Correction of quark mass misstuning

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$$f_\pi = 130.5 \text{ MeV} \qquad M_\pi = 135.0 \text{ MeV} \qquad M_K = 494.6 \text{ MeV} \qquad M_{D_s} = 1967 \text{ MeV}\,.$$

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- Mismatch iso/sim in the valence
  - Unitary analysis for f_π and M_π
  - For Other observable many valence point and interpolation

Mismatch iso/sim in the sea

- Loops

$$\langle O\rangle_{\mu^{iso}} = \langle O\rangle_{\mu^{sim}} + (\mu^{iso} - \mu^{sim})\left(\langle OS[\mu^{sim}]\rangle_{\mu^{sim}} - \langle O\rangle\langle S[\mu^{sim}]\rangle_{\mu^{sim}}\right)$$

  - Derivative at μ^{sim}

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- Reweighting

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  - Derivative at μ^*

$$\left.\frac{\partial\langle O\rangle_\mu}{\partial_\mu}\right|_{\mu^{sim}} = \frac{1}{\epsilon}\left(\langle O\frac{\det D(\mu^{sim}+\epsilon)}{\det D(\mu^{sim})}\rangle_{\mu^{sim}} - \langle O\rangle_{\mu^{sim}}\right)$$

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- The non-degenerate doublet has parameters μ_σ and μ_δ which are matched to μ_s and μ_c as

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$$\cancel{\mu_s = \mu_\sigma - \frac{Z_P}{Z_S}\mu_\delta}$$

$$aM_K^{Unitary}(\mu_\sigma, \mu_\delta) = aM_K^{OS}(\mu_s)$$

- Both loops and reweighing use

$$\frac{\det D_{tm}(\mu_s, \mu_c)}{\det D_{OS}(\mu_s)\det D_{OS}(\mu_c)} = 1 + \mathcal{O}(a^2)$$

scale setting

- To correct f_π and M_π in m_0 we use the analytic formulae

$$F_\pi(m_{\mathrm{PCAC}}=0) = F_\pi(m_{\mathrm{PCAC}})\sqrt{1+(Z_A m_{\mathrm{PCAC}}/m_\ell)^2}$$
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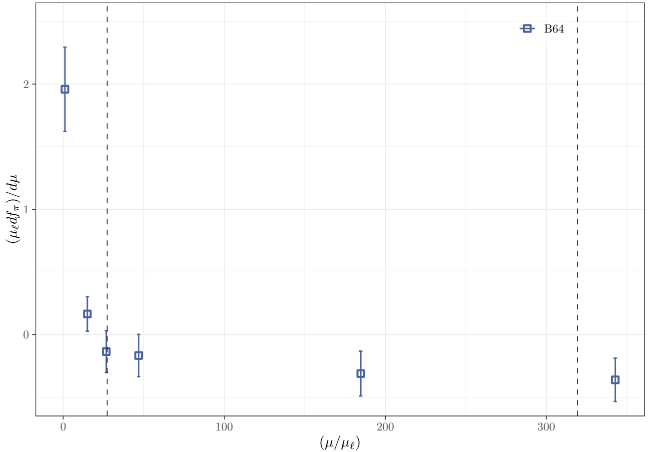
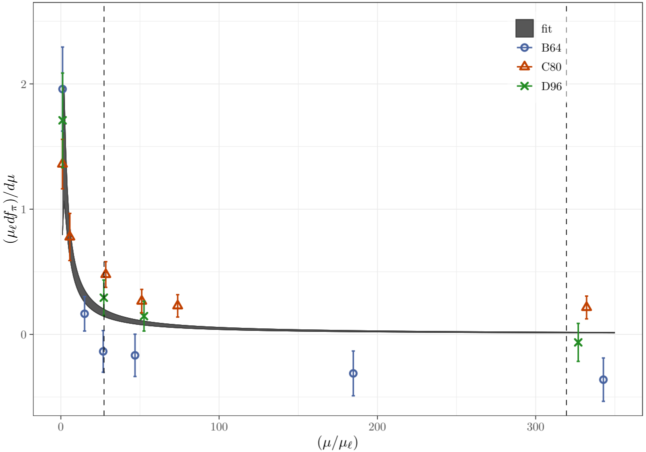
- To correct in μ_s and μ_c we evaluated the pseudo-derivative at several μ



- renormalized quantity $\mu_\ell \frac{\partial f_\pi}{\partial \mu}$ computed for more lattice spacing

- fit $\mu_\ell \frac{\partial f_\pi}{\partial \mu} = P_0 \frac{1}{\mu} + P_1 \frac{1}{\mu^2} + P_2 \frac{1}{\mu^2}$
- $\chi^2/d.o.f = 2$



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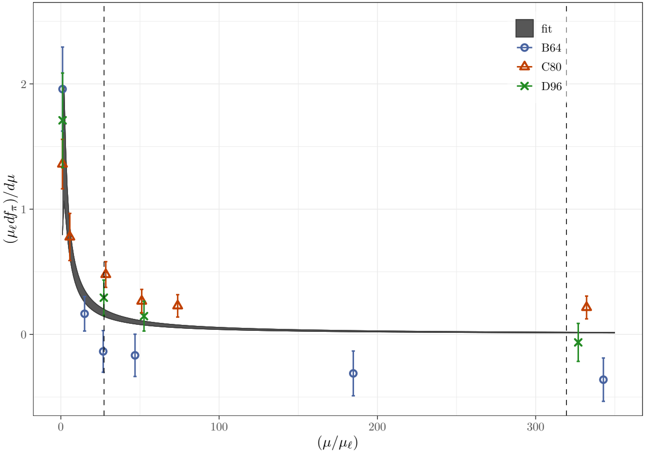
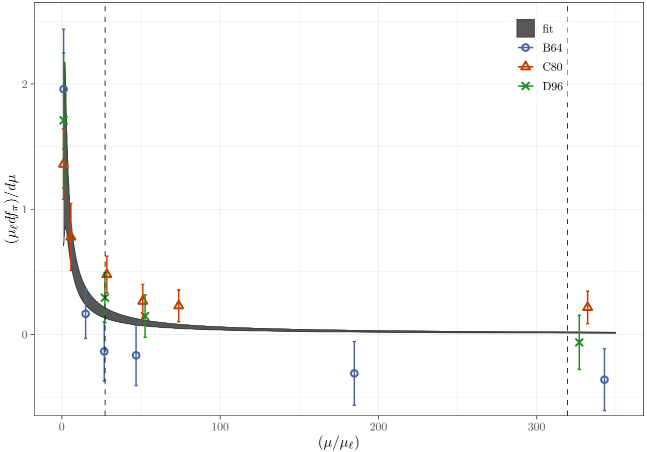
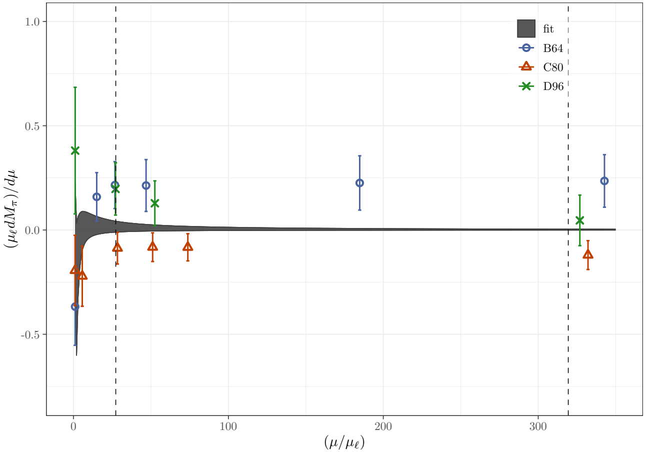


