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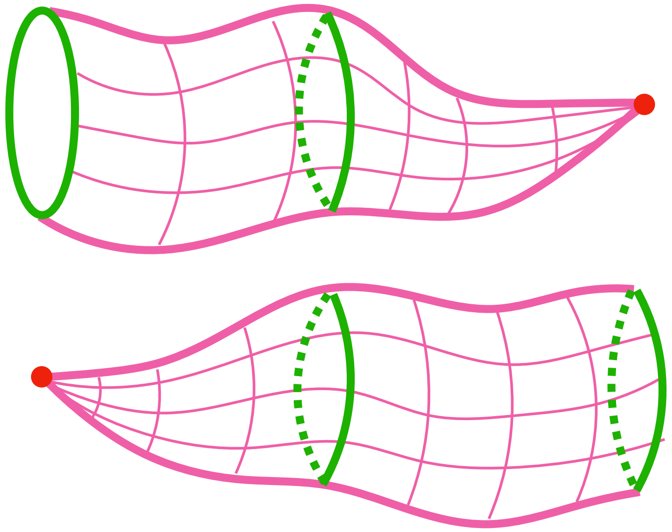
Periods of fibre products of elliptic surfaces

arxiv:2505.07685



Periods of algebraic varieties

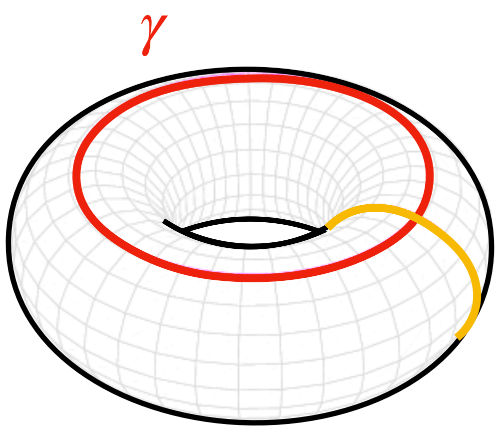
A **period** of an algebraic variety is the integral of a rational form of the variety on a cycle.

Some integration domain without boundary

$$\int_{T(\gamma)} \frac{A}{P^k}$$

A is a polynomial

P defines a smooth variety
$V(P) = \{z \mid P(z) = 0\}$



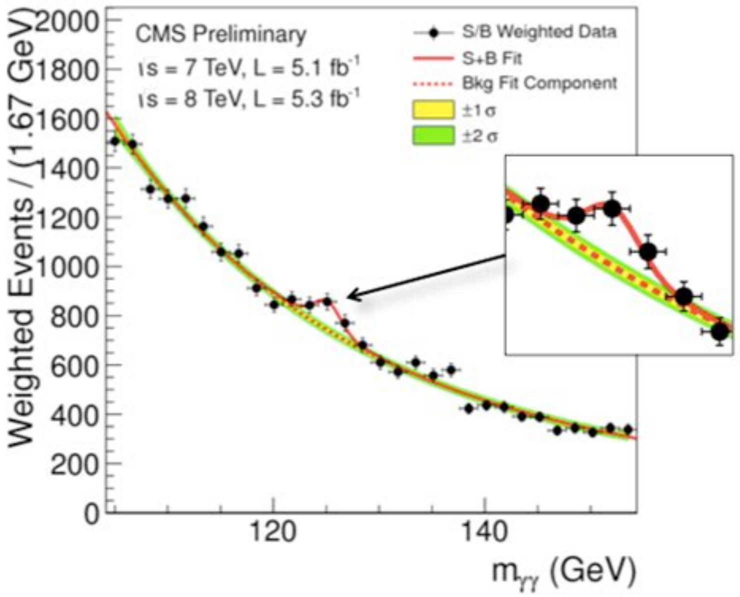
An elliptic curve

Torelli-type theorem for K3 surfaces:
Two K3 surfaces are isomorphic if and only if they have “the same” periods.

They describe the comparison between **topological data** (cycles)
and **algebraic data** (algebraic De Rham forms).

$$H_n(S, \mathbb{Z}) \times H^n_{DR}(S) \to \mathbb{C} \qquad \gamma, \omega \mapsto \int_\gamma \omega$$

Motivation and goals



Periods appear in diverse fields of mathematics and physics, such as **Quantum field theory** (Feynman integrals), **Hodge theory**, **motives**, **number theory** (BSD conjecture) …

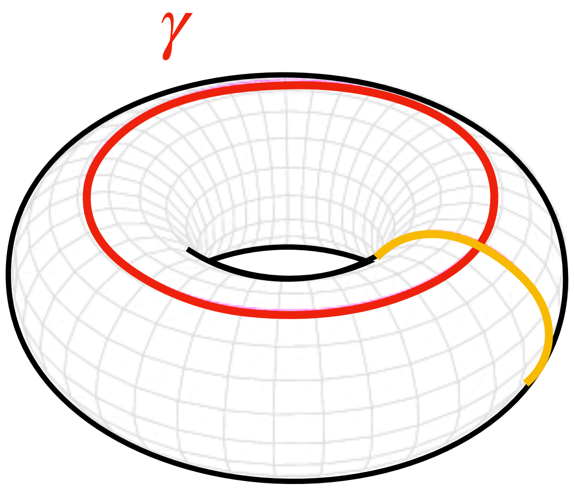
Hundreds of digits

Sufficiently many to recover algebraic invariants

Goal: compute numerical approximations of these integrals with **large precision.**

For this, we need an appropriate description of the integrals.

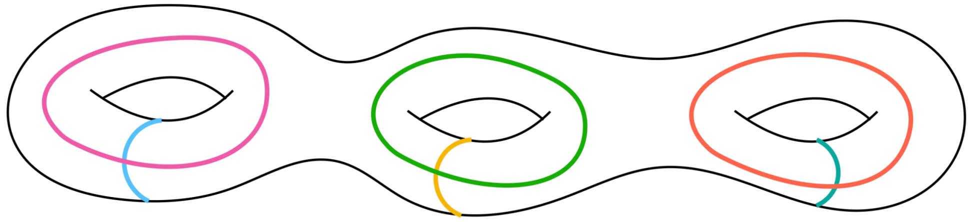
In particular we will focus on **understanding the cycles of integration** (the homology), how to represent them in a way that makes integration possible, and how to compute a basis of them.



Furthermore we want this to be **effective** and **efficient**.

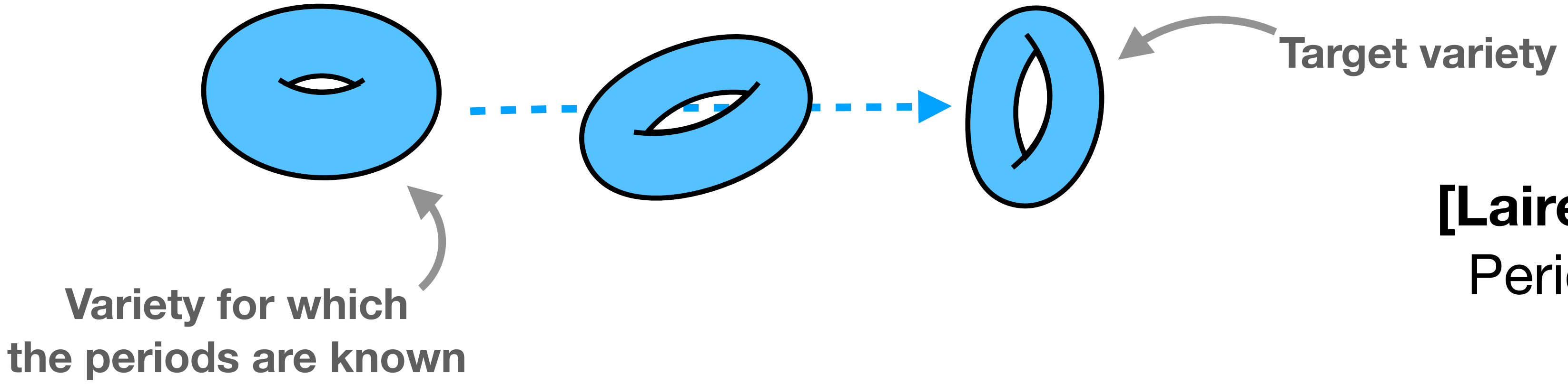
Previous works on period computations

[Deconinck, van Hoeij 2001], [Bruin, Sijsling, Zotine 2018], [Molin, Neurohr 2017]:
Algebraic curves (Riemann surfaces)



[Elsenhans, Jahnel 2018], [Cynk, van Straten 2019]:
Higher dimensional varieties
(double covers of $\mathbb{P}^2/\mathbb{P}^3$ ramified along a hyperplane arrangement)

[Sertöz 2019]: compute the period matrix of smooth projective hypersurfaces by **deformation**.



[Lairez PP Vanhove 2025]:
Periods of hypersurfaces

[PP 2025 x2]:
Periods of elliptic surfaces and fibre products of elliptic surfaces

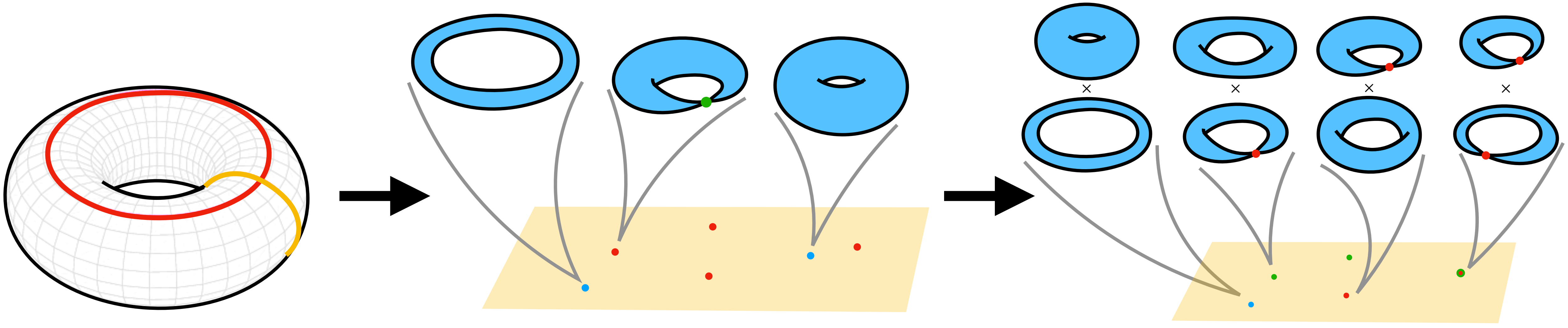
[Đonlagić 2025]:
Periods of fibre products of elliptic surfaces with semi-simple singular fibres

In this talk

Elliptic curves

and ~~beyond~~ higher dimensions

Algebraic curves | **Elliptic (K3) surfaces** | **Calabi-Yau threefolds**



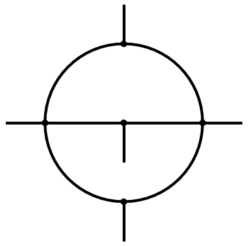
[Deconinck, van Hoeij 2001]

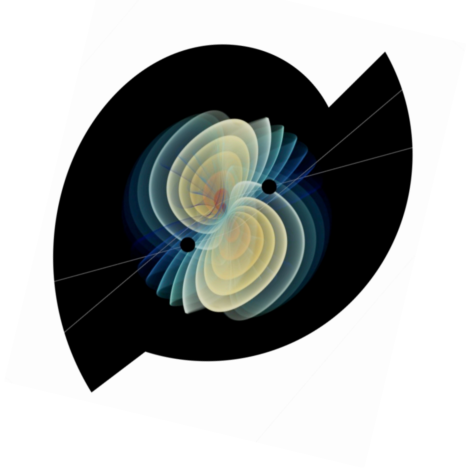
[Lairez, PP, Vanhove 2025]

arxiv:2306.05263

[PP 2025]

arxiv:2505.07685





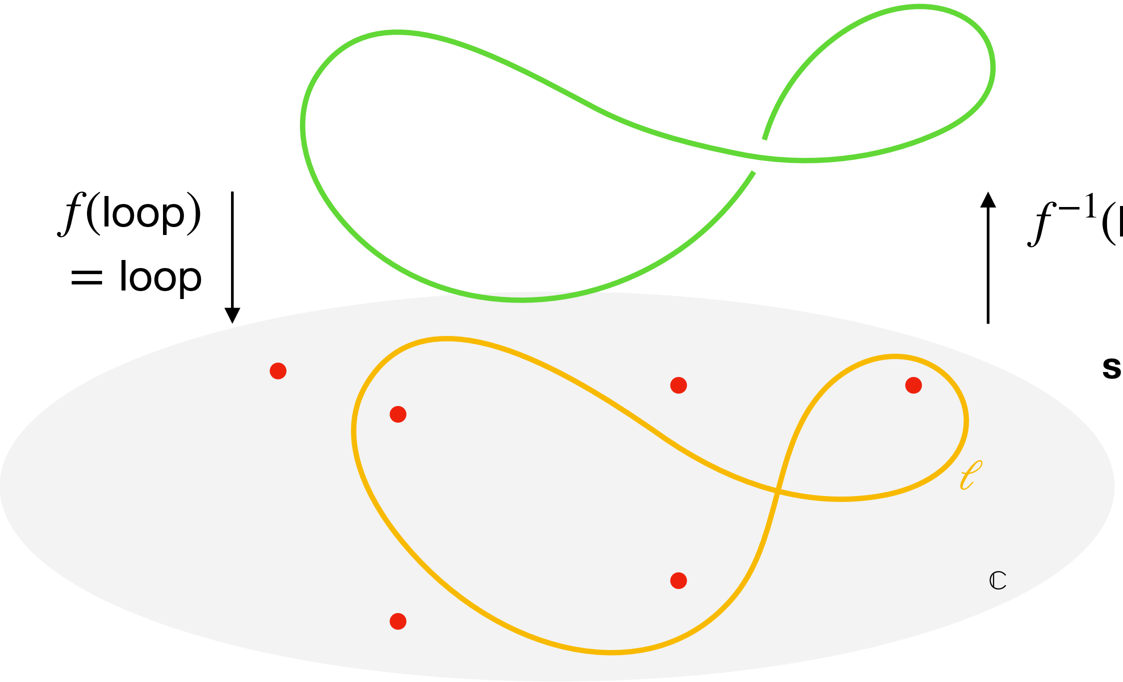
Periods of algebraic curves

First example: an elliptic curves

Let $\mathscr{X}$ be the elliptic curve defined by
$P = y^3 + x^3 + 1 = 0$
and let $f : (x, y) \mapsto y/(2x + 1)$ be a generic projection.

The fibre above $t \in \mathbb{C}$ is $\mathscr{X}_t = f^{-1}(t)$
$= \{(x, t(2x + 1)) \mid P\left(x, t(2x + 1)\right) = 0\}$.
It deforms continuously with respect to t.

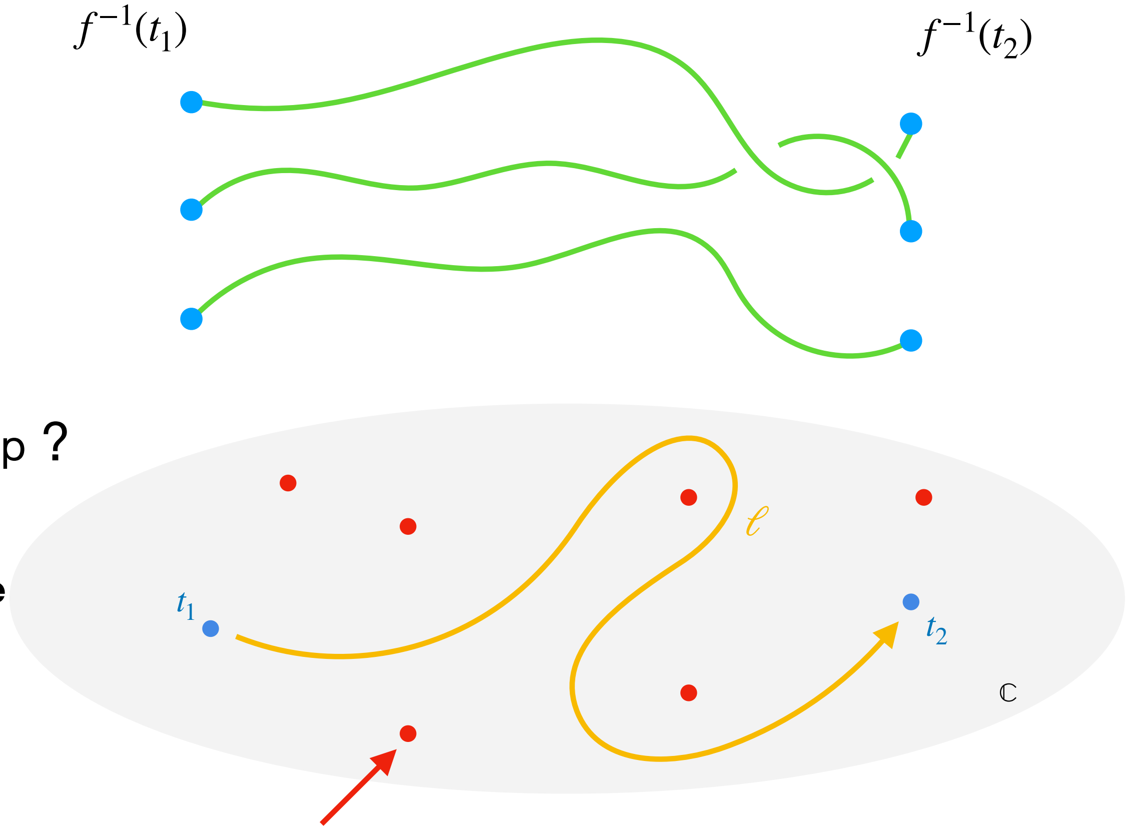
We are looking for closed paths in $\mathscr{X}$, up to deformation (1-cycles).



$f(\text{loop})$
$= \text{loop}$

$f^{-1}(\text{loop}) = \text{loop}$?

