Recent Developments for Multiple Eisenstein Series

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	Numbers	Functions
"single" version	Riemann zeta values $\zeta(k) = \sum_{m>0} \frac{1}{m^k}$	Eisenstein series $(q = e^{2\pi i\tau})$ $\mathbb{G}_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$
"multiple" version	Multiple zeta values $\zeta(k_1,\dots,k_r) = \sum_{m_1>\dots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$	Multiple Eisenstein series $\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau+\mathbb{Z}}} \frac{1}{\lambda_1^{k_1}\cdots\lambda_r^{k_r}} = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_n q^n$
relations	Harmonic product + Shuffle product = Double shuffle relations	Harmonic product + involution invariance
related Lie algebras	$\mathrm{Lie}(\sigma_3, \sigma_5, \sigma_7, \dots)$ $\mathfrak{dm}_0$	$\mathrm{Lie}(\delta, \omega_3, \omega_5, \omega_7, \dots) \big/$ cusp form relations $\mathfrak{bm}_0$ $\delta \in \mathfrak{sl}_2$

Definition

For an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ with $k_1 \geq 2, k_2, \dots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By $\operatorname{dep}(\mathbf{k}) = r$ we denote its **depth** and $\operatorname{wt}(\mathbf{k}) = k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs
- $\mathcal{Z}_k$: $\mathbb{Q}$-vector space of MZVs of weight k.

- In the case $r = 1$ these are just the classical Riemann zeta values

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k}\,, \qquad \zeta(2) = \frac{\pi^2}{6}\,, \qquad \zeta(3) \notin \mathbb{Q}\,, \qquad \zeta(4) = \frac{\pi^4}{90}\,, \quad \dots .$$

- MZVs were first studied by Euler ($r = 2$) and for general depth, they reappeared prominently around 1990 in various areas of mathematics and physics.

MZVs can also be written as **iterated integrals**:

Proposition

The MZV $\zeta(k_1, ..., k_r)$ of weight $k = k_1 + ... + k_r$ can be written as an iterated integral

$$\zeta(k_1, ..., k_r) = \int_{1>t_1>\cdots>t_k>0} \omega_1(t_1)\cdots\omega_k(t_k)\,,$$

where

$$\omega_j(t) = \begin{cases} \frac{dt}{1-t} & \text{if } j \in \{k_1, k_1+k_2, \ldots, k_1+\cdots+k_r\} \\ \frac{dt}{t} & \text{else} \end{cases}.$$

Example

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}.$$

There are two different ways to express the product of MZVs in terms of MZVs.

Harmonic product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}}\\ &= \sum_{m>n>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{n>m>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}}\\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

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Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \int \dots \cdot \int \dots = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j,k_1+k_2-j)\,.$$

① MZV - Double shuffle relations

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

Example

$$\zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(2,3) + \zeta(3,2) + \zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2) + 6\zeta(4,1) \overset{\textbf{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZVs, e.g.:

$$\sum_{m>n>0} \frac{1}{m^2 n} = \zeta(2,1) = \zeta(3) = \sum_{n>0} \frac{1}{n^3}.$$

These follow from regularizing the double shuffle relations and they are called **extended double shuffle relations**.

Conjectures

The extended double shuffle relations give all linear relations among MZV and

$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k \,,$$

i.e. there are no relations between MZV of different weight.

This leads to the definition of the algebra of **formal multiple zeta values** $\mathcal{Z}^f$

$$\mathcal{Z}^f := \mathfrak{H}^0_* \big/ \mathrm{EDS}_*,$$

given by symbols $\zeta^f(k_1, \dots, k_r)$ satisfying exactly the extended double shuffle relations (EDS).
(We will make the definition of $\mathcal{Z}^f$ precise later)

$$\text{Conjecture} \implies \mathcal{Z} \cong \mathcal{Z}^f.$$

Conjectures

- (Zagier) The dimension of the spaces $\mathcal{Z}_k$ is given by

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k \, X^k = \frac{1}{1 - X^2 - X^3} \, .$$

- (Hoffman) The following set gives a basis of $\mathcal{Z}$

$$\{\zeta(k_1, \ldots, k_r) \mid r \geq 0, k_1, \ldots, k_r \in \{2, 3\}\} \, .$$

Theorem (Brown, 2012)

The space $\mathcal{Z}$ is spanned by $\zeta(k_1, \ldots, k_r)$ with $r \geq 0$ and $k_1, \ldots, k_r \in \{2, 3\}$.