- Rescale errors such $\chi^2/d.o.f = 1$
- Equivalent of rescaling the fit parameters error $\tilde{P} = P\sqrt{\chi^2/d.o.f}$



- Same procedure for M_π

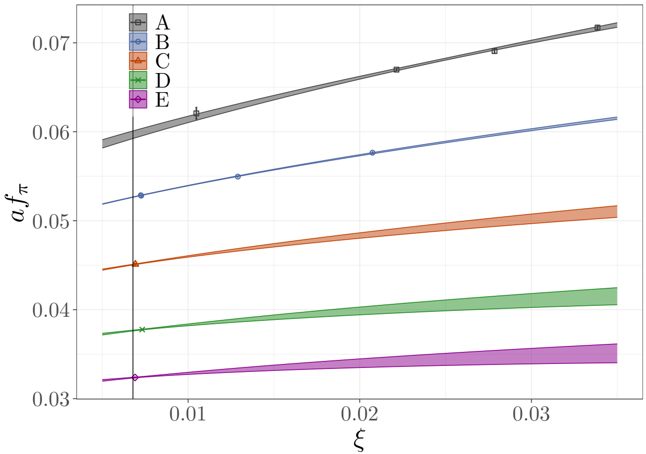


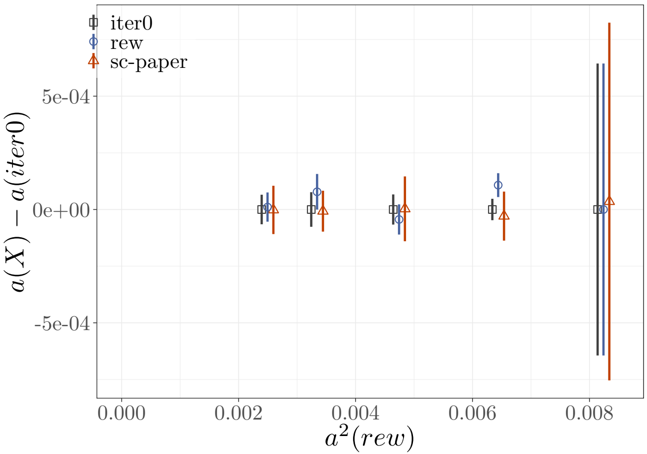
ensemble	$\Delta_s f_\pi/\sigma_{f_\pi}$	$\Delta_c f_\pi/\sigma_{f_\pi}$
B64	0.69	0.36
C80	-1.2	0.67
D96	0.46	0.89
E112	0.45	-0.27

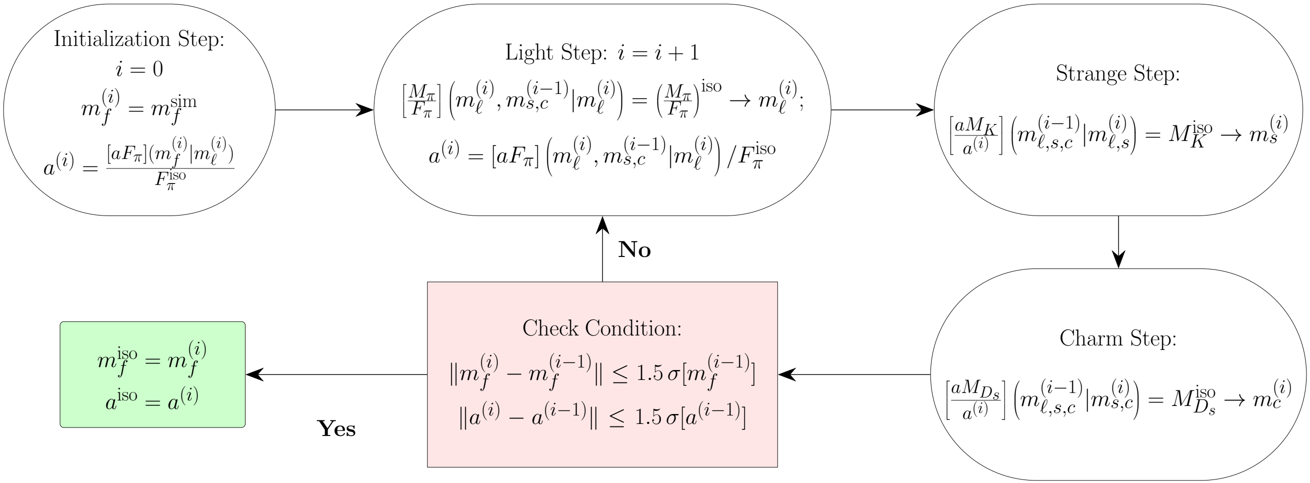
ensemble	$\Delta_s M_\pi/\sigma_{M_\pi}$	$\Delta_c M_\pi/\sigma_{M_\pi}$
B64	0.10	0.06
C80	-0.16	0.11
D96	0.05	0.13
E112	0.05	-0.04

- Unitary analysis

$$aF_\pi(\xi_\pi,\beta) = a\overline{F}_\pi(\beta)\cdot\left\{1-2\xi_\pi\log(\xi_\pi/\xi_\pi^{\rm iso}) + [P+P_{disc}(aF_\pi(\xi_\pi,\beta))^2](\xi_\pi-\xi_\pi^{\rm iso})\right\}$$







