Wall crossing structure from quantum phenomena to Feynman Integrals

Anthony Massidda

Elliptics and Beyond - Bonn 03/09/2025

Based on

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UNIVERSITÀ DEGLI STUDI DELL'INSUBRIA

INFN
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- Introduction and motivations

- Exponential integrals for holomorphic functions: Pearcey Integral

- Exponential integrals for closed forms: Legendre Family

- Conclusions and Outlooks

Consider the exponential integral:

$$I(\gamma) = \int_\Gamma e^{-\gamma f} \mu$$

Where:

- $f : X \to \mathbb{C}$
- Γ *is a n-chain with* $D_0 \supset \partial\Gamma \neq 0$
- $\gamma \in \mathbb{C}^*$
- μ *is an algebraic volume n-form*

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- *Analyze the dependence on* γ : *wall crossing structure*

Multiloop Feynman integrals in Baikov representation

$$I = \int_\Delta \mathscr{B}(x_i)^{-\gamma} \omega$$

- $\mathscr{B}(x_i)$ *is the Baikov Polynomial*
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$$\int exp[f]\mu \,:\, H_\bullet^{Betti,global}(X, D_0, f) \otimes H_{dR,global}^\bullet(X, D_0, f) \to \mathbb{C}$$

[Kontsevich and Soibelman: arXiv:2402.07343, 2024]

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Global Betti homology

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Global Twisted de Rham cohomology

$$H_{dR}^\bullet(X, D_0, f) \cong \mathbb{H}^\bullet(X, \Omega_{X,D_0}^\bullet, \nabla_f)$$

With

$$(\Omega_{X,D_0}^\bullet, \nabla_f) : \; \Omega_{X,D_0}^0 \xrightarrow{\nabla_f} \Omega_{X,D_0}^1 \xrightarrow{\nabla_f} \dots \xrightarrow{\nabla_f} \Omega_{X,D_0}^n$$

$$\nabla_f = d - df \wedge$$

Let $f : X \mapsto \mathbb{C}$ holomorphic and $X \cong \mathbb{C}^n$.
We want to find a basis of integration contours such that