Let $X = \{x_0, x_1\}$ and $Y = \{y_1, y_2, \ldots\}$. Then, we have an embedding

$$\iota : \mathbb{Q}\langle Y\rangle \hookrightarrow \mathbb{Q}\langle X\rangle$$
$$y_{k_1} \cdots y_{k_r} \mapsto x_0^{k_1-1}x_1 \cdots x_0^{k_r-1}x_1,$$

and a canonical projection

$$\Pi_Y : \mathbb{Q}\langle X\rangle \to \mathbb{Q}\langle Y\rangle,$$
$$wx_0 \mapsto 0,$$
$$x_0^{k_1-1}x_1 \cdots x_0^{k_r-1}x_1 \mapsto y_{k_1} \cdots y_{k_r}.$$

Interpretation

Harmonic side: $\mathbb{Q}\langle Y\rangle$, $\quad k \leftrightarrow y_k$ $\qquad$ $x_0 \leftrightarrow \frac{dt}{t}$, $\quad x_1 \leftrightarrow \frac{dt}{1-t}$, $\quad \mathbb{Q}\langle X\rangle$: **Shuffle side**

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}$$

$$\iota : y_2 y_3 \mapsto x_0 x_1 x_0 x_0 x_1$$

By $\Delta_{ш} : \mathbb{Q}\langle X\rangle \to \mathbb{Q}\langle X\rangle$ we denote the **shuffle coproduct**

$$\Delta_{ш}(x_i) = x_i \otimes 1 + 1 \otimes x_i, \qquad (i = 0, 1)$$

and, by $\Delta_* : \mathbb{Q}\langle Y\rangle \to \mathbb{Q}\langle Y\rangle$ we denote the **harmonic coproduct**

$$\Delta_*(y_i) = y_i \otimes 1 + 1 \otimes y_i + \sum_{k+l=i} y_k \otimes y_l, \qquad (i \geq 1).$$

Interpretation

Let $\bullet \in \{ш, *\}$.

Coefficients of ψ satisfy the $\bullet$-product formula $\quad\longleftrightarrow\quad \Delta_\bullet(\psi) = \psi \otimes \psi$

Coefficients of ψ satisfy the $\bullet$-product formula mod. products $\quad\longleftrightarrow\quad \Delta_\bullet(\psi) = \psi \otimes 1 + 1 \otimes \psi$

Let K be a field and A a K-algebra.

- **Derivations on A:** $\mathrm{Der}(A) = \{\, K$-linear maps $d : A \to A$ satisfying $d(ab) = d(a)b + ad(b)\}$.
- **Lie algebra**: K-vector space V with a K-bilinear map $[\cdot,\cdot] : V \times V \to V$ satisfying $[x,x] = 0$ and the Jacobi identity $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$ for all $x, y, z \in V$.

Examples of Lie algebras

- $\mathrm{Der}(A)$ with the Lie bracket $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$.
- The space of all matrices $K^{2\times 2}$ with the Lie bracket $[M, N] = MN - NM$ and its subspace

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K^{2\times 2} \mid a + d = 0 \right\}.$$

For $f \in \mathbb{Q}\langle X\rangle$ and a word $w \in \mathbb{Q}\langle X\rangle$, we denote by $(f \mid w)$ the coefficient of w in f.

Definition (Racinet)

Let $\mathfrak{dm}_0$ be the set of all $\psi \in \mathbb{Q}\langle X\rangle$, such that

(i) $(\psi \mid x_0) = (\psi \mid x_1) = (\psi \mid x_0x_1) = 0$,

(ii) $\Delta_{\sqcup\!\sqcup}\psi = \psi \otimes 1 + 1 \otimes \psi$,

(iii) $\Delta_*\psi_* = \psi_* \otimes 1 + 1 \otimes \psi_*$,

where $\psi_* = \Pi_Y(\psi) +$ correction terms.

The **Ihara bracket** is

$$\{f, g\} = d_f(g) - d_g(f) + [f, g], \qquad f, g \in \mathbb{Q}\langle \mathcal{X}\rangle,$$

where d_f is the derivation on $\mathbb{Q}\langle X\rangle$ with $d_f(x_0) = 0$ and $d_f(x_1) = [x_1, f]$.

Theorem (Racinet)

$(\mathfrak{dm}_0, \{-,-\})$ is a Lie algebra.