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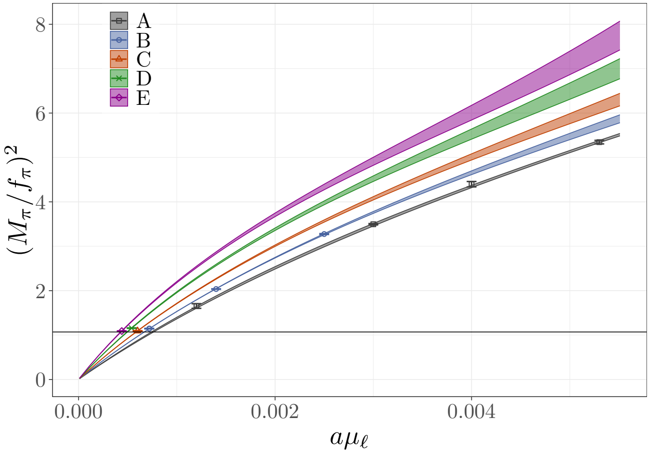
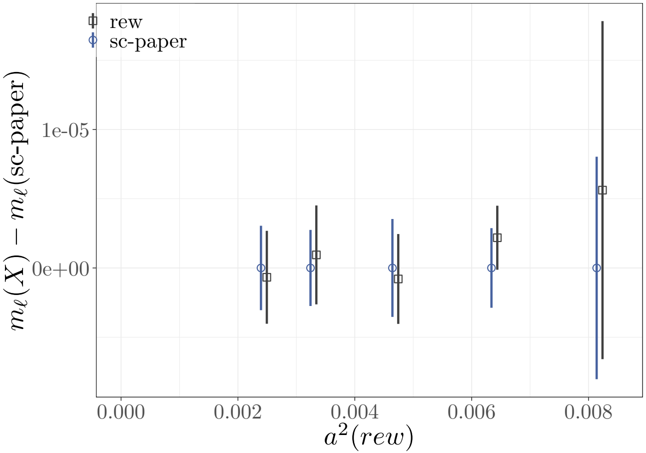
- Unitary analysis

$$\frac{M_\pi}{f_\pi} = 2P_0(a)a\mu_\ell \cdot \left\{1 + 5\xi_\pi \log(\xi_\pi) + P_1\xi_\pi + P_2 a\mu_\ell a^2\right\}$$





- Scale setting trough w_0 BMW (Ω)

$$W(t) = t\frac{\partial(t^2\langle E\rangle)}{\partial t} = t^2\left(2\langle E\rangle + t\frac{\partial\langle E\rangle}{\partial t}\right)$$
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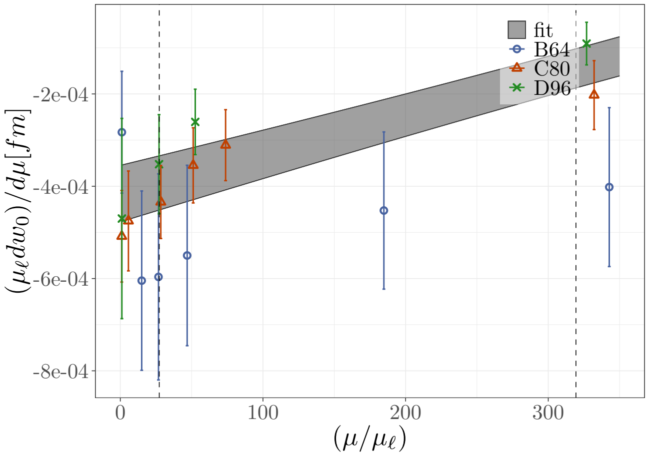
- Quark mass correction with reweighting



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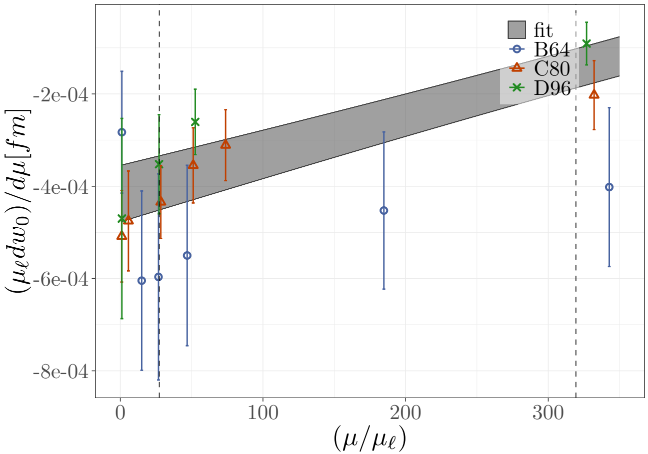

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- Strong correction due to charm misstuning

ensemble	$\Delta_\ell w_0/\sigma_{w_0}$	$\Delta_s w_0/\sigma_{w_0}$	$\Delta_c w_0/\sigma_{w_0}$
B64	0.2	-1.7	-3.5
C80	0.1	3.7	-7.9
D96	0.4	-1.2	-10.2
E112	-0.04	-1.1	2.7

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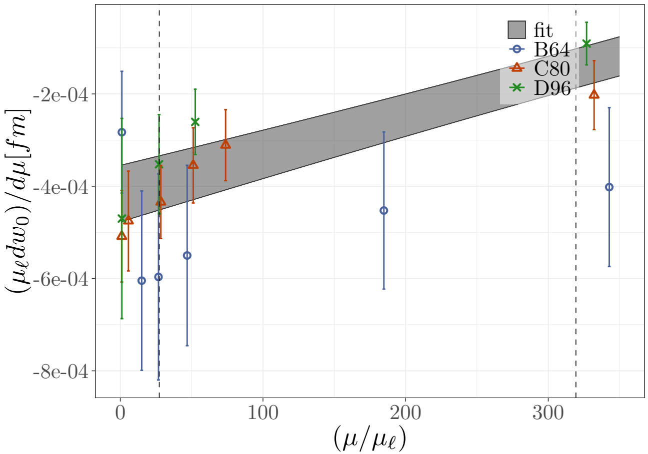

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- m_0 correction calculation in progress

insignificance of sea mass mistunings on M_K and M_{D_s}

Same procedure for M_K and M_{D_s}, but slightly different implementation:

- constant fit between the three (available) ensembles for ℓ- and s-contributions
- fit $\mu_\ell \frac{\partial M}{\partial \mu} = P_0 \frac{1}{\mu^2} + P_1 \frac{1}{\mu^3}$ in the range $\mu_s \leq \mu \leq \mu_c$ to evaluate the c-contribution
- all the errors rescaled with $\sqrt{\chi^2_{\text{red}}}$ when $\chi^2_{\text{red}} > 1$

Ensemble	$\Delta_\ell M_K$ [MeV]	$\Delta_\ell M_K/\sigma_{M_K}$
B64	$0.0275(245)$	0.0773
C80	$0.0080(71)$	0.0147
D96	$0.0327(291)$	0.0457