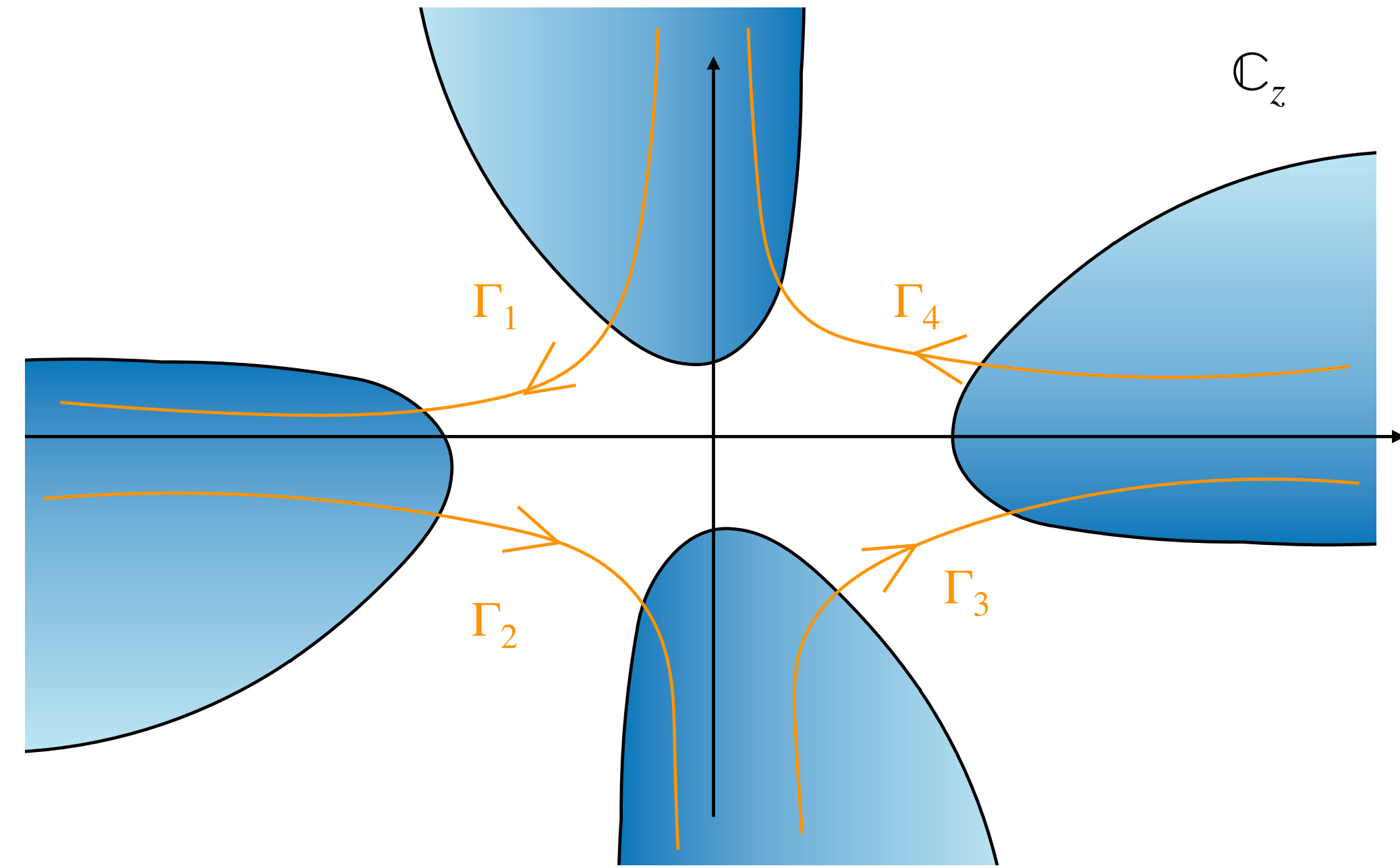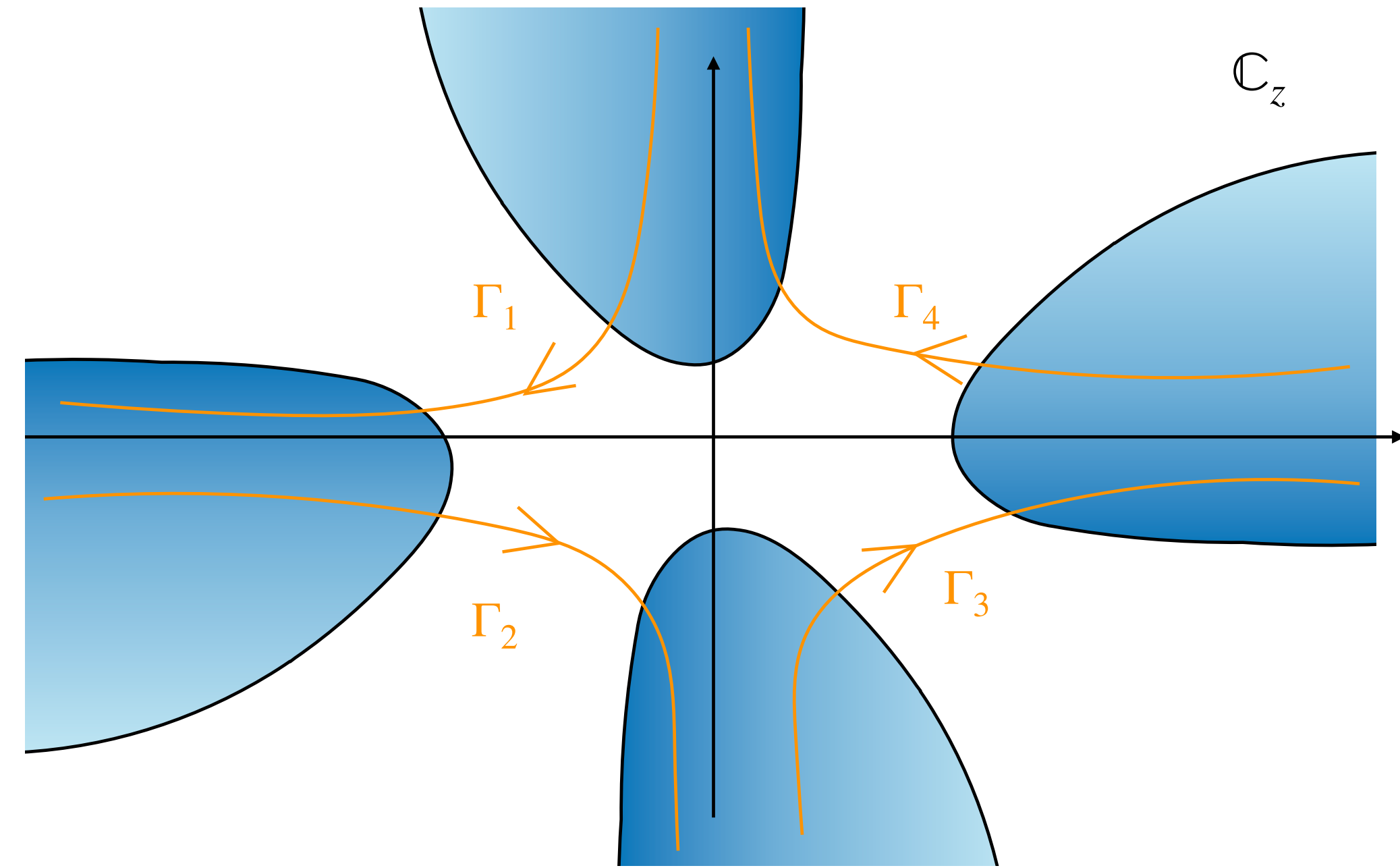$$I(\gamma) = \int_\Gamma e^{-\gamma f(z)} \mu$$

converges.