Not always,
see next slide

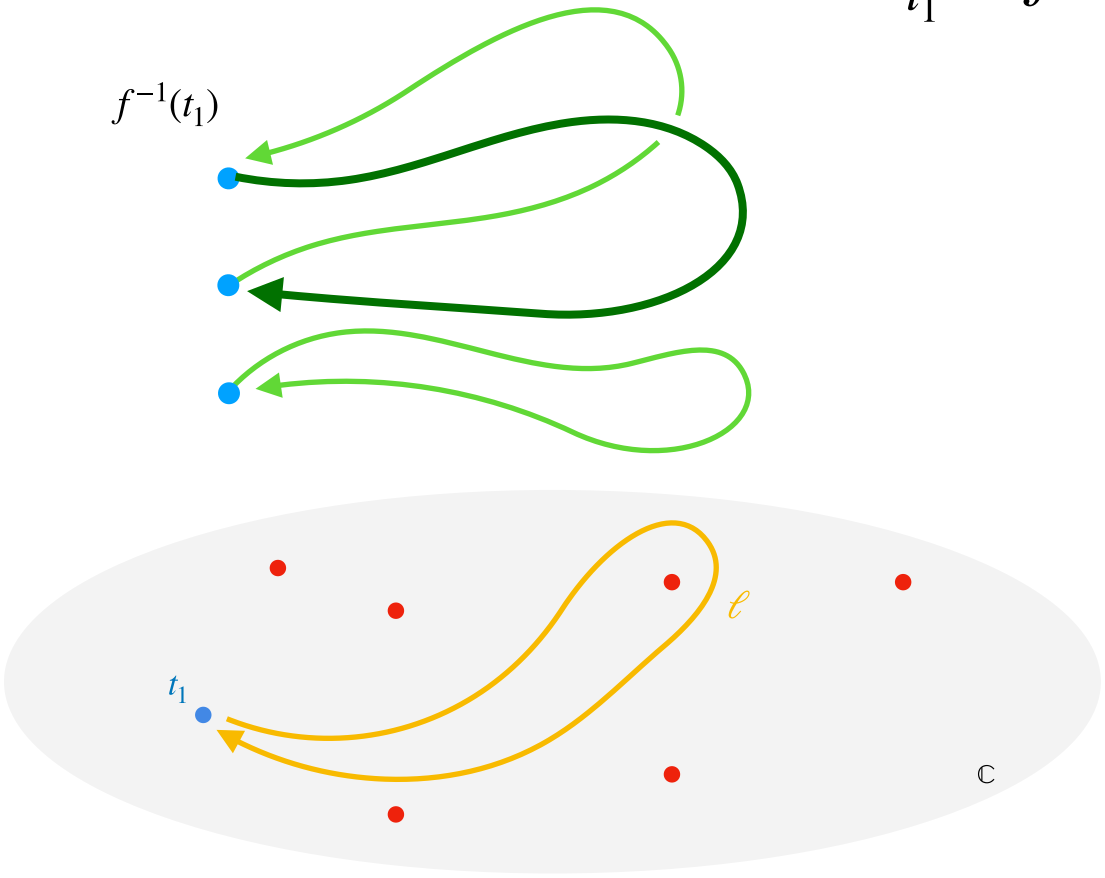


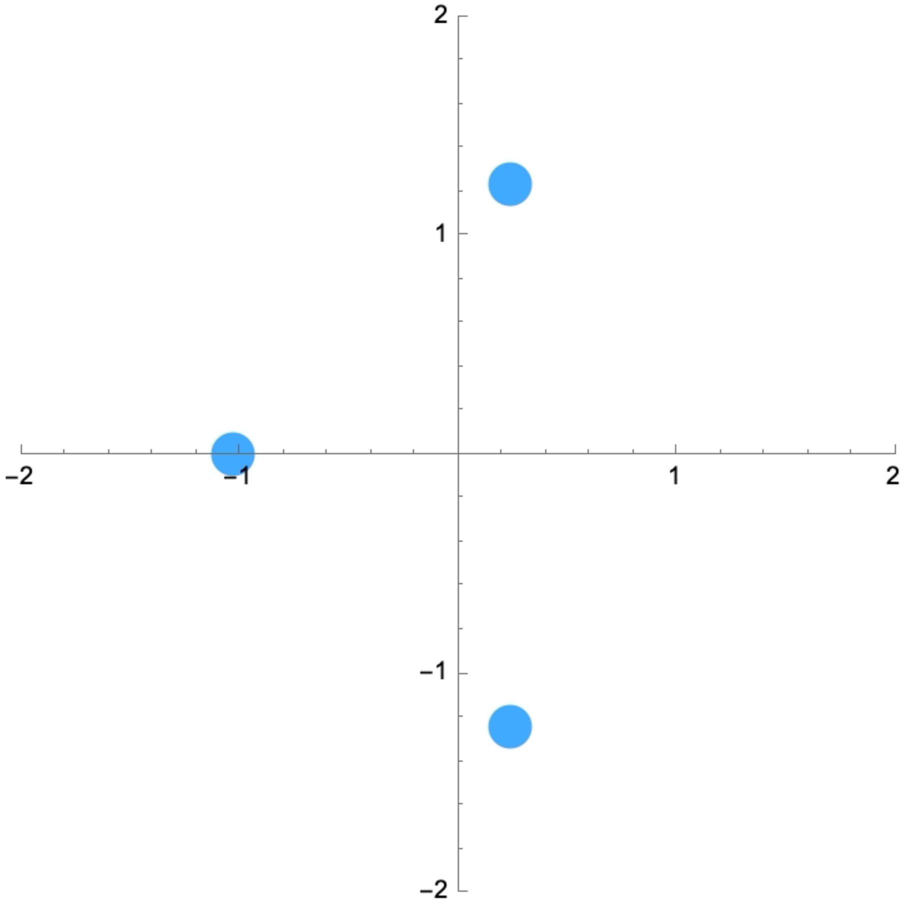
Values of t for which
$P(x, t(2x + 1)) = t^3(2x + 1)^3 + x^3 + 1$
has a double root (critical values)

What happens when you loop around a critical point?

A loop ℓ in $\mathbb{C}$ pointed at t_1 induces a permutation of

$$\mathcal{X}_{t_1} = f^{-1}(t_1).$$





This permutation is called the **action of monodromy along ℓ** on $\mathcal{X}_{t_1}$.

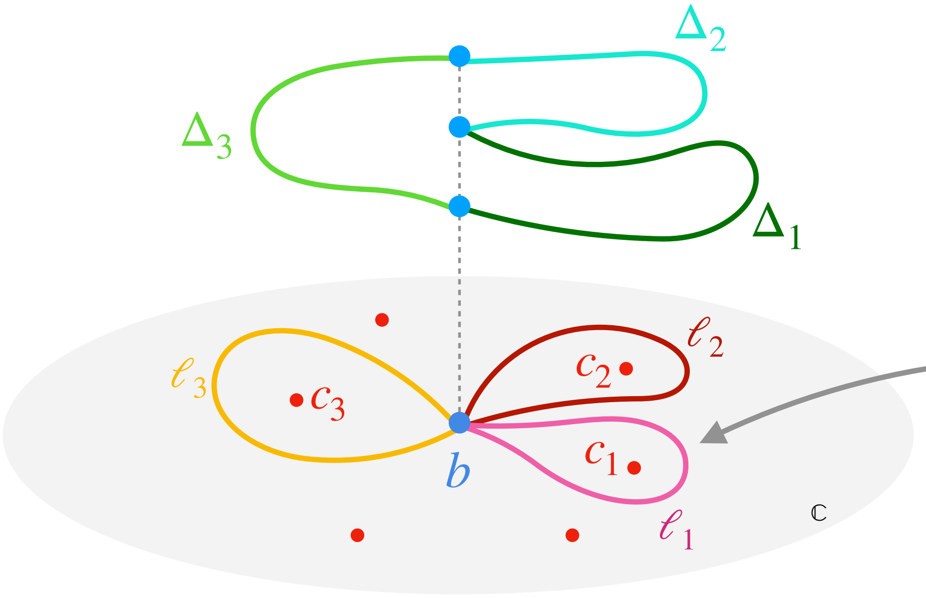
It is denoted ℓ_*.

If ℓ is a simple loop around a critical value, ℓ_* is a transposition.

Periods of algebraic curves

The lift of a simple loop ℓ around a critical value c that has a non-trivial boundary in $\mathcal{X}_b$ is called the **thimble** of c. It is an element of $H_1(\mathcal{X}, \mathcal{X}_b)$.

Relative homology of the pair $(\mathcal{X}, \mathcal{X}_b)$



Simple loop around c_1

Thimbles serve as building blocks to recover $H_1(\mathcal{X})$.
It is sufficient to glue thimbles together in a way such that their boundaries cancels.

Concretely, we take the kernel of the boundary map

$$\delta : H_1(\mathcal{X}, \mathcal{X}_b) \to H_0(\mathcal{X}_b)$$

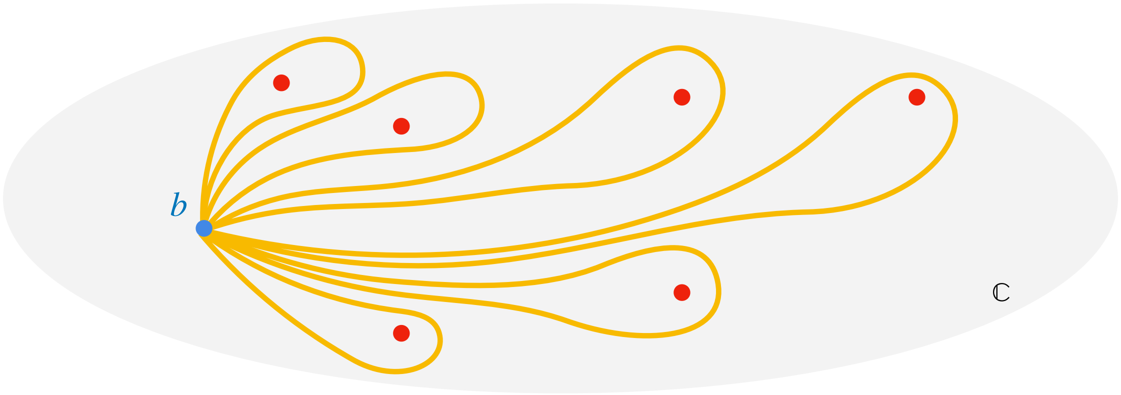
Fact: all of $H_1(\mathcal{X})$ can be recovered this way.

$$0 \to H_1(\mathcal{X}) \to H_1(\mathcal{X}, \mathcal{X}_b) \to H_0(\mathcal{X}_b)$$

Generated by thimbles

Computing periods of algebraic curves

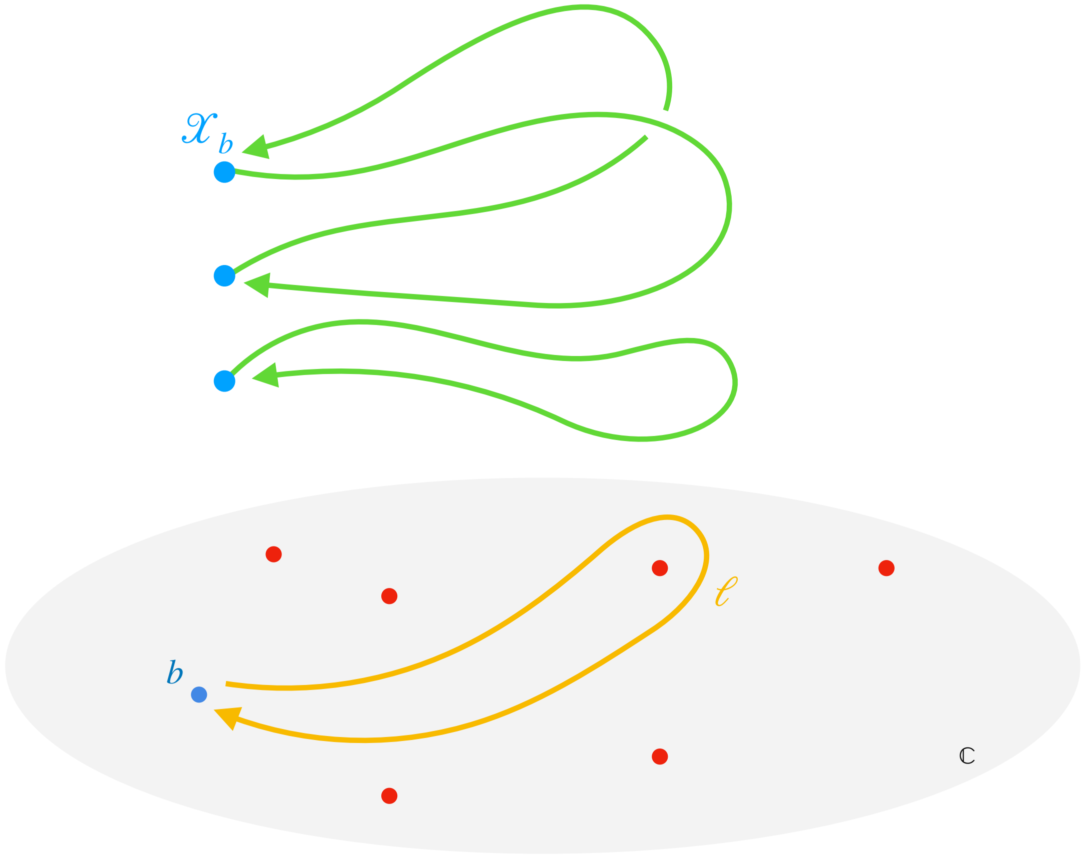
1. Compute simple loops $\ell_1, \ldots, \ell_{\text{\#crit.}}$ around the critical values — basis of $\pi_1(\mathbb{C} \setminus \{\text{crit. val.}\})$



Computing periods of algebraic curves

1. Compute simple loops $\ell_1, \dots, \ell_{\#\text{crit.}}$ around the critical values — basis of $\pi_1(\mathbb{C}\setminus\{\text{crit. val.}\})$

2. For each i compute the action of monodromy along ℓ_i on $\mathscr{X}_b$ (transposition)



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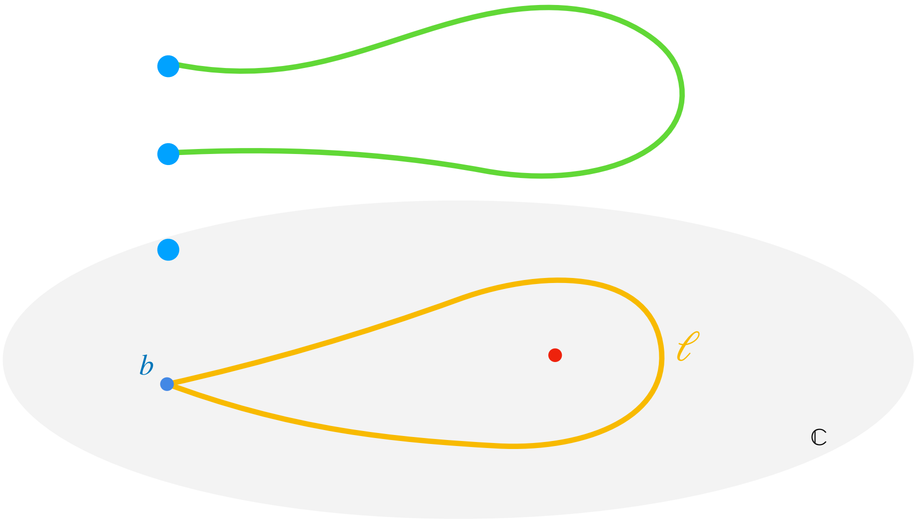
3. This provides the corresponding thimble Δ_i. Its boundary is the difference of the two points of $\mathcal{X}_b$ that are permuted.



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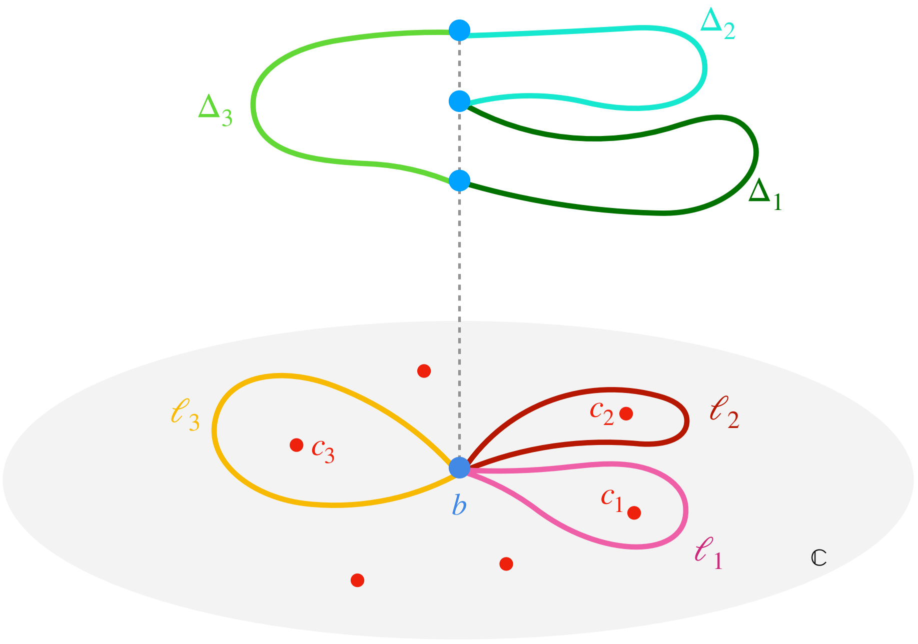
4. Compute sums of thimbles without boundary → basis of $H_1(\mathscr{X})$



Computing periods of algebraic curves

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4. Compute sums of thimbles without boundary → basis of $H_1(\mathcal{X})$

5. Periods are integrals along these loops
→ we have an explicit parametrisation of these paths → numerical integration.

$$\int_\gamma \omega = \int_\ell \omega_t$$

Elliptic surfaces

[Lairez PP Vanhove 2024]

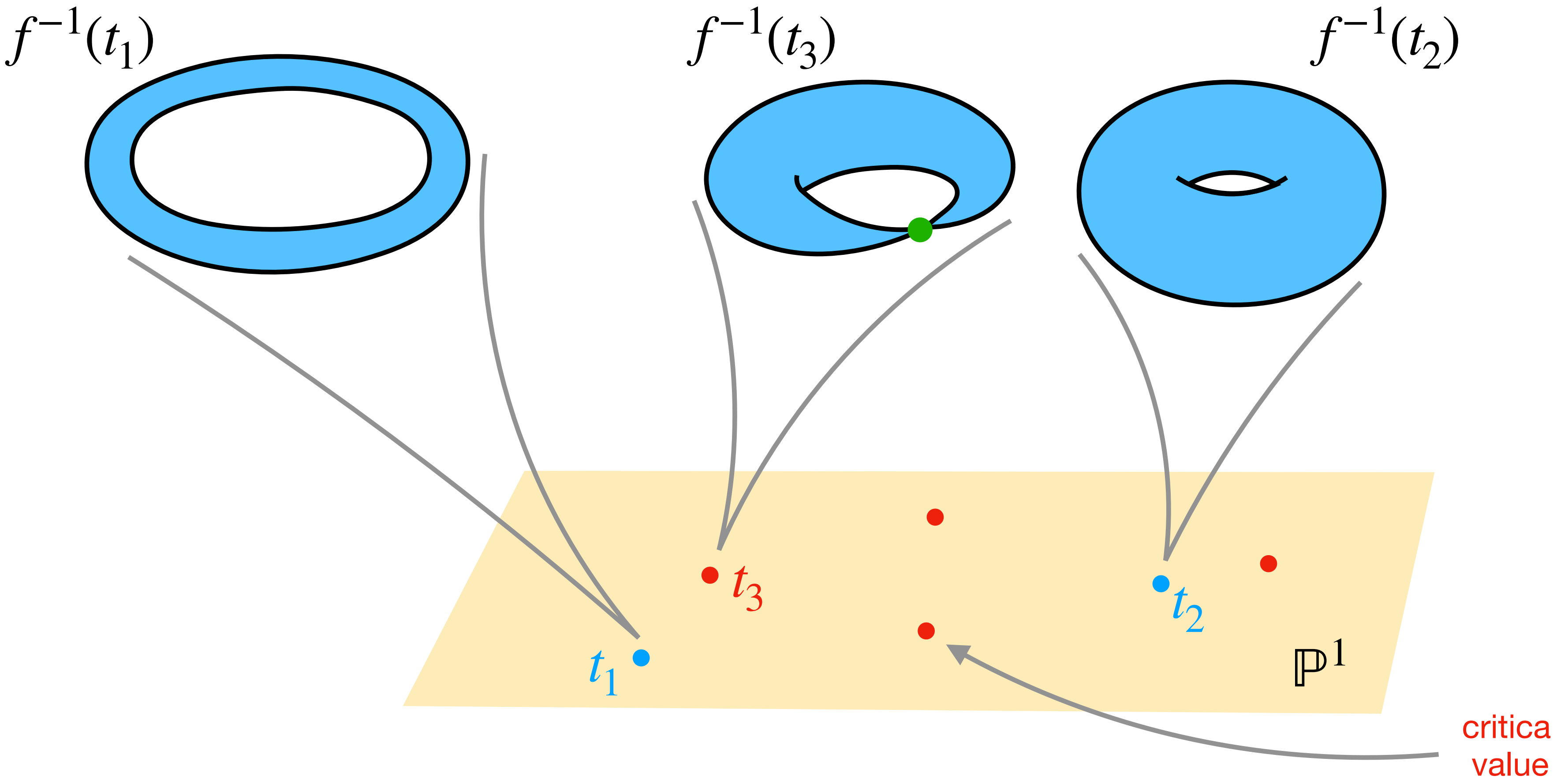
[PP 2024]

Elliptic surfaces

An **elliptic surface** S is a smooth algebraic surface equipped with a map to the projective line

$$f : S \to \mathbb{P}^1$$

such that all but finitely many fibres $f^{-1}(t)$ are elliptic curves.