Ensemble	$\Delta_\ell M_{D_s}$ [MeV]	$\Delta_\ell M_{D_s}/\sigma_{M_{D_s}}$
B64	$0.0340(333)$	0.02766
C80	$0.0099(97)$	0.00495
D96	$0.0404(395)$	0.01473

Ensemble	$\Delta_s M_K$ [MeV]	$\Delta_s M_K/\sigma_{M_K}$
B64	$0.0689(543)$	0.193
C80	$-0.111(87)$	-0.203
D96	$0.0361(284)$	0.0504

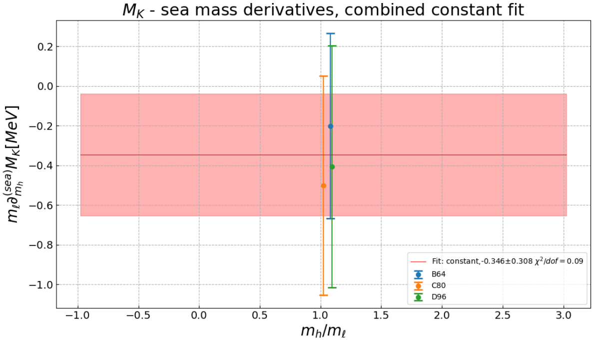
Ensemble	$\Delta_s M_{D_s}$ [MeV]	$\Delta_s M_{D_s}/\sigma_{M_{D_s}}$
B64	$0.288(125)$	0.234
C80	$-0.463(200)$	-0.231
D96	$0.151(65)$	0.05499

Ensemble	$\Delta_c M_K$ [MeV]	$\Delta_c M_K/\sigma_{M_K}$
B64	$0.0071(221)$	0.0200
C80	$0.0097(303)$	0.0178
D96	$-0.0111(346)$	-0.0155

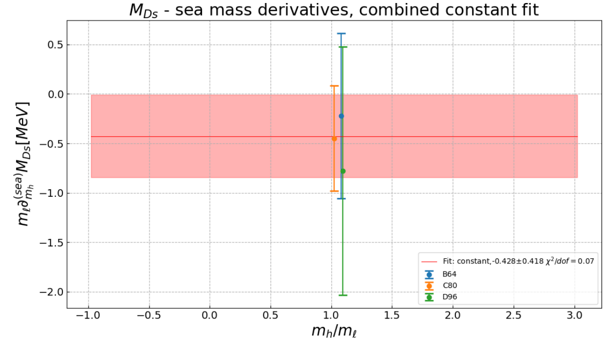
Ensemble	$\Delta_c M_{D_s}$ [MeV]	$\Delta_c M_{D_s}/\sigma_{M_{D_s}}$
B64	$0.0642(603)$	0.05215
C80	$0.0879(83)$	0.0439
D96	$-0.100(94)$	-0.0366

- As shown, all the contributions are negligible wrt the errors on the meson masses (the sum of all the contributions is negligible as well).

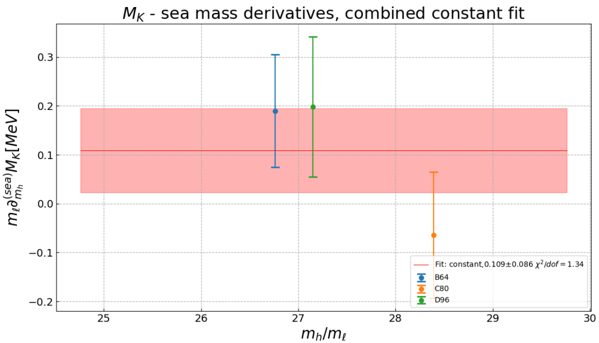
$$\mu_\ell \frac{\partial M_K}{\partial \mu_\ell} = -0.35(31); \qquad \frac{\chi^2}{\text{d.o.f}} = 0.1$$



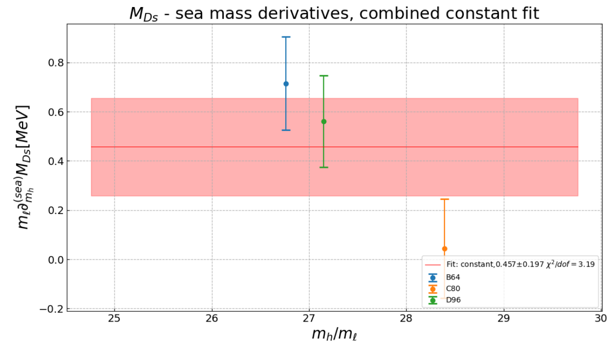
$$\mu_\ell \frac{\partial M_{D_s}}{\partial \mu_\ell} = -0.43(42); \qquad \frac{\chi^2}{\text{d.o.f}} = 0.1$$



$$\mu_\ell \frac{\partial M_K}{\partial \mu_s} = -0.11(9); \qquad \frac{\chi^2}{\text{d.o.f}} = 1.3$$

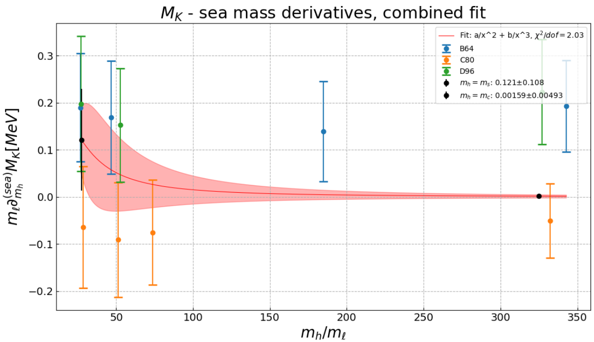


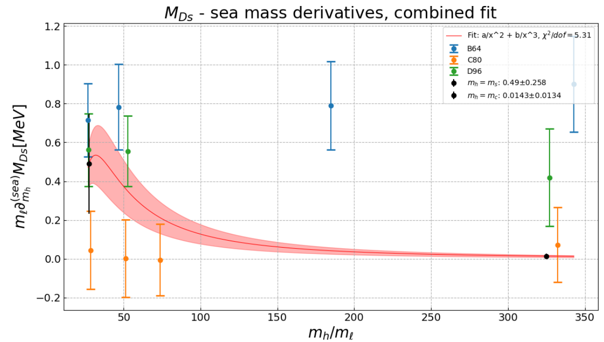
$$\mu_\ell \frac{\partial M_{D_s}}{\partial \mu_s} = -0.46(20); \qquad \frac{\chi^2}{\text{d.o.f}} = 3.19$$



$$\mu_\ell \frac{\partial M_K}{\partial \mu_c} = -0.0016(49); \qquad \frac{\chi^2}{\mathrm{d.o.f}} = 2.0$$

$$\mu_\ell \frac{\partial M_{D_s}}{\partial \mu_c} = -0.014(13); \qquad \frac{\chi^2}{\mathrm{d.o.f}} = 5.3$$





