Bunches of bananas from Birmingham and surprising evasions of elliptic obstructions

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In **1968**, I met an **elliptic integral** when studying the decay of the **eta meson**. Yet an apparent elliptic **obstruction** was **evaded** when evaluating the **magnetic moment** of the electron at **two loops**. In **1998**, I expected elliptic obstructions when evaluating **3-loop tadpoles**. Yet empirical fits to numerical integrations gave results in terms of **polylogarithms**. Recently I have found more examples of **evasion**, with **integrals of elliptic integrals** multiplied by polylogarithms reducible to **classical tetralogarithms**. I describe how **Steven Charlton** and I have proved these results. The basic idea is to transform a **double** integral of a products of logs, to **rationalize** an obstructing **square root of a quartic** polynomial by a **pair** of Euler transformations.

1. **Pendulums, perimeters and bunches of bananas from Birmingham**
2. **Elliptically obstructed kites and tadpoles**
3. **Three notable evasions, with proofs by Steven Charlton**

Pendulums: 60 years ago, I met a complete **elliptic integral** of the **first** kind:

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi/2}{\mathrm{AGM}(1,\sqrt{1-k^2})}$$

when studying the **period** $T(\theta)$ of a pendulum swinging with angular amplitude θ:

$$T(\theta) = \frac{T(0)}{\mathrm{AGM}(1,\cos(\theta/2))}, \quad \mathrm{AGM}(a,b) = \mathrm{AGM}((a+b)/2,\sqrt{ab})$$

with an **amazingly fast** evaluation of an **arithmetic-geometric mean**.
At **90** degrees and **30** degrees, we encounter the **first** and **third singular values**

$$\frac{T(\pi/2)}{T(0)} = \frac{\Gamma^2(\frac{1}{4})}{2\pi^{3/2}} \approx 1.180340599, \quad \frac{T(\pi/6)}{T(0)} = \frac{3^{1/4}\Gamma^3(\frac{1}{3})}{2^{4/3}\pi^2} \approx 1.017408798.$$

More generally, at the r-th singular value, with $T(\pi-\theta_r) = \sqrt{r}T(\theta_r)$, we obtain an **algebraic multiple** of a quotient of Γ values. At the **sixth** singular value, we have

$$\sin(\theta_6/2) = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+2}, \quad \frac{T(\theta_6)}{T(0)} = \left[\frac{(\sqrt{2}-1)(\sqrt{3}+1)}{96(3\sqrt{2}-2\sqrt{3})}\right]^{1/2} \frac{\Gamma(\frac{1}{24})\Gamma(\frac{11}{24})}{\Gamma^3(\frac{1}{2})} \approx 1.001820672.$$

Perimeters: 60 years ago, I met a complete **elliptic integral** of the **second** kind:

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \phi}\, \mathrm{d}\phi$$

when evaluating the **perimeter** $4aE(\varepsilon)$ of an ellipse $(x/a)^2 + (y/b)^2 = 1$, with semi-major axis a and **eccentricity** $\varepsilon = \sqrt{1 - b^2/a^2}$. Later, the brothers **Borwein**, **Jonathan** and **Peter**, taught me to evaluate **singular** perimeters, such as

$$E(\sin(\pi/4)) = \frac{\Gamma^2(\frac{1}{4}) + 4\,\Gamma^2(\frac{3}{4})}{8\sqrt{\pi}}$$

$$E(\sin(\pi/12)) = \frac{(\sqrt{3}+1)\Gamma^3(\frac{1}{3}) + 2^{2/3}3\,\Gamma^3(\frac{2}{3})}{2^{10/3}3^{1/4}\pi}.$$

In **2007**, Stefano **Laporta** and I encountered the **fifteenth** singular value, where

$$\sin(\theta_{15}/2) = \frac{(3-\sqrt{5})(2-\sqrt{3})}{4\sqrt{2}(\sqrt{5}+\sqrt{3})}, \quad \frac{T(\theta_{15})}{T(0)} = \left[\frac{\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})}{15(\sqrt{5}-1)\pi^3}\right]^{1/2} \approx 1.000020790$$

$$2\sqrt{15}E(\sin(\theta_{15}/2)) - (\sqrt{15}+\sqrt{5}+1)K(\sin(\theta_{15}/2)) = \frac{T(0)}{T(\theta_{15})} = 1 - 4q + 12q^2 + O(q^3)$$

with $q = e^{-\pi\sqrt{15}}$ from my talk on ***Reciprocal*** *PSLQ and the* ***Tiny Nome*** *of Bologna.*