Proposition (Racinet)

We have
$$\mathcal{Z}^{\mathrm{f}} \cong \mathbb{Q}[\zeta^{\mathrm{f}}(2)] \otimes \mathcal{U}(\mathfrak{dm}_0)^{\vee}$$

It is known by results of Drinfeld, Brown, and Furusho, that there is an embedding
$$\mathrm{Lie}(\sigma_3, \sigma_5, \sigma_7, \ldots) \hookrightarrow \mathfrak{dm}_0.$$

Conjecture

We have $\mathfrak{dm}_0 \cong \mathrm{Lie}(\sigma_3, \sigma_5, \sigma_7, \ldots)$.

Conjecture + Proposition $\implies \sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{Z}^f_k X^k = \frac{1}{1-X^2} \cdot \frac{1}{1-X^3-X^5-X^7-\ldots}$

$$= \frac{1}{1-X^2-X^3}.$$

Definition

An $\mathfrak{sl}_2$**-algebra** is an algebra A together with a Lie algebra homomorphism $\mathfrak{sl}_2 \to \mathrm{Der}(A)$.

$\mathfrak{sl}_2$ is three-dimensional, spanned by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

These fulfill the commutator relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [Y, X] = H.$$

Definition (rephrased)

An $\mathfrak{sl}_2$-algebra is an algebra A together with three derivations $D, W, \delta \in \mathrm{Der}(A)$, satisfying

$$[\mathrm{W}, \mathrm{D}] = 2\,\mathrm{D}, \quad [\mathrm{W}, \delta] = -2\delta, \quad [\delta, \mathrm{D}] = \mathrm{W}\,.$$

In this case, (D, W, δ) is also called an $\mathfrak{sl}_2$**-triple**.

For $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ write $q = e^{2\pi i \tau}$.

- **Eisenstein series:** For even $k \geq 2$ define

$$\mathbb{G}_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \overset{k>2}{=} \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k},$$

  where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the generalized sum-of-divisors function.
- **Modular forms:** $\mathcal{M} = \mathbb{Q}[\mathbb{G}_4, \mathbb{G}_6, \mathbb{G}_8, \dots] = \mathbb{Q}[\mathbb{G}_4, \mathbb{G}_6]$.
- **Quasimodular forms:** $\widetilde{\mathcal{M}} = \mathbb{Q}[\mathbb{G}_2, \mathbb{G}_4, \mathbb{G}_6]$.

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Proposition

$\widetilde{\mathcal{M}}$ is an $\mathfrak{sl}_2$-algebra and the derivations $D, W, \delta \in \operatorname{Der}(\widetilde{\mathcal{M}})$ defined by

$$D(\mathbb{G}_k) = (2\pi i)\frac{d}{d\tau}\mathbb{G}_k, \qquad W(\mathbb{G}_k) = k\mathbb{G}_k, \qquad \delta(\mathbb{G}_k) = \begin{cases} -\frac{1}{2}, & k = 2, \\ 0, & k \geq 4. \end{cases}$$

form an $\mathfrak{sl}_2$-triple (D, W, δ).

Summary so far...

related
Lie Algebras

Modular forms $\mathbb{Q}[\mathbb{G}_4, \mathbb{G}_6] \subset$ Quasimodular forms $\mathbb{Q}[\mathbb{G}_2, \mathbb{G}_4, \mathbb{G}_6] \subset$? ?

$\mathfrak{sl}_2$

$\tau \to i\infty$

$\mathbb{Q}[\zeta(2)] \subset \mathcal{Z}$ $\mathfrak{dm}_0$

$\zeta(k_1, \ldots, k_r)$

Multiple Zeta Values

Definition

For $k_1, k_2, \dots, k_r \geq 2$ and $\tau \in \mathbb{H}$ the **multiple Eisenstein series** are defined by[a]

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) := \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}} \,,$$

where the order $\succ$ on the lattice $\mathbb{Z}\tau + \mathbb{Z}$ is defined as the lexicographical order given by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Longleftrightarrow \quad m_1 > m_2 \text{ or } m_1 = m_2 \wedge n_1 > n_2 .$$

[a] In the case $k_1 = 2$ use Eisenstein summation.

We denote the $\mathbb{Q}$-vector space spanned by all MES by

$$\mathcal{E} = \langle \mathbb{G}_{k_1,\dots,k_r} \mid r \geq 0, k_1, \dots, k_r \geq 2 \rangle_{\mathbb{Q}} \,,$$

where we set $\mathbb{G}_{k_1,\dots,k_r} = 1$ for $r = 0$.

- Multiple Eisenstein series are holomorphic function in the upper-half plane.
- They are bounded as $\tau \to i\infty$ and we have

$$\lim_{\tau \to i\infty} \mathbb{G}_{k_1,\dots,k_r}(\tau) = \lim_{\tau \to i\infty} \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}} = \zeta(k_1, \dots, k_r).$$

In particular, we obtain a (surjective) $\mathbb{Q}$-linear map $\mathcal{E} \to \mathcal{Z}$.