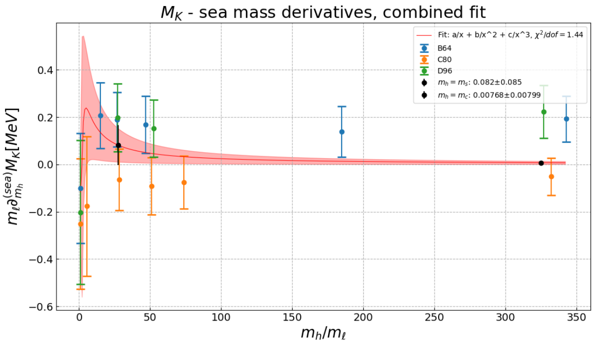
These results are obtained using the fit ansatz:

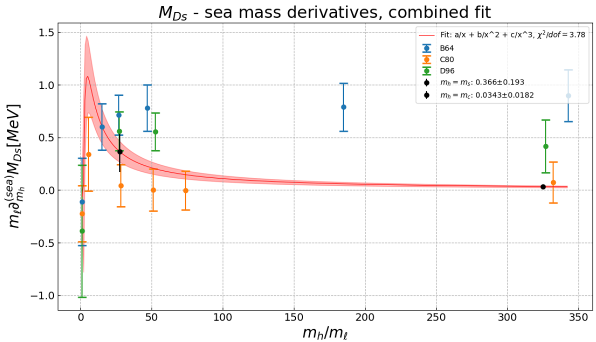
$$\mu_\ell \frac{\partial M}{\partial \mu} = P_0 \frac{1}{\mu^2} + P_1 \frac{1}{\mu^3},$$

and are those used to produce the tables above. For comparison, we also tried to fit with the same ansatz used for M_π and f_π.

$$\mu_\ell \frac{\partial M_K}{\partial \mu_c} = -0.0077(80); \qquad \frac{\chi^2}{\text{d.o.f}} = 1.44$$

$$\mu_\ell \frac{\partial M_{D_s}}{\partial \mu_c} = 0.034(18); \qquad \frac{\chi^2}{\text{d.o.f}} = 3.78$$





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$$\mu_\ell \frac{\partial M}{\partial \mu} = P_0 \frac{1}{\mu} + P_1 \frac{1}{\mu^2} + P_2 \frac{1}{\mu^3}.$$

We obtain similar results for the derivative w.r.t μ_s; for the derivatives w.r.t. μ_c, these results are, respectively, 5 times and 3 times larger for M_K and M_{D_s}. Even so, almost all the shifts would be negligible w.r.t. the errors on the meson masses. The only exception being the shift on M_K in the B64 ensemble: $\Delta_{\ell+s+c} M_K \simeq 0.4\sigma_{M_K}$.

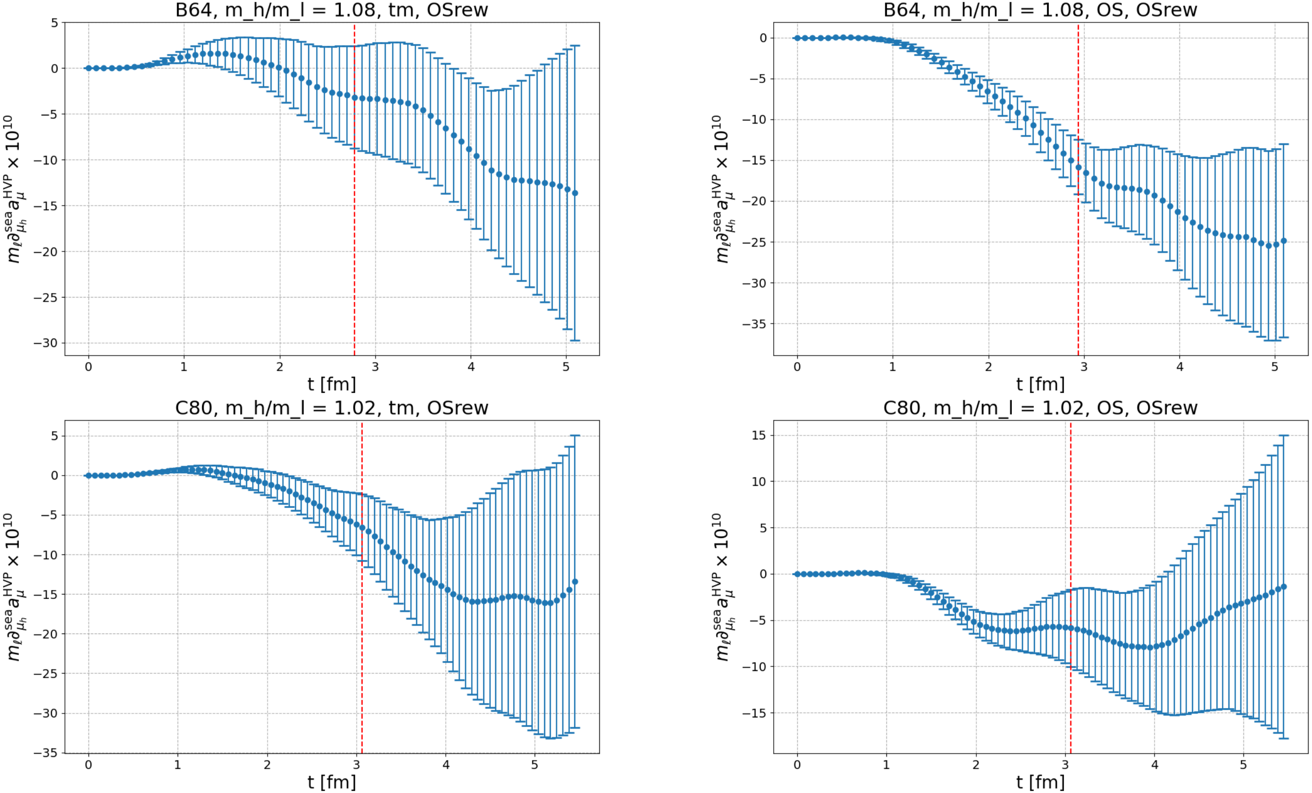
Sea mass mistunings on $a_\mu^{\mathrm{HVP}}(\ell)^{\mathrm{conn}}$

- the derivative of $a_\mu^{\mathrm{HVP}}(\ell)^{\mathrm{conn}}$ is given by

$$\partial^{\mathrm{sea}}_{\mu_q} a_\mu^{\mathrm{HVP}}(\ell)^{\mathrm{conn}} = \int_0^\infty dt\, t^2\, K(t\; m_\mu)\; \partial^{\mathrm{sea}}_{\mu_q} < V_k(\ell)\, V_k(\ell) >^{\mathrm{conn}}$$

- $\partial^{\mathrm{sea}}_{\mu_q} < V_k(\ell)\, V_k(\ell) >^{\mathrm{conn}}$ is computed using the reweighting factors;
- the time integration must be performed carefully, to avoid putting in too much noise;
- then, the renormalized quantity $\mu_\ell \frac{\partial a_\mu^{\mathrm{HVP}}(\ell)^{\mathrm{conn}}}{\partial \mu}$ can be compared among different ensembles (as before);
- we use results from both the TM and OS discretizations to increase statistics (taking care of the correlations);
- we perform a constant fit of the derivatives for the ℓ and s contributions, and a "decoupling-inspired" fit for the c contribution.

