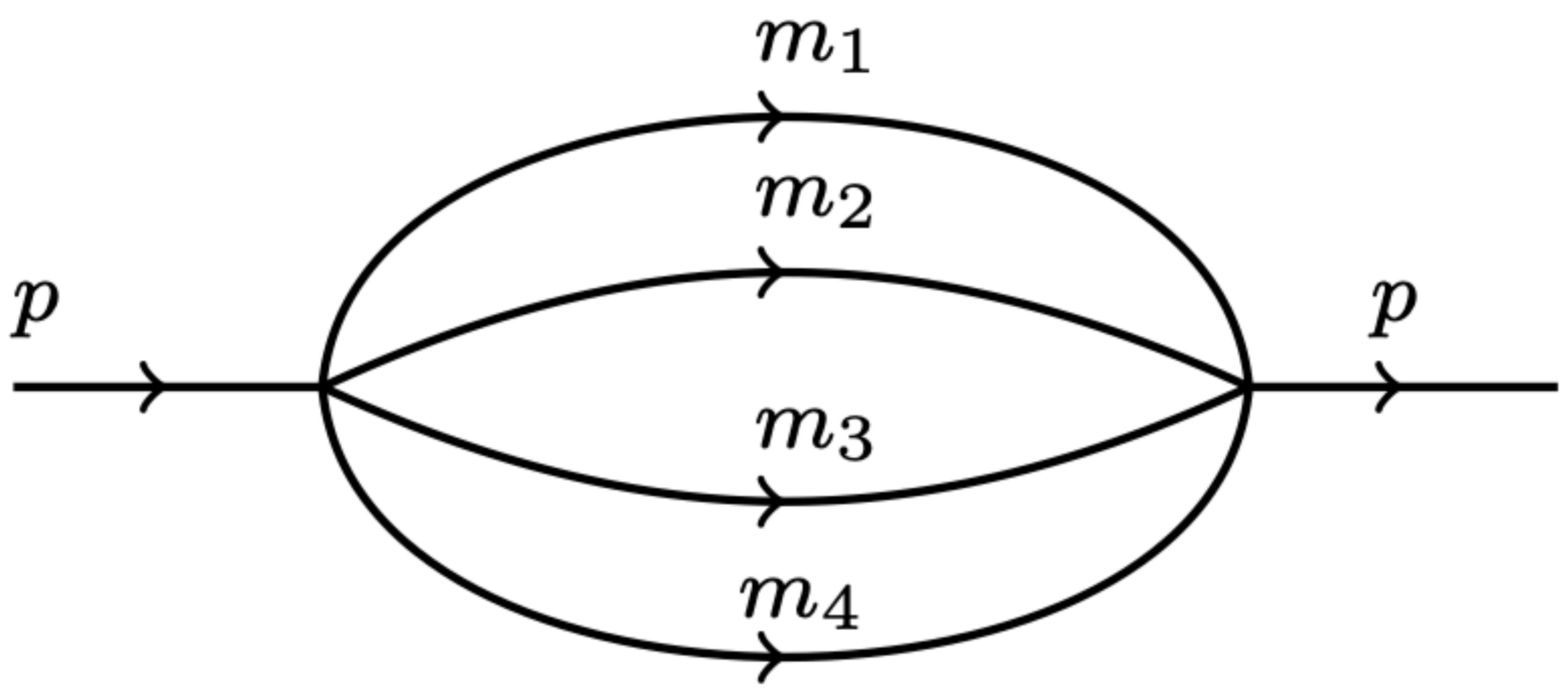
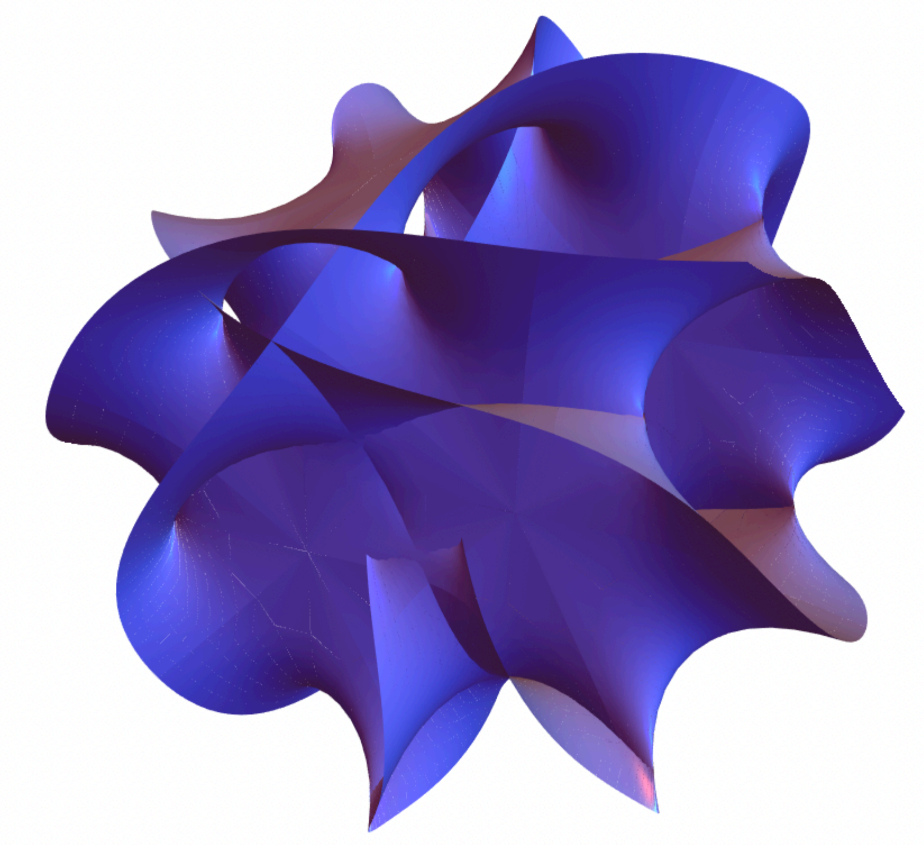
Three-loop banana integrals with four unequal masses