Theorem (Gangl-Kaneko-Zagier 2006, B. 2012)

We have

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n>0} a_{k_1,\dots,k_r}(n) q^n$$

for some (explicit) $a_{k_1,\dots,k_r}(n) \in \mathcal{Z}[\pi i]$.

Natural questions

Algebra structure? Modularity? $\mathfrak{sl}_2$-action? Relations? Dimensions? Related Lie algebras?

$\mathcal{E}_k$: subspace of multiple Eisenstein series $\mathbb{G}_{k_1,\dots,k_r}$ of weight $k = k_1 + \cdots + k_r$.

Proposition

The space $\mathcal{E}$ is a $\mathbb{Q}$-algebra, and we have for $k_1, k_2 \geq 0$,

$$\mathcal{E}_{k_1} \cdot \mathcal{E}_{k_2} \subset \mathcal{E}_{k_1+k_2}.$$

Moreover, $\mathbb{Q}[\mathbb{G}_2, \mathbb{G}_4, \mathbb{G}_6]$ and $\mathbb{Q}[\mathbb{G}_4, \mathbb{G}_6]$ are subalgebras of $\mathcal{E}$.

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Conjecture

The space $\mathcal{E}$ is graded by weight, i.e.,

$$\mathcal{E} = \bigoplus_{k \geq 0} \mathcal{E}_k.$$

In contrast to the corresponding conjecture for multiple zeta values, the conjecture above seems more accessible.

Conjecture

- The maps $\mathrm{D}, \mathrm{W}, \delta$ defined on the generators of $\mathcal{E}$ via

$$\mathrm{D} : \mathbb{G}_{k_1,\dots,k_r} \longmapsto (2\pi i)\frac{d}{d\tau}\mathbb{G}_{k_1,\dots,k_r},$$

$$\mathrm{W} : \mathbb{G}_{k_1,\dots,k_r} \longmapsto (k_1 + \dots + k_r)\mathbb{G}_{k_1,\dots,k_r},$$

$$\delta : \mathbb{G}_{k_1,\dots,k_r} \longmapsto \begin{cases} -\frac{1}{2}\mathbb{G}_{k_2,\dots,k_r}, & k_1 = 2, \\ 0, & k_1 > 2, \end{cases}$$

  give well-defined $\mathbb{Q}$-linear maps $\mathcal{E} \to \mathcal{E}$.
- The maps $\mathrm{D}, \mathrm{W}, \delta$ are derivations on $\mathcal{E}$.
- $(\mathrm{D}, \mathrm{W}, \delta)$ forms an $\mathfrak{sl}_2$-triple, i.e. we have the commutator relations

$$[\mathrm{W}, \mathrm{D}] = 2\,\mathrm{D}, \quad [\mathrm{W}, \delta] = -2\delta, \quad [\delta, \mathrm{D}] = \mathrm{W}$$

  and thus $\mathcal{E}$ is an $\mathfrak{sl}_2$-algebra.

Recall Zagier's dimension conjecture:

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k X^k \stackrel{?}{=} \frac{1}{1-X^2-X^3}.$$

Setting

$$\mathsf{E}(X) = \frac{1}{1-X^2} = 1 + X^2 + X^4 + \dots, \quad \mathsf{O}(X) = \frac{X^3}{1-X^2} = X^3 + X^5 + \dots.$$

we can rewrite this to

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k X^k \stackrel{?}{=} \frac{1}{1-X^2-X^3} = \mathsf{E}(X) \cdot \frac{1}{1-\mathsf{O}(X)}.$$

From the perspective of Hilbert–Poincaré series for graded algebras, this suggests

$$\mathcal{Z} \stackrel{?}{\cong} \mathbb{Q}[f_2] \otimes \mathbb{Q}\langle f_3, f_5, \dots \rangle \cong \mathbb{Q}[f_2] \otimes U(\mathrm{Lie}(\sigma_3, \sigma_5, \sigma_7, \dots))^\vee.$$

The Hilbert–Poincaré series of modular forms and cusp forms are given by

$$\mathsf{M}(X) = \frac{1}{(1-X^4)(1-X^6)}, \qquad \mathsf{S}(X) = X^{12}\mathsf{M}(X) = \frac{X^{12}}{(1-X^4)(1-X^6)}.$$

Conjecture

We have

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{E}_k X^k = \mathsf{M}(X) \cdot \frac{1}{1 - X^2 - \mathsf{O}(X) + 2\mathsf{S}(X)}.$$

In particular,

$$\mathcal{E} \cong \mathcal{M} \otimes U(\mathfrak{E})^{\vee},$$

where $\mathfrak{E}$ is a Lie algebra generated by $\delta_2, \sigma_3, \sigma_5, \sigma_7, \ldots$ subject to relations arising from cusp forms.