For the ℓ-contribution, we integrate up to the t-cut selected for $a_\mu^{\mathrm{HVP}}(\ell)^{\mathrm{SIM}}$ with the ETM bounding method.



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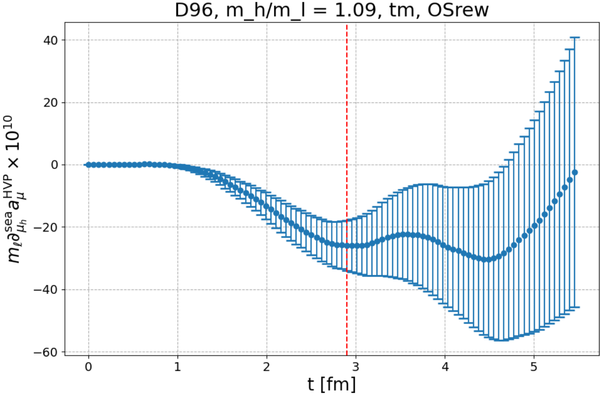
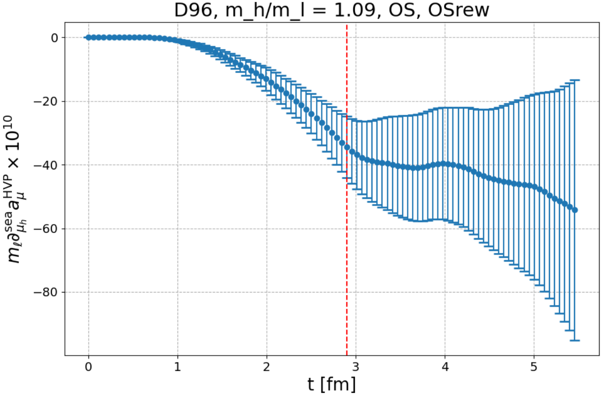
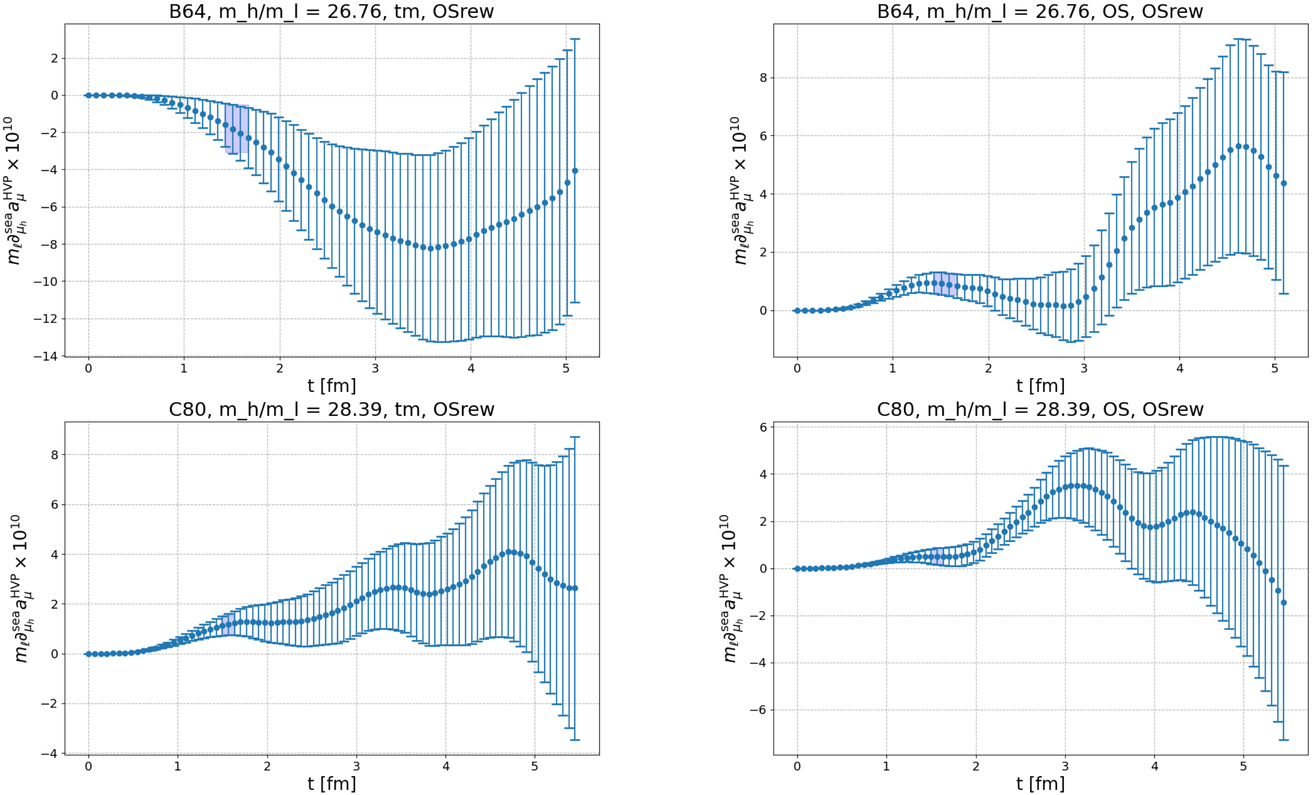
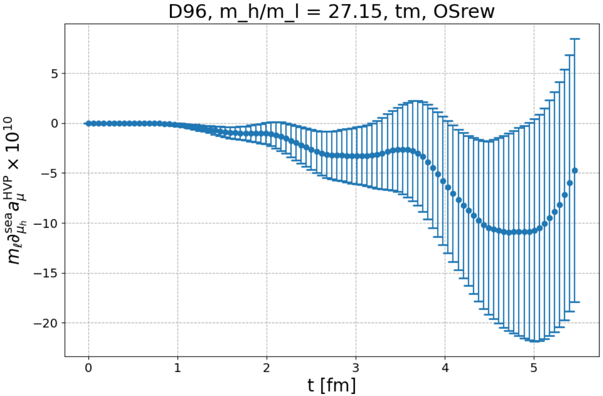