Journey to Birmingham: In **1939**, G.H. Hardy was asked to evaluate the integrals

$$I_1 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w}{1-\cos u\cos v\cos w}$$

$$I_2 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w}{3-\cos v\cos w-\cos w\cos u-\cos u\cos v}$$

$$I_3 = \frac{1}{\pi^3}\int_0^\pi\int_0^\pi\int_0^\pi \frac{\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w}{3-\cos u-\cos v-\cos w}$$

arising from **ferromagnetism** in face-centred, body-centred and simple **cubic lattices**. They were evaluated by G.N. **Watson**: "The problem of evaluating them was proposed by Kramers to R.H. Fowler who communicated them to G.H. Hardy. The problem then became common knowledge first in Cambridge and subsequently in Oxford, whence it made the **journey to Birmingham** without difficulty."

They yield **squares of periods** at the **first**, **third** and **sixth** singular values:

$$I_1 = \left(\frac{T(\pi/2)}{T(0)}\right)^2 \approx 1.393203930, \quad I_2 = \frac{\sqrt{3}}{4}\left(\frac{T(\pi/6)}{T(0)}\right)^2 \approx 0.4482203944$$

$$I_3 = \frac{\Gamma^2(\frac{1}{24})\Gamma^2(\frac{11}{24})}{48(\sqrt{3}+1)\pi^3} \approx 0.5054620197,$$

with a neat reduction to Γ values in I_3 obtained by Jon **Borwein** and John **Zucker**.

On-shell banana: I found that the **three-loop** equal-mass **banana** integral

$$J(t)=\int_0^\infty\int_0^\infty\int_0^\infty\frac{1}{(1+a+b+c)(1+1/a+1/b+1/c)-t}\,\frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{abc}$$

gives the **on-shell** value $J(1)=\Gamma(\frac{1}{15})\Gamma(\frac{2}{15})\Gamma(\frac{4}{15})\Gamma(\frac{8}{15})/(30\sqrt{5})$.

With Jon **Borwein**, I found that the **Bessel moments** $M_n=8\int_0^\infty I_0(x)K_0^4(x)x^{2n-1}dx$ are combinations of $M_1=J(1)$ and its **reciprocal** in

$$R_1=(2\pi)^4/M_1=30\sqrt{5}\,\Gamma(\tfrac{7}{15})\Gamma(\tfrac{11}{15})\Gamma(\tfrac{13}{15})\Gamma(\tfrac{14}{15})$$

with $M_2=(260M_1-8R_1)/(3^2 5^3)$, $M_3=(13760M_1-608R_1)/(3^3 5^4)$ and so on.

As shown by **Bloch**, **Kerr** and **Vanhove**, $J(t)$ is given by the square of an elliptic integral, multiplied by an **elliptic trilogarithm**. On-shell, the latter is an algebraic multiple of π^3 and we obtain the square of the period of a **pendulum** displaced by $\theta_{15}\approx 1.044975628^\circ$, were I now use **degrees**, in Babylonian measure.

Bunches of bananas from Birmingham: $\pi^3 I_1$, with a pendulum displaced by 90°, yields the imaginary part of $J(16+8\sqrt{3})$. $\pi^3 I_2$, with a pendulum displaced by 30°, appears in combinations of a **bunch of 4 bananas**: $J(t)$ with $t\in\{-32,-2,4,16\}$. $\pi^3 I_3$, with a pendulum displaced by $\theta_6\approx 9.770937965^\circ$, appears in combinations of a **bunch of 5 bananas**: $J(t)$ with $t\in\{-8,\,-8t_1,\,-8t_2\,,t_1,\,t_2\}$ and $t_{1,2}=10\mp 6\sqrt{3}$.

Watson's bunch of 5 bananas: $J(t)$ with $t \in \{-8,\, -8t_1,\, -8t_2\,, t_1,\, t_2\}$ satisfy

$$\begin{aligned} 2(1+\sqrt{3})J(t_1) - (2+\sqrt{3})J(-8) &= 2J(-8t_1) \\ t_1 J(t_1) + (t_1+\sqrt{3})J(-8) &= 2J(-8t_2) \\ \frac{(1-\sqrt{3})J(t_1) + (1+\sqrt{3})J(t_2 \pm \mathrm{i}\varepsilon) + 3J(-8)}{3-\sqrt{3}} &= \left(1 \pm \frac{\mathrm{i}}{\sqrt{6}}\right) \frac{\Gamma^2(\frac{1}{24})\Gamma^2(\frac{11}{24})}{48} \end{aligned}$$

with $t_{1,2} = 10 \mp 6\sqrt{3}$ and hence $t_2 > 16$ on the cut with branchpoint at $t = 16$.
Diego **Chicharro** recently found a **bunch of 3 bananas** that yield the square of the period of a pendulum at the **second** singular value, obtaining the relation

$$4J(8) + (2+\sqrt{6})J(t_3) + (2-\sqrt{6})J(t_4) = \frac{\Gamma^2(\frac{1}{8})\Gamma^2(\frac{3}{8})}{4\sqrt{2}}$$