with **Claude Duhr, Franziska Porkert, Cathrin Semper, Sven F. Stawinski**

Based on: arXiv 2507.23061





Sara Maggio | Elliptics and Beyond ‘25, BCTP | 04/09/2025

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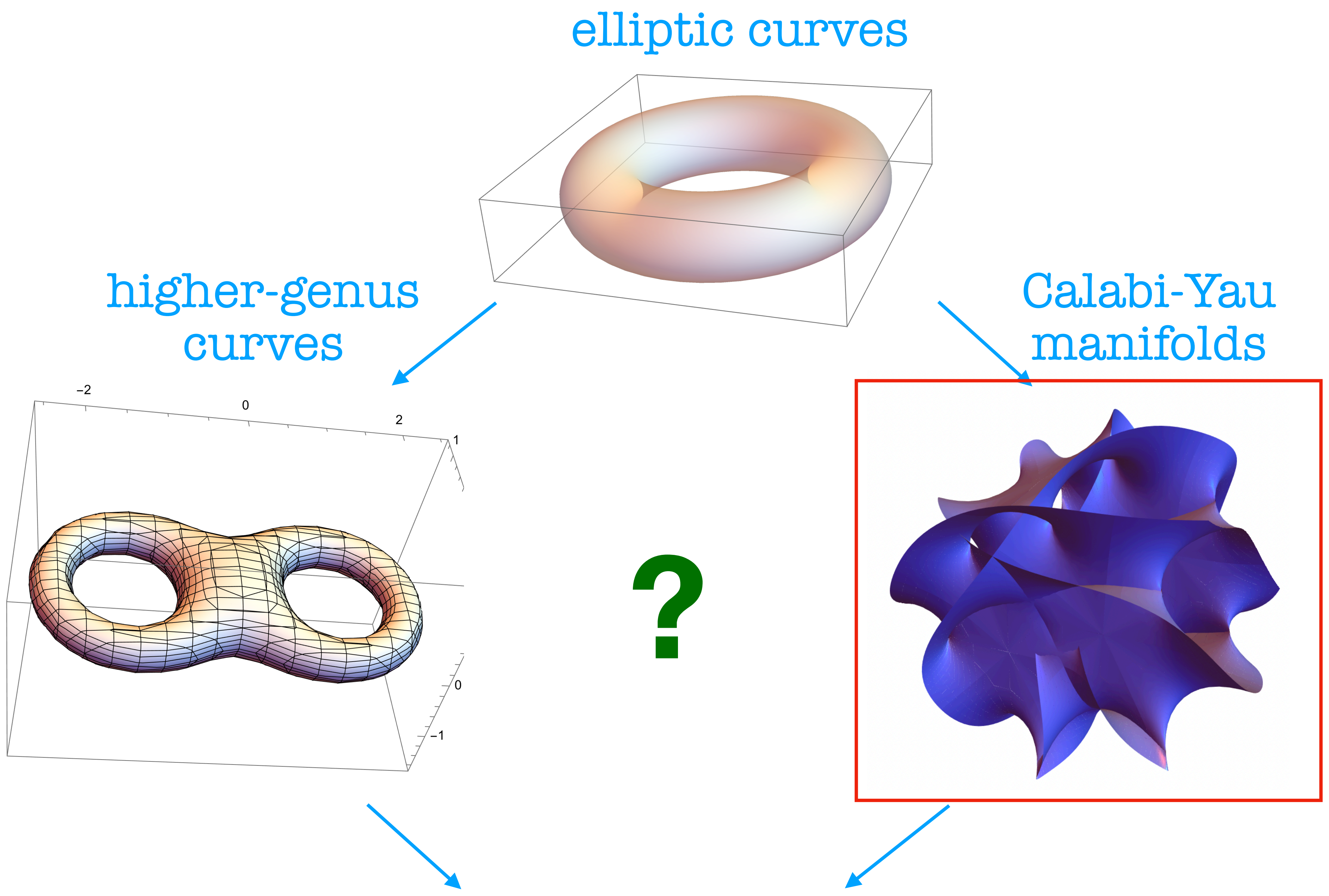
Motivation

- Feynman integrals are known to have other underlying geometries than the Riemann sphere

➡ *Elliptics and Beyond '25*



powerful way to solve them: *canonical* DE

- Canonical DE?
- functions in the DE *not* expressible in terms of rational functions and periods, but as *integrals* over them;
- are these really *new functions*?