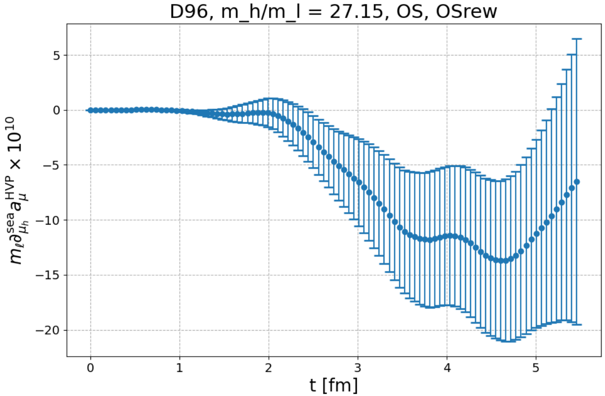


Heavier quarks contributions: we fit a plateau in a_μ vs t_c where we can see a signal.

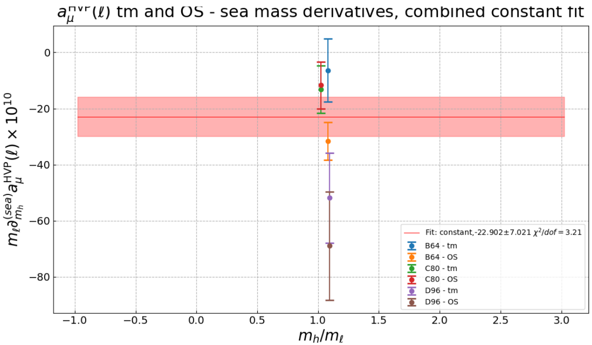


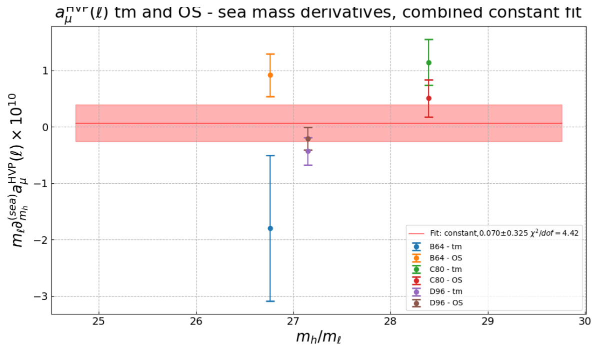
Heavier quarks contributions: we fit a plateau in a_μ vs t_c where we can see a signal.





For the ℓ- and s-contributions, we performed a constant fit





$$\mu_\ell \times \partial^{\rm sea}_{\mu_\ell} a_\mu^{\rm HVP}(\ell)^{\rm conn} \times 10^{10} = -22.9(7.0)$$

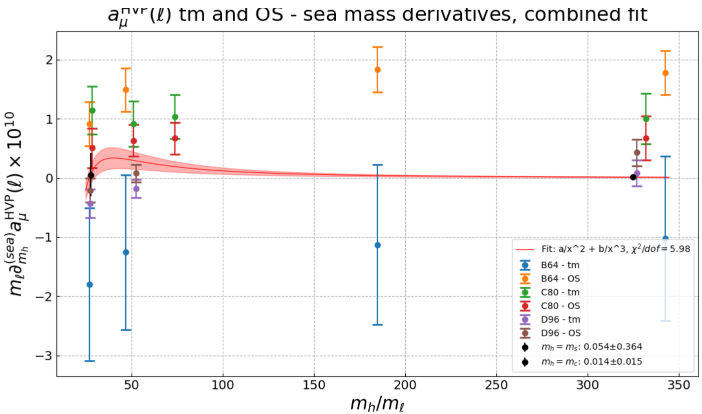
$$\mu_\ell \times \partial^{\rm sea}_{\mu_s} a_\mu^{\rm HVP}(\ell)^{\rm conn} \times 10^{10} = 0.070(325)$$

$$\chi^2_{\rm red} = 3.2 \qquad \chi^2_{\rm red} = 4.4$$

All the errors have already been rescaled with $\sqrt{\chi^2_{\rm red}}$

For the c-contribution, we performed a decoupling-inspired fit with the ansatz

$$\mu_\ell \frac{\partial a_\mu^{\mathrm{HVP}}(\ell)^{\mathrm{conn}}}{\partial \mu} = P_0 \frac{1}{\mu^2} + P_1 \frac{1}{\mu^3},$$



$$\mu_\ell \times \partial^{\mathrm{sea}}_{\mu_c} a_\mu^{\mathrm{HVP}}(\ell)^{\mathrm{conn}} \times 10^{10} = 0.014(15)$$

$$\chi^2_{\mathrm{red}} = 5.98$$

All the errors have already been rescaled with $\sqrt{\chi^2_{\mathrm{red}}}$

Summarizing, the computed derivatives are

$$\mu_\ell \times \partial^{\text{sea}}_{\mu_\ell} a_\mu^{\text{HVP}}(\ell)^{\text{conn}} \times 10^{10} = -22.9(7.0),$$
$$\mu_\ell \times \partial^{\text{sea}}_{\mu_s} a_\mu^{\text{HVP}}(\ell)^{\text{conn}} \times 10^{10} = 0.070(325),$$
$$\mu_\ell \times \partial^{\text{sea}}_{\mu_c} a_\mu^{\text{HVP}}(\ell)^{\text{conn}} \times 10^{10} = 0.014(15).$$

from which one can compute the shifts on the observable, listed in the table.