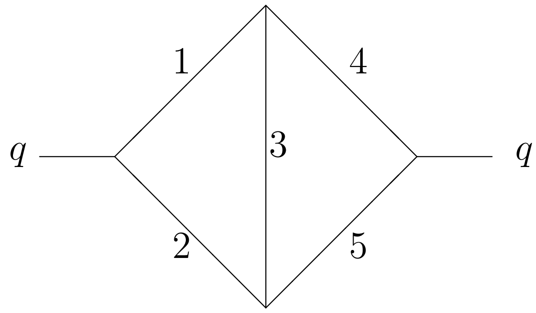
with $t_{3.4} = -8(\sqrt{3} \pm \sqrt{2})^2$. The corresponding **pendulum** has $T(\pi - \theta_2) = \sqrt{2}T(\theta_2)$,

$$\sin(\theta_2/2) = \sqrt{2} - 1, \quad \frac{T(\theta_2)}{T(0)} = \sqrt{1 + \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{1}{8})\Gamma(\frac{3}{8})}{4\pi^{3/2}}} \approx 1.047601377.$$

This concludes my prelude, on **bananas** that yield Γ products, thanks to **Watson**.

Eta decay: 60 years ago, T.D. **Lee** suggested, in `Phys.Rev.139(1965)B1415`, the possibility of a **charge asymmetry** between the energy distributions of $\pi^\pm$ in the **Dalitz plot** for the decay $\eta^0 \to \pi^+ + \pi^- + \pi^0$. In **1968**, Don **Perkins** (1925–2022) told me that the area of this plot is an **elliptic integral**. This prepared me to find an **integral of an elliptic integral**, in the 2-loop **electron propagator**.

The generic kite: Consider the **generic** 2-loop scalar **kite** with 5 internal masses:



$$I(q^2) = -\frac{q^2}{\pi^4}\int \mathrm{d}^4 l \int \mathrm{d}^4 k \prod_{j=1}^{5} \frac{1}{p_j^2 - m_j^2 - \mathrm{i}\epsilon} = \int_{s_0}^{\infty} \sigma'(s) \log\left(1 - \frac{q^2}{s}\right) \mathrm{d}s$$

$$(p_1,\, p_2,\, p_3,\, p_4,\, p_5) = (l,\, l-q,\, l-k,\, k,\, k-q)$$

$$s_0 = \min(s_{1,2},\, s_{4,5},\, s_{2,3,4},\, s_{1,3,5}), \quad s_{j,k} = (m_j + m_k)^2, \quad s_{i,j,k} = (m_i + m_j + m_k)^2.$$

Discontinuity: We have reduced the problem to finding the **derivative** of the discontinuity $I(s+\mathrm{i}\epsilon)-I(s-\mathrm{i}\epsilon)=2\pi\mathrm{i}\sigma(s)$ of the kite, which splits into $\sigma=\sigma_a+\sigma_b$, with **2-particle** cuts in $\sigma_a=\sigma_{1,2}+\sigma_{4,5}$ and **3-particle** cuts in $\sigma_b=\sigma_{2,3,4}+\sigma_{1,3,5}$.

Electron propagator: Here the most demanding master integral is the scalar kite with $m_2=m_3=m_4=1$ and $m_1=m_5=0$. It has an **elliptic obstruction** from $\sigma'_{2,3,4}(s)$ with branchpoint $s=9$.

On-shell evasion: In `Z.Phys.C47(1990)115`, I proved that the integral of an elliptic integral gives **trilogarithms** at the on-shell point $q^2=1$, where we obtain

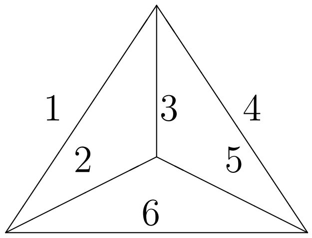
$$\begin{aligned} I(1) &= \frac{4\pi}{3}\int_3^\infty \frac{w^2-9}{\mathrm{AGM}(\sqrt{16w},\sqrt{(w+3)(w-1)^3})}\log\left(1-\frac{1}{w^2}\right)\frac{w\,\mathrm{d}w}{w^2-1} \;-\; 4\zeta_3 \\ &= \tfrac{3}{2}\zeta_3-\pi^2\log 2 = -\mathtt{5.03800310911772511674787191168692108515229996}\ldots \end{aligned}$$

