Lattice QCD for Dummies

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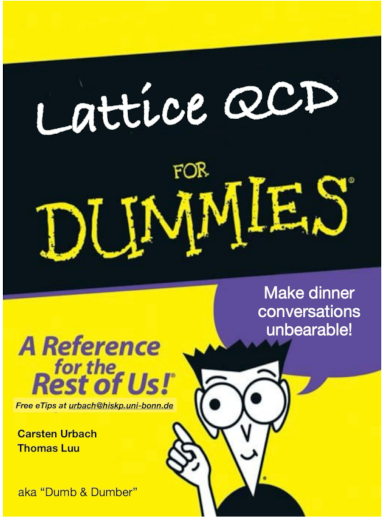


Classical Electrodynamics

- described in $\vec{E}$ and $\vec{B}$ fields and potentials $\phi, \vec{A}$ ($\hbar = c = 1$)

$$\vec{E} \;=\; -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\,, \qquad \vec{B} = \mathrm{rot}\vec{A}$$

- can be writen as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (F = \mathrm{d}A)$$

  with 4-vector potential $A = (\phi, \vec{A})$

- F in components

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Classical Electrodynamics

- two inhomogeneous Maxwell Equations (no charges) become

$$\partial_\mu F^{\mu\nu} = 0$$

- which correspond to the Euler Lagrange equations of the Lagrangian

$$\mathcal{L}_{\mathsf{EM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

- thus, the corresponding action reads in Minkowski space

$$S_{\mathsf{EM}} \;=\; \int \mathsf{d}^4 x \; \mathcal{L}_{\mathsf{EM}}$$

Action of Quantum Chromodynamics (QCD)

- analogously, the Lagrange density for QCD, now including fermionic matter

$$\mathcal{L}[\bar{\psi}, \psi, A_\mu] \;=\; -\frac{1}{4} G^a_{\mu\nu} G_a^{\mu\nu} + \bar{\psi}\,(\mathrm{i}\gamma^\mu D_\mu - m_q)\,\psi$$

- field strength tensor

$$G^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g f^{abc} A^b_\mu(x) A^c_\nu(x)$$

- gauge covariant derivative

$$D_\mu(x) \;=\; \partial_\mu - \mathrm{i}\, g\, T_a A^a_\mu(x)$$

Action of QCD

- $\psi(x)$: matter fields (quarks) in the fundamental representation of SU(3)
  $\bar{\psi} = \psi^\dagger \gamma_0$
  anticommuting Graßmann numbers
- $A_\mu^a(x)$: gauge potential or gauge fields
- g: bare coupling
- a: colour index; μ: spinor index
- T_a: generators of SU(3) with $[T_a, T_b] = \sum_c f_{abc} T_c$,
  f^{abc}: structure constants of SU(3)
- $U \in$ SU(3), then $U^{-1} = U^\dagger$ and det $U = 1$, $\quad U = e^{\mathrm{i} \sum_a \alpha_a T_a}$
- γ_μ: Dirac matrices with $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

Why gauge invariance??

- the Δ^{++} baryon is naïvely composed of three up quarks with parallel spin

- Pauli exclusion principle forbids this!
  Δ^{++} is a fermion, wave function must be antisymmetric

- solved by introducing an additional quantum number: “colour”

  [Greenberg, Han, Nambu (1964-65)]

$\Rightarrow$ additional $\mathrm{SU}(3)$ gauge degree of freedom

- quarks interact via an octet of vector gauge bosons

Quantisation

- QCD (like other field theories) quantised via path integral

$$\mathcal{Z} = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}A_\mu\, e^{iS}$$

[Feynman]

- later more...

