B01: T-Designs

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B01: Quasi-Particle Dynamics in Low-Dimensional Topological Systems

- PLs: Thomas Luu & Ulf Meißner
- Postdoc started last October/November
  - Could not attend due to medical appointment (that could not be rescheduled in a reasonable amount of time)
- **B01** poster: engineering spin centers in 1-D chains arXiv:2507.18806

Lin Wang

T-Designs

- Came about from discussions at NuMeriQS-sponsored ECT* workshop on *Hamiltonian Lattice Gauge Theories*
- Preliminary learning stage
    - potential relevance for NuMeriQS?

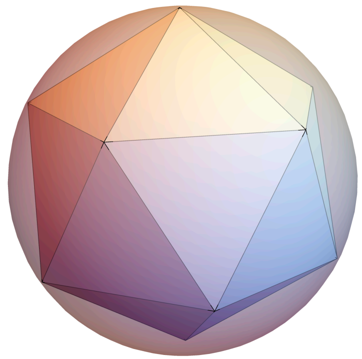
Spherical T-Design

- Given: Any polonomial f of degree $\leq t$ residing on a d-dimensional unit sphere S^d
- $\exists$ a set of N-points Ω_i on S^d where

$$\frac{1}{|S^d|}\int_{\Omega\in S^d} d\Omega f(\Omega) = \frac{1}{N}\sum_i^N f(\Omega_i)$$

- Similar to ***Quadrature Integration*** $\int dx p(x) \approx \sum_i \omega_i p(x_i)$
  - but here all weights $\omega_i = 1/N$
- Existence proof that N exists for any d, t combination (Seymour & Zaslavsky)

Example: Spherical 5-design



Points defined by icosohedron vertices (snub tetrahedron)

$$f(x, y, z) = \sqrt{3}x^4 + 11xyz + \frac{xy}{\sqrt{3}} + \sqrt{5}y^2$$

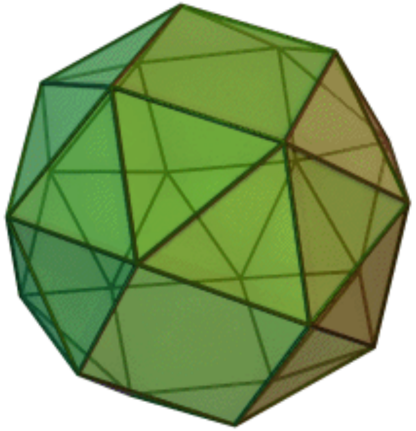
$$\frac{1}{4\pi}\int_{\Omega\in S^3} d\Omega f(\Omega) = \frac{1}{12}\sum_i^{12} f(\Omega_i) = \frac{\sqrt{3}}{5} + \frac{\sqrt{5}}{3}$$

- Don't believe me? Download this file and try yourself!

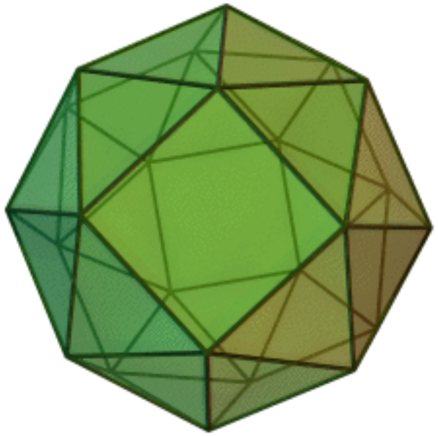
Warning

There exists no simple algorithm to find these N points!

Connection to Snub Polygons?



Left handed snub cube



Right handed snub cube

Quantum T-Design

- Can generalize to qubits defined on a Bloch sphere (ie the projective Hilbert space $P(H)$)
- A ***quantum t-design*** is a finite set of N quantum states that reproduces statistical properties of the uniform Haar distribution, up to t-th moments
  - ***state t-design***

$$\int d\psi \; |\psi\rangle\langle\psi|^{\otimes t} = \sum_i^N \omega_i |\psi_i\rangle\langle\psi_i|^{\otimes t}$$

  - ***unitary t-design***

$$\int dU \; U^{\otimes t} \rho (U^\dagger)^{\otimes t} = \frac{1}{|\{\mathcal{U}\}|} \sum_{U_i \in \{\mathcal{U}\}} U_i^{\otimes t} \rho (U_i^\dagger)^{\otimes t}$$

  for any operator ρ.

> **Example: Application to qubits**
>
> The 6 Pauli-eigenstates of σ_z, σ_+, and σ_- form the simplest quantum state (qubit) 2-design

Where do we go from here?

- Is this useful for lattice gauge theories?
  - digitization of gauge fields?
  - discrete integration of gauge-invariant quantities?
- Is it interesting for quantum computing?
  - Yes, in *quantum information/cryptography/tomography*
  - Interesting for NuMeriQS?
- Connection to Majorana stars?
- Anything else?