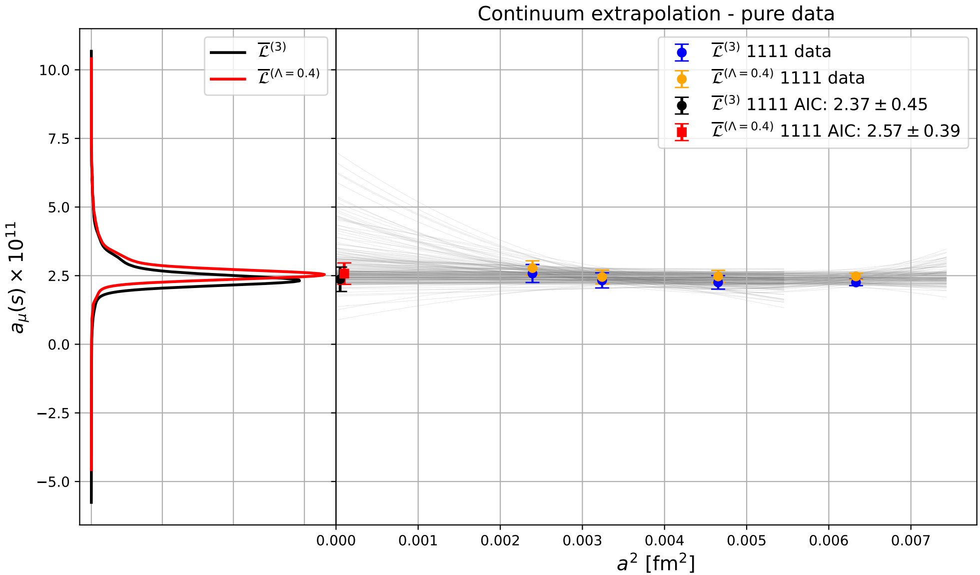
Strange contribution (connected) - $\mathcal{L}^{(\Lambda=0.4)}$



Strange contribution: continuum extrapolation



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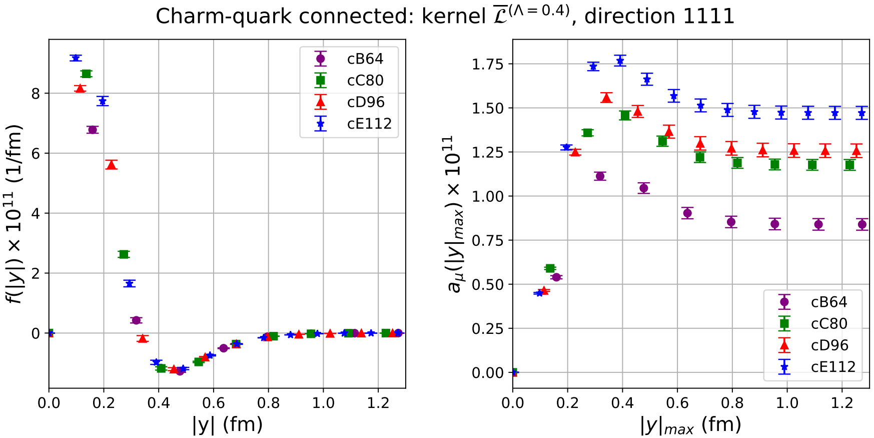
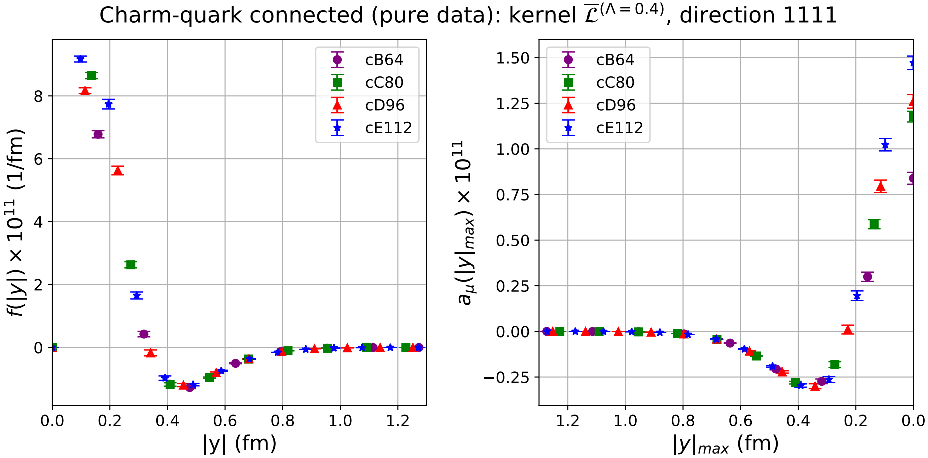
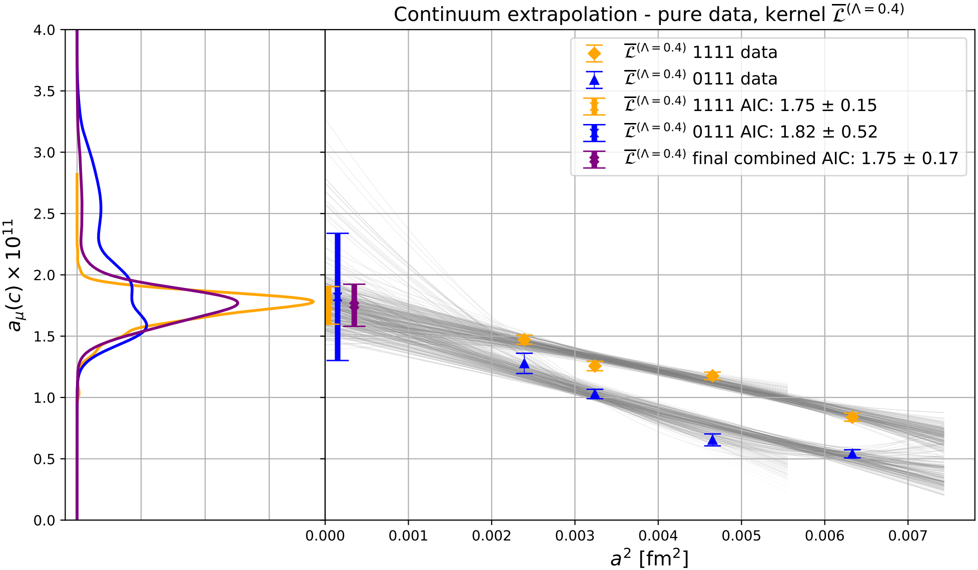
Charm contribution (connected) - $\mathcal{L}^{(\Lambda=0.4)}$