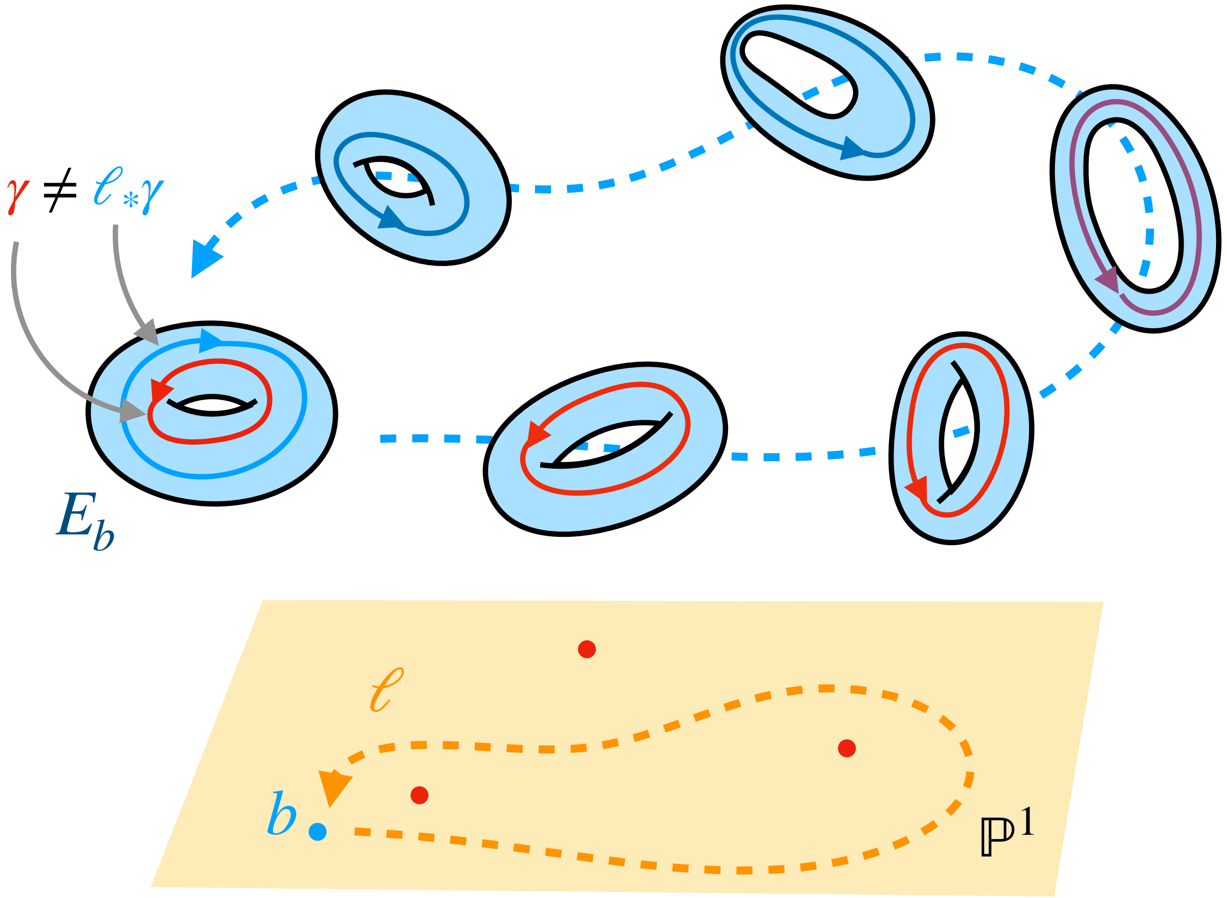


Monodromy

Let $\mathscr{X}$ be a smooth surface in $\mathbb{P}^3$. We consider a projection $\mathscr{X} \to \mathbb{P}^1$.
The fibre $\mathscr{X}_t = f^{-1}(t)$ is a curve, which deforms continuously as t moves in $\mathbb{P}^1$.

Ehresmann's
fibration theorem

The map $\ell_* : H_1(\mathscr{X}_b) \to H_1(\mathscr{X}_b)$ induced by this deformation along a loop ℓ is called the **monodromy along ℓ**.



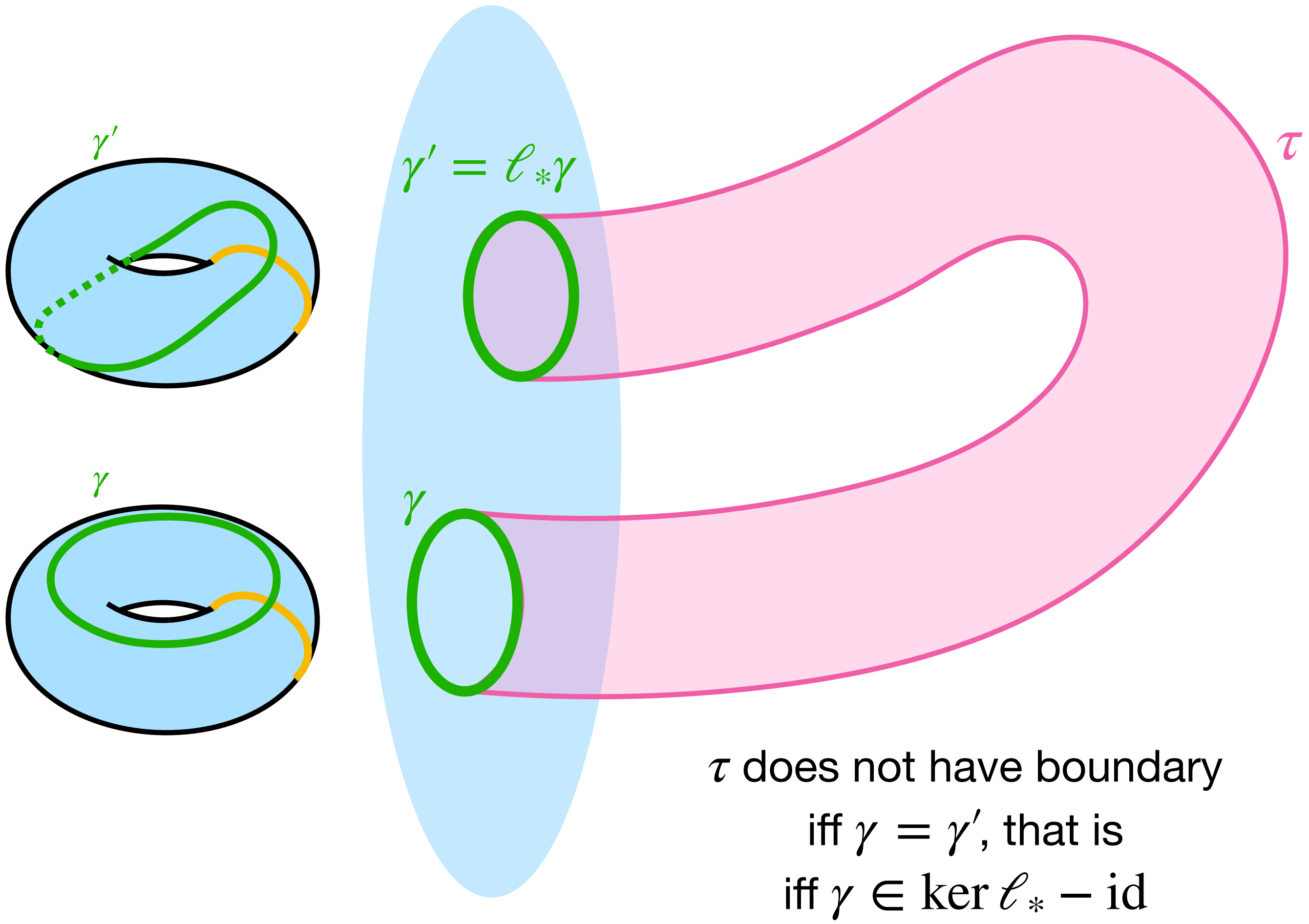
Extensions

We can recover **2-cycles** of surfaces as **extensions** of 1-cycles of the fibre.

$$\pi_1(\mathbb{P}^1 \backslash \Sigma, b) \times H_1(E_b) \to H_2(S, E_b)$$

$$\ell, \gamma \mapsto \tau_\ell(\gamma)$$

$$\partial \tau_\ell(\gamma) = \gamma' - \gamma$$

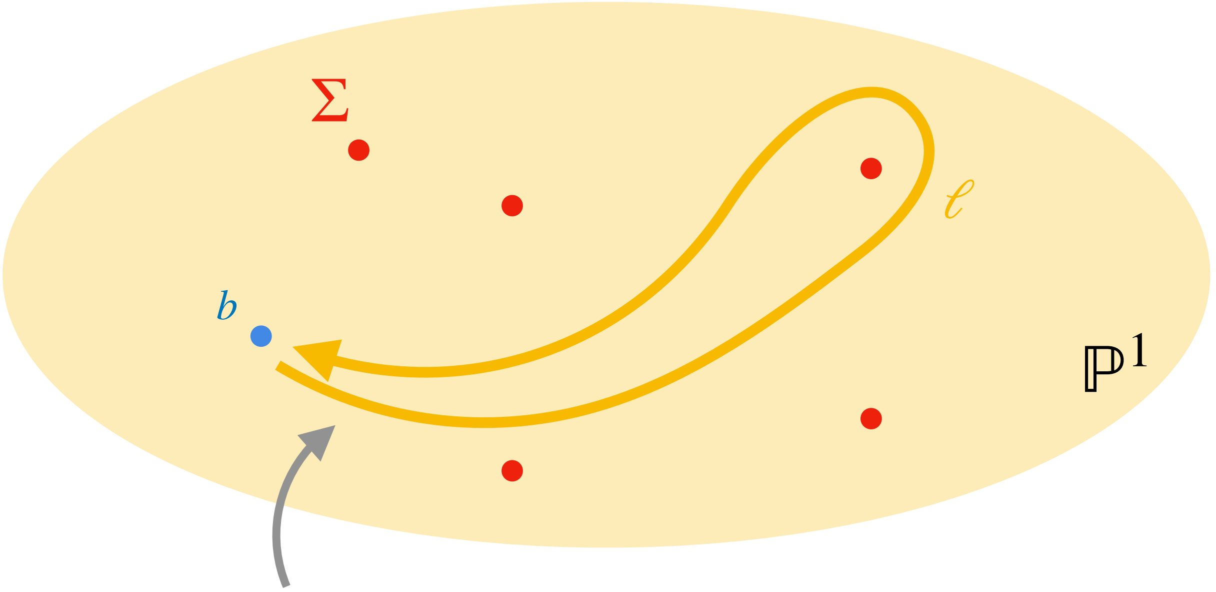


τ does not have boundary
iff $\gamma = \gamma'$, that is
iff $\gamma \in \ker \ell_* - \mathrm{id}$

This description of cycles is well-suited for integrating the periods:

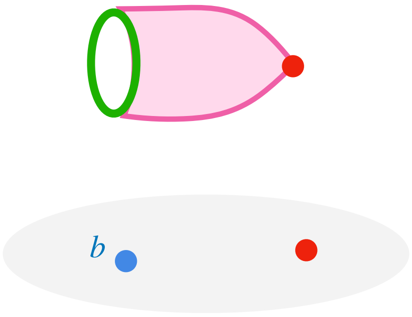
$$\int_{\tau_\ell(\gamma)} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_\ell \left(\int_\gamma f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$$

Two line integrals:
we know how to compute these efficiently!
[Chudnovsky², Van der Hoeven, Mezzarobba]

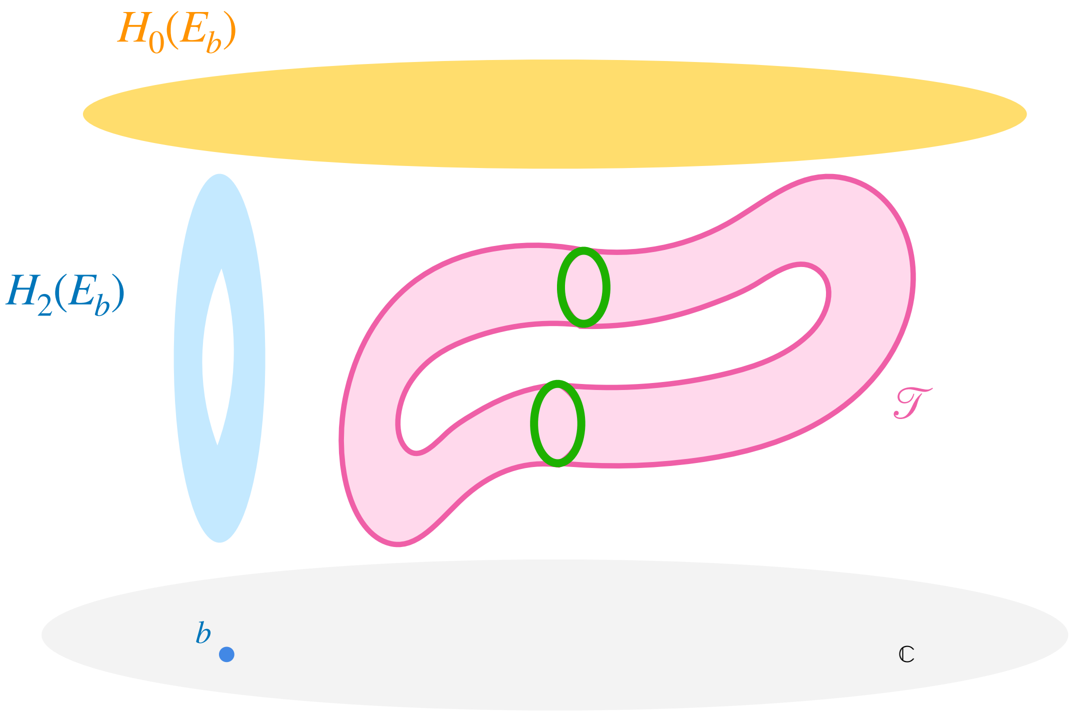


such a path is called a **simple loop**

Parabolic homology



Each **simple loop** ℓ contributes relative homology classes, called **thimbles**, in $H_2(S, E_b)$.



Thimbles serve as building blocks for extensions: we can glue thimbles together in a way that matches their **boundary** to obtain closed cycles.

Obtained from the monodromy :

$$\partial\tau_\ell(\gamma) = \ell_*\gamma - \gamma$$

Furthermore $H_2(S)$ is generated by **extensions**, **fibre components**, and a **section**.

Their periods are zero.
We only need to compute periods of extensions.

The **parabolic homology** is the lattice generated by **extensions**.

Algorithm for computing periods of elliptic surfaces

1. Compute the set Σ of **critical values**.
2. Compute a basis of **simple loops** $\ell_1, \ldots, \ell_r$ of $\pi_1(\mathbb{P}^1 \backslash \Sigma, b)$.
3. For each $1 \leq i \leq r$, compute the **monodromy map** ℓ_{i*}.
4. Match **boundaries** of thimbles together to obtain **extensions**.
5. Integrate the **periods** on the parabolic homology.

Gauss-Manin connection

The cohomology sheaf $\mathscr{H}^n_{DR}\left(E_t/\mathbb{Q}(t)\right)$ inherits a connection from the derivation in the base $\mathbb{P}^1$: the **Gauss-Manin connection** **[Katz Oda 1968]**.

Period functions $\int_{\gamma_t} \omega_t$ are solutions to a Fuchsian differential equation: **the Picard-Fuchs equation**.

This connection can be computed explicitly via the Griffiths—Dwork reduction **[Griffiths 1969, Dwork 1964]**.

Example: Let $\mathscr{X}_t = V(X^3 + Y^3 + Z^3 + tXYZ)$ be an elliptic surface.
A basis of the De Rham cohomology sheaf is given by the residues of

$$\omega_1(t) = \frac{\Omega}{X^3 + Y^3 + Z^3 + tXYZ} \text{ and } \omega_2(t) = \frac{X^3\Omega}{(X^3 + Y^3 + Z^3 + tXYZ)^2}.$$

Let $\mathscr{L} = (t^3 + 27)\partial_t^2 + 3t^2\partial_t + t$. Then $\mathscr{L}\omega_1$ is an exact differential. In particular for any cycle γ_t the period function $\pi(t) = \int_{\gamma_t} \operatorname{res} \omega_1(t)$ is a solution of $\mathscr{L}$.