Electron $g-2$: In quantum electrodynamics, we obtain

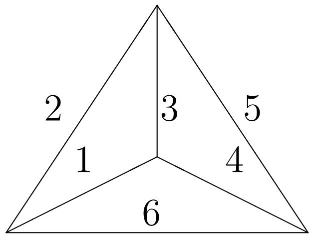
$$g-2=\alpha/\pi+(I(1)+\zeta_2+\tfrac{197}{72})\alpha^2/\pi^2+\ldots$$

for the gyro-magnetic ratio g of electron spin. **Petermann**, in 1957, corrected the erroneous result of **Karplus** and **Kroll**, from 1950, by recalculating those 2 out of 5 terms in which ζ_3 occurs. In both of these terms it is accompanied by $\pi^2\log 2$ in the same proportions as in $I(1)=\frac{3}{2}\zeta_3-\pi^2\log 2$. **Sommerfield** confirmed the total result.

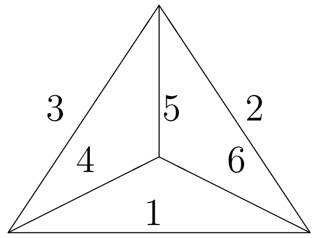
Tadpoles: Now **close the kite** with a **sixth** propagator $1/(q^2 - m_6^2)$ to obtain



tadpole



same tadpole



same tadpole

with the symmetry group S_4 of the **tetrahedron** giving **12 elliptic obstructions**. The tadpole has a logarithmic divergence that we regulate in $D = 4 - 2\varepsilon$ dimensions

$$T_{1,2,3}^{5,4,6} = \left(\frac{1}{3\varepsilon} + 1 \right) 6\zeta_3 + 3\zeta_4 - F_{1,2,3}^{5,4,6} + O(\epsilon)$$

with a **finite part** F that depends on the **six ratios** m_k/μ, where μ is the **scale** of dimensional regularization. The rescaling $m_k \to \lambda m_k$ gives $F \to F + 12\zeta_3 \log(\lambda)$. Without loss of generality, choose m_6 to be the **largest** mass and set $\mu = m_6 = 1$.

With $\mu = m_6 = 1$, **Schwinger** parametrization gives the **5-dimensional** integral

$$F_{1,2,3}^{5,4,6} = \int_0^\infty \mathrm{d}x_1 \ldots \int_0^\infty \mathrm{d}x_5 \, \frac{1}{U^2} \log\left(1 + \sum_{k=1}^{5} x_k m_k^2\right)$$

after setting $x_6 = 1$ in the **Symanzik** polynomial of the **tetrahedron**

$$\begin{aligned} U = {} & x_3(x_1x_2 + x_4x_5) + x_6(x_1x_4 + x_2x_5) + x_3x_6(x_1 + x_2 + x_4 + x_5) \\ & + x_2x_4(x_1 + x_3 + x_5 + x_6) + x_1x_5(x_2 + x_3 + x_4 + x_6). \end{aligned}$$

I reduce this to a **single** integral of a **dilogarithm** against the derivative of $\sigma = \sigma_a + \sigma_b$,

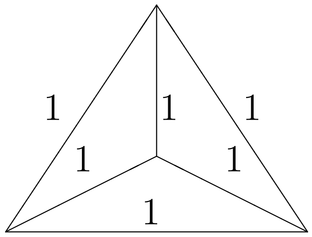
$$F_{1,2,3}^{5,4,6} = -\int_{s_0}^{\infty} (\sigma_a'(s) + \sigma_b'(s)) \mathrm{Li}_2(1-s) \, \mathrm{d}s$$

where σ_b' contains, generically, **complete** elliptic integrals of the **third kind**

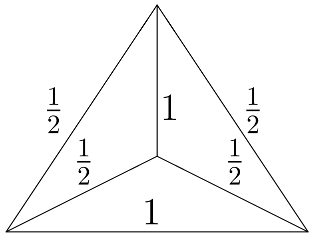
$$\Pi(n,k) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{(1 - n\sin^2\phi)\sqrt{1 - k^2\sin^2\phi}}$$

for which I gave in `arXiv:2212.01962v2` a very efficient AGM process.

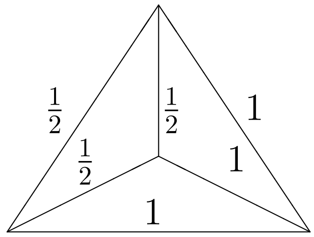
Surprising reductions to polylogs: When all **6** masses are non-zero, there is no non-elliptic route. Yet in 3 cases, I found **empirical** reductions to **polylogs**.