- simplest class (CY): *Banana integrals*
- equal masses ✓ [Pögel, Wang, Weinzierl '22]
- two different masses ✓ [Maggio, Sohnle, '25]
- all unequal masses ✓ this talk & [Pögel, Teschke, Wang, Weinzierl '25]

Outline

talk by Christoph

Canonical form

+

talk by Sven

Twisted cohomology

[1] [Görges, Nega, Tancredi, Wagner, '23]
[2] [Duhr, **S.M**, Nega, Sauer, Tancredi, Wagner, '25]

[3] [Duhr, Porkert, Semper, Stawinski, '24]
[4] [Duhr, **S.M,** Porkert, Semper, Sohnle, Stawinski, '25]

- Compact analytic solution
- Study the power of these tools & the functions that appear

! The three-loop unequal masses banana integral is just an example, we expect these methods to be applicable to other multiloop, multiscale Feynman integrals attached to non-trivial geometries **!**

Other methods to achieve an (almost) ε-form: talk by Vasily

talk by Pouria → ε-form for the three-loop unequal masses banana

Quick review of the method from [1],[2]

- Good initial basis: *integrand analysis* in integer dimensions. [2]: choose integrals aligned with the geometry associated to the maximal cuts at $\varepsilon = 0$.

- Rotate the initial basis $I(x, \varepsilon)$ by a sequence of rotations:

$$J(x,\varepsilon) = U_{\mathrm{t}}(x,\varepsilon)\,U_{\varepsilon}(\varepsilon)\,U_{\mathrm{ss}}(x)\,I(x,\varepsilon)$$

to get $\varepsilon-$form:

$$\mathrm{d}J(x,\varepsilon) = \varepsilon A(x) J(x,\varepsilon)$$

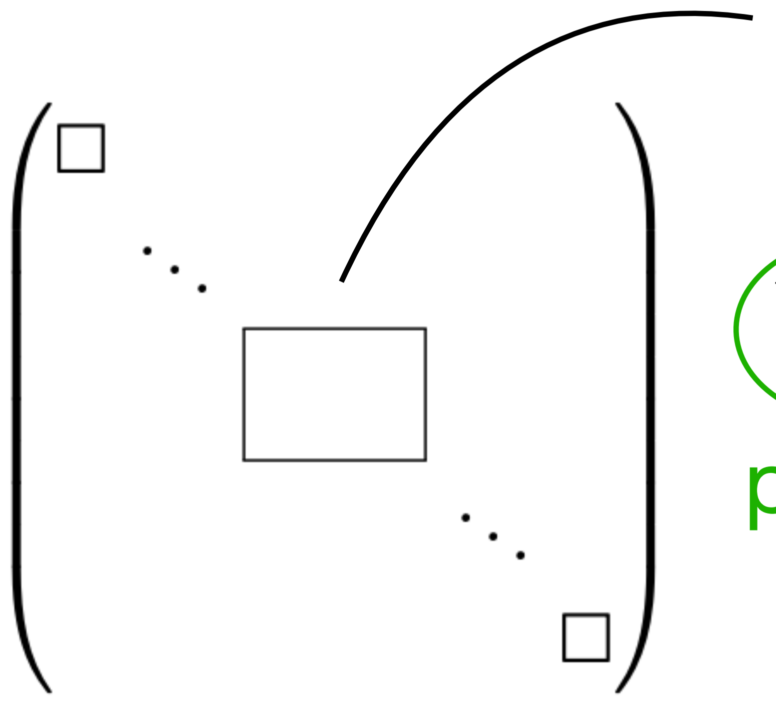
$$\frac{1}{\varepsilon^{n-1}} U_{\mathrm{t}}^{(1-n)}(x) + \ldots + \frac{1}{\varepsilon} U_{\mathrm{t}}^{(-1)}(x) + U_{\mathrm{t}}^{(0)}(x)$$

$U_{\mathrm{t}}^{(i)}$: strictly lower-triangular

dimension of the geometry

- $\varepsilon-$scaling to realign the transcendental weight

- the non linear in ε parts of the DE are in a strictly lower-triangular matrix



W_{ss}^{-1}

$W(x) = W_{\mathrm{ss}}(x) W_{\mathrm{u}}(x)$

period matrix

disentangles the logs

Quick review of the tools from [3],[4]

multi-valued function

single-valued differential form

- Feynman integrals (MC), in dim reg, are *multi-valued* differential forms: $\int_C \Psi \phi$

singularities at the branch point of Ψ

- Twisted cohomology group: vector space generated by a well-defined set of ϕ

basis ~ master integrals $I(x, \varepsilon)$

dual: $\int_\gamma \Psi^{-1} \check{\phi}$

Cohomology intersection matrix:

$$C(x, \varepsilon)_{ij} = \frac{1}{(2\pi i)^n} \left(\int_X \phi_i \wedge \check{\phi}_j \right)$$