Conjecture (Same as on the previous slide)

We have

$$\sum_{k\geq 0} \dim_{\mathbb{Q}} \mathcal{E}_k X^k = \mathsf{M}(X) \cdot \frac{1}{1 - X^2 - \mathsf{O}(X) + 2\mathsf{S}(X)}$$

$$= \frac{1}{1 - X^2 - X^3 - X^4 - X^5 + X^8 + X^9 + X^{10} + X^{11} + X^{12}}.$$

We obtain the following table:

weight k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
# of indices	1	0	1	1	2	3	5	8	13	21	34	55	89	144	233
# of relations $\stackrel{?}{=}$	0	0	0	0	0	0	1	1	4	6	13	23	42	74	129
$\dim_{\mathbb{Q}} \mathcal{E}_k \stackrel{?}{=}$	1	0	1	1	2	3	4	7	9	15	21	32	47	70	104

Summary so far...

related
Lie Algebras

Multiple Eisenstein series

$\mathbb{G}_{k_1,\dots,k_r}$

$\mathfrak{E} \subset ?$

$\cup$

$?$

$\cup$

Modular forms — Quasimodular forms

$\mathbb{Q}[\mathbb{G}_4, \mathbb{G}_6] \subset \mathbb{Q}[\mathbb{G}_2, \mathbb{G}_4, \mathbb{G}_6] \subset \mathcal{E} \subset ?$

$\mathfrak{sl}_2$ $\quad \tau \to i\infty \quad$ $\mathfrak{sl}_2$?

$\mathbb{Q}[\zeta(2)] \subset \mathcal{Z}$

$\mathfrak{dm}_0$

$\zeta(k_1, \dots, k_r)$

Multiple Zeta Values

Now: Introduce the formal analogues

Set
$$\mathfrak{H}^0 = \mathbb{Q} + x_0\mathfrak{H}x_1 \quad \subset \quad \mathfrak{H}^1 = \mathbb{Q} + \mathfrak{H}x_1 \subset \quad \mathfrak{H} = \mathbb{Q}\langle x_0, x_1\rangle$$
and write $y_k = x_0^{k-1}x_1$ for $k \geq 1$. Note that $\mathfrak{H}^1 = \mathbb{Q}\langle y_1, y_2, \ldots\rangle$.

Definition

- $* : \mathfrak{H}^1 \otimes \mathfrak{H}^1 \to \mathfrak{H}^1$: **harmonic product**. For words $w, v \in \mathfrak{H}^1$ and $r, s \geq 1$,
$$\mathbf{1} * w = w = w * \mathbf{1},$$
$$y_r w * y_s v = y_r(w * y_s v) + y_s(y_r w * v) + y_{r+s}(w * v).$$
- $⧢ : \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H}$: **shuffle product**. For words $w, v \in \mathfrak{H}$ and $a, b \in \{x_0, x_1\}$,
$$\mathbf{1} ⧢ w = w = w ⧢ \mathbf{1},$$
$$aw ⧢ bv = a(w ⧢ bv) + b(aw ⧢ v).$$
- For $\bullet \in \{*, ⧢\}$, we denote by $\mathfrak{H}^1_\bullet$ and $\mathfrak{H}^0_\bullet$ the algebras $(\mathfrak{H}^1, \bullet)$ and $(\mathfrak{H}^0, \bullet)$.

Double shuffle relations: Viewing ζ as a $\mathbb{Q}$-linear map $\mathfrak{H}^0 \to \mathbb{R}$, we have, for all $w, v \in \mathfrak{H}^0$,
$$\zeta(w)\,\zeta(v) \;=\; \zeta(w * v) \;=\; \zeta(w ⧢ v)\,.$$

Let $\mathrm{reg}_* : \mathfrak{H}^1_* \cong \mathfrak{H}^0_*[y_1] \to \mathfrak{H}^0_*$ be the alg. hom. that is the identity on $\mathfrak{H}^0$ and maps y_1 to 0.

Definition

The algebra of **formal multiple zeta values** is defined by

$$\mathcal{Z}^f := \mathfrak{H}^0_* \big/ \mathrm{EDS}_*,$$

where EDS_* is the ideal in $\mathfrak{H}^0_*$ generated by $\mathrm{reg}_*(w * v - w \sqcup\!\sqcup\, v)$ for all $w \in \mathfrak{H}^0, v \in \mathfrak{H}^1$.