- *The boundary of Γ must lie on :*

$$X \supset D_N = \{ z \in \mathbb{C}^n \,|\, \mathrm{Re}(\gamma f(z)) \geq N \}$$

- *Γ should connect two distinct regions of D_N*

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$\mathbb{C}_z$

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$$\Gamma \in H_n(X, D_N, \mathbb{Z})$$

- *The $\mathrm{Im}(\gamma f)$ must remain constant along Γ to avoid oscillations*

A good basis for $H_n(X, D_N, \mathbb{Z})$ is provided by the so called Lefschetz thimbles:
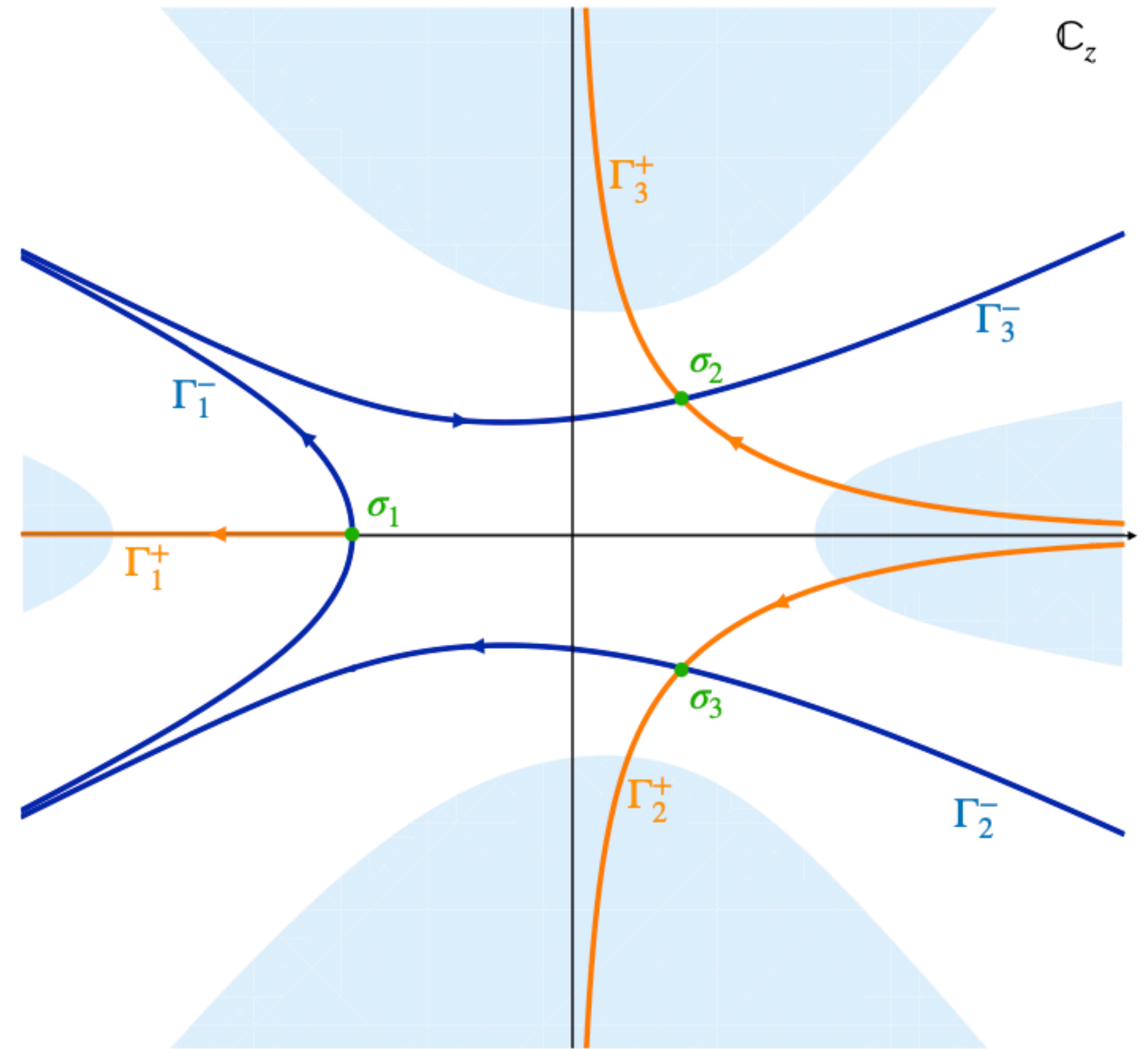
- *Gradient flow lines with constant phase passing through the critical points of $h = \mathrm{Re}(\gamma f(z))$*
- *Ascendent paths Γ_i^+ and descendent paths Γ_i^-*

$$rank[H_k(X, D_N, \mathbb{Z})] = \begin{cases} 0, & k < n, \\ \