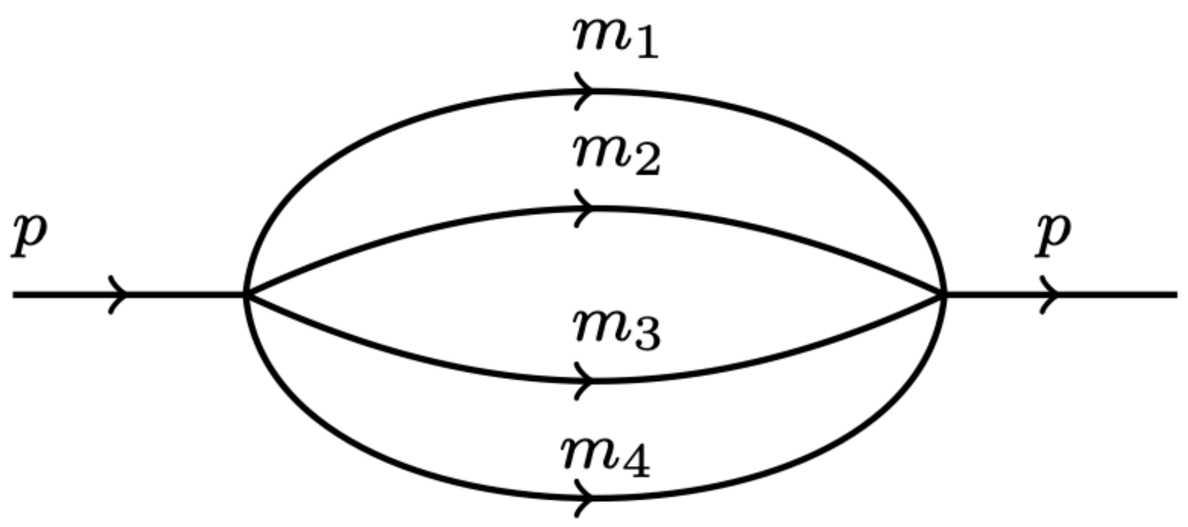
[3]: For maximal cuts: $\check{I}(x, \varepsilon) = I(x, -\varepsilon)$

constant matrix

- [3]: Rotating to canonical form: $C(\varepsilon) \to \mathrm{d}C = 0 \to C = f(\varepsilon)\Delta$

- [4]: $U_{\mathrm{t}}^{(0)}(x) = O(x)\, R(x)$: $\Delta\, O^T\, \Delta^{-1} = O^{-1}\,, \Delta\, R^T\, \Delta^{-1} = R$

The three-loop banana integral



$D = 2 - 2\epsilon$

$$I_{\nu_1,\ldots,\nu_9} = e^{3\gamma_E \epsilon} \int \left(\prod_{a=1}^{3} \frac{d^D k_a}{i\pi^{\frac{D}{2}}} \right) \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5} D_6^{\nu_6} D_7^{\nu_7} D_8^{\nu_8} D_9^{\nu_9}},$$

$D_1 = k_1^2 - m_1^2, \quad D_2 = k_2^2 - m_2^2,$
$D_3 = (k_1 - k_3)^2 - m_3^2, \quad D_4 = (k_2 - k_3 - p)^2 - m_4^2,$
$D_5 = k_3^2, \quad D_6 = k_3 \cdot p,$
$D_7 = k_1 \cdot p, \quad D_8 = k_2 \cdot p,$
$D_9 = k_1 \cdot k_2.$

$x_i = \frac{m_i^2}{p^2}$

$\mathrm{MC}\left(I_{1,1,1,1}\right) \sim \int \mathrm{d}z_1\, \mathrm{d}z_2\, \Psi(z_1, z_2; x)$ $\quad y^2 = \Psi(z_1, z_2; x)_{|\varepsilon=0}$ (Hulek-Verril) 4-parameters K3 variety

[2]: Basis compatible with the geometry at $\varepsilon = 0$

middle cohomology: $H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$

(1) (4) (1)

How can we span the middle cohomology?

Griffiths transversality!