For $k_1 \geq 2$ and $k_2, ..., k_r \geq 1$, we denote the class of $z_{k_1} \cdots z_{k_r}$ in $\mathcal{Z}^f$ by $\zeta^f(k_1, ..., k_r)$.

Conjecture

The linear map

$$\mathcal{Z}^f \longrightarrow \mathcal{Z}$$

$$\zeta^f(k_1, \ldots, k_r) \longmapsto \zeta(k_1, \ldots, k_r)$$

is an algebra isomorphism.

Let $\mathcal{B} = \{b_0, b_1, \ldots\}$.

Definition

Define on $\mathbb{Q}\langle\mathcal{B}\rangle$ the **harmonic product** $*$ as the $\mathbb{Q}$-bilinear product satisfying for $i, j \geq 0$

$$b_i u * b_j v = b_i(u * b_j v) + b_j(b_i u * v) + \delta_{ij>0}\, b_{i+j}(u * v)$$

and $1 * w = w * 1 = w$ for any $u, v, w \in \mathbb{Q}\langle\mathcal{B}\rangle$.

- $\mathbb{Q}\langle\mathcal{B}\rangle^0$: subspace spanned by words not starting in b_0.
- $\mathbb{Q}\langle\mathcal{B}\rangle^0_* = (\mathbb{Q}\langle\mathcal{B}\rangle^0, *)$ is a commutative $\mathbb{Q}$-algebra.

Definition

On $\mathbb{Q}\langle\mathcal{B}\rangle^0$ we define the $\mathbb{Q}$-linear involution $\quad (k_1, \ldots, k_s \geq 1, m_1, \ldots, m_s \geq 0)$

$$\tau : \mathbb{Q}\langle\mathcal{B}\rangle^0 \longrightarrow \mathbb{Q}\langle\mathcal{B}\rangle^0$$

$$b_{k_1} b_0^{m_1} \cdots b_{k_r} b_0^{m_r} \longmapsto b_{m_r+1} b_0^{k_r-1} \cdots b_{m_1+1} b_0^{k_1-1}.$$

Definition

The algebra of **formal multiple Eisenstein series** $\mathcal{G}^f$ is the $\mathbb{Q}$-algebra defined by

$$\mathcal{G}^f := {}^{\mathbb{Q}\langle\mathcal{B}\rangle_*^0}\!\big/\!{}_{\mathcal{T}}\,,$$

where $\mathcal{T}$ is the ideal in $\mathbb{Q}\langle\mathcal{B}\rangle_*^0$ generated by $\tau(w) - w$ for all $w \in \mathbb{Q}\langle\mathcal{B}\rangle^0$.

By $G^f(k_1, \ldots, k_r)$ we denote the class of $z_{k_1} \cdots z_{k_r}$ for $k_1 \geq 1, k_2, \ldots, k_r \geq 0$.
Claim: The $G^f(k_1, \ldots, k_r)$ satisfy the same relations as (regularized) MES. In particular, define

$$\mathcal{E}^f = \langle G^f(k_1, \ldots, k_r) \mid r \geq 0, k_1, \ldots, k_r \geq 2\rangle_{\mathbb{Q}}\,.$$

Conjecture

We have an isomorphism of $\mathbb{Q}$-algebras

$$\begin{aligned} \mathcal{E}^f &\longrightarrow \mathcal{E} \\ G^f(k_1, \ldots, k_r) &\longmapsto \mathbb{G}_{k_1,\ldots,k_r}\,. \end{aligned}$$

Theorem (B.-van-Ittersum, 2023+)

There exist explicit derivations W, D, δ *on* $\mathcal{G}^f$ *such that*

- $\mathcal{G}^f$ *is an* $\mathfrak{sl}_2$*-algebra;*
- *the subalgebra* $\mathbb{Q}[G^f(2), G^f(4), G^f(6)] \subset \mathcal{G}^f$ *is isomorphic to* $\widetilde{\mathcal{M}}$ *as an* $\mathfrak{sl}_2$*-algebra.*

Theorem (B.-van-Ittersum, 2023+)

There exists a surjective algebra homomorphism (The "formal projection to the constant term")

$$\pi : \mathcal{G}^f \to \mathcal{Z}^f,$$

with $\pi(G^f(k_1, \ldots, k_r)) = \zeta^f(k_1, \ldots, k_r)$*. The kernel of* π *can be described explicitly.*