Charm contribution (connected) - $\mathcal{L}^{(\Lambda=0.4)}$: reverse integration



Charm contribution: continuum extrapolation



Charm quark: improving the lattice artifacts

Subtracting the quark loop contribution

We subtract the quark loop contribution as in Ref. [4]:

$$a_\mu^{\text{HLbL, lat}}(c) \to a_\mu^{\text{HLbL, lat}}(c) + \left[a_\mu^{\text{HLbL loop, cont}}(c) - a_\mu^{\text{HLbL loop},a\neq 0}(c)\right]$$

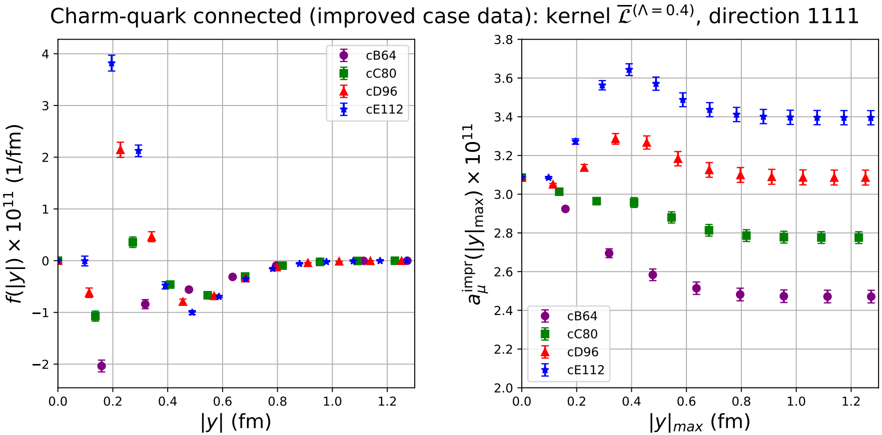
Setup

- Charm quark mass in the $\overline{\text{MS}}$ scheme: $\overline{m}_c(\overline{m}_c) = 1.27$ GeV.
- Continuum value found by adapting the calculation of τ lepton loop (see Ref. [5]).
- Lattice quark loop curve found by turning off the strong interactions: $U_\mu(x) = \mathbb{1}$.

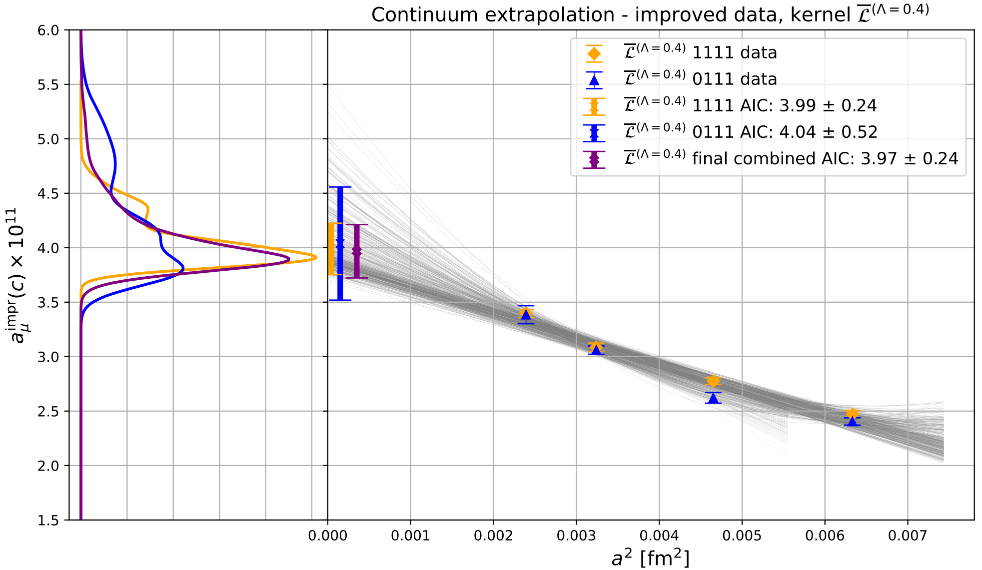
Final goal

Reduction of lattice artifacts → flatter continuum extrapolation

Charm quark: improved lattice artifacts, $\bar{\mathcal{L}}^{(\Lambda=0.4)}$



Charm quark: improved continuum extrapolation



Charm contribution: “improved” version

Final result

	$\bar{\mathcal{L}}^{(3)}$	$\bar{\mathcal{L}}^{(\Lambda=0.4)}$
$a_\mu^{\text{HLbL, conn, improved}}(c)$	3.88(25)	3.97(24)