F_6



G_6



H_6

A binary surprise: Dressings of the tetrahedron with $m_k \in \{0,1\}$ give rational linear combinations of **4 tetralogarithms**: $\zeta_4 = \pi^4/90$, $\mathrm{Cl}_2^2(\pi/3)$, $U_{3,1}$ and $V_{3,1}$, with $\mathrm{Cl}_2(\pi/3) = \Im \mathrm{Li}_2(\lambda)$, where $\lambda = \frac{1}{2}(1+\sqrt{-3})$, and **reducible** double sums

$$U_{3,1} = \sum_{m>n>0} \frac{(-1)^{m+n}}{m^3 n} = \tfrac{1}{2}\zeta_4 + \tfrac{1}{2}\zeta_2 \log^2(2) - \tfrac{1}{12}\log^4(2) - 2\,\mathrm{Li}_4(\tfrac{1}{2})$$

$$V_{3,1} = \sum_{m>n>0} \frac{(-1)^m \cos(2\pi n/3)}{m^3 n} = -\tfrac{145}{432}\zeta_4 + \tfrac{1}{8}\zeta_2 \log^2(3) - \tfrac{1}{96}\log^4(3) + \tfrac{1}{32}\mathrm{Li}_4(\tfrac{1}{9}) - \tfrac{3}{4}\mathrm{Li}_4(\tfrac{1}{3}) + \tfrac{1}{3}\mathrm{Cl}_2^2(\pi/3).$$

With only **5** unit masses, there was a **non-elliptic** route to my empirical result

$$F_5 = \tfrac{550}{27}\zeta_4 + 16V_{3,1} - \tfrac{8}{3}\mathrm{Cl}_2^2(\pi/3) \tag{1}$$

which Yajun **Zhou** and (independently) Steven **Charlton** proved, by applying `HyperInt` from Erik **Panzer** and `MZIntegrate` from Kam Cheong **Au**, using the representation

$$F_5 = 4\int_0^1 \frac{g(y^2)-g(-y)}{y}\left(\frac{1-y^2}{1+y+y^2}\right)\log\left(\frac{(1+y)^2}{y}\right)\mathrm{d}y$$

with $g(y) = \mathrm{Li}_2(y) + \frac{1}{2}\log(y^2)\log(1-y)$, which I obtained by setting $m_3 = 0$.
More **surprising** was my very simple **empirical** result for the **totally massive** case

$$F_6 = 16\,\zeta_4 + 8\,U_{3,1} + 4\,\mathrm{Cl}_2^2(\pi/3). \tag{2}$$

Divide and conquer: We may now **separate** 2-particle and 3-particle cuts in the **difference** $F_6 - F_5 = F_a + F_b$, obtaining **convergent** integrals in

$$F_a = -2\int_0^1 \left(\frac{(1-y^2)\log(y)}{y} + \frac{2\pi}{\sqrt{3}}\right)\left(\mathrm{Li}_2\left(\frac{1+y+y^2}{(1+y)^2}\right) - \zeta_2\right)\frac{\mathrm{d}y}{1-y+y^2}$$

$$F_b = \int_9^\infty \sigma_b'(s)(\mathrm{Li}_2(1-s^{-1}) - \zeta_2)\,\mathrm{d}s$$

with a transformation $s = (1+y)^2/y$ for the 2-particle cut with branchpoint $s = 4$.

Tetralogarithms of sixth roots of unity: With $\lambda = \frac{1}{2}(1+\sqrt{-3})$, I obtained

$$F_a = -\tfrac{422}{81}\zeta_4 + \tfrac{16}{9}U_{3,1} - \tfrac{32}{3}V_{3,1} + \tfrac{50}{9}\mathrm{Cl}_2^2(\pi/3) + \pi\left(\tfrac{16}{3}\Im\mathrm{Li}_3(\lambda/2) - \tfrac{64}{15}\Im\mathrm{Li}_3(\mathrm{i}/\sqrt{3})\right) + \pi^2\left(\tfrac{14}{45}\log^2(3) - \tfrac{8}{9}\log^2(2) - \tfrac{2}{9}\mathrm{Li}_2(\tfrac{1}{4})\right) \qquad (3)$$

empirically. This has been **proved** by Charlton, using Au's `MultipleZetaValues`.