Theorem (B.-Burmester, 2023)

There exists an algebra homomorphism $\mathcal{G}^f \to \mathbb{Q}[\![q]\!]$ *with* $G^f(k) \mapsto (-2\pi i)^{-k}\mathbb{G}_k$*.*

Definition

For $l \geq 1$ denote by $\mathfrak{D}_l$ all $\mathbb{Q}$-linear maps $d : \mathbb{Q}\langle\mathcal{B}\rangle^0 \to \mathbb{Q}\langle\mathcal{B}\rangle^0$ such that for all $w, v \in \mathbb{Q}\langle\mathcal{B}\rangle^0$

- $d(w * v) = d(w) * v + w * d(v)$.
- $\tau(d(w)) = d(\tau(w))$.
- For $k \geq 0$ we have $d\mathbb{Q}\langle\mathcal{B}\rangle^0_k \subseteq \mathbb{Q}\langle\mathcal{B}\rangle^0_{k-l}$, where we set $\mathbb{Q}\langle\mathcal{B}\rangle^0_0 = \mathbb{Q}$ and $\mathbb{Q}\langle\mathcal{B}\rangle^0_m = 0$ for $m < 0$.

Set

$$\mathfrak{D} = \sum_{l \geq 1} \mathfrak{D}_l.$$

By definition one checks easily that we have the following:

Proposition

$\mathfrak{D}$ is a Lie subalgebra of $\mathrm{Der}(\mathbb{Q}\langle\mathcal{B}\rangle^0_*)$.

Theorem (B.-van-Ittersum, 2023)

There exists explicit non-zero elements

$$\omega_1 \in \mathfrak{D}_1, \qquad \delta \in \mathfrak{D}_2.$$

Conjecturally, these are (up to multiples) the only elements in $\mathfrak{D}_1$ (resp. $\mathfrak{D}_2$)

Work in progress (B.-Burmester-van-Ittersum)

We expect/have an embedding $\iota : \mathfrak{dm}_0 \hookrightarrow \mathfrak{D}$.

Assuming the conjecture $\mathfrak{dm}_0 \cong \mathrm{Lie}(\sigma_3, \sigma_5, \sigma_7, \ldots)$ we therefore also expect that in each odd weight $s \geq 3$ we have elements

$$\omega_s = \iota(\sigma_s) \in \mathfrak{D}_s.$$

Conjecture

The Lie algebra $\mathfrak{D}$ is generated by δ and ω_s for odd $s \geq 1$.

Definition (Burmester, 2022)

The space $\mathfrak{bm}_0$ consists of all $\Psi \in \mathbb{Q}\langle\mathcal{B}\rangle$, such that

- $(\Psi \mid b_k) = 0$ for $k = 0, 2, 4, 6,$
- $\Delta_b(\Psi) = \Psi \otimes 1 + 1 \otimes \Psi,$
- $\tau(\Pi_0(\Psi)) = \Pi_0(\Psi).$

Theorem (Burmester, 2022)

There exists an (explicit) embedding $\iota : \mathfrak{dm}_0 \hookrightarrow \mathfrak{bm}_0$.

Burmester gives an explicit formula for a bracket $\{-,-\}_q$ and conjectures the following:

Conjecture (Burmester, 2022)

- $(\mathfrak{bm}_0, \{-,-\}_q)$ is a Lie algebra.
- We have $\mathcal{G}^f \cong \widetilde{\mathcal{M}} \otimes \mathcal{U}(\mathfrak{bm}_0)^\vee$.

Define $\mathfrak{B} = \mathfrak{D}_1 \oplus \bigoplus_{l\geq 3} \mathfrak{D}_l$, i.e. $\mathfrak{D} = \mathfrak{B} \oplus \mathfrak{D}_2$.

Proposition

If $\mathfrak{D}_1 = \mathbb{Q}\omega_1$ then $\mathfrak{B}$ is a Lie subalgebra of $\mathfrak{D}$.

Conjecture

We have $\mathfrak{B} \cong \mathfrak{bm}_0$ as Lie algebras.

- In other words, we expect that $\mathfrak{D} \cong \delta\mathbb{Q} \oplus \mathfrak{bm}_0$.
- There seem to be relations among the brackets of $\omega_1, \delta, \omega_3, \omega_5, \dots$ related to modular forms.
- We expect that the Lie subalgebra $\mathfrak{E}$ generated by $\delta, \omega_3, \omega_5, \dots$ "corresponds" to the subspace $\mathcal{E}$.