Comments and open points

- Compatibility with Ref. [4]:

$$a_\mu^{(c),\text{Ref. [4]}} = 3.916(12)_{\text{stat}}(20)_{\text{syst}}(72)_{\text{cont}}(250)_{\text{kernel}}10^{-11}$$

- Continuum limit significantly different if we include the improvement
- Choice of renormalization scale $\overline{m}_c$: justification?

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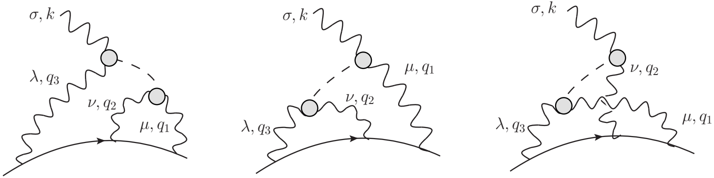
Light contribution and meson poles

Leading contribution at long distance: meson poles [6, 7]



Effective models

PQChPT: connection with Wick contractions [8]:

$$a_\mu^{\text{HLbL, conn.}}(|y|) \approx \frac{34}{9} a_\mu^{\text{HLbL},\pi^0}(|y|)$$

$$a_\mu^{\text{HLbL},(2+2)}(|y|) \approx -\frac{25}{9} a_\mu^{\text{HLbL},\pi^0}(|y|) + a_\mu^{\text{HLbL},\eta}(|y|) + a_\mu^{\text{HLbL},\eta'}(|y|)$$

Pole contribution (finite volume)

$$\Pi_{\mu\nu\sigma\lambda}(x,y,z) = \Pi^{(1)}_{\mu\nu\sigma\lambda}(x,y,z) + \Pi^{(2)}_{\mu\nu\sigma\lambda}(x,y,z) + \Pi^{(3)}_{\mu\nu\sigma\lambda}(x,y,z)\,,$$

where, e.g., (see Refs [4, 9]):

$$\Pi^{(1)}_{\mu\nu\sigma\lambda}(x,y,z) = -\int_{u,v} \widetilde{M}_{\mu\nu}(u,x,y) \left[\frac{e^{ipx}}{p^2+m_P^2}\right] \widetilde{M}_{\sigma\lambda}(v,z,0)\,,$$

and

$$\begin{aligned}\tilde{M}_{\mu\nu}(u,x,y) &= \int_{q_1,q_2} M_{\mu\nu}(q_1,q_2)\, e^{iq_1(x-u)} e^{iq_2(y-u)} \\ &= -\epsilon_{\mu\nu\alpha\beta}\,\partial_{x_\alpha}\partial_{y_\beta} \int_{q_1,q_2} \mathcal{F}_{P\gamma^*\gamma^*}(-q_1^2,-q_2^2)\, e^{iq_1(x-u)} e^{iq_2(y-u)}\,,\end{aligned} \tag{6}$$

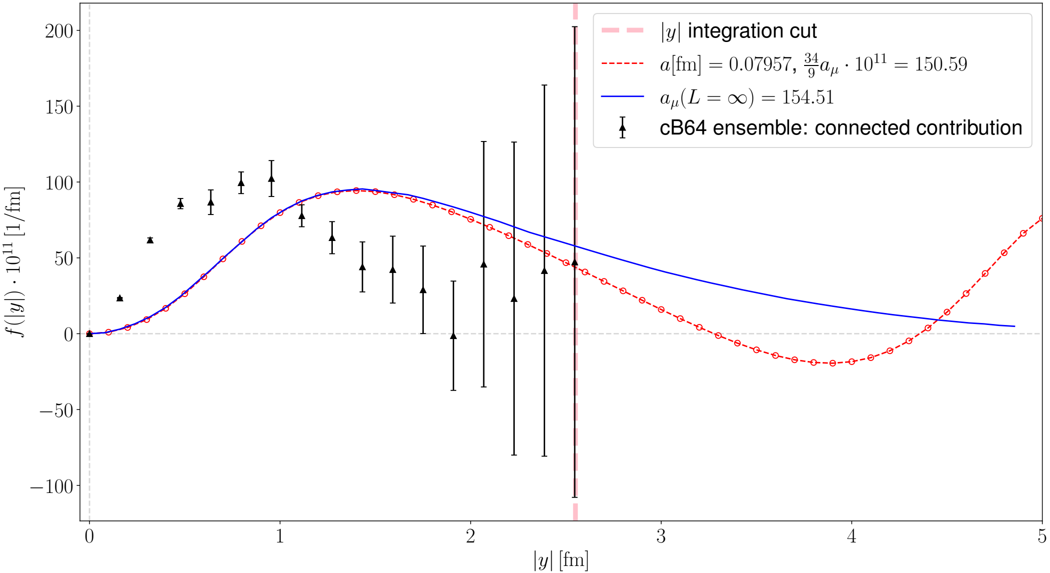
Pole contribution (VMD, infinite volume)