$\Omega(x) \in H^{2,0}$ unique holomorphic (2,0)-form $\quad \Omega(x) = \Psi(z_1, z_2; x)_{|\varepsilon=0}\, \mathrm{d}z_1\, \mathrm{d}z_2$

period vector $\quad \psi(x) = \left(\psi_0(x), \psi_1^{(1)}(x), \psi_1^{(2)}(x), \psi_1^{(3)}(x), \psi_1^{(4)}(x), \psi_2(x)\right)^T = \left(\int_{\Gamma_0} \Omega, \ldots, \int_{\Gamma_5} \Omega\right)^T$

$\partial_{x_j} \Omega(x) \in H^{2,0} \oplus H^{1,1}$ four first derivatives

$\partial_{x_j} \partial_{x_k} \Omega(x) \in H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ one double derivative

Good choice of basis

- [2]: basis of MIs on the maximal cut at least 6-dimensional
- IBP algorithms: 15 MIs

$$I_1 = I_{0,1,1,1,0,0,0,0,0}\,,\quad I_2 = I_{1,0,1,1,0,0,0,0,0}\,,\quad I_3 = I_{1,1,0,1,0,0,0,0,0}\,,\quad I_4 = I_{1,1,1,0,0,0,0,0,0}\,,$$

tadpoles

$$I_5 = I_{1,1,1,1,0,0,0,0,0}\,,$$
$$I_6 = I_{2,1,1,1,0,0,0,0,0} = \partial_{m_1^2} I_5\,,$$
$$I_7 = I_{1,2,1,1,0,0,0,0,0} = \partial_{m_2^2} I_5\,,$$
$$I_8 = I_{1,1,2,1,0,0,0,0,0} = \partial_{m_3^2} I_5\,,$$
$$I_9 = I_{1,1,1,2,0,0,0,0,0} = \partial_{m_4^2} I_5\,,$$
$$I_{15} = I_{3,1,1,1,0,0,0,0,0} = \frac{1}{2}\partial^2_{m_1^2} I_5$$

K3 block

$$I_{10} = I_{1,1,1,1,-1,0,0,0,0}\,,$$
$$I_{11} = I_{1,1,1,1,0,-1,0,0,0}\,,$$
$$I_{12} = I_{1,1,1,1,0,0,-1,0,0}\,,$$
$$I_{13} = I_{1,1,1,1,0,0,0,-1,0}\,,$$
$$I_{14} = I_{1,1,1,1,0,0,0,0,-1}\,,$$

ISPs “3rd kind”

Rotation to canonical form: $U_\varepsilon(\varepsilon)U_{\text{ss}}(x)$

$\mathbf{j}(\tau) = \left(\frac{\partial x_i}{\partial \tau_j}\right)_{1\le i,j\le 4}.$

- U_{ss}:

tadpoles 4x4 block & LS(integrands)=1

K3 6x6 block & non-trivial geometry

$W(x) = W_{\text{ss}}(x)W_{\text{u}}(x)$

CY multi-scale: [Maggio, Sohnle, '25]

ISPs 5x5 block & LS(integrands)=1

$$\begin{pmatrix} \mathbb{1}_{4\times4} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\psi_0} & 0 & 0 & 0 \\ 0 & \partial_{\boldsymbol{\tau}}\frac{1}{\psi_0} & \frac{\boldsymbol{j}^T}{\psi_0} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{5\times5} & 0 \\ 0 & f_1 & \boldsymbol{f} & 0 & f_6 \end{pmatrix}$$

$\tau_r = \frac{\psi_1^{(r)}}{\psi_0}$

$$W_{\text{ss}}(x)\Sigma W_{\text{ss}}(x)^T = Z(x)$$

intersection paring $C(x)^{-1}_{|\epsilon=0}$

- U_ε:

tadpoles & I_5

$$\begin{pmatrix} \varepsilon^3\mathbb{1}_{5\times5} & 0 & 0 & 0 \\ 0 & \varepsilon^2\mathbb{1}_{4\times4} & 0 & 0 \\ 0 & 0 & \varepsilon^3\mathbb{1}_{5\times5} & 0 \\ 0 & 0 & 0 & \varepsilon \end{pmatrix}$$

4 single derivatives integrals

ISPs

1 double derivatives integral

disentangles the logs ✓

transcendental weight ✓

Rotation to canonical form: $U_{\rm t}(x,\varepsilon)$

K3: $n=2$

$$U_{\rm t}(x,\varepsilon) = \frac{1}{\varepsilon} U_{\rm t}^{(-1)}(x) + U_{\rm t}^{(0)}(x)$$

$$\begin{pmatrix} \mathbb{1}_{4\times4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{9\times9} & 0 \\ 0 & G_0(\boldsymbol{x}) & 0 & 1 \end{pmatrix}$$

1 "new" function

9 $\varepsilon-$functions

MC: 19 $\varepsilon-$ functions

$$\begin{pmatrix} \mathbb{1}_{4\times4} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \mathbf{t}_5(\boldsymbol{x}) & \mathbb{1}_{9\times9} & 0 \\ \mathbf{t}_{15,t}(\boldsymbol{x}) & t_{15,5}(\boldsymbol{x}) & \mathbf{t}_{15}(\boldsymbol{x}) & 1 \end{pmatrix}$$

4 $\varepsilon-$functions

9 $\varepsilon-$functions

1 $\varepsilon-$function

- Determine them such that $\mathrm{d}J(x,\varepsilon) = \varepsilon A(x) J(x,\varepsilon)$;
- Defined by first-order differential equations;
- Solve them by series expansion;
- 24 ε-functions in total; 23 appear in $A(x)$.