Computing monodromy

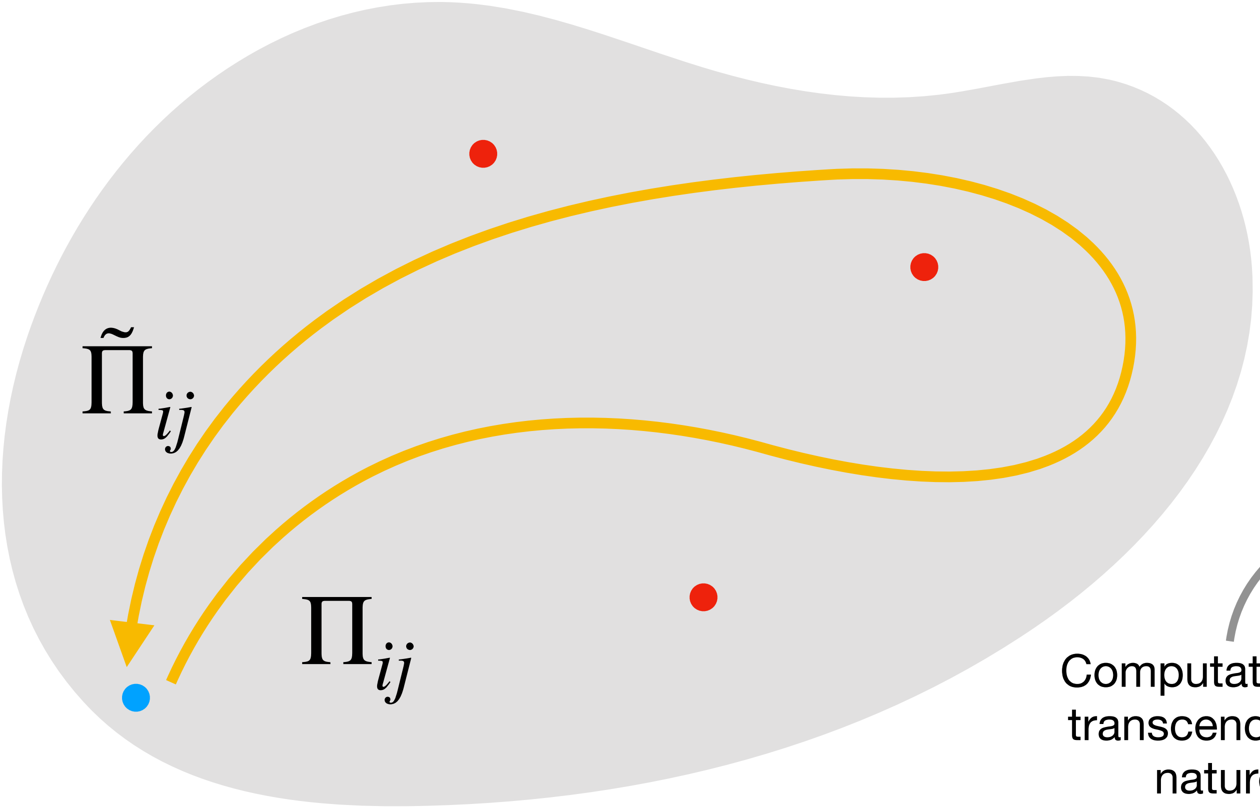
Globally defined
$\Longrightarrow$ no monodromy

$$\Pi_{ij} = \int_{\gamma_j} \partial_t^i \omega_t \xrightarrow[\text{[Chudnovsky}^2\text{ 90, Van der Hoeven 99, Mezzarobba 2010]}]{\text{Analytic continuation}} \tilde{\Pi}_{ij} = \int_{\sum_k c_{kj}\gamma_k} \partial_t^i \omega_t = \sum_k c_{kj} \int_{\gamma_k} \partial_t^i \omega_t$$

Solution to
Picard-Fuchs
equation of ω_t

$$\tilde{\gamma}_j = \sum_k c_{kj}\gamma_k$$

The c_{kj}'s are integers



Thus $\tilde{\Pi} = \Pi C$ i.e.

$$\boxed{\Pi^{-1}\tilde{\Pi} = C \in \mathrm{GL}_2(\mathbb{Z})}$$

Computation of
transcendental
nature

It is sufficient to carry out this
computation with precision $< 1/2$
to recover C exactly.

Computing monodromy of differential operators

[Chudnovsky² 90, Van der Hoeven 99, Mezzarobba 2010]

In a small radius around α:

$$\left| f(t) - \sum_{k=0}^{m} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \right| \leq \mathscr{P}(m) 2^{-m}$$

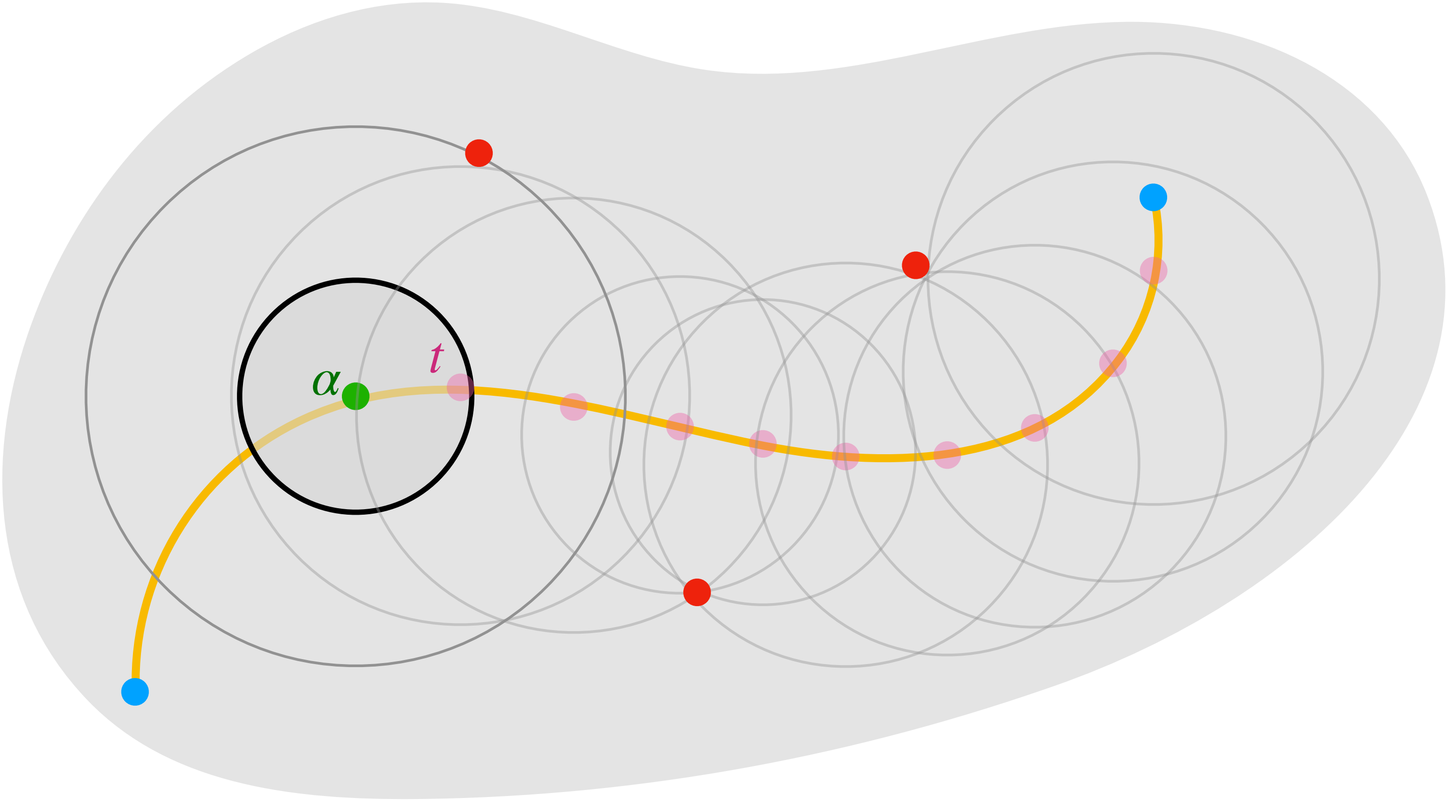
polynomial
in m (effective)
[Mezzarobba Salvy 2009]

We compute $f^{(k)}(\alpha)$ from $\mathscr{L}$.

In a disk around α, the precision given by the Taylor formula is exponential in its order.

From the derivatives at α,
we can recover the derivatives at t.

Linear complexity:
recover m digits in $\mathcal{O}(m)$ operations
(using binary splitting)



Recovering certain algebraic invariants

Theorem [Doran Harder PP Vanhove 2024]: The Tardigrade hypersurface has the same motivic geometry as a quartic K3 surface with six A_1 singularities.

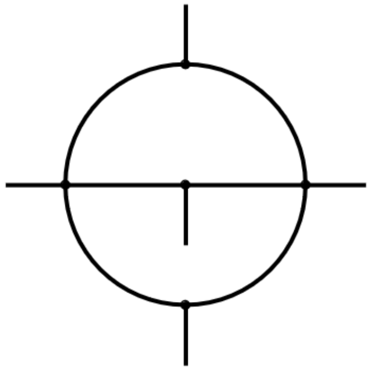


FIGURE 13. The tardigrade graph

Our methods allow to compute the periods of this quartic K3 surface.

From the periods, we recover numerically that **its Néron-Severi rank is 11** for generic values of the mass parameters.

Lefschetz's theorem on (1,1) classes:

A homology class $\gamma \in H_2(S)$ is in the Néron-Severi group $NS(S)$ iff the periods of holomorphic forms on γ vanish.

Using the LLL algorithm, we can heuristically recover this kernel by finding integer linear relations between the periods.

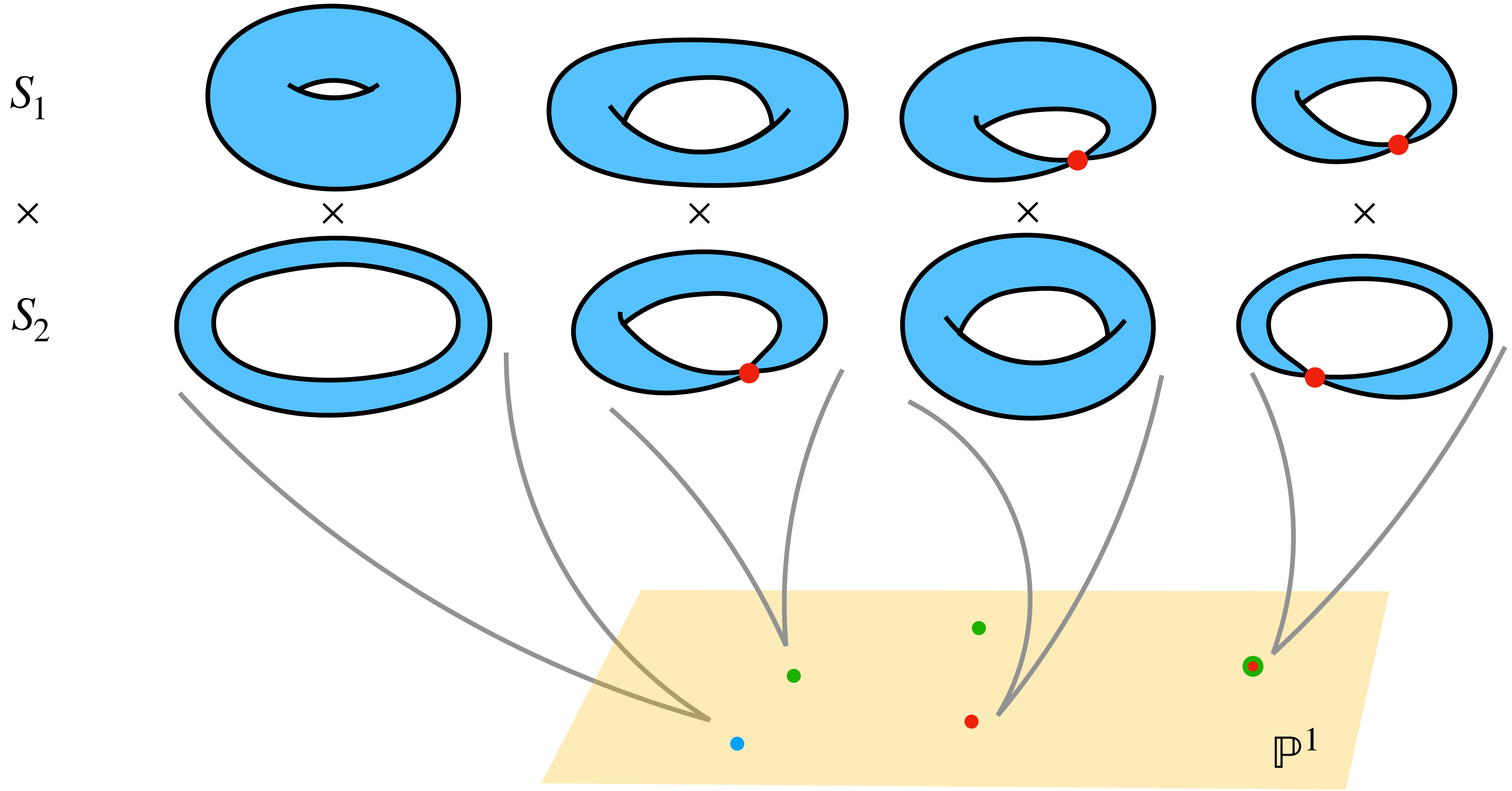
Fibre products

of elliptic surfaces

Schoen's construction

The fibre product $T = S_1 \times_{\mathbb{P}^1} S_2$ of two rational elliptic surface with disjoint critical values yields a smooth **Calabi-Yau threefold. [Schoen 1988]**

"[…] a class of such threefolds which is large enough to exhibit many of the phenomena which one wants to study, yet is special enough to be quite tractable."



When critical values coincide, we obtain a singular threefold.
Under certain conditions, the singularities admit a crepant resolution:
we obtain a smooth Calabi-Yau threefold. **[Kapustka² 2009]**

Goal: We want to compute periods of such threefolds

Homology of smooth fibre products

We can use the same construction to compute the parabolic homology.

By the Künneth formula, the homology of the fibre is

$$H_2(E_1 \times E_2) = H_0(E_1) \otimes H_2(E_2) \oplus \boxed{H_1(E_1) \otimes H_1(E_2)} \oplus H_2(E_1) \otimes H_0(E_2)$$

only component with monodromy

Periods of the fibres are products of periods

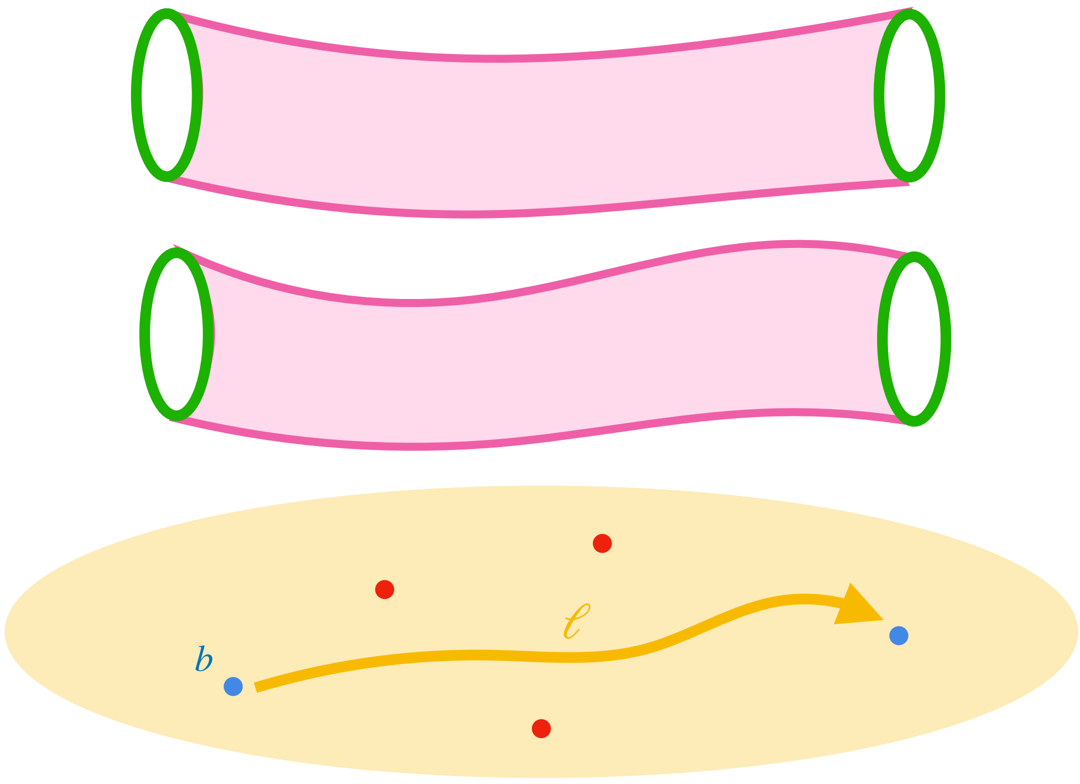
$$\int_{\gamma_1 \times \gamma_2} \omega_1 \otimes \omega_2 = \int_{\gamma_1} \omega_1 \int_{\gamma_2} \omega_2$$

The monodromy representation on $H_2(E_1 \times E_2)$ is the tensor product of the monodromy representations

$$M_\ell = M_{1\ell} \otimes M_{2\ell} \in \mathrm{GL}_4(Z)$$

$$\int_{\tau_\ell(\gamma_1 \times \gamma_2)} \omega_1 \otimes \omega_2 \wedge \mathrm{d}t = \int_\ell \left(\int_{\gamma_1} \omega_1 \int_{\gamma_2} \omega_2 \right) \mathrm{d}t$$

Extensions are fibre products of extensions of the elliptic surfaces



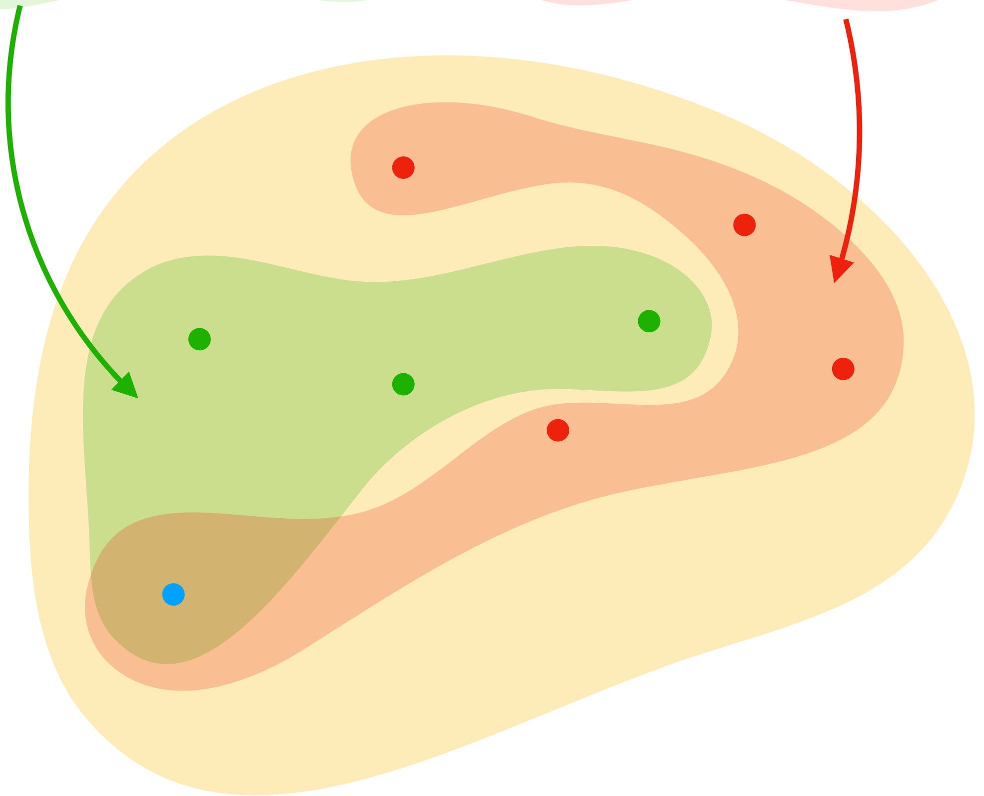
Homology of smooth fibre products

The homology group $H_3(T)$ is generated by **extensions**, and **fibre components** of the elliptic surfaces.