Evasion: The elliptic **obstruction** is in

$$F_b = \int_9^\infty \sigma_b'(s)(\mathrm{Li}_2(1-s^{-1}) - \zeta_2)\,\mathrm{d}s$$

$$\sigma_b'(s) = \Re\left(\frac{32}{\widehat{s}}\left(1+\frac{3s}{t}\right)\frac{\Pi(\widehat{n},k) - \Pi(n,k)}{\sqrt{(w+3)(w-1)^3}}\right)$$

$$k^2 = \frac{(w-3)(w+1)^3}{(w+3)(w-1)^3}, \quad n = \frac{(3-w)(w+1)}{3(w-1)^2}, \quad \frac{\widehat{n}}{n} = \frac{6+3\widehat{s}}{6-\widehat{s}}$$

with $t = \sqrt{3s(4-s)}$, $\widehat{s} = s+t$ and $w = \sqrt{s} > 3$.

The **empirical** relation $F_6 = F_5 + F_a + F_b = 16\zeta_4 + 8U_{3,1} + 4\mathrm{Cl}_2^2(\pi/3)$ can now be checked at **250-digit** precision in **6 seconds**, thanks to the extraordinary efficiency of the AGM process for complete elliptic integrals of the **third kind**.

At MPIM Bonn, in August 2024, Steven **Charlton** and I found a **proof**, as follows.

Taming the square root of a quartic: Leonhard **Euler** showed how to handle integrals over the square root of a **quadratic**, by a **transformation** of the variable of integration. In F_b we have an **integral** of an **elliptic integral** that results from integration over 3-body phase space, obstructed by the square root of a **quartic** that is a **product** of quadratics. With **two** variables of integration, it may be possible to **tame the quartic**. We succeeded in doing this, obtaining

$$F_b = 2\int_0^1 \mathrm{d}y \left(\frac{\mathrm{d}A(y)}{\mathrm{d}y}\right) \int_0^1 \mathrm{d}v \left(\frac{\partial B(y,v)}{\partial v}\right) C(y,v)D(y,v)$$

with **logarithms** in $\{B, C, D\}$ whose arguments are very conveniently **linear** in v:

$$\begin{aligned}
A(y) &= \log(y^2) - \log(y^2 - \lambda^2) - \log(y^2 - \lambda^{-2})\\
B(y,v) &= \log\left(v + \frac{y}{1+y^2}\right) + \log\left(v + \frac{1+y^2}{y}\right) - \log(v+y) - \log(v+y^{-1})\\
C(y,v) &= \log\left(v + \frac{y}{1+y^2}\right) + \log\left(v + \frac{1+y^2}{y}\right) - \log\left(\frac{vy}{1+y^2}\right)\\
D(y,v) &= \log(vy+\lambda) + \log(vy+\lambda^{-1}) - \log(v+y\lambda) - \log(v+y\lambda^{-1})
\end{aligned}$$

and were then able to **prove** that $F_6 = 16\zeta_4 + 8U_{3,1} + 4\mathrm{Cl}_2^2(\pi/3)$.

Two more surprising evasions: In G_6 we have $m_1 = m_2 = m_4 = m_5 = \frac{1}{2}$ and $m_3 = m_6 = 1$. In H_6 we have $m_1 = m_2 = m_3 = \frac{1}{2}$ and $m_4 = m_5 = m_6 = 1$. Set $m_3 = 0$ in G_6 to get G_5 and set $m_4 = 0$ in H_6 to get H_5, There are **non-elliptic** routes to

$$G_5 = \int_0^1 \left(\frac{1}{y} - \frac{2}{1+y}\right) (2L_3(y) - L_3(y^2))\, \mathrm{d}y$$
$$H_5 = \int_0^{\frac{1}{2}} \left(\frac{1}{y} + \frac{2}{1-y}\right) (L_3(2y) + L_3(y/2) - L_3(y^2))\, \mathrm{d}y - 12\,\zeta_3 \log(2)$$

where $L_3(y) = 6\,\mathrm{Li}_3(y) - 4\log(y)\,\mathrm{Li}_2(y) - \log^2(y)\log(1-y)$ and the subtraction of $12\,\zeta_3\log(2)$ in H_5 results from a rescaling of masses. Then we were able to **prove** that

$$G_5 = -16\zeta_4 + 30\zeta_3 \log(2) - 32U_{3,1} \tag{4}$$
$$H_5 = 37\zeta_4 - 18\zeta_3 \log(2) + 32U_{3,1} - \tfrac{1}{2}B \tag{5}$$

with a notable combination of **classical tertralogarithms** in

$$B = 6(2\zeta_4 - 3\mathrm{Li}_4(\tfrac{1}{4})) + 8(2\zeta_3 - 3\mathrm{Li}_3(\tfrac{1}{4}))\log(2) - 12\mathrm{Li}_2(\tfrac{1}{4})\log^2(2) - 4\log^4(2).$$

It remains to deal with 2-particle cuts (with subscript a) and 3-particle cuts (with subscript b) in the **differences** $G_6 - G_5 = G_a + G_b$ and $H_6 - H_5 = H_a + H_b$.