→ $A(x)$ has only simple-poles ✓

$\varepsilon\, A(x)$ is ϵ-factorised ✓

- Analytic representation?
- Are there relations?

another approach: [Pögel, Wang, Weinzierl, Wu, Zu]

Splitting of $U_{\rm t}^{(0)}(x,\varepsilon)$

On the maximal cut, rotate $C(x,\epsilon)$ by $U(x,\epsilon)$:

$$U(x,\varepsilon)C(x,\varepsilon)U(x,-\varepsilon)^T = -\frac{\varepsilon^4}{4}\Delta, \qquad \boldsymbol{\Delta} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\boldsymbol{S} & 0 & 0 \\ 0 & 0 & \boldsymbol{E} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

matrices of numbers

[3] linear independence of differential forms ✓

$\Delta\, O^T \Delta^{-1} = O^{-1}$, $\quad \Delta\, R^T \Delta^{-1} = R$

[4] If $\Delta\,(U_{\rm t}^{(0)})^T \Delta^{-1} \sim U_{\rm t}^{(0)}$ ➡ $U_{\rm t}^{(0)} = O R$ —[4]→ rational functions, periods, and their derivatives!

$$O = \begin{pmatrix} 1 & 0 & 0 \\ \boldsymbol{G}(\boldsymbol{x}) & \mathbb{1} & 0 \\ \tilde{G}_0(\boldsymbol{x}) & -\boldsymbol{\rho}(\boldsymbol{G}(\boldsymbol{x}))^T & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 & 0 \\ \boldsymbol{s}(\boldsymbol{x}) & \mathbb{1} & 0 \\ s_{10}(\boldsymbol{x}) & \boldsymbol{\rho}(\boldsymbol{s}(\boldsymbol{x}))^T & 1 \end{pmatrix}$$

9 functions   9 functions

➡ 9 (+4) functions to determine!

(-2) from comparing their DE

$\epsilon-$functions

- What about $G_0, \dots, G_{13}$?

Not fixed by the previous constraints

Fixed by solving the differential equations

$$
\begin{aligned}
G_9(\boldsymbol{x}) =& \frac{\boldsymbol{C}^{(0)}_{10,11}(\boldsymbol{x})}{2\,\boldsymbol{C}^{(0)}_{1,11}(\boldsymbol{x})}\psi_0(\boldsymbol{x}) + \int_{\xi_1}^{x_1} \mathrm{d}y_1\, x_2\partial_{x_2}\psi_0(y_1,x_2,x_3,x_4) + c_{G_9}\,, \\
G_8(\boldsymbol{x}) =& \frac{\boldsymbol{C}^{(0)}_{9,11}(\boldsymbol{x})}{2\,\boldsymbol{C}^{(0)}_{1,11(\boldsymbol{x})}}\psi_0(\boldsymbol{x}) - x_2\psi_0(\boldsymbol{x}) - \int_{\xi_2}^{x_2} \mathrm{d}y_2\,(x_4\partial_{x_4}\psi_0 + x_3\partial_{x_3}\psi_0 + x_1\partial_{x_1}\psi_0) + c_{G_8}\,, \\
G_7(\boldsymbol{x}) =& \frac{\boldsymbol{C}^{(0)}_{8,11}(\boldsymbol{x})}{2\,\boldsymbol{C}^{(0)}_{1,11}(\boldsymbol{x})}\psi_0(\boldsymbol{x}) + x_1\psi_0(\boldsymbol{x}) + \int_{\xi_1}^{x_1} \mathrm{d}y_1\,(x_4\partial_{x_4}\psi_0 + x_3\partial_{x_3}\psi_0 + x_2\partial_{x_2}\psi_0) + c_{G_7}\,, \\
G_6(\boldsymbol{x}) =& \frac{\boldsymbol{C}^{(0)}_{7,11}(\boldsymbol{x})}{2\,\boldsymbol{C}^{(0)}_{1,11}(\boldsymbol{x})}\psi_0(\boldsymbol{x}) + (x_1+x_3)\psi_0(\boldsymbol{x}) + \int_{\xi_1}^{x_1} \mathrm{d}y_1\,(x_4\partial_{x_4}\psi_0 \\
& + x_2\partial_{x_2}\psi_0 + x_3\partial_{x_3}\psi_0) + \int_{\xi_3}^{x_3} \mathrm{d}y_3\,(x_4\partial_{x_4}\psi_0 + x_2\partial_{x_2}\psi_0 + x_1\partial_{x_1}\psi_0) + c_{G_6}\,, \\
G_5(\boldsymbol{x}) =& \frac{\boldsymbol{C}^{(0)}_{6,11}(\boldsymbol{x})}{2\,\boldsymbol{C}^{(0)}_{1,11}(\boldsymbol{x})}\psi_0(\boldsymbol{x}) - (x_1+x_3)\psi_0(\boldsymbol{x}) - \int_{\xi_1}^{x_1} \mathrm{d}y_1\, x_3\partial_{x_3}\psi_0 - \int_{\xi_3}^{x_3} \mathrm{d}y_3\, x_1\partial_{x_1}\psi_0 + c_{G_5}\,, \\
G_{10}(\boldsymbol{x}) =& -\big((\boldsymbol{\Omega}^{(2)}_1(\boldsymbol{x}))_{7,1}\boldsymbol{Z}^{-1}_{2,3}(\boldsymbol{x}) + (\boldsymbol{\Omega}^{(2)}_1(\boldsymbol{x}))_{8,1}\boldsymbol{Z}^{-1}_{2,4}(\boldsymbol{x}) + (\boldsymbol{\Omega}^{(2)}_1(\boldsymbol{x}))_{9,1}\boldsymbol{Z}^{-1}_{2,5}(\boldsymbol{x})\big)\psi_0(\boldsymbol{x}) \\
& - 4\int_{\xi_1}^{x_1} \mathrm{d}y_1\,\,(x_4\partial_{x_4}\psi_0 + x_3\partial_{x_3}\psi_0) - 2\int_{\xi_2}^{x_2} \mathrm{d}y_2\,\,(x_4\partial_{x_4}\psi_0 + x_3\partial_{x_3}\psi_0 + x_1\partial_{x_1}\psi_0) \\
& + \frac{2(1-x_1-3\,x_2)}{3}\psi_0(\boldsymbol{x}) + c_{G_{10}}\,, \\
G_{11}(\boldsymbol{x}) =& G_{10}[x_1 \leftrightarrow x_2]\,, \\
G_{12}(\boldsymbol{x}) =& G_{10}[x_1 \leftrightarrow x_3]\,, \\
G_{13}(\boldsymbol{x}) =& G_{11}[x_2 \leftrightarrow x_4]\,.
\end{aligned}
$$