Define the Hilbert–Poincaré series of the space of period polynomials W_k with even $k \geq 2$ by

$$\mathsf{W}(X) = \sum_{\substack{k \geq 2 \\ k \text{ even}}} \dim_{\mathbb{Q}} W_k X^k = \mathsf{M}(X) + \mathsf{S}(X) - 1 = \frac{X^4}{1 - X^2} + 2\,\mathsf{S}(X)$$

and recall that $\mathsf{D}(X) = \frac{1}{1-X^2}$, $\mathsf{O}(X) = \frac{X^3}{1-X^2}$, $\mathsf{M}(X) = \frac{1}{(1-X^4)(1-X^6)}$, $\mathsf{S}(X) = X^{12}\mathsf{M}(X)$.

Conjecture

We have

$$\sum_{k \geq 0} \dim_{\mathbb{Q}} \mathcal{G}_k^f X^k = \mathsf{M}(X) \cdot \frac{1}{1 - X - X^2 - \mathsf{O}(X) + \mathsf{W}(X)},$$

$$= \frac{1}{1 - X - X^2 - X^3 + X^6 + X^7 + X^8 + X^9}.$$

Overview

related
Lie Algebras

Multiple Eisenstein series

$\mathbb{G}_{k_1,\ldots,k_r}$

regularized
MES

Modular forms

Quasimodular forms

$$\mathbb{Q}[\mathbb{G}_4, \mathbb{G}_6] \subset \mathbb{Q}[\mathbb{G}_2, \mathbb{G}_4, \mathbb{G}_6] \subset \mathcal{E} \subset \mathcal{G}\ ?$$

$$\mathfrak{E} \subset \mathfrak{D} \cup ?\ \mathfrak{bm}_0 \cup \mathfrak{dm}_0$$

$\mathfrak{sl}_2$ (acting on $\mathbb{Q}[\mathbb{G}_2, \mathbb{G}_4, \mathbb{G}_6]$), $\mathfrak{sl}_2$? (acting on $\mathcal{E}$), $\mathfrak{sl}_2$? (acting on $\mathcal{G}$)

$\tau \to i\infty$

$$\mathbb{Q}[\zeta(2)] \subset \mathcal{Z}$$

$\zeta(k_1, \ldots, k_r)$

Multiple Zeta Values

Overview – Formal version

related
Lie Algebras

Formal multiple Eisenstein series

$G^f(k_1,\dots,k_r)$

$\mathfrak{E} \subset \mathfrak{D}$

formal
Modular forms

formal
Quasimodular forms

$\mathbb{Q}[G^f(4),G^f(6)] \subset \mathbb{Q}[G^f(2),G^f(4),G^f(6)] \subset \mathcal{E}^f \subset \mathcal{G}^f$

$\cup$?

$\mathfrak{bm}_0$

$\mathfrak{sl}_2$
B.-van-Ittersum

π
B.-van-Ittersum
Burmester

$\mathfrak{sl}_2$?

$\mathfrak{sl}_2$
B.-van-Ittersum

$\cup$

$\mathfrak{dm}_0$

$\mathbb{Q}[\zeta^f(2)] \subset \mathcal{Z}^f$

$\zeta^f(k_1,\dots,k_r)$

Formal multiple Zeta Values

	Numbers	Functions
"single" version	Riemann zeta values $$\zeta(k)=\sum_{m>0}\frac{1}{m^k}$$	Eisenstein series $(q=e^{2\pi i\tau})$ $$\mathbb{G}_k(\tau)=\zeta(k)+\frac{(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$
"multiple" version	Multiple zeta values $$\zeta(k_1,\dots,k_r)=\sum_{m_1>\cdots>m_r>0}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$	Multiple Eisenstein series $$\mathbb{G}_{k_1,\dots,k_r}(\tau)=\sum_{\substack{\lambda_1\succ\cdots\succ\lambda_r\succ 0\\ \lambda_i\in\mathbb{Z}\tau+\mathbb{Z}}}\frac{1}{\lambda_1^{k_1}\cdots\lambda_r^{k_r}}=\zeta(k_1,\dots,k_r)+\sum_{n>0}a_nq^n$$
relations	Harmonic product + Shuffle product = Double shuffle relations	Harmonic product + involution invariance
related Lie algebras	$\mathrm{Lie}(\sigma_3,\sigma_5,\sigma_7,\dots)$ $\mathfrak{dm}_0$	$\mathrm{Lie}(\delta,\omega_3,\omega_5,\omega_7,\dots)\,/$ cusp form relations $\mathfrak{bm}_0$ $\quad\delta\in\mathfrak{sl}_2$