Two-particle cuts: Surprisingly, $G_a = \mathbf{0}$. This is a special case of a remarkable **functional identity**, which we discovered when studying a more general case with $m_1 = m_5 = 1/(1+r)$, $m_2 = m_4 = r/(1+r)$ and $m_3 = m_6 = 1$. For $r > 0$, let $h = (1-r)/(1+r)$ and

$$\begin{aligned} g(r,y) = & \left(\frac{1}{1-y} - \frac{1}{1+y} - \frac{h}{1-hy} + \frac{h}{1+hy}\right) \log\left(\frac{1+y}{1-y}\right) \\ & - \left(\frac{h}{1-hy} + \frac{h}{1+hy}\right) \log(r) + \left(\frac{1}{1-y} + \frac{1}{1+y}\right) \log\left(\frac{r}{(1+r)^2}\right). \end{aligned}$$

Then we have a relation between a **tetralogarithm** and the **square of a dilogarithm**:

$$\begin{aligned} G_a(r) &= 2\int_0^1 g(r,y)\left(\mathrm{Li}_2\left(1 - \frac{1-y^2}{1-h^2y^2}\right) - \zeta_2\right) \mathrm{d}y \\ &= -\left(2\log(r)\log(1+r) + \mathrm{Li}_2(1-r^2)\right)^{\mathbf{2}} = G_a(1/r) \end{aligned} \tag{6}$$

and in particular $G_a = G_a(1) = 0$. A **proof** of this identity was achieved using `hyperint`. Its **existence**, revealed by studies of Feynman integrals, indicates structure that is even richer than that explored here, for the 3-loop tadpole with $r = 1$.

There are 2-particle cuts with **branchpoints** at $s=1$ and $s=4$ in the case of

$$H_a = \int_4^\infty \frac{\operatorname{arccosh}(2s-1)+\operatorname{arccosh}(s/2-1)}{\sqrt{(s-1)(s-4)}}\left(1+\frac{2}{s}\right)L_2(s)\,\mathrm{d}s$$
$$-\int_1^4 \frac{\arccos(1-s/2)}{\sqrt{(s-1)(4-s)}}\left(1+\frac{2}{s}\right)L_2(s)\,\mathrm{d}s + \int_1^\infty \frac{2\log(2)L_2(s)}{\sqrt{s(s-1)}}\,\mathrm{d}s$$

with $L_2(s) = \mathrm{Li}_2(1-s^{-1})-\zeta_2$. After a suitable transformation of variable, we obtained

$$H_a = -23\,\zeta_4 - \tfrac{5}{3}\,\zeta_3\log(2) - 20\,U_{3,1} + 10\,\mathrm{Cl}_2^2(\pi/3) + \tfrac{1}{3}B. \tag{7}$$

Taming 3-particle cuts: We **eventually** evaluated the **doubly** transformed integrals

$$G_b = \int_0^1\int_0^1 P(y,v)\log\left(\frac{1+av}{v+a}\right)\mathrm{d}v\,\mathrm{d}y = 16\,\zeta_4 - 30\,\zeta_3\log(2) + 32\,U_{3,1} + B \tag{8}$$

$$H_b = \int_0^1\int_0^1 P(y,v)\log\left(\frac{1-bv}{|v-b|}\right)\mathrm{d}v\,\mathrm{d}y = -4\zeta_4 + \tfrac{58}{3}\zeta_3\log(2) - 16\,U_{3,1} - \tfrac{1}{3}B \tag{9}$$

$$P(y,v) = \frac{8y}{1-y^2}\left(\frac{1}{v+u}+\frac{u}{1+uv}-\frac{1}{v+b}-\frac{b}{1+bv}\right)\log\left(\frac{(v+u)(1+uv)}{4u^2v}\right)$$

$$\frac{a}{u} = \frac{3+y}{1+y},\quad \frac{b}{u} = \frac{3-y}{1+y},\quad u = \left(\frac{1-y^2}{9-y^2}\right)^{1/2}.$$