$$
\int_{\xi_i}^{x_i} \mathrm{d}y_i\, x_j\partial_{x_j}\psi_0
$$

$$
\begin{aligned}
G_1(\boldsymbol{x}) =& \sum_{i=1}^{4}\frac{\boldsymbol{C}^{(1)}_{i+1,11}(\boldsymbol{x})\,\boldsymbol{j}_{i,1}(\boldsymbol{x})}{2\,\boldsymbol{C}^{(0)}_{1,11}(\boldsymbol{x})} - \int_{\xi_1}^{x_1} \mathrm{d}y_1\Big\{-\boldsymbol{j}_{1,1}\frac{8\,\boldsymbol{C}^{(0)}_{1,11}\,G_0}{\psi_0^2} \\
& + \boldsymbol{j}_{2,1}\,h_1 + \boldsymbol{j}_{3,1}\,h_1[x_2 \leftrightarrow x_3] + \boldsymbol{j}_{4,1}\,h_1[x_2 \leftrightarrow x_4]\Big\} + c_{G_1}\,, \\
G_2(\boldsymbol{x}) =& \sum_{i=1}^{4}\frac{\boldsymbol{C}^{(1)}_{i+1,11}(\boldsymbol{x})\,\boldsymbol{j}_{i,2}(\boldsymbol{x})}{2\,\boldsymbol{C}^{(0)}_{1,11}(\boldsymbol{x})} - \int_{\xi_2}^{x_2} \mathrm{d}y_2\Big\{\boldsymbol{j}_{2,2}\Big[-\frac{8\,x_1\,\boldsymbol{C}^{(0)}_{1,11}\,G_0}{y_2\,\psi_0^2} + \frac{\partial_{x_1}\psi_0 - \partial_{y_2}\psi_0}{y_2\,\psi_0}\Big] \\
& + \boldsymbol{j}_{1,2}\,h_1 + \boldsymbol{j}_{3,2}\,h_1[x_1 \leftrightarrow x_3] + \boldsymbol{j}_{4,2}\,h_1[x_1 \leftrightarrow x_4]\Big\} + c_{G_2}\,, \\
G_3(\boldsymbol{x}) =& \sum_{i=1}^{4}\frac{\boldsymbol{C}^{(1)}_{i+1,11}(\boldsymbol{x})\,\boldsymbol{j}_{i,3}(\boldsymbol{x})}{2\,\boldsymbol{C}^{(0)}_{1,11}(\boldsymbol{x})} - \int_{\xi_3}^{x_3} \mathrm{d}y_3\Big\{\boldsymbol{j}_{3,3}\Big[-\frac{8\,x_1\,\boldsymbol{C}^{(0)}_{1,11}\,G_0}{y_3\,\psi_0^2} + \frac{\partial_{x_1}\psi_0 - \partial_{y_3}\psi_0}{y_3\,\psi_0}\Big] \\
& + \boldsymbol{j}_{1,3}\,h_1[x_2 \leftrightarrow y_3] + \boldsymbol{j}_{2,3}\,h_1[x_1 \leftrightarrow y_3] + \boldsymbol{j}_{4,3}\,h_1[x_2 \leftrightarrow x_4, x_1 \leftrightarrow y_3]\Big\} + c_{G_3}\,, \\
G_4(\boldsymbol{x}) =& \sum_{i=1}^{4}\frac{\boldsymbol{C}^{(1)}_{i+1,11}(\boldsymbol{x})\,\boldsymbol{j}_{i,4}(\boldsymbol{x})}{2\,\boldsymbol{C}^{(0)}_{1,11}(\boldsymbol{x})} - \int_{\xi_1}^{x_4} \mathrm{d}y_4\Big\{\boldsymbol{j}_{4,4}\Big[-\frac{8\,x_1\,\boldsymbol{C}^{(0)}_{1,11}\,G_0}{y_4\,\psi_0^2} + \frac{\partial_{x_1}\psi_0 - \partial_{y_4}\psi_0}{y_4\,\psi_0}\Big] \\
& + \boldsymbol{j}_{1,4}\,h_1[x_2 \leftrightarrow y_4] + \boldsymbol{j}_{2,4}\,h_1[x_1 \leftrightarrow y_4] + \boldsymbol{j}_{3,4}\,h_1[x_2 \leftrightarrow x_3, x_1 \leftrightarrow y_4]\Big\} + c_{G_4}\,,
\end{aligned}
$$