More important QCD properties

- the coupling strength depends on the energy scale
  so-called **running coupling**

- at low energy:
  Colour confinement: only colour singlet states can be observed (so called hadrons)
  quarks and gluons not observable

- at large energy:
  Asymptotic freedom: quarks and gluons become free particles

Better defined in a space-time Box

- lattice regularisation:

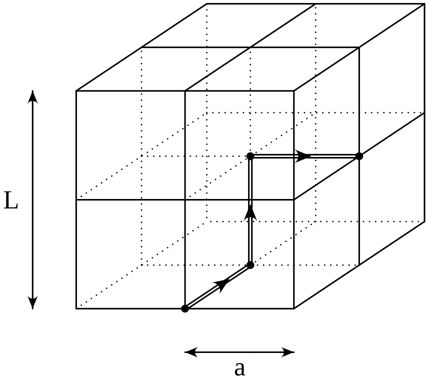
$\Rightarrow$ discretise space-time

- hyper-cubic $L^3 \times T$-lattice with lattice spacing a

$\Rightarrow$ momentum cut-off: $k_{\text{max}} \propto 1/a$

$\Rightarrow$ infrared cut-off: $1/L$

- derivatives $\Rightarrow$ finite differences
- integrals $\Rightarrow$ sums
- gauge potentials A_μ in $G_{\mu\nu}$ $\Rightarrow$ link matrices U_μ ('•⟹•')



Lattice Regularisation

- construction of lattice action: maintain local gauge invariance
- fermionic fields ψ at discrete sites x transform

$$\psi(x) \quad \rightarrow \quad V(x)\psi(x)\,, \quad V \in \mathsf{SU}(3)$$

- then the mass term is gauge invariant

$$m_q\bar{\psi}(x)\psi(x) \quad \rightarrow \quad m_q\bar{\psi}(x)V^\dagger(x)V(x)\psi(x) = m_q\bar{\psi}(x)\psi(x)$$

- but the naïve derivative part is not!

$$\bar{\psi}(x)(\psi(x+a\hat{\mu}) - \psi(x)) \quad \rightarrow \quad \bar{\psi}(x)V^\dagger(x)(V(x+a\hat{\mu})\psi(x+a\hat{\mu}) - V(x)\psi(x))$$

Lattice Regularisation

$\Rightarrow$ need an "interpreter" between sites x and $x + a\hat{\mu}$:

$$U(x, x + a\hat{\mu}) \quad \equiv \quad U_\mu(x) \; \in \; \mathsf{SU}(3)$$

- gauge covariant difference operator

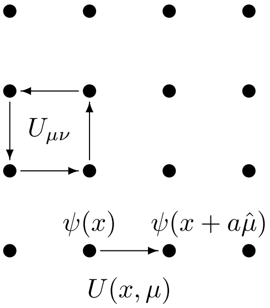
$$D_\mu \; = \; \frac{1}{a}(U(x, x + a\hat{\mu})\psi(x + a\hat{\mu}) - \psi(x))$$

- derivative term transforms correctly if we demand

$$U(x, y) \quad \to \quad V(x)U(x, y)V^\dagger(y)$$

- the U must be related to the gauge potential A_μ!

Lattice Regularisation



- gauge invariant objects from Us?

$$U(x,y)\,U(y,z)\,U(z,u)\cdots U(w,v)$$

- not a scalar!
- and the outermost Vs do not cancel

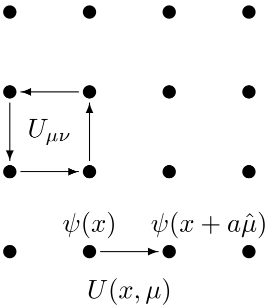
$$V(x)\ldots V^{\dagger}(v)$$

$\Rightarrow$ take the trace of closed loops

$$\mathrm{Tr}(U(x,y)U(y,z)U(z,u)\cdots U(w,x))$$

- gauge invariant!

Lattice Regularisation



- identify:

$$U_\mu(x) \;\to\; e^{iA_\mu(x+\hat{\mu}a/2)}$$

- one can show

$$\mathrm{Tr}(U_{\mu\nu}) = 3 - \frac{3}{4}g^2a^2(G^c_{\mu\nu})^2 + \mathcal{O}(a^6)$$

  by expanding in a

- $U_{\mu\nu}$ the smallest closed loop $\to$ plaquette

- this defines Wilson's plaquette gauge action!