From the result for G_b we obtain the remarkable evaluation

$$
\begin{gathered}
4\int_0^1 (1-y)U(y)T_3(y)\,\mathrm{d}y = 16\,\zeta_4 - 30\,\zeta_3\log(2) + 32\,U_{3,1} + B \\
U(y) = \frac{y(1+y)K(k)+E(k)}{y(1+y+y^2)\sqrt{1+y}}\,, \quad k^2 = 1-y^3 \\
T_3(y) = \mathrm{Li}_3(u) - \tfrac{1}{2}\,\mathrm{Li}_2(u)\log(u)\,, \quad u = \frac{y}{(1+y)^2},
\end{gathered}
$$

integrating complete elliptic integrals of **first** and **second** kinds against a **trilogarithm**. Combing the results for G_b and H_b we obtain

$$
\begin{gathered}
4\int_2^\infty V(w)\left(\mathrm{Li}_2(1-w^{-2}) - \zeta_2\right)\,\mathrm{d}w = \zeta_4 + 7\zeta_3\log(2) - 4U_{3,1} \\
V(w) = \frac{\Pi(0,k) - \Pi(n,k) - 6\,\Pi(\widehat{n},k)}{w(w-1)\sqrt{w^2+2w}} \\
k^2 = 1 - \frac{4}{(w-1)^2(w+2)}, \quad n = 1 - \frac{1}{(w-1)^2}, \quad \widehat{n} = 1 - \frac{2}{w(w-1)}
\end{gathered}
$$

integrating complete elliptic integrals of the **third** kind against a **dilogarithm**.

Comments: Combining **9** evaluations, we **prove** 3 results for **3-loop tadpoles**:

$$F_6 = F_5 + F_a + F_b = 16\,\zeta_4 + 8\,U_{3,1} + 4\,\mathrm{Cl}_2^2(\pi/3) \tag{10}$$

$$G_6 = G_5 + G_a + G_b = B \tag{11}$$

$$H_6 = H_5 + H_a + H_b = 10\,\zeta_4 + 3\,\zeta_3\log(2) - 4\,U_{3,1} + 10\,\mathrm{Cl}_2^2(\pi/3) - \tfrac{1}{2}B \tag{12}$$

in terms of **5** constants reducible to **polylogarithms** of **depth 1** and their products. The first result was conjectured **27 years** ago; the others merely **2 years** ago.

Steven **Charlton** was able to prove these results, thanks to his mastery of methods that have been automated by Erik **Panzer** and by Kam Cheong **Au**. It was a privilege to work with Steven at **MPIM Bonn**. We are currently considering the possibility of **evasion** of elliptic obstructions in **families** $G_6(r) = G_5(r) + G_a(r) + G_b(r)$, in the light of the **amazing** reduction of $G_a(r)$ to a **square of a dilogarithm**.

I like to think that our 3 surprising results bear comparison with the 3 notable results of George Neville **Watson** (1886–1965) in **Birmingham**, which I have here related to **bunches** of **3-loop bananas** in response to recent work by Diego **Chicharro**.

At bottom, this present work came from mentors in **1968**: Donald **Perkins** (1925–2022), in Oxford, and Gabriel **Barton** (1934–2022), in Sussex. I like to think that they would have been pleased to know how long their kind and wise advice has endured.

Appendix: A fragment of the magnetic moment of the electron

$$
\begin{aligned}
I(1) &= \frac{4\pi}{3}\int_3^\infty \frac{w^2-9}{\mathrm{AGM}(\sqrt{16w},\sqrt{(w+3)(w-1)^3})}\log\left(1-\frac{1}{w^2}\right)\frac{w\,\mathrm{d}w}{w^2-1} \;-\; 4\zeta_3\\
&= \tfrac{3}{2}\zeta_3-\pi^2\log 2 = -5.03800310911772511674787191168692108515229 96\ldots
\end{aligned}
$$

To integrate, commence at value three,
concluding at infinity, as close you dare,
an integral, with respect to this, the chosen variable,
here designated w, whatever that might be,
of an integrand, richly composed as follows.
First, from the square of w, take nine,
so that the threshold is suppressed;
straightway, divide by this fine AGM:
the arithmetic-geometric mean of two square roots,
with arguments both linear and quartic.

Next take a logarithm, from **John Napier**,
falling as w tends to the firmament,
and take, from **Gottfried Leibniz**,
an increment so infinitesimally small
that **Isaac Newton** would approve it.
Adjoin one power of guiding w
and make a quotient with its square,
diminished by a simple unity,
to conclude the chosen integrand:
yet still we are not done.
Take, with all this, a product with a factor
that measures out the volume of a sphere
and last of all subtract, with factor four,
a *zeta* value, whose argument is three,
defined by **Bernhard Riemann's** single sum.
The whole appears, in an almighty sum
of terms that **Richard Feynman** still dictates,
determining, as best we may, the magnetism
of a point-like spinning charge
that clings to amber as *ήλεκτρον*.