Proposition: the homology of the fibre product can be recovered from the monodromy representation of the elliptic surfaces.

$$H_3(T^*, F_b) =$$

$$H_2(S_1^*, E_{1b}) \otimes H_1(E_{2b}) \oplus H_2(S_2^*, E_{2b}) \otimes H_1(E_{1b})$$

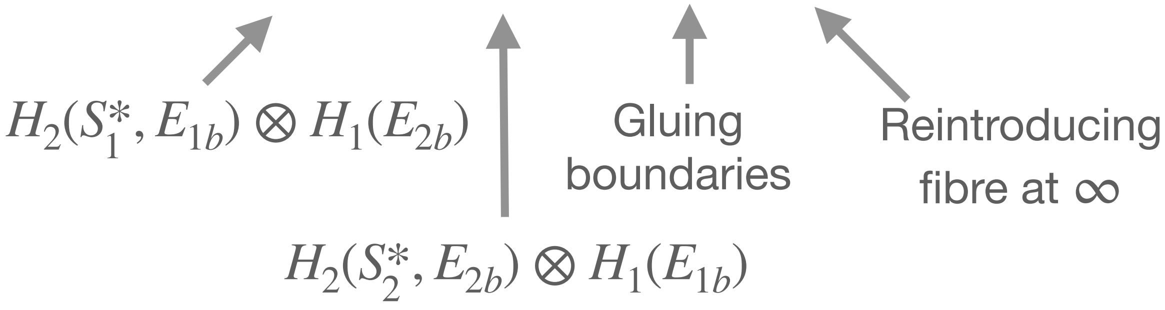


Here $*$ means that we removed one fibre “at infinity”

In the case of rational surfaces, we recover a result of Schoen:

$H_3(T)$ has rank

$$12 \times 2 + 12 \times 2 - 4 - 4 = 40.$$



$H_2(S_1^*, E_{1b}) \otimes H_1(E_{2b})$

Gluing boundaries

Reintroducing fibre at ∞

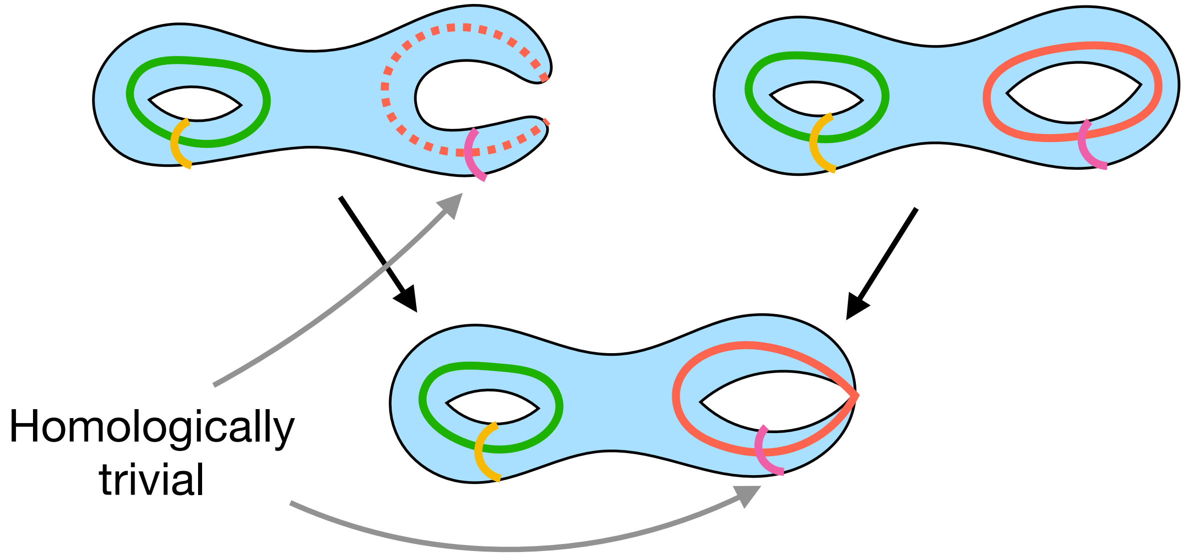
$H_2(S_2^*, E_{2b}) \otimes H_1(E_{1b})$

We have an explicit description of these cycles. We can perform the same integration methods

$$\int_{\tau_\ell(\gamma_1\times\gamma_2)} \omega_1 \otimes \omega_2 \wedge \mathrm{d}t = \int_\ell \left(\int_{\gamma_1} \omega_1 \int_{\gamma_2} \omega_2 \right) \mathrm{d}t$$

Smoothings and vanishing cycles

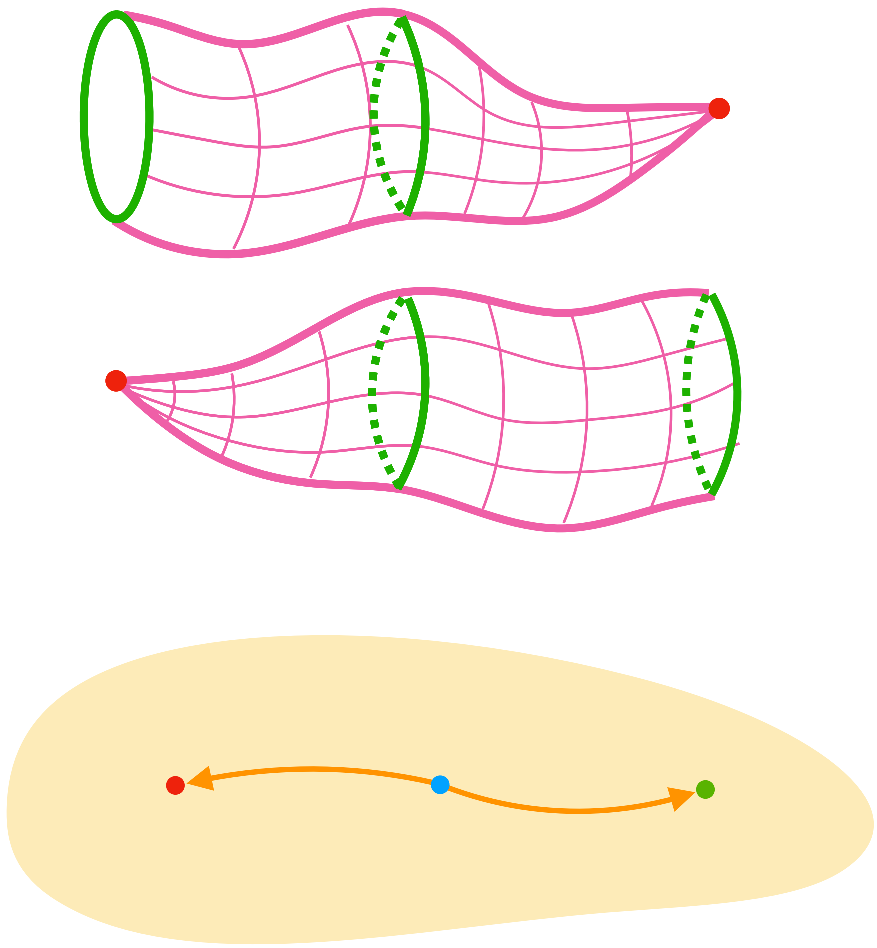
The one dimensional case



We reduce to the smooth case by **smoothing** the variety.

This creates new cycles which collapse in the singular limit: the **vanishing cycles**.

Our threefolds



We can compute the periods on the orthogonal complement of the lattice of vanishing cycles.

Calabi-Yau operators

Hadamard products

The **Hadamard product** of two elliptic surfaces S_1 and S_2 is the family of threefolds

$$T_u = S_1 \times_u S_2 := S_1 \times_{\mathbb{P}^1} \varphi_u^* S_2$$

where $\varphi_u \colon \mathbb{P}^1 \to \mathbb{P}^1$, $\phi_u(t) = u/t$.

T_u is equipped with a map to $\mathbb{P}^1$ with fibre $E_{1t} \times E_{2\frac{u}{t}}$

Under some condition on the fibres of S_1 and S_2 at 0 and ∞, there is a **maximal unipotent monodromy point** at 0.

The holomorphic period of is the Hadamard product of the periods of the underlying surfaces:

$$\pi(u) = \sum_i a_i b_i u^i \qquad \text{where} \qquad \pi_1(t) = \sum_i a_i t^i \text{ and } \pi_2(t) = \sum_i b_i t^i$$

We are interested in the case where the Picard-Fuchs equation $\mathscr{L}^{\mathrm{Had}}$ has order 4 — see next slides.

T_u is generically singular, but we can smooth homogeneously in u.
It is not known whether we can resolve homogeneously (and crepantly!) in u.

[Golyshev van Straten 2023]

Calabi-Yau operators

Calabi-Yau operators are differential operators in one variable satisfying certain conditions:

→ be **Fuchsian**, i.e. having solutions that have "nice" singularities

→ be **self-dual**, some technical notion stemming from mirror symmetry

→ have a **Maximal unipotent monodromy** point, i.e. such that the monodromy around it satisfies $(M-1)^n = 0$ and $(M-1)^k \neq 0, \forall k < n$

order

→ certain **integrality** conditions on the holomorphic solution, instanton numbers and q-coordinates

These operators are expected to be Picard–Fuchs equations of families of algebraic varieties.

Calabi-Yau operators of order 4

[AESZ 2010] gave a list of around 500 Calabi-Yau (CY) operators of order 4 obtained partially through an extensive computer search.

They are conjectured to be the Picard-Fuchs equations of varieties carrying a motive of type (1,1,1,1), that is families with one degree of freedom.

There are 14 hypergeometric Calabi-Yau operators of order 4 and 105 Hadamard products of elliptic surfaces.

In many cases, a geometric realisation is not known, and in some cases a smooth geometric realisation is not known (e.g. 14th hypergeometric operator).

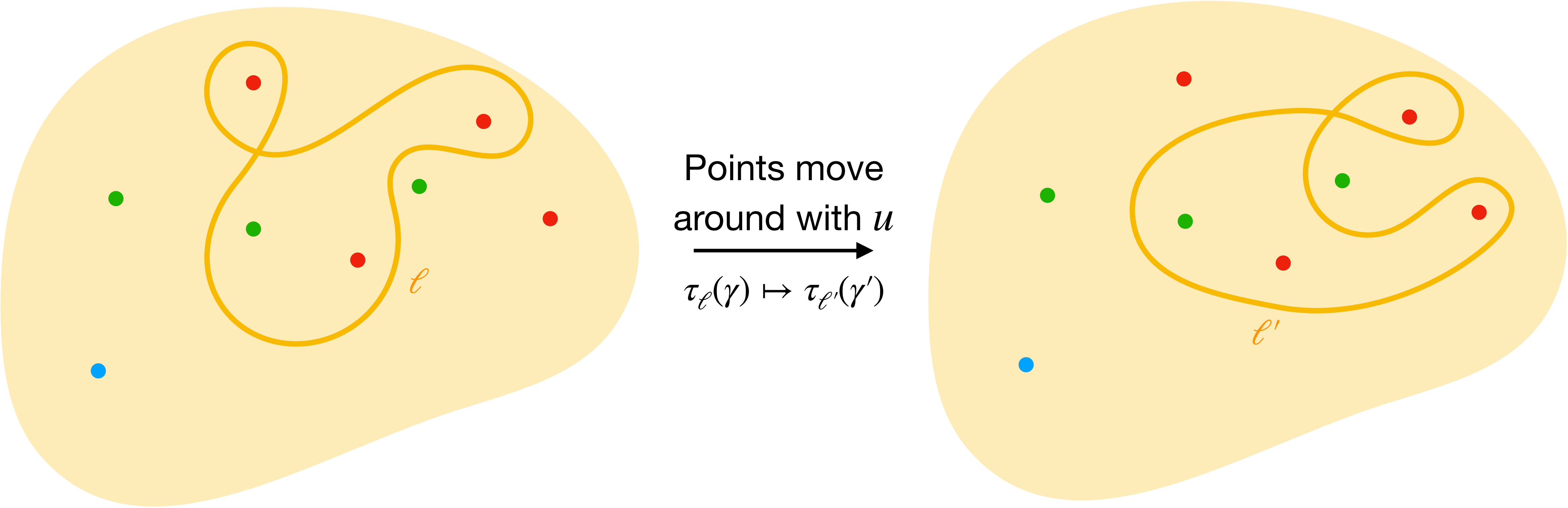
In particular this motivates using smoothings instead of looking for resolutions.

$$
\begin{aligned}
A &: y^2 - yx - ty = x^3 + tx^2\,,\\
B &: y^2 = x^3 - (12t-1)\,x^2 + 48t^2x - 64t^3\,,\\
C &: y^2 = x^3 + (144t-3)\,x - 144t + 2\,,\\
D &: y^2 = x^3 - 3x + 1728t - 2\,,\\
a &: y^2 - (2t-1)\,yx + 3t^2x^2 + 2t^3y + \left(-3t^4\right)x + t^6 = x^3\,,\\
b &: y^2 - (t+11)\,yx - ty = x^3 + tx^2\,,\\
c &: y^2 - (3t-1)\,yx + 3t^2x^2 + 2t^3y = x^3 + 3t^4x - t^6\,,\\
d &: y^2 - (4t-1)\,yx + 2t^2x^2 = x^3 - 4t^4x - (8t^2-8t+1)t^4\,,\\
e &: (16t-1)\,y^2 - (16t-1)\,yx - ty = (16t-1)\,x^3 + tx^2\,,\\
f &: y^2 - (3t-1)\,yx + 9t^3y = x^3 - t^3(6t-1)\left(9t^2-3t+1\right)\,,\\
g &: y^2 - (6t-1)\,yx - 2t^3y = x^3 - 3t^2x^2 + 3t^4x - t^6\,,\\
h &: y^2 = 9x^3 - 3(-1+3t)(-1+27t)^3 - 6(27t-1)^4\left(27t^2+18t-1\right)\,,\\
i &: (64t-1)\,y^2 = (64t-1)\,x^3 - (48t-3)\,x - 16t - 2\,,\\
j &: (432t-1)\,y^2 = (432t-1)\,x^3 - (1296t-3)\,x + 864t + 2\,.
\end{aligned}
$$

Parabolic homology

In all considered cases, $H_3^{\mathrm{para}}(T_u)$ has rank 4 and carries precisely the $(1,1,1,1)$ motive.

The monodromy with respect to u acts by a braid action on $\pi_1(\mathbb{P}^1 \backslash \Sigma_u)$.



Points move around with u

$$\xrightarrow{}$$

$$\tau_\ell(\gamma) \mapsto \tau_{\ell'}(\gamma')$$

The parabolic homology $H_3^{\mathrm{para}}(T_u)$ is stable under monodromy.

In particular the monodromy matrices have integer coefficients.
In other words this realisation of the $(1,1,1,1)$ motive carries a local system defined over $\mathbb{Z}$.

A new Gamma-class formula

See also works
of Katz, Klemm,
Schimannek, Sharpe

[Candelas, De la Ossa, Green, Parkes] An ansatz for the period matrix Π can also be obtained from **topological invariants** of the (mirror) family from the formula

$$(2\pi i)^3 \begin{pmatrix} \frac{\zeta(3)}{(2i\pi)^3}\chi & \frac{c_2H}{24} & 0 & \frac{H^3}{6} \\ \frac{c_2H}{24} & \frac{\sigma}{2} & -\frac{H^3}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varpi_0 \\ \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix}$$

where $(2i\pi)^i \varpi_i = \sum_{k=0}^{i} h_k(t) \frac{\log^k(t)}{k!}$ form the Frobenius basis at the MUM point

and χ, c_2H, H^3 are the Euler characteristic, the second Chern class and the triple intersection numbers of the mirror threefold.

Using our methods, we can compute this matrix numerically with **very high precision** (several hundred digits) in reasonable time for Hadamard products.