	$\Delta^{\text{sea}}_{\mu_l}$	$\sigma(\Delta^{\text{sea}}_{\mu_l})$	$\Delta^{\text{sea}}_{\mu_s}$	$\sigma(\Delta^{\text{sea}}_{\mu_s})$	$\Delta^{\text{sea}}_{\mu_c}$	$\sigma(\Delta^{\text{sea}}_{\mu_c})$
B64	1.8235	0.5590	0.0443	0.2051	0.0632	0.0694
C80	0.5312	0.1628	-0.0712	0.3295	0.0865	0.0951
D96	2.1630	0.6631	0.0232	0.1073	-0.0987	0.1085
E112	0.4999	0.1533	-0.0473	0.2191	-0.0473	0.0520

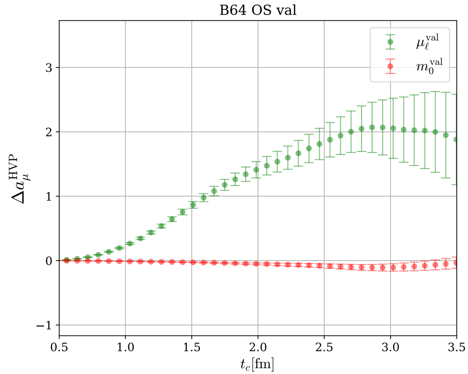
m_0 misstuning

$$2m_{\text{PCAC}}(m_f^{\text{sim}}, m_{\text{cr}}) = \frac{\sum_{\vec{x}} \left\langle [\partial_0 \bar{\chi}_\ell \gamma_5 \gamma_0 \tau^1 \chi_\ell](t,\vec{x}) [\bar{\chi}_\ell \gamma_5 \tau^1 \chi_\ell](0) \right\rangle}{\sum_{\vec{x}} \langle [\bar{\chi}_\ell \gamma_5 \gamma_0 \tau^1 \chi_\ell](t,\vec{x}) [\bar{\chi}_\ell \gamma_5 \gamma_0 \tau^1 \chi_\ell](0) \rangle}$$

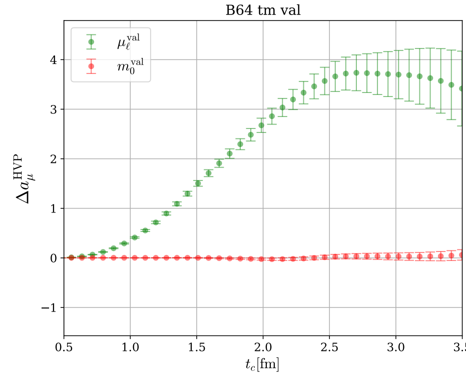
$$\chi_\ell(x) = \exp\left(-i\frac{\pi}{4}\gamma_5\tau^3\right)\Psi_\ell(x)\,, \qquad \bar{\chi}_\ell(x) = \bar{\Psi}_\ell(x)\exp\left(-i\frac{\pi}{4}\gamma_5\tau^3\right)\,,$$

- Valence and sea effect in $a_\mu^{\text{HVP}}(\ell)$
- $\frac{\partial O}{\partial m_{\text{PCAC}}} = \frac{\partial O}{\partial m_0}\frac{\partial m_0}{\partial m_{\text{PCAC}}}$

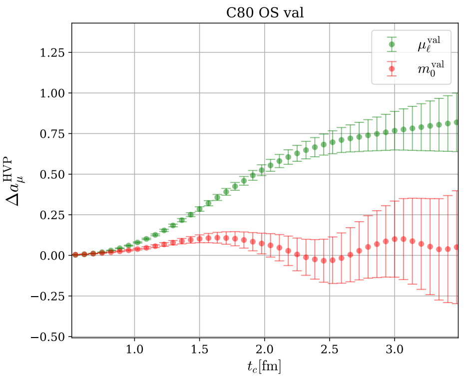
Valence



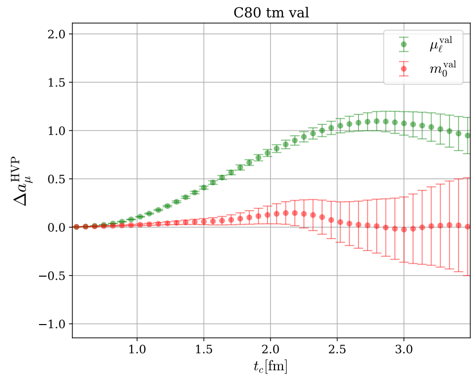
B64 OS val
$\mu_\ell^{\rm val}$
$m_0^{\rm val}$
$\Delta a_\mu^{\rm HVP}$
t_c[fm]



B64 tm val
$\mu_\ell^{\rm val}$
$m_0^{\rm val}$
$\Delta a_\mu^{\rm HVP}$
t_c[fm]



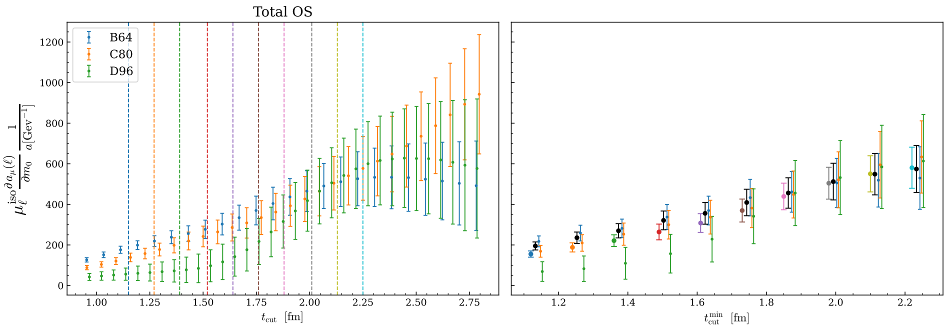
C80 OS val
$\mu_\ell^{\rm val}$
$m_0^{\rm val}$
$\Delta a_\mu^{\rm HVP}$
t_c[fm]

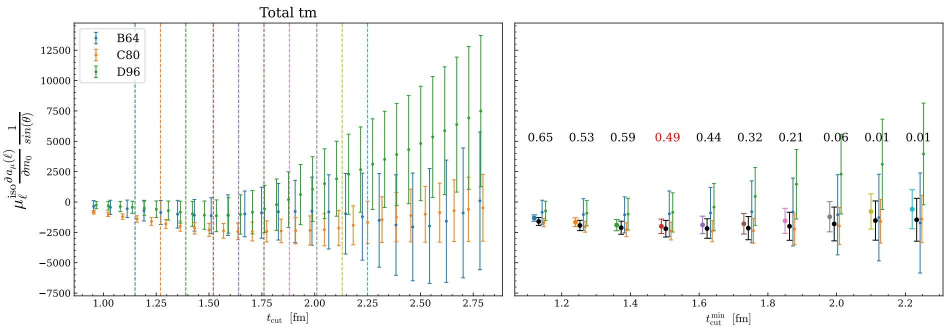


C80 tm val
$\mu_\ell^{\rm val}$
$m_0^{\rm val}$
$\Delta a_\mu^{\rm HVP}$
t_c[fm]

Sea

- Need to correct for m_0^{sea} light, strange and charm
- Constant fit in the region $[t_{cut}, t_{cut} + 0.4\text{fm}]$





To do list

- Check that m_0^{seq} strange and charm effect are negligible in M_π and f_π
- m_0 correction in w_0
- Check m_0 correction in M_K and M_{D_s}
- Run the analysis with the last values of a and m_ℓ