  [Wilson, 1974, 1975]

Lattice Action

- with this, a possible lattice action reads

$$S = \beta \sum_x \sum_{\mu<\nu} \mathrm{Tr}\,\mathrm{Re}(1 - U_{\mu\nu}) + \bar{\psi}(i\gamma^\mu D_\mu - m_0)\psi$$

[Wilson (1974)]

- $\beta = 6/g^2$ the bare inverse square coupling (mind the $\sum_{\mu<\nu}$!)
- can invent different versions of S, as long as only higher orders in a are changed
- it has the clasically correct limit as $a \to 0$

Quantisation

- QCD (like other field theories) quantised via path integral

$$\mathcal{Z} = \int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}A_\mu\, e^{iS}$$

[Feynman]

- well defined on the lattice

$$\int \mathcal{D}\psi\, \mathcal{D}\bar{\psi}\, \mathcal{D}A_\mu \quad \rightarrow \quad \prod_x \prod_\mu \int \mathrm{d}\bar{\psi}(x)\, \mathrm{d}\psi(x)\, \mathrm{d}U_\mu(x)$$

- dU: group invariant Haar measure

How to solve the path integral?

- analytically continue to complex times
- work with Euclidean times $\tau \to -\mathrm{i}t$

$$Z_{\mathsf{E}} = \int \mathcal{D}\psi \, \mathcal{D}\bar{\psi} \, \mathcal{D}A_\mu \, e^{-S_{\mathsf{E}}}$$

- with Euclidean action S_{E} real and bounded from below (at least for some situations of interest!)

$\Rightarrow$ can use Markov chain Monte Carlo to compute Z_{E}!

Measuring things with the Path Integral (PI)

- Integrate out the fermion fields

$$Z_{\mathsf{E}} \;=\; \int \prod_{x,\mu} \mathsf{d}\bar{\psi}(x)\, \mathsf{d}\psi(x)\, \mathsf{d}U_\mu(x)\; e^{-S_{\mathsf{E}}} = \int \prod_{x,\mu} \mathsf{d}U_\mu(x)\, \mathsf{det}\, D[U]\; e^{-S_g[U]} \equiv \int \mathcal{D}U e^{-S_{\mathsf{eff}}[U]}$$

- Want to measure some quantity $\hat{O}$? Calculate

$$\langle \hat{O} \rangle \;=\; \frac{1}{Z_{\mathsf{E}}} \int \mathcal{D}U\, O[U] e^{-S_{\mathsf{eff}}[U]}$$

Look familiar?

This is the analog of the *thermal expectation value of* O in statistical physics:

$$\langle O \rangle_T = \frac{1}{Z}\, \mathsf{Tr}\, O e^{-H/T}$$

Monte Carlo Estimates of $\hat{O}$

- Generate collection, or *ensemble*, of configurations $\{U\}$ s.t. the probablity of any U follows

$$P(U) = e^{-S_{\text{eff}}[U]}$$

(for example, via Markov Chain Monte Carlo (MCMC))

- Monte Carlo estimate

$$\langle \hat{O} \rangle \approx \frac{1}{N_{\text{cfg}}} \sum_{\{U\}} O[U] + \mathcal{O}\left(N_{\text{cfg}}^{-1/2} \right)$$

Example: Pion correlator

- Observable build from time-dependent *interpolating operators*:

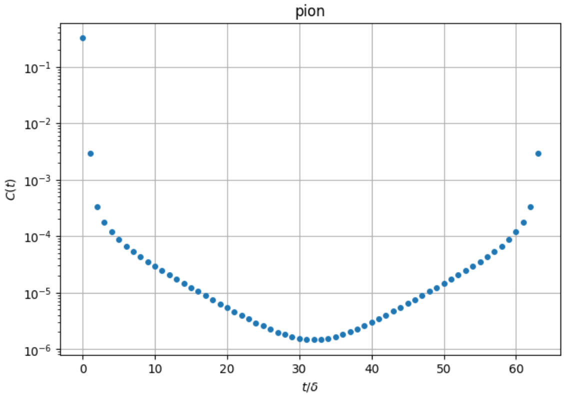
$$\langle \hat{O} \rangle = \langle \pi_y(t) \bar{\pi}_x(0) \rangle \equiv C_{yx}(t)$$

where $\bar{\pi}_x(0) = \bar{u}(x,0)\gamma_5 d(x,0)$

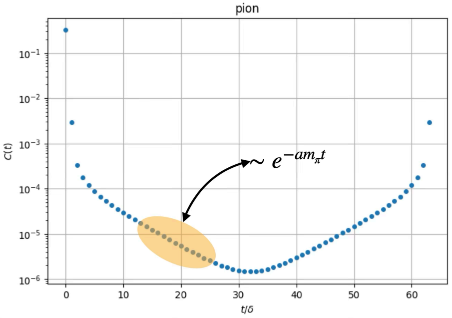
- Wick contraction

$$\langle d(x,t)\gamma_5 u(x,t)\overline{u}(x,0)\gamma_5 d(x,0)\rangle$$

$$= \frac{-1}{N_{\text{cfgs}}} \sum_{\{U\}} \text{Tr}\left(\gamma_5 D^{-1}_{u,xy;t}[U]\gamma_5 D^{-1}_{d,yx;-t}[U]\right)$$



Why are correlators useful?



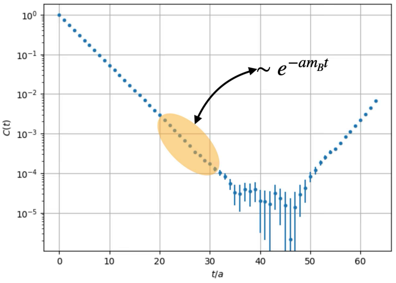
Spectral Decomposition Theorem

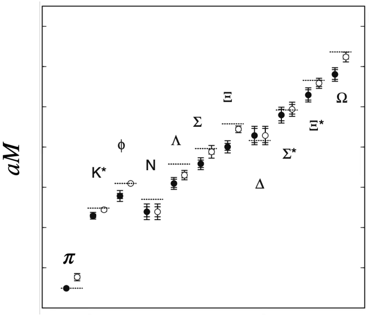
$$C(t) = A_\pi e^{-am_\pi t} + A_{3\pi} e^{-3am_\pi t} + A_{2\pi 1K} e^{-a(2m_\pi + m_K)t} + A_{\pi^*} e^{-am_{\pi^*} t} + \dots$$

$$\xrightarrow[t\to\infty]{} A_\pi e^{-am_\pi t}$$

Noise makes life hard

- Correlators tend to be noisy (stochasticity!)

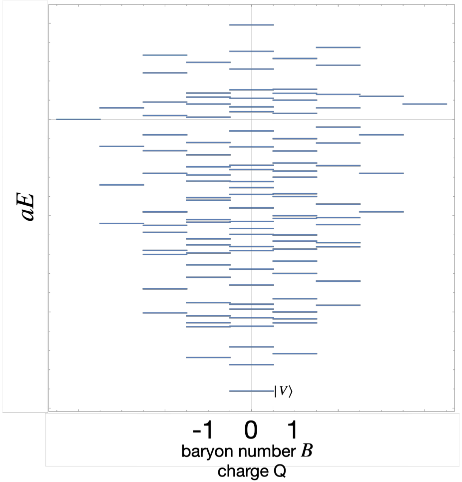




- Fitting in the presence of noise is a non-trivial thing! Talk to Johann and Carsten about this!

What actually is this mass?