We find a slightly different version which we conjecture to be general:

$$(2\pi i)^3 \begin{pmatrix} \frac{\zeta(3)}{(2i\pi)^3}\chi - \frac{\alpha}{2}\frac{c_2H}{24} - \frac{\delta}{2} & \frac{c_2H}{24} & \frac{\alpha}{2}\frac{H^3}{2} & M\frac{H^3}{6} \\ \frac{c_2H}{24} & N\frac{\sigma}{2} & -\frac{H^3}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \alpha\frac{N}{M} & N & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varpi_0 \\ \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix}$$

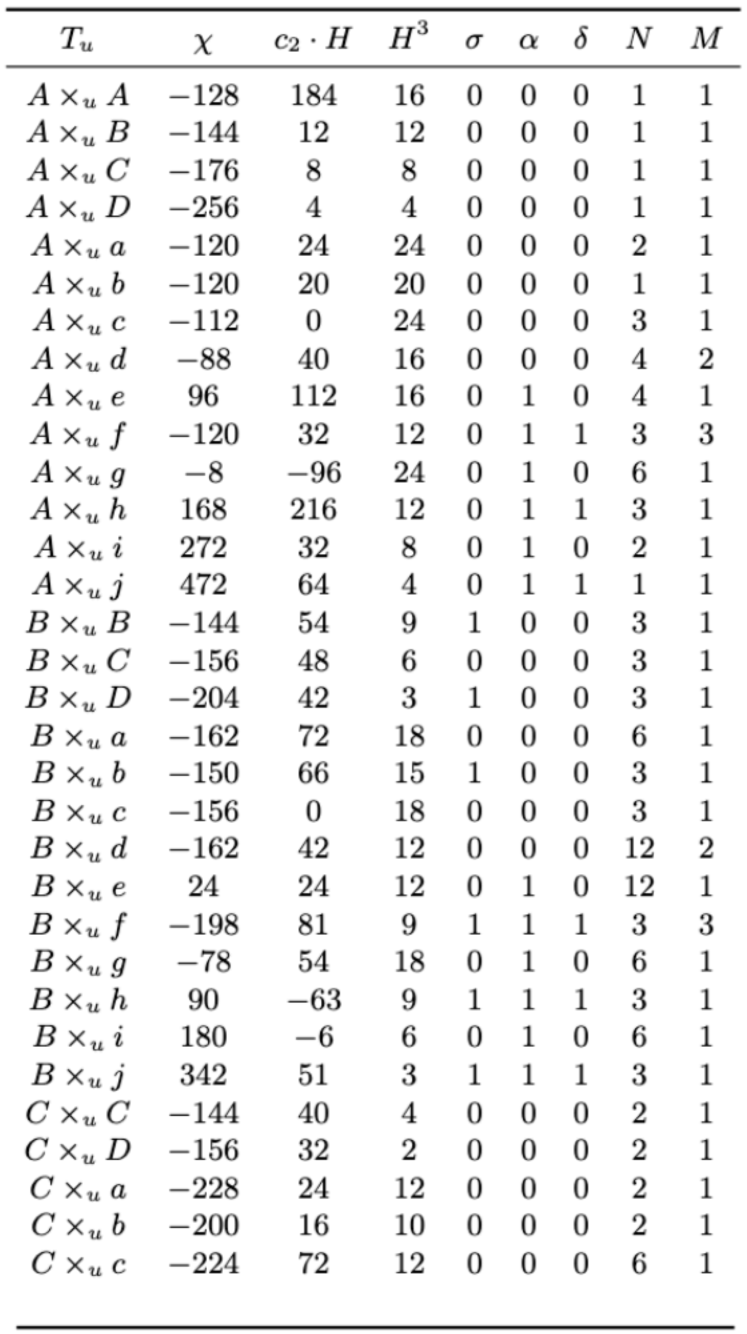
with intersection product:

$$\begin{pmatrix} 0 & 0 & M & 0 \\ 0 & 0 & 0 & N \\ -M & 0 & 0 & 0 \\ 0 & -N & 0 & 0 \end{pmatrix}$$

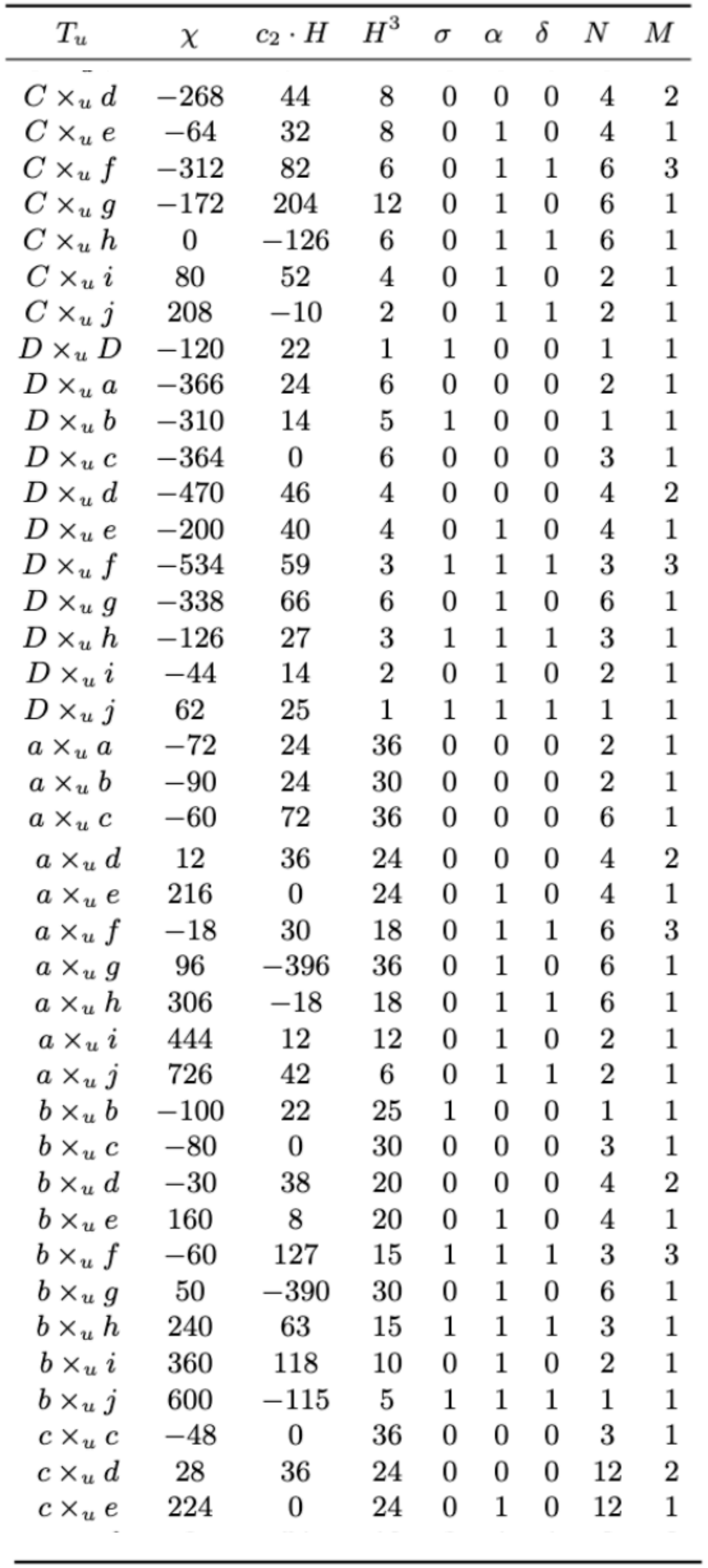
where $\chi, c_2H, H^3 \in \mathbb{Z}, \quad \alpha, \delta, \sigma \in \{0,1\}$ and $M, N \in \mathbb{N}$.

It agrees with the above when $M = N = 1$ and $\delta = \alpha = 0$

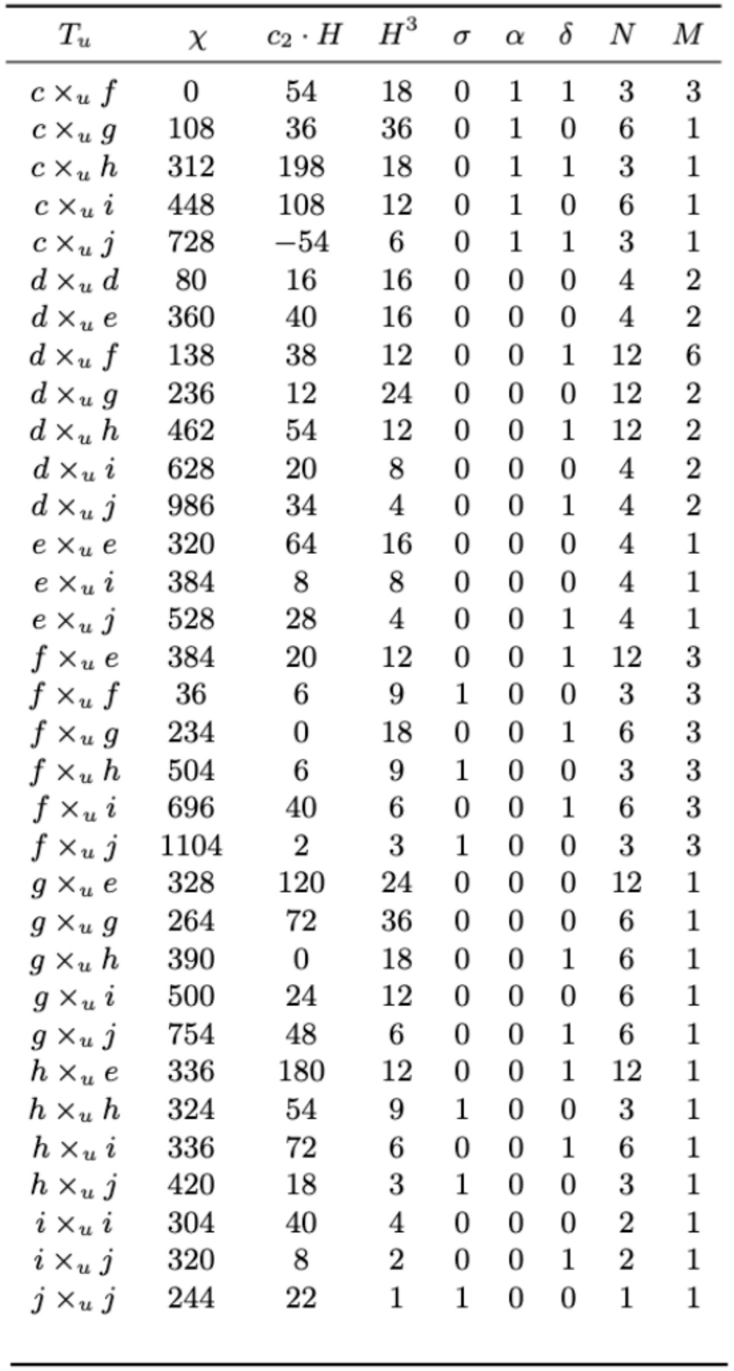
The 105 Hadamard products



T_u	χ	$c_2 \cdot H$	H^3	σ	α	δ	N	M
$A \times_u A$	-128	184	16	0	0	0	1	1
$A \times_u B$	-144	12	12	0	0	0	1	1
$A \times_u C$	-176	8	8	0	0	0	1	1
$A \times_u D$	-256	4	4	0	0	0	1	1
$A \times_u a$	-120	24	24	0	0	0	2	1
$A \times_u b$	-120	20	20	0	0	0	1	1
$A \times_u c$	-112	0	24	0	0	0	3	1
$A \times_u d$	-88	40	16	0	0	0	4	2
$A \times_u e$	96	112	16	0	1	0	4	1
$A \times_u f$	-120	32	12	0	1	1	3	3
$A \times_u g$	-8	-96	24	0	1	0	6	1
$A \times_u h$	168	216	12	0	1	1	3	1
$A \times_u i$	272	32	8	0	1	0	2	1
$A \times_u j$	472	64	4	0	1	1	1	1
$B \times_u B$	-144	54	9	1	0	0	3	1
$B \times_u C$	-156	48	6	0	0	0	3	1
$B \times_u D$	-204	42	3	1	0	0	3	1
$B \times_u a$	-162	72	18	0	0	0	6	1
$B \times_u b$	-150	66	15	1	0	0	3	1
$B \times_u c$	-156	0	18	0	0	0	3	1
$B \times_u d$	-162	42	12	0	0	0	12	2
$B \times_u e$	24	24	12	0	1	0	12	1
$B \times_u f$	-198	81	9	1	1	1	3	3
$B \times_u g$	-78	54	18	0	1	0	6	1
$B \times_u h$	90	-63	9	1	1	1	3	1
$B \times_u i$	180	-6	6	0	1	0	6	1
$B \times_u j$	342	51	3	1	1	1	3	1
$C \times_u C$	-144	40	4	0	0	0	2	1
$C \times_u D$	-156	32	2	0	0	0	2	1
$C \times_u a$	-228	24	12	0	0	0	2	1
$C \times_u b$	-200	16	10	0	0	0	2	1
$C \times_u c$	-224	72	12	0	0	0	6	1



T_u	χ	$c_2 \cdot H$	H^3	σ	α	δ	N	M
$C \times_u d$	-268	44	8	0	0	0	4	2
$C \times_u e$	-64	32	8	0	1	0	4	1
$C \times_u f$	-312	82	6	0	1	1	6	3
$C \times_u g$	-172	204	12	0	1	0	6	1
$C \times_u h$	0	-126	6	0	1	1	6	1
$C \times_u i$	80	52	4	0	1	0	2	1
$C \times_u j$	208	-10	2	0	1	1	2	1
$D \times_u D$	-120	22	1	1	0	0	1	1
$D \times_u a$	-366	24	6	0	0	0	2	1
$D \times_u b$	-310	14	5	1	0	0	1	1
$D \times_u c$	-364	0	6	0	0	0	3	1
$D \times_u d$	-470	46	4	0	0	0	4	2
$D \times_u e$	-200	40	4	0	1	0	4	1
$D \times_u f$	-534	59	3	1	1	1	3	3
$D \times_u g$	-338	66	6	0	1	0	6	1
$D \times_u h$	-126	27	3	1	1	1	3	1
$D \times_u i$	-44	14	2	0	1	0	2	1
$D \times_u j$	62	25	1	1	1	1	1	1
$a \times_u a$	-72	24	36	0	0	0	2	1
$a \times_u b$	-90	24	30	0	0	0	2	1
$a \times_u c$	-60	72	36	0	0	0	6	1
$a \times_u d$	12	36	24	0	0	0	4	2
$a \times_u e$	216	0	24	0	1	0	4	1
$a \times_u f$	-18	30	18	0	1	1	6	3
$a \times_u g$	96	-396	36	0	1	0	6	1
$a \times_u h$	306	-18	18	0	1	1	6	1
$a \times_u i$	444	12	12	0	1	0	2	1
$a \times_u j$	726	42	6	0	1	1	2	1
$b \times_u b$	-100	22	25	1	0	0	1	1
$b \times_u c$	-80	0	30	0	0	0	3	1
$b \times_u d$	-30	38	20	0	0	0	4	2
$b \times_u e$	160	8	20	0	1	0	4	1
$b \times_u f$	-60	127	15	1	1	1	3	3
$b \times_u g$	50	-390	30	0	1	0	6	1
$b \times_u h$	240	63	15	1	1	1	3	1
$b \times_u i$	360	118	10	0	1	0	2	1
$b \times_u j$	600	-115	5	1	1	1	1	1
$c \times_u c$	-48	0	36	0	0	0	3	1
$c \times_u d$	28	36	24	0	0	0	12	2
$c \times_u e$	224	0	24	0	1	0	12	1



T_u	χ	$c_2 \cdot H$	H^3	σ	α	δ	N	M
$c \times_u f$	0	54	18	0	1	1	3	3
$c \times_u g$	108	36	36	0	1	0	6	1
$c \times_u h$	312	198	18	0	1	1	3	1
$c \times_u i$	448	108	12	0	1	0	6	1
$c \times_u j$	728	-54	6	0	1	1	3	1
$d \times_u d$	80	16	16	0	0	0	4	2
$d \times_u e$	360	40	16	0	0	0	4	2
$d \times_u f$	138	38	12	0	0	1	12	6
$d \times_u g$	236	12	24	0	0	0	12	2
$d \times_u h$	462	54	12	0	0	1	12	2
$d \times_u i$	628	20	8	0	0	0	4	2
$d \times_u j$	986	34	4	0	0	1	4	2
$e \times_u e$	320	64	16	0	0	0	4	1
$e \times_u i$	384	8	8	0	0	0	4	1
$e \times_u j$	528	28	4	0	0	1	4	1
$f \times_u e$	384	20	12	0	0	1	12	3
$f \times_u f$	36	6	9	1	0	0	3	3
$f \times_u g$	234	0	18	0	0	1	6	3
$f \times_u h$	504	6	9	1	0	0	3	3
$f \times_u i$	696	40	6	0	0	1	6	3
$f \times_u j$	1104	2	3	1	0	0	3	3
$g \times_u e$	328	120	24	0	0	0	12	1
$g \times_u g$	264	72	36	0	0	0	6	1
$g \times_u h$	390	0	18	0	0	1	6	1
$g \times_u i$	500	24	12	0	0	0	6	1
$g \times_u j$	754	48	6	0	0	1	6	1
$h \times_u e$	336	180	12	0	0	1	12	1
$h \times_u h$	324	54	9	1	0	0	3	1
$h \times_u i$	336	72	6	0	0	1	6	1
$h \times_u j$	420	18	3	1	0	0	3	1
$i \times_u i$	304	40	4	0	0	0	2	1
$i \times_u j$	320	8	2	0	0	1	2	1
$j \times_u j$	244	22	1	1	0	0	1	1

The AESZ list

The Gamma-class formula

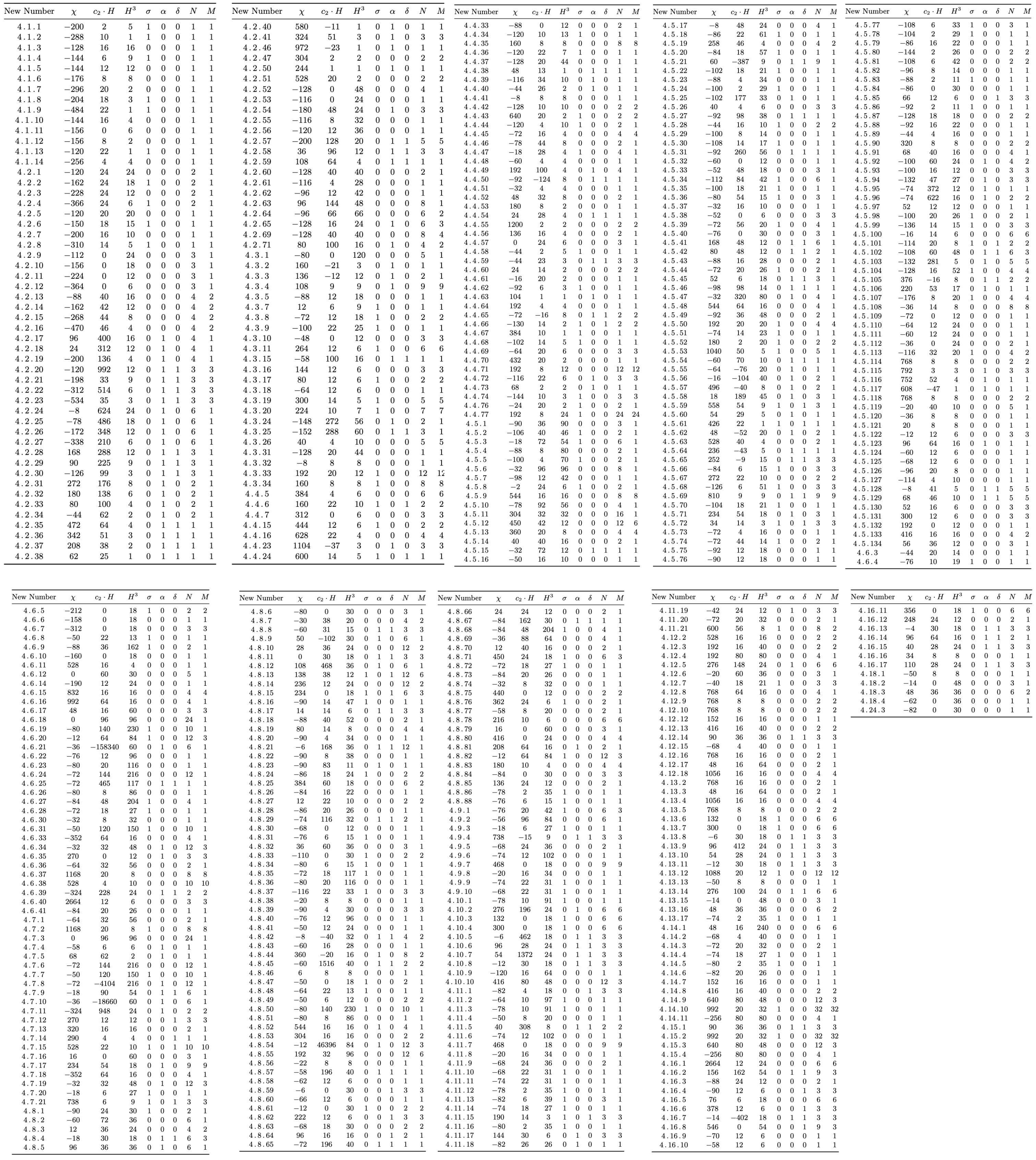
$$(2\pi i)^3 \begin{pmatrix} \frac{\zeta(3)}{(2i\pi)^3}\chi - \frac{\alpha}{2}\frac{c_2H}{24} - \frac{\delta}{2} & \frac{c_2H}{24} & \frac{\alpha}{2}\frac{H^3}{2} & M\frac{H^3}{6} \\ \frac{c_2H}{24} & N\frac{\sigma}{2} & -\frac{H^3}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \alpha\frac{N}{M} & N & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varpi_0 \\ \varpi_1 \\ \varpi_2 \\ \varpi_3 \end{pmatrix}$$

seems to apply to **all** operators in the CYDB (at least up to order 20).