$$
\begin{aligned}
h_1(\boldsymbol{x}) =& (\boldsymbol{\Omega}^{(1)}_1(\boldsymbol{x}))_{7,5} + 4\frac{\boldsymbol{C}^{(0)}_{2,3}(\boldsymbol{x})}{\psi_0(\boldsymbol{x})^2}G_0(\boldsymbol{x}) + \frac{1}{\psi_0(\boldsymbol{x})}\Big[(\boldsymbol{\Omega}^{(1)}_1(\boldsymbol{x}))_{7,9}\,\partial_{x_4}\psi_0(\boldsymbol{x}) \\
& + (\boldsymbol{\Omega}^{(1)}_1(\boldsymbol{x}))_{7,8}\,\partial_{x_3}\psi_0(\boldsymbol{x}) + (\boldsymbol{\Omega}^{(1)}_1(\boldsymbol{x}))_{7,7}\,\partial_{x_2}\psi_0(\boldsymbol{x}) + (\boldsymbol{\Omega}^{(1)}_1(\boldsymbol{x}))_{7,6}\,\partial_{x_1}\psi_0(\boldsymbol{x})\Big]
\end{aligned}
$$

$\varepsilon-$functions

- Can they be rewritten more compactly? ➡ Yes!

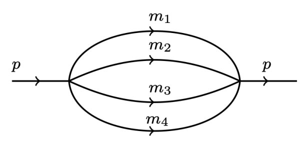
  Let us define

$$\mathcal{I}_1(x_i,x_j,x_k,x_l) = \int_0^{x_i} \mathrm{d}y_i\, x_j \partial_{x_j} \psi_0(y_i,x_j,x_k,x_l)\,,$$

$$\mathcal{I}_2(x_i,x_j,x_k,x_l) = \int_0^{x_i} \mathrm{d}y_i \Bigg[-\boldsymbol{j}_{1,1}(y_i,x_j,x_k,x_l) \frac{8\, \boldsymbol{C}^{(0)}_{1,11}(y_i,x_j,x_k,x_l)\, G_0(y_i,x_j,x_k,x_l)}{\psi_0(y_i,x_j,x_k,x_l)^2} + \sum_\sigma \boldsymbol{j}_{2,1}(y_i,x_{\sigma(j)},x_{\sigma(k)},x_{\sigma(l)})\, h_1(y_i,x_{\sigma(j)},x_{\sigma(k)},x_{\sigma(l)}) \Bigg]\,,$$

$\sigma \in \{e,(j\,k),(j\,l)\}$

➡ All the ε-functions can be rewritten using only these 2 integrals!

constant Δ | symmetries of 

23 ε-functions ➡ 13 ε-functions ➡ 2 ε-functions (to evaluate at different kinematics)

11 ε-functions

Summary and outlooks

Canonical form + **Twisted cohomology**

[1], [2]: $J(x,\varepsilon) = U_t(x,\varepsilon) U_\varepsilon(\varepsilon) U_{ss}(x) I(x,\varepsilon)$

$\frac{1}{\varepsilon} U_t^{(-1)}(x) + U_t^{(0)}(x)$ geometric input

23 ε—functions

[3], [4]: $U_t^{(0)} = OR$

13 ε-functions to determine

rational functions, periods, and their derivatives

→ canonical DE + reduced the number of independent ε-functions

- Despite the complexity arisen from the high number of loops and kinematics we still get a compact analytic solution.
- Do the functions in O always live in a function space beyond the periods and their derivatives?
- What about the functions beyond the maximal cut?
- Can we use these tools from twisted cohomology to show that the method we used to get the ε-form delivers always a canonical form?

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Thank you for your attention!