At finite volume we sum over lattice points explicitly.
In the $L \to \infty$ limit we match the known expression from the VMD model [1]:

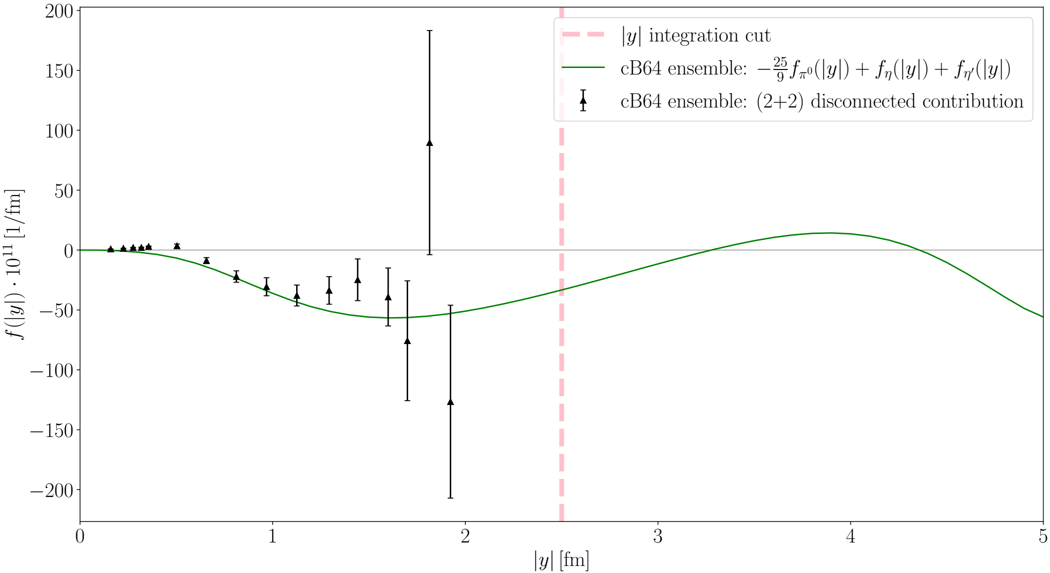
$$i\hat{\Pi}^{\mathrm{VMD}}_{\mu\nu\lambda\sigma}(x,y) = \frac{c_\pi^2}{M_V^2(M_V^2 - M_\pi^2)} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial y_\beta} \times$$
$$\left[\epsilon_{\mu\nu\alpha\beta}\epsilon_{\sigma\lambda\rho\gamma}\left(\frac{\partial}{\partial x_\gamma} + \frac{\partial}{\partial y_\gamma}\right) K_\pi(x,y) + \epsilon_{\mu\lambda\alpha\beta}\epsilon_{\nu\sigma\gamma\rho}\frac{\partial}{\partial y_\gamma} K_\pi(y-x,y)\right.$$
$$\left. + \epsilon_{\mu\sigma\alpha\rho}\epsilon_{\nu\lambda\beta\gamma}\frac{\partial}{\partial x_\gamma} K_\pi(x, x-y)\right] ,$$

$$K_\pi(x,y) = \int_u \left(G_{m_P}(u) - G_{m_V}(u)\right) G_{m_V}(x-u) G_{m_V}(y-u)$$

Pole contribution VS lattice data: connected contribution



Pole contribution VS lattice data: 2+2 disconnected



Treating the tail of the integrand

High noise-to-signal ration in the $|y|$ tail

- It seems we cannot use the π^0 pole
- *Workaround*: similarly to Ref. [8], we replace the tail with the ansatz:

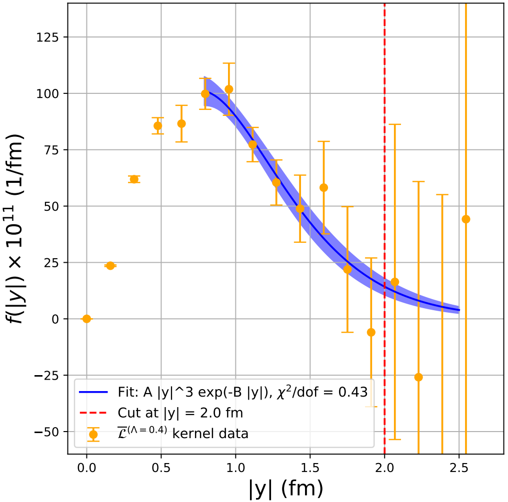
$$A|y|^3 e^{-B|y|} \tag{7}$$

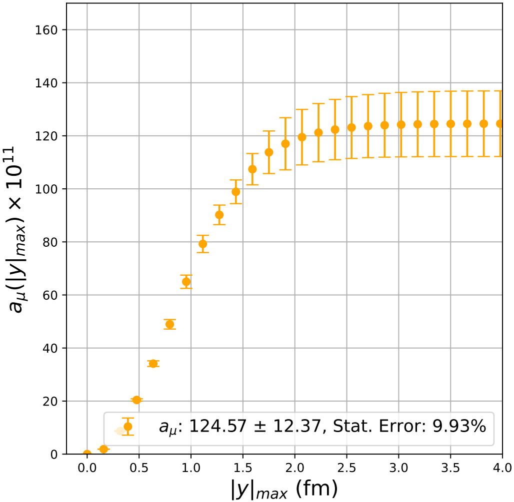
A, B are free parameters of a fit to the tail data.