"[…] a class of such threefolds which is large enough to exhibit many of the phenomena which one wants to study, yet is special enough to be quite tractable."

In some cases $M \neq N$ – in particular the operator seems to not have $\mathrm{Sp}_4(\mathbb{Z})$-integral monodromy.

$$\begin{pmatrix} 0 & 0 & M & 0 \\ 0 & 0 & 0 & N \\ -M & 0 & 0 & 0 \\ 0 & -N & 0 & 0 \end{pmatrix}$$



New Number	χ	$c_2 \cdot H$	H^3	σ	α	δ	N	M
4.1.1	−200	2	5	1	0	0	1	1
4.1.2	−288	10	1	1	0	0	1	1
4.1.3	−128	16	16	0	0	0	1	1
4.1.4	−144	6	9	1	0	0	1	1
4.1.5	−144	12	12	0	0	0	1	1
4.1.6	−176	8	8	0	0	0	1	1
4.1.7	−296	20	2	0	0	0	1	1
4.1.8	−204	18	3	1	0	0	1	1
4.1.9	−484	22	1	1	0	0	1	1
4.1.10	−144	16	4	0	0	0	1	1
4.1.11	−156	0	6	0	0	0	1	1
4.1.12	−156	8	2	0	0	0	1	1
4.1.13	−120	22	1	1	0	0	1	1
4.1.14	−256	4	4	0	0	0	1	1
4.2.1	−120	24	24	0	0	0	2	1
4.2.2	−162	24	18	1	0	0	2	1
4.2.3	−228	24	12	0	0	0	2	1
4.2.4	−366	24	6	1	0	0	2	1
4.2.5	−120	20	20	0	0	0	1	1
4.2.6	−150	18	15	1	0	0	1	1
4.2.7	−200	16	10	0	0	0	1	1
4.2.8	−310	14	5	1	0	0	1	1
4.2.9	−112	0	24	0	0	0	3	1
4.2.10	−156	0	18	0	0	0	3	1
4.2.11	−224	0	12	0	0	0	3	1
4.2.12	−364	0	6	0	0	0	3	1
4.2.13	−88	40	16	0	0	0	4	2
4.2.14	−162	42	12	0	0	0	4	2
4.2.15	−268	44	8	0	0	0	4	2
4.2.16	−470	46	4	0	0	0	4	2
4.2.17	96	400	16	0	1	0	4	1
4.2.18	24	312	12	0	1	0	4	1
4.2.19	−200	136	4	0	1	0	4	1
4.2.20	−120	992	12	0	1	1	3	3
4.2.21	−198	33	9	0	1	1	3	3
4.2.22	−312	514	6	0	1	1	3	3
4.2.23	−534	35	3	0	1	1	3	3
4.2.24	−8	624	24	0	1	0	6	1
4.2.25	−78	486	18	0	1	0	6	1
4.2.26	−172	348	12	0	1	0	6	1
4.2.27	−338	210	6	0	1	0	6	1
4.2.28	168	288	12	0	1	1	3	1
4.2.29	90	225	9	0	1	1	3	1
4.2.30	−126	99	3	0	1	1	3	1
4.2.31	272	176	8	0	1	0	2	1
4.2.32	180	138	6	0	1	0	2	1
4.2.33	80	100	4	0	1	0	2	1
4.2.34	−44	62	2	0	1	0	2	1
4.2.35	472	64	4	0	1	1	1	1
4.2.36	342	51	3	0	1	1	1	1
4.2.37	208	38	2	0	1	1	1	1
4.2.38	62	25	1	0	1	1	1	1
4.2.40	580	−11	1	0	1	0	1	1
4.2.41	324	51	3	0	1	0	3	3
4.2.46	972	−23	1	0	1	0	1	1
4.2.47	304	2	2	0	0	0	2	2
4.2.50	244	1	1	0	1	0	1	1
4.2.51	528	20	2	0	0	0	2	2
4.2.52	−128	0	48	0	0	0	4	1
4.2.53	−116	0	24	0	0	0	1	1
4.2.54	−180	48	24	0	1	0	3	3
4.2.55	−116	8	32	0	0	0	1	1
4.2.56	−120	12	36	0	0	0	1	1
4.2.57	−200	128	20	0	1	1	5	5
4.2.58	36	96	12	0	1	1	3	3
4.2.59	108	64	4	0	1	1	1	1
4.2.60	−128	40	40	0	0	0	2	1
4.2.61	−116	4	28	0	0	0	1	1
4.2.62	−96	12	42	0	0	0	1	1
4.2.63	96	144	48	0	0	0	8	1
4.2.64	−96	66	66	0	0	0	6	2
4.2.65	−128	16	24	0	1	0	6	3
4.2.69	−128	40	40	0	0	0	8	4
4.2.71	80	100	16	0	1	0	4	2
4.3.1	−80	0	120	0	0	0	5	1
4.3.2	160	−21	3	0	1	0	1	1
4.3.3	136	−12	12	0	1	0	2	1
4.3.4	108	9	9	0	1	0	9	9
4.3.5	−88	12	18	0	0	0	1	1
4.3.7	12	6	9	1	0	0	1	1
4.3.8	−72	12	18	1	0	0	2	2
4.3.9	−100	22	25	1	0	0	1	1
4.3.10	−48	0	12	0	0	0	3	3
4.3.11	264	12	6	1	0	0	6	6
4.3.15	−58	100	16	0	1	1	1	1
4.3.16	144	12	6	0	0	0	3	3
4.3.17	80	12	6	1	0	0	2	2
4.3.18	−64	12	6	0	0	0	1	1
4.3.19	300	14	5	1	0	0	5	5
4.3.20	224	10	7	1	0	0	7	7
4.3.24	−148	272	56	0	1	0	2	1
4.3.25	−152	288	60	0	1	1	3	1
4.3.26	40	4	10	0	0	0	5	5
4.3.31	−128	20	44	0	0	0	1	1
4.3.32	−8	8	8	0	0	0	1	1
4.3.33	192	20	12	1	0	0	12	12
4.3.34	160	8	8	1	0	0	8	8
4.4.5	384	4	6	0	0	0	6	6
4.4.6	160	22	10	1	0	1	2	2
4.4.7	312	0	6	0	0	0	3	3
4.4.15	444	12	6	1	0	0	2	2
4.4.16	628	22	4	0	0	0	4	4
4.4.23	1104	−37	3	0	1	0	3	3
4.4.24	600	14	5	1	0	1	1	1
4.4.33	−88	0	12	0	0	0	2	1
4.4.34	−120	10	13	1	0	0	1	1
4.4.35	160	8	8	0	0	0	8	8
4.4.36	−120	22	7	1	0	0	1	1
4.4.37	−128	20	44	0	0	0	1	1
4.4.38	48	13	1	0	1	1	1	1
4.4.39	−116	34	10	0	1	0	1	1
4.4.40	−44	26	2	0	1	0	1	1
4.4.41	−8	8	8	0	0	0	1	1
4.4.42	−128	10	10	0	0	0	2	2
4.4.43	640	20	2	1	0	0	2	2
4.4.44	−120	4	10	1	0	0	2	1
4.4.45	−72	16	4	0	0	0	4	4
4.4.46	−78	44	8	0	0	0	2	1
4.4.47	−18	28	4	1	0	0	4	1
4.4.48	−60	4	4	0	0	0	1	1
4.4.49	192	100	4	0	1	0	4	1
4.4.50	−92	−124	8	0	1	1	1	1
4.4.51	−32	4	4	0	0	0	1	1
4.4.52	48	32	8	0	0	0	2	1
4.4.53	180	8	2	0	0	0	1	1
4.4.54	24	28	4	0	1	1	1	1
4.4.55	1200	2	2	0	0	0	2	2
4.4.56	136	16	4	0	0	0	2	1
4.4.57	0	24	6	0	0	0	3	1
4.4.58	−44	2	5	1	0	0	1	1
4.4.59	−44	23	3	0	1	1	3	3
4.4.60	24	14	2	0	0	0	2	2
4.4.61	−16	20	2	0	0	0	1	1
4.4.62	−92	6	3	1	0	0	1	1
4.4.63	104	1	1	0	1	0	1	1
4.4.64	192	4	4	0	0	0	1	1
4.4.65	−72	−16	8	0	1	1	2	2
4.4.66	−130	14	2	1	0	1	2	2
4.4.67	384	10	1	1	0	0	1	1
4.4.68	−102	14	5	1	0	0	1	1
4.4.69	−64	20	6	0	0	0	3	3
4.4.70	432	20	2	0	0	0	1	1
4.4.71	192	8	12	0	0	0	12	12
4.4.72	−116	22	6	0	1	0	3	3
4.4.73	68	2	2	0	1	0	1	1
4.4.74	−144	10	3	1	0	0	3	3
4.4.76	−24	20	2	1	0	0	2	1
4.4.77	192	8	24	1	0	0	24	24
4.5.1	−90	36	90	0	0	0	3	1
4.5.2	−106	40	46	1	0	0	2	1
4.5.3	−18	72	54	1	0	0	6	1
4.5.4	−88	8	80	0	0	0	2	1
4.5.5	−100	4	70	1	0	0	2	1
4.5.6	−32	96	96	0	0	0	8	1
4.5.7	−98	12	42	0	0	0	1	1
4.5.8	−2	24	6	1	0	0	2	1
4.5.9	544	16	16	0	0	0	8	8
4.5.10	−78	92	56	0	0	0	4	1
4.5.11	304	32	32	0	0	0	16	1
4.5.12	450	42	12	0	0	0	12	6
4.5.13	360	20	8	0	0	0	4	4
4.5.14	40	40	16	0	0	0	2	1
4.5.15	−32	72	12	0	1	1	1	1
4.5.16	−50	16	10	0	0	0	1	1
4.5.17	−8	48	24	0	0	0	4	1
4.5.18	−86	22	61	1	0	0	1	1
4.5.19	258	46	4	0	0	0	4	2
4.5.20	−84	18	57	1	0	0	1	1
4.5.21	60	−387	9	0	1	1	9	1
4.5.22	−102	18	21	1	0	0	1	1
4.5.23	−88	4	34	0	0	0	1	1
4.5.24	−100	2	29	1	0	0	1	1
4.5.25	−102	177	33	0	1	0	1	1
4.5.26	40	4	6	0	0	0	3	3
4.5.27	−92	98	38	0	1	1	1	1
4.5.28	−44	16	10	1	0	0	2	2
4.5.29	−100	8	14	0	0	0	1	1
4.5.30	−108	14	17	1	0	0	1	1
4.5.31	−92	260	56	0	1	1	1	1
4.5.32	−60	0	12	0	0	0	1	1
4.5.33	−52	48	18	0	0	0	3	1
4.5.34	−112	84	42	1	0	0	6	1
4.5.35	−100	18	21	1	0	0	1	1
4.5.36	−80	54	15	1	0	0	3	1
4.5.37	−32	16	10	0	0	0	1	1
4.5.38	−52	0	6	0	0	0	3	3
4.5.39	−72	56	20	1	0	0	4	1
4.5.40	−76	0	30	0	0	0	3	1
4.5.41	168	48	12	0	1	1	6	1
4.5.42	80	48	12	0	1	1	2	1
4.5.43	−88	16	28	0	0	0	2	1
4.5.44	−72	20	26	1	0	0	2	1
4.5.45	52	6	18	0	1	1	3	1
4.5.46	−98	98	14	0	1	1	1	1
4.5.47	−32	320	80	0	1	0	4	1
4.5.48	544	64	16	0	0	0	4	1
4.5.49	−92	36	48	0	0	0	2	1
4.5.50	192	20	20	1	0	0	4	4
4.5.51	−74	14	23	1	0	0	1	1
4.5.52	180	2	20	1	0	0	2	2
4.5.53	1040	50	5	1	0	0	5	1
4.5.54	−60	70	10	0	1	1	1	1
4.5.55	−64	−76	20	0	1	0	1	1
4.5.56	−16	−104	40	0	1	0	2	1
4.5.57	496	−40	8	0	1	0	2	1
4.5.58	18	189	45	0	1	0	3	1
4.5.59	558	54	9	1	0	1	3	1
4.5.60	54	29	5	0	1	0	1	1
4.5.61	426	22	1	1	0	1	1	1
4.5.62	48	−52	20	0	1	0	2	1
4.5.63	528	40	4	0	0	0	2	1
4.5.64	236	−43	5	0	1	1	1	1
4.5.65	252	−9	15	0	1	1	3	3
4.5.66	−84	6	15	1	0	0	3	3
4.5.67	272	22	10	0	0	0	2	2
4.5.68	−126	6	51	1	0	0	3	3
4.5.69	810	9	9	0	1	1	9	9
4.5.70	−104	18	21	1	0	0	1	1
4.5.71	234	54	18	0	1	0	3	1
4.5.72	34	14	3	1	0	1	3	3
4.5.73	−72	4	16	0	0	0	1	1
4.5.74	−72	44	14	1	0	0	2	1
4.5.75	−92	12	18	0	0	0	1	1
4.5.76	−90	12	18	0	0	0	1	1
4.5.77	−108	6	33	1	0	0	3	1
4.5.78	−104	2	29	1	0	0	1	1
4.5.79	−86	16	22	0	0	0	1	1
4.5.80	−144	2	26	0	0	0	2	2
4.5.81	−108	6	42	0	0	0	2	2
4.5.82	−96	8	14	0	0	0	1	1
4.5.83	−88	2	11	1	0	0	1	1
4.5.84	−86	0	30	0	0	0	1	1
4.5.85	66	12	6	0	0	1	3	3
4.5.86	−92	2	11	1	0	0	1	1
4.5.87	−128	18	18	0	0	0	2	2
4.5.88	−92	16	22	0	0	0	1	1
4.5.89	−44	4	16	0	0	0	1	1
4.5.90	320	8	8	0	0	0	2	2
4.5.91	68	40	16	0	0	0	4	1
4.5.92	−100	60	24	0	1	0	4	2
4.5.93	−100	16	12	0	0	0	3	3
4.5.94	−132	47	27	0	1	0	3	3
4.5.95	−74	372	12	0	1	0	1	1
4.5.96	−74	622	16	0	1	0	2	2
4.5.97	52	12	12	0	0	0	1	1
4.5.98	−100	20	26	1	0	0	2	1
4.5.99	−136	14	15	1	0	0	3	3
4.5.100	−16	14	6	0	0	0	6	6
4.5.101	−114	20	8	1	0	1	2	2
4.5.102	−108	60	48	0	1	1	6	3
4.5.103	−132	281	5	0	1	0	5	5
4.5.104	−128	16	52	1	0	0	4	4
4.5.105	376	−16	8	0	1	1	2	2
4.5.106	220	53	17	0	1	0	1	1
4.5.107	−176	8	20	1	0	0	4	4
4.5.108	−36	14	8	0	0	0	8	8
4.5.109	−72	0	12	0	0	0	1	1
4.5.110	−64	12	24	0	0	0	1	1
4.5.111	−60	12	24	0	0	0	1	1
4.5.112	−36	0	24	0	0	0	2	1
4.5.113	−116	32	20	1	0	0	4	2
4.5.114	768	8	8	0	0	0	2	2
4.5.115	792	3	3	0	1	0	3	3
4.5.116	752	52	4	0	1	0	1	1
4.5.117	608	−47	1	0	1	0	1	1
4.5.118	768	8	8	0	0	0	2	2
4.5.119	−20	40	10	0	0	0	5	1
4.5.120	−36	8	8	0	0	0	1	1
4.5.121	20	8	8	0	0	0	1	1
4.5.122	−12	12	6	0	0	0	3	3
4.5.123	96	64	16	0	1	0	1	1
4.5.124	−60	12	6	0	0	0	1	1
4.5.125	−68	12	6	0	0	0	1	1
4.5.126	−96	20	8	0	0	0	1	1
4.5.127	−114	4	10	0	0	0	1	1
4.5.128	−8	41	5	0	1	1	5	5
4.5.129	68	46	10	0	1	1	5	5
4.5.130	52	16	6	0	0	0	3	3
4.5.131	300	12	6	0	0	0	3	3
4.5.132	192	0	12	0	0	0	1	1
4.5.133	416	16	16	0	0	0	4	2
4.5.134	56	36	12	0	0	0	3	1
4.6.3	−44	20	14	0	0	0	1	1
4.6.4	−76	10	19	1	0	0	1	1
4.6.5	−212	0	18	1	0	0	2	2
4.6.6	−158	0	18	0	0	0	1	1
4.6.7	−312	0	18	0	0	0	3	3
4.6.8	−50	22	13	1	0	0	1	1
4.6.9	−88	36	162	1	0	0	2	1
4.6.10	−160	0	18	0	0	0	1	1
4.6.11	528	16	4	0	0	0	1	1
4.6.12	0	60	30	0	0	0	5	1
4.6.14	−190	12	24	0	0	0	1	1
4.6.15	832	16	16	0	0	0	4	4
4.6.16	992	64	16	0	0	0	4	1
4.6.17	48	16	60	0	0	0	3	3
4.6.18	0	96	96	0	0	0	24	1
4.6.19	−80	140	230	1	0	0	10	1
4.6.20	−12	64	84	1	0	0	12	3
4.6.21	−36	−158340	60	0	1	0	6	1
4.6.22	−76	12	96	0	0	0	1	1
4.6.23	−80	20	116	0	0	0	1	1
4.6.24	−72	144	216	0	0	0	12	1
4.6.25	−72	465	117	0	1	1	1	1
4.6.26	−80	8	86	0	0	0	1	1
4.6.27	−84	48	204	1	0	0	4	1
4.6.28	−72	18	27	1	0	0	1	1
4.6.30	−32	8	32	0	0	0	1	1
4.6.31	−50	120	150	1	0	0	10	1
4.6.33	−352	64	16	0	0	0	4	1
4.6.34	−32	32	48	0	1	0	12	3
4.6.35	270	0	12	0	1	0	3	3
4.6.36	−64	32	56	0	0	0	2	1
4.6.37	1168	20	8	0	0	0	8	8
4.6.38	528	4	10	0	0	0	10	10
4.6.39	−324	228	24	0	1	1	2	2
4.6.40	2664	12	6	0	0	0	3	3
4.6.41	−84	20	26	0	0	0	1	1
4.7.1	−64	32	56	0	0	0	2	1
4.7.2	1168	20	8	1	0	0	8	8
4.7.3	0	96	96	0	0	0	24	1
4.7.4	−58	6	6	0	1	0	1	1
4.7.5	68	62	2	0	1	0	1	1
4.7.6	−72	144	216	0	0	0	12	1
4.7.7	−50	120	150	1	0	0	10	1
4.7.8	−72	−4104	216	0	1	0	12	1
4.7.9	−18	90	54	0	1	1	6	1
4.7.10	−36	−18660	60	0	1	0	6	1
4.7.11	−324	948	24	0	1	0	2	2
4.7.12	270	12	12	0	0	1	3	3
4.7.13	320	16	16	0	0	0	2	1
4.7.14	290	4	4	0	0	1	1	1
4.7.15	528	22	10	1	0	1	10	10
4.7.16	16	0	60	0	0	0	3	1
4.7.17	234	54	18	0	1	0	9	9
4.7.18	−352	64	16	0	0	0	4	1
4.7.19	−32	32	48	0	1	0	12	3
4.7.20	−18	6	27	1	0	0	1	1
4.7.21	738	6	9	1	0	1	3	3
4.8.1	−90	24	30	1	0	0	2	1
4.8.2	−60	72	36	0	0	0	6	1
4.8.3	12	36	24	0	0	0	4	2
4.8.4	−18	30	18	0	1	1	6	3
4.8.5	96	36	36	0	1	0	6	1
4.8.6	−80	0	30	0	0	0	3	1
4.8.7	−30	38	20	0	0	0	4	2
4.8.8	−60	31	15	0	1	1	3	3
4.8.9	50	−102	30	0	1	0	6	1
4.8.10	28	36	24	0	0	0	12	2
4.8.11	0	30	18	0	1	1	3	3
4.8.12	108	468	36	0	1	0	6	1
4.8.13	138	38	12	1	0	1	12	6
4.8.14	236	12	24	0	0	0	12	2
4.8.15	234	0	18	1	0	1	6	3
4.8.16	−90	14	47	1	0	0	1	1
4.8.17	14	14	6	0	1	1	3	3
4.8.18	−88	40	52	0	0	0	2	1
4.8.19	80	14	8	0	0	0	4	4
4.8.20	−90	4	34	0	0	0	1	1
4.8.21	−6	168	36	0	1	1	12	1
4.8.22	−90	8	38	0	0	0	1	1
4.8.23	−90	83	11	0	1	0	1	1
4.8.24	−86	18	24	1	0	0	2	2
4.8.25	384	60	18	0	0	0	6	2
4.8.26	−84	16	22	0	0	0	1	1
4.8.27	12	22	10	0	0	0	2	2
4.8.28	−86	20	26	0	0	0	1	1
4.8.29	−74	116	32	0	1	1	2	1
4.8.30	−68	0	12	0	0	0	1	1
4.8.31	−76	6	15	1	0	0	1	1
4.8.32	36	60	36	0	0	0	3	1
4.8.33	−110	0	30	1	0	0	2	2
4.8.34	−80	6	15	1	0	0	1	1
4.8.35	−72	18	117	1	0	0	1	1
4.8.36	−80	20	116	0	0	0	1	1
4.8.37	−116	22	33	1	0	0	3	3
4.8.38	−20	8	8	0	0	0	1	1
4.8.39	−90	4	30	0	0	0	3	3
4.8.40	−76	12	96	0	0	0	1	1
4.8.41	−50	12	24	0	0	0	1	1
4.8.42	−8	−40	32	0	1	1	4	2
4.8.43	−60	16	28	0	0	0	1	1
4.8.44	360	−20	16	0	1	0	8	2
4.8.45	−60	1516	40	0	1	1	2	2
4.8.46	6	8	8	0	0	0	1	1
4.8.47	−50	0	18	1	0	0	2	1
4.8.48	−64	22	13	1	0	0	1	1
4.8.49	−50	6	12	0	0	0	2	2
4.8.50	−80	140	230	1	0	0	10	1
4.8.51	−80	8	86	0	0	0	1	1
4.8.52	544	16	16	0	1	0	4	1
4.8.53	304	16	16	0	0	0	2	2
4.8.54	−12	46396	84	0	1	0	12	3
4.8.55	192	32	96	0	0	0	12	6
4.8.56	−22	8	8	0	0	0	1	1
4.8.57	−58	196	40	0	1	1	1	1
4.8.58	−62	12	6	0	0	0	1	1
4.8.59	−6	0	30	0	0	1	3	3
4.8.60	−66	12	6	0	0	0	1	1
4.8.61	−12	0	30	1	0	0	2	2
4.8.62	222	12	6	0	0	1	3	3
4.8.63	−68	18	30	0	0	0	2	2
4.8.64	96	16	16	0	0	1	2	1
4.8.65	−72	196	40	0	1	1	1	1
4.8.66	24	24	12	0	0	0	2	1
4.8.67	−84	162	30	0	1	1	1	1
4.8.68	−84	48	204	1	0	0	4	1
4.8.69	−36	88	64	0	0	0	4	1
4.8.70	12	40	16	0	0	0	2	1
4.8.71	450	24	18	1	0	0	6	3
4.8.72	−72	18	27	1	0	0	1	1
4.8.73	−84	20	26	0	0	0	1	1
4.8.74	−32	8	32	0	0	0	1	1
4.8.75	440	0	12	0	0	0	2	2
4.8.76	362	24	6	1	0	0	2	1
4.8.77	−58	8	20	0	0	0	2	1
4.8.78	216	10	6	0	0	0	6	6
4.8.79	16	0	60	0	0	0	3	1
4.8.80	416	0	24	0	0	0	4	4
4.8.81	208	64	16	0	1	0	2	1
4.8.82	−12	64	84	1	0	0	12	3
4.8.83	180	10	4	0	0	0	4	4
4.8.84	−84	0	30	0	0	0	3	3
4.8.85	136	24	12	0	0	0	2	1
4.8.86	−78	2	35	1	0	0	1	1
4.8.88	−76	6	15	1	0	0	1	1
4.9.1	−76	20	42	1	0	0	6	3
4.9.2	−56	96	84	0	0	0	6	1
4.9.3	−18	6	27	1	0	0	1	1
4.9.4	738	−15	9	0	1	1	3	3
4.9.5	−68	24	36	0	0	0	2	1
4.9.6	−74	12	102	0	0	0	1	1
4.9.7	468	0	18	0	0	0	9	9
4.9.8	−20	16	34	0	0	0	1	1
4.9.9	−74	22	31	1	0	0	1	1
4.9.10	−68	22	31	1	0	0	1	1
4.10.1	−78	10	91	1	0	0	1	1
4.10.2	276	196	24	0	1	0	6	6
4.10.3	132	0	18	1	0	0	6	6
4.10.4	300	0	18	1	0	0	6	6
4.10.5	−6	462	18	0	1	1	3	3
4.10.6	96	28	24	0	1	1	3	3
4.10.7	54	1372	24	0	1	1	3	3
4.10.8	−12	30	18	0	1	1	3	3
4.10.9	−120	16	64	0	0	0	1	1
4.10.10	416	80	48	0	0	0	12	3
4.11.1	−82	4	18	0	0	1	3	3
4.11.2	−64	10	97	1	0	0	1	1
4.11.3	−78	10	91	1	0	0	1	1
4.11.4	−50	8	20	0	0	0	1	1
4.11.5	40	308	8	0	1	1	2	2
4.11.6	−74	12	102	0	0	0	1	1
4.11.7	468	0	18	0	0	0	9	9
4.11.8	−20	16	34	0	0	0	1	1
4.11.9	−68	24	36	0	0	0	2	1
4.11.10	−68	22	31	1	0	0	1	1
4.11.11	−74	22	31	1	0	0	1	1
4.11.12	−78	2	35	1	0	0	1	1
4.11.13	−82	6	39	1	0	0	3	1
4.11.14	−74	18	27	1	0	0	1	1
4.11.15	190	14	3	1	0	1	3	3
4.11.16	−80	2	35	1	0	0	1	1
4.11.17	144	30	6	0	1	0	3	3
4.11.18	−82	26	26	0	1	0	1	1
4.11.19	−42	24	12	0	1	0	3	3
4.11.20	−72	20	32	0	0	0	2	1
4.11.21	600	56	8	1	0	0	8	2
4.12.2	528	16	16	0	0	0	2	2
4.12.3	192	16	40	0	0	0	2	2
4.12.4	192	80	80	0	0	0	4	1
4.12.5	276	148	24	0	1	0	6	6
4.12.6	−20	60	36	0	0	0	3	1
4.12.7	−40	18	21	1	0	0	3	3
4.12.8	768	64	16	0	0	0	4	1
4.12.9	768	8	8	0	0	0	2	2
4.12.10	768	8	8	0	0	0	2	2
4.12.12	152	16	16	0	0	0	1	1
4.12.13	416	16	40	0	0	0	2	2
4.12.14	90	36	36	0	1	1	3	3
4.12.15	−68	4	40	0	0	0	1	1
4.12.16	768	16	16	0	0	0	2	1
4.12.17	48	16	64	0	0	0	2	1
4.12.18	1056	16	16	0	0	0	4	4
4.13.2	768	16	16	0	0	0	2	1
4.13.3	48	16	64	0	0	0	2	1
4.13.4	1056	16	16	0	0	0	4	4
4.13.5	768	8	8	0	0	0	2	2
4.13.6	132	0	18	1	0	0	6	6
4.13.7	300	0	18	1	0	0	6	6
4.13.8	−6	30	18	0	1	1	3	3
4.13.9	96	412	24	0	1	1	3	3
4.13.10	54	28	24	0	1	1	3	3
4.13.11	−12	30	18	0	1	1	3	3
4.13.12	1088	20	12	1	0	0	12	12
4.13.13	−50	8	8	0	0	0	1	1
4.13.14	276	100	24	0	1	1	6	6
4.13.15	−14	0	48	0	0	0	3	1
4.13.16	48	36	36	0	0	0	6	2
4.13.17	−74	2	35	1	0	0	1	1
4.14.1	48	16	240	0	0	0	6	6
4.14.2	−68	4	40	0	0	0	1	1
4.14.3	−72	20	32	0	0	0	2	1
4.14.4	−74	18	27	1	0	0	1	1
4.14.5	−80	2	35	1	0	0	1	1
4.14.6	−82	20	26	0	0	0	1	1
4.14.7	152	16	16	0	0	0	1	1
4.14.8	416	16	40	0	0	0	2	2
4.14.9	640	80	48	0	0	0	12	3
4.14.10	992	20	32	1	0	0	32	32
4.14.11	−256	80	80	0	0	0	4	1
4.15.1	90	36	36	0	1	1	3	3
4.15.2	992	20	32	1	0	0	32	32
4.15.3	640	80	48	0	0	0	12	3
4.15.4	−256	80	80	0	0	0	4	1
4.16.1	2664	12	24	0	0	0	6	6
4.16.2	156	162	54	0	1	1	9	3
4.16.3	−88	24	12	0	0	0	2	1
4.16.4	−90	12	6	0	0	1	3	3
4.16.5	76	6	18	0	0	0	6	6
4.16.6	378	12	6	0	0	1	3	3
4.16.7	−14	−402	18	0	1	1	3	3
4.16.8	546	0	54	0	0	1	9	3
4.16.9	−70	12	6	0	0	0	1	1
4.16.10	−58	12	6	0	0	0	1	1
4.16.11	356	0	18	1	0	0	6	6
4.16.12	248	24	12	0	0	0	2	1
4.16.13	−4	30	18	0	1	1	3	3
4.16.14	96	64	16	0	1	1	2	1
4.16.15	40	28	24	0	1	1	3	3
4.16.16	34	8	8	0	0	0	1	1
4.16.17	110	28	24	0	1	1	3	3
4.18.1	−50	8	8	0	0	0	1	1
4.18.2	−14	0	48	0	0	0	3	1
4.18.3	48	36	36	0	0	0	6	2
4.18.4	−62	0	36	0	0	0	1	1
4.24.3	−82	0	30	0	0	0	1	1