- strictly speaking, we measure energy *differences*
  - relative to global ground state $|V\rangle$



Other things we can measure

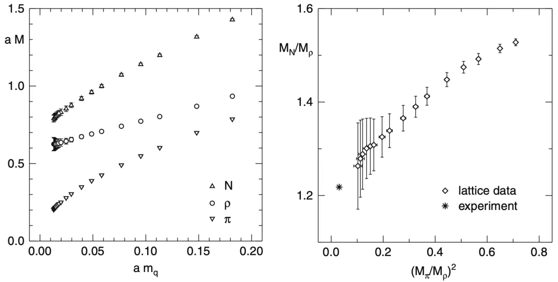
- Euclidean time $\implies$ no distinction between time and space: statistical physics (in equilibrium)
- spectroscopy (ie masses)
- multi-hadron systems
  - interaction parameters, decay constants (a la Lüscher)

. . . but

- non-equilibrium processes
- scattering of particles
- real-time evolution
- sign problem: $P \sim e^{-S}$ when $S \in \mathbb{C}$

Reality check!

- The Lattice QCD action $S_{\text{eff}}[U]$ is dimensionless
- Input (bare) parameters to Lattice QCD: am_q, g
- $\Longrightarrow$ All quantities calculated in Lattice QCD are dimensionless



[Gattringer & Lang]

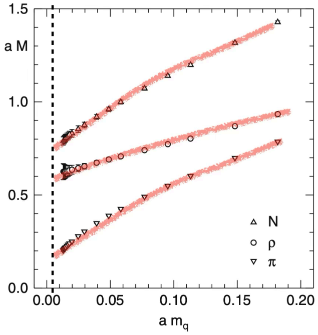
How do we make connection to the real world?

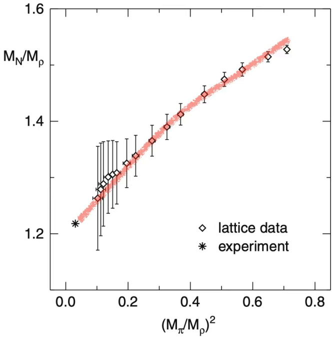
Setting the scale

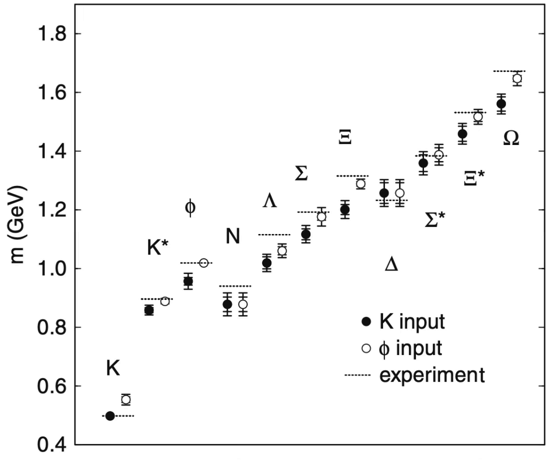
- In order to compare with the experiment, one has to relate a mass or length unit a to an experimental value

- Once a is physically known, for example, we can assign physical values to all observables

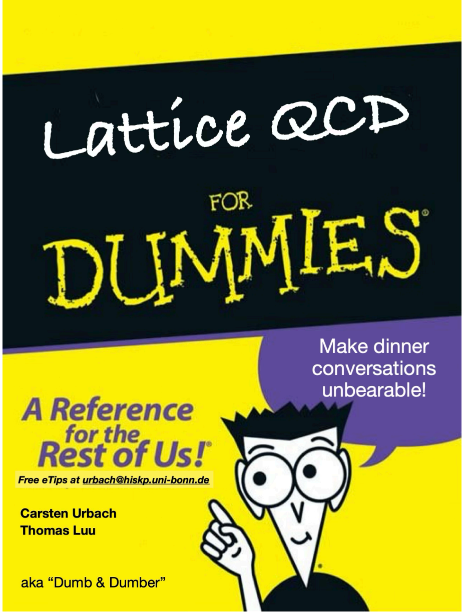
$$M_{\text{physical}} = \frac{(aM)_{\text{lat}}}{a}$$







[S. Aoki et al., Phys. Rev. D 67, 034503 (2003)]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