Steps of the calculation

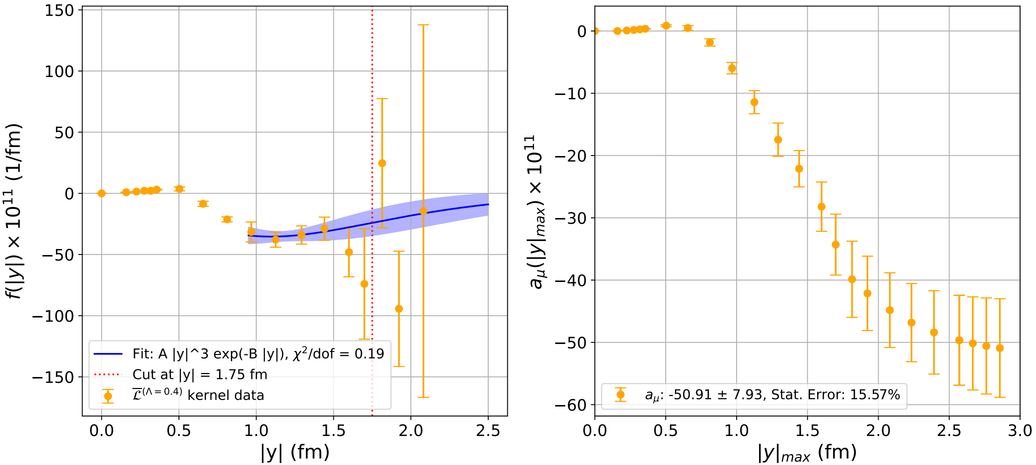
1. Choose the values $|y|_1$ and $|y|_2$ and fit the lattice data in between
2. Replace the lattice data with the ansatz for $|y| > |y|_2$

Light contribution (connected) - cB64 ensemble





Light contribution (2+2 disconnected) - cB64



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Conclusion and outlook

Contribution	Diagrams	N_{confs}	N_{src}	Result $(a \to 0)$
Strange	connected	~ 65	1	2.37(45), 2.57(39)
Charm	connected	~ 260	1	1.75(17), 1.70(18)
Light (cB64 ensemble)	connected	~ 780	$\|y\|$-dependent	124(12)
Light (cB64 ensemble)	$(2+2)$ disc.	~ 780	$\|y\|$-dependent	-50.9(7.9)

Open points

- Charm: quark loop and discretization improvement on the integrand.
- Light: pole contribution, matching the tail of the integrand.

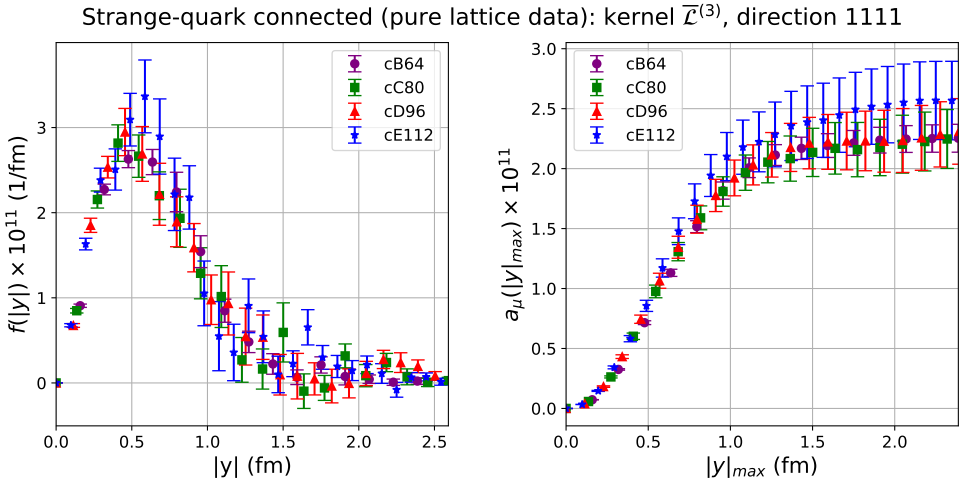
Future directions

- Strange and charm: nearing publication (stay tuned)
- Light: continuum limit
- Flavor-mixing disconnected contributions.

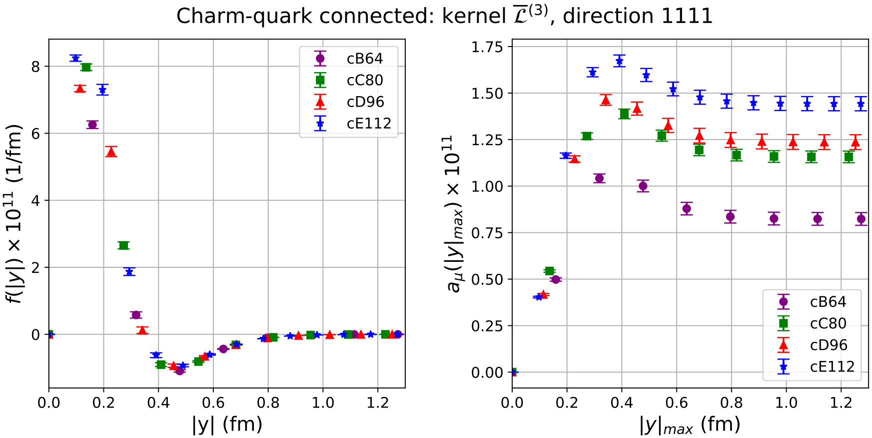
Thank you for the attention!

Backup slides

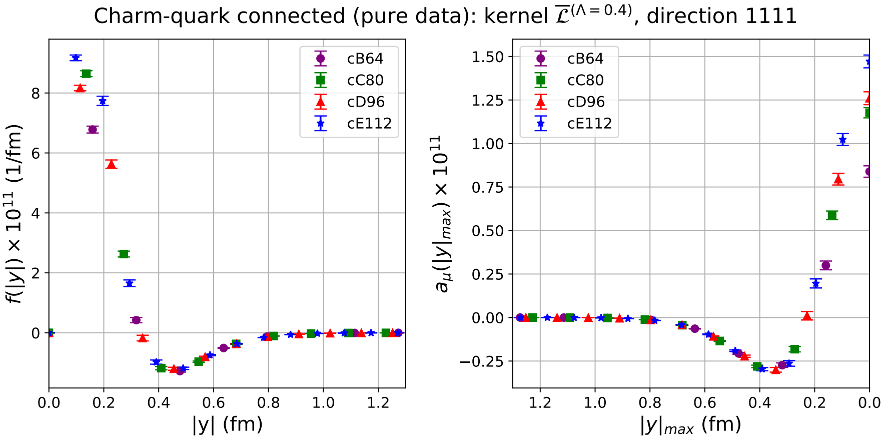
Strange contribution (connected) - $\mathcal{L}^{(3)}$



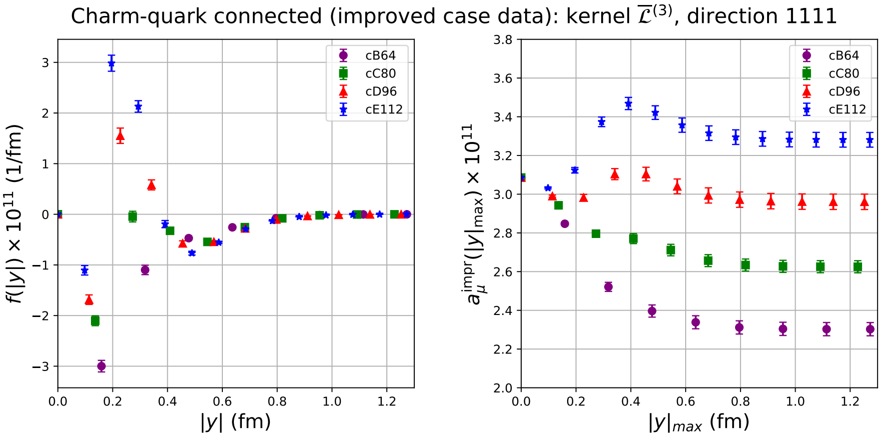
Charm contribution (connected) - $\mathcal{L}^{(3)}$



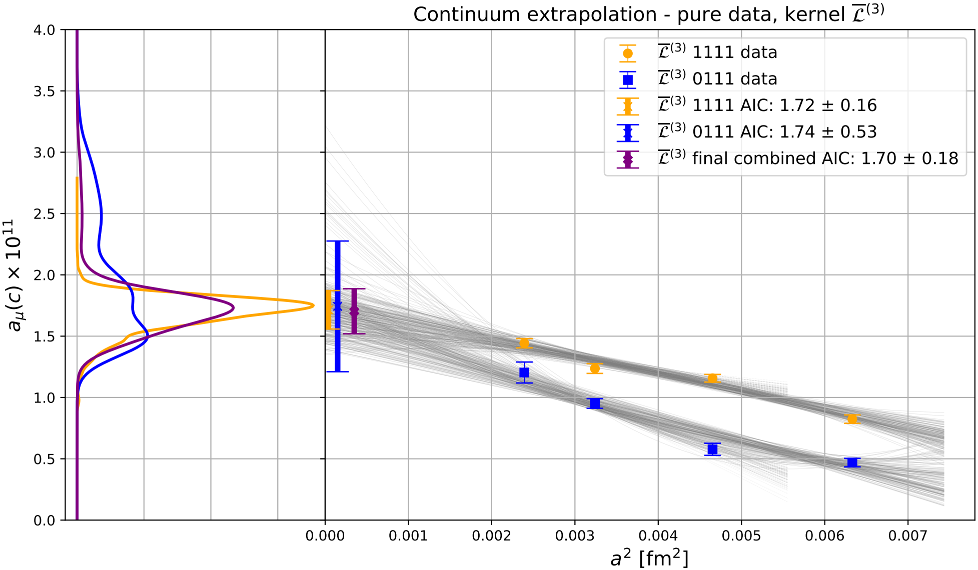
Charm contribution (connected) - $\mathcal{L}^{(3)}$: reverse integration



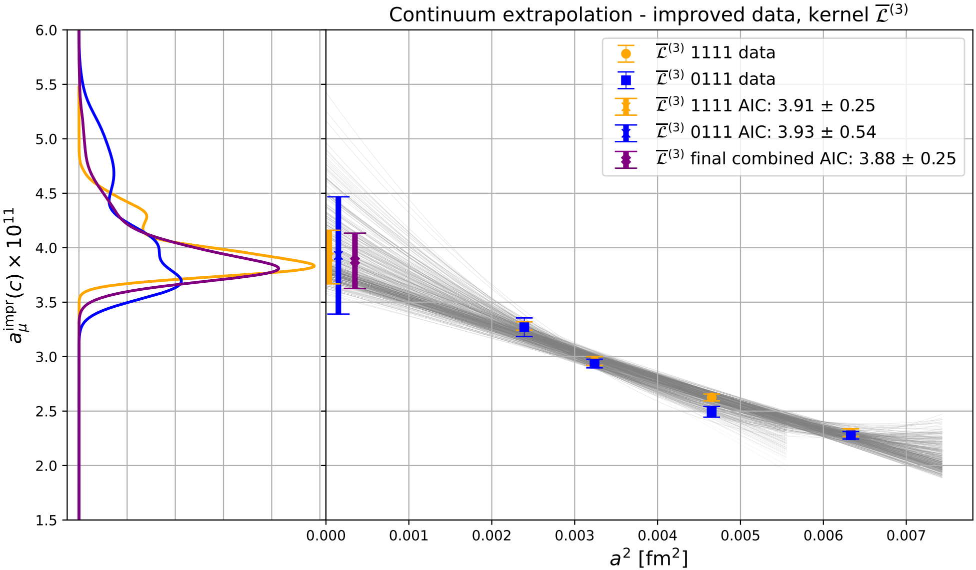
Charm quark: improved lattice artifacts, $\bar{\mathcal{L}}^{(3)}$



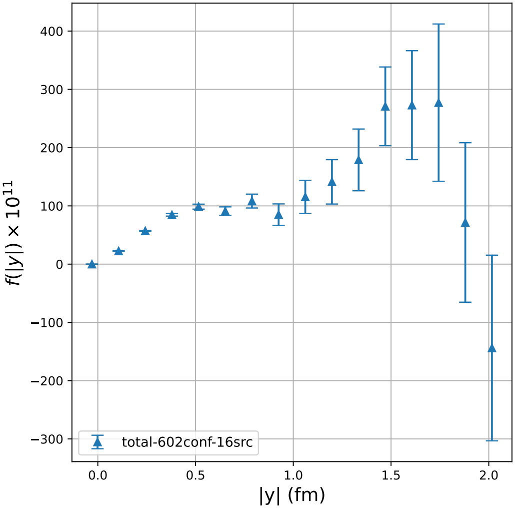
Charm contribution: continuum extrapolation