See also **[Yang 2020]**

The Deligne conjecture

with Nutsa Gegelia
and Duco van Straten
(JGU Mainz)

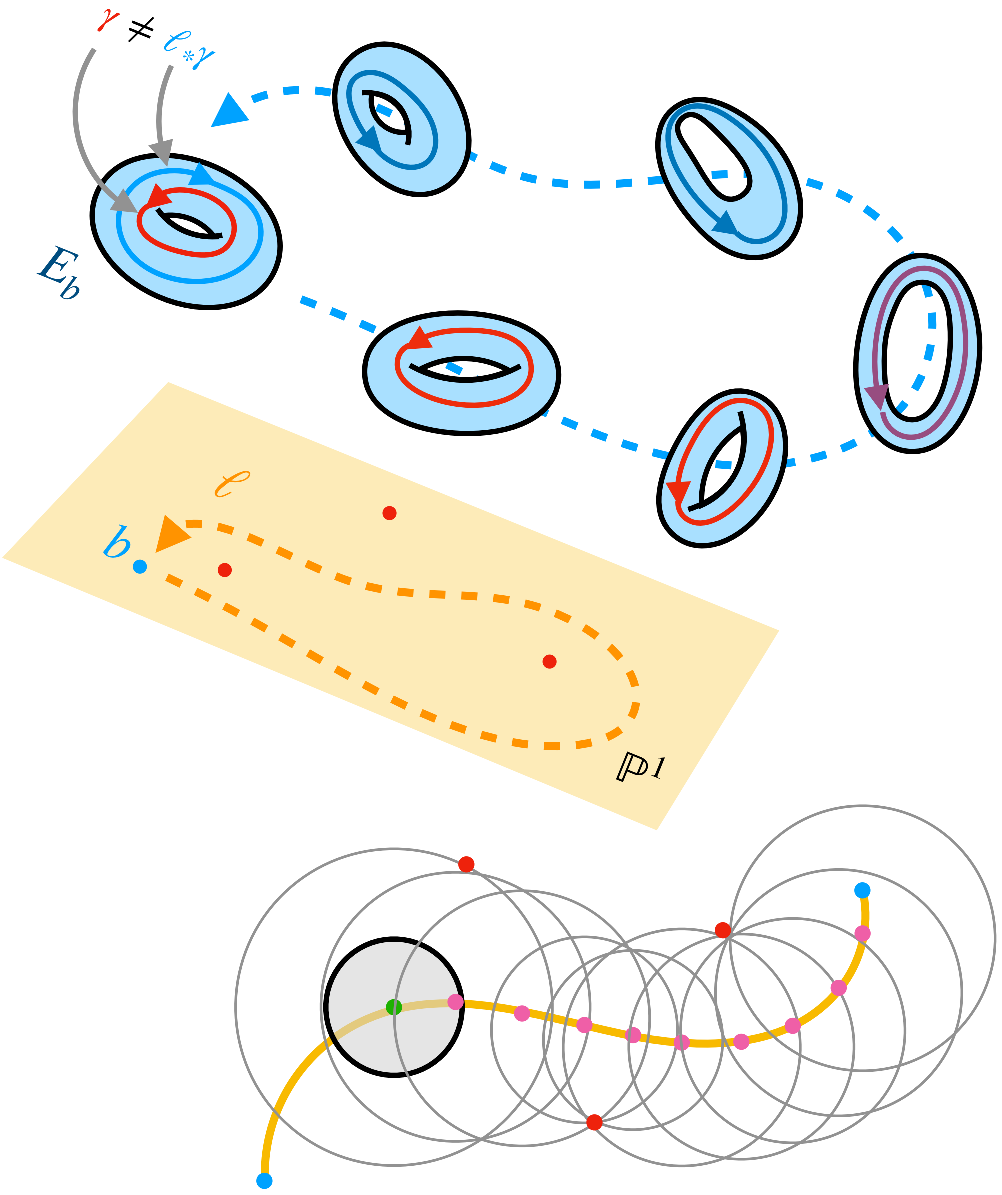
Another application concerns **numerical checks of the Deligne conjecture (1979)**, which relates a minor c^+ of the period matrix to the value $L(2)$ of the L-function via the formula $L(2) = qc^+$, where $q \in \mathbb{Q}$.

Fibre product	ratio
$A \times_1 A$	-2^{-4}
$A \times_1 B$	$2^2 \cdot 3^{-2}$
$A \times_1 c$	3^{-1}
$A \times_1 d$	2^{-2}
$B \times_1 B$	$2^8 \cdot 3^{-5}$
$B \times_1 c$	$-2^5 \cdot 3^{-3}$
$A \times_{-1} A$	-2^{-4}

Fibre product	ratio
$A \times_{-1} B$	$2^2 \cdot 3^{-2}$
$A \times_{-1} b$	2^{-5}
$A \times_{-1} c$	3^{-1}
$A \times_{-1} f$	$-2 \cdot 3^{-1}$
$B \times_{-1} B$	$2^8 \cdot 3^{-4}$
$B \times_{-1} c$	$2^6 \cdot 3^{-3}$
$B \times_{-1/8} a$	$7 \cdot 3^2 \cdot 2^{-2}$

We are able to numerically recover the value of q for several examples with many digits of precision.

Thank you!



$$\begin{pmatrix} \frac{\zeta(3)}{(2i\pi)^3}\chi - \frac{\alpha}{2}\frac{c_2H}{24} - \frac{\delta}{2} & \frac{c_2H}{24} & \frac{\alpha}{2}\frac{H^3}{2} & M\frac{H^3}{6} \\ \frac{c_2H}{24} & N\frac{\sigma}{2} & -\frac{H^3}{2} & 0 \\ 1 & 0 & 0 & 0 \\ \alpha\frac{N}{M} & N & 0 & 0 \end{pmatrix}$$