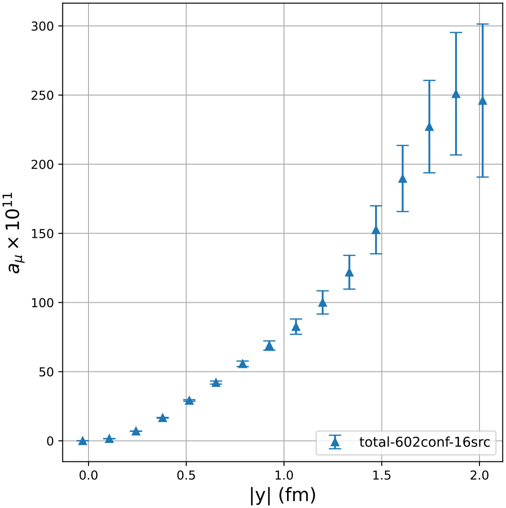


Charm quark: improved continuum extrapolation

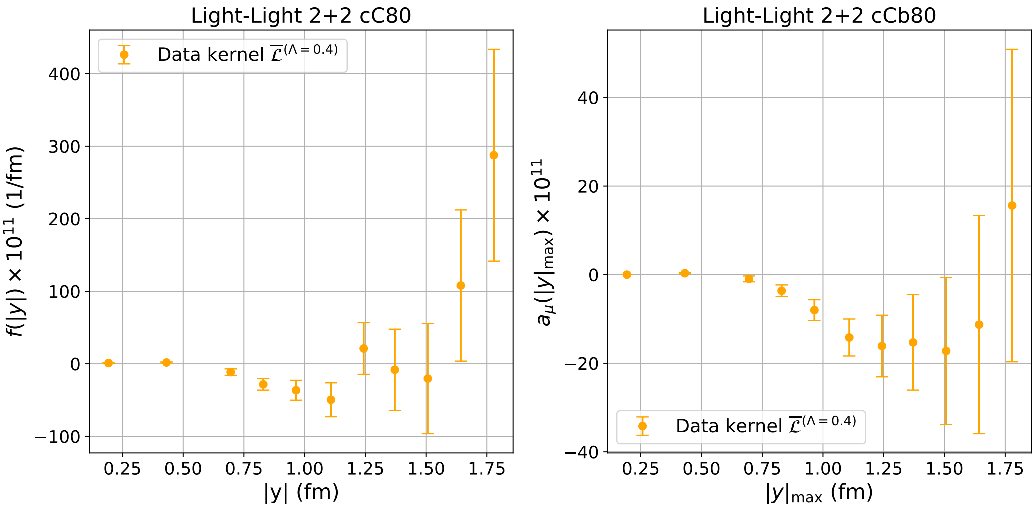


Light contribution (connected) - cC80 ensemble

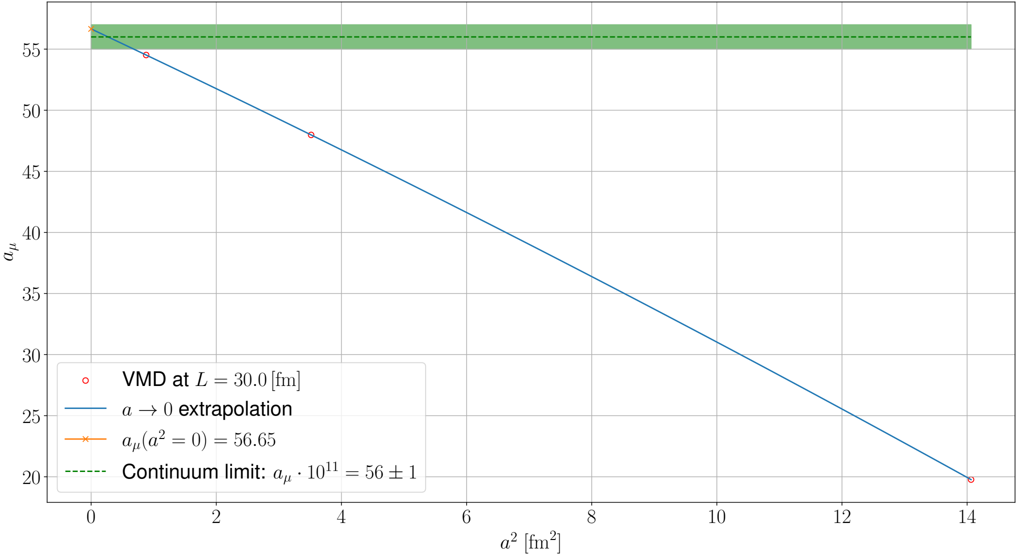




Light contribution (2+2 disconnected) - cC80



π^0-pole contribution (consistent continuum limit)



Pole contribution: other collaborations' findings

VMD and LMD

Good description of TFF at small momenta.
VMD and LMD in position space differ only at short distances.

Comparing with the literature

- Ref. [2] (Mainz 2020): $SU(3)_f$-symmetric point, $m_{\pi,K,\eta} \approx 410 - 420\text{MeV}$.
  $\pi^0 + \eta$ VMD integrand is slightly above the lattice data, FVEs are non-negligible.
- Ref. [1, 8] (Mainz 2021, 2023): VMD,
  $(34/9) \cdot a_\mu^{\text{HLbL},\pi^0}(|y|) > a_\mu^{\text{HLbL, light, conn.}}(|y|)$.
  Including π^+ (scalar QED) improves the matching of the tail for one ensemble.
- Ref. [4] (BMW 2024): $290 < M_\pi\,[\text{MeV}] < 430$.
  Good agreement of the VMD/LMD tail for connected and disconnected integrands.

AIC model averaging

We combine different fit anśatze using the (modified) Akaike criterion [10].

CDF from bootstraps

The Cumulative Density Function is given by a weighted histogram over the bootstrap samples.
The weights of each model i is:

$$w_i = \exp\left[-(\chi_i^2 + 2n_{\mathrm{par}} - n_{\mathrm{data}})/2\right]$$

Total error

The total error is found from quantiles:

$$\sigma_{\mathrm{tot}} = \sqrt{\sigma_{\mathrm{stat}}^2 + \sigma_{\mathrm{syst.}}^2} = (y_{84} - y_{16})/2$$

Bibliography I

[1] Nils Asmussen, En-Hung Chao, Antoine Gérardin, Jeremy R. Green, Renwick J. Hudspith, Harvey B. Meyer, and Andreas Nyffeler. "Hadronic light-by-light scattering contribution to the muon g − 2 from lattice QCD: semi-analytical calculation of the QED kernel". In: *JHEP* 04 (2023), p. 040. DOI: 10.1007/JHEP04(2023)040. arXiv: 2210.12263 [hep-lat] (cit. on pp. 4, 5, 12, 27, 46).

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[3] Thomas Blum, Norman Christ, Masashi Hayakawa, Taku Izubuchi, Luchang Jin, Chulwoo Jung, and Christoph Lehner. "Connected and Leading Disconnected Hadronic Light-by-Light Contribution to the Muon Anomalous Magnetic Moment with a Physical Pion Mass". In: *Phys. Rev. Lett.* 118.2 (2017), p. 022005. DOI: 10.1103/PhysRevLett.118.022005. arXiv: 1610.04603 [hep-lat] (cit. on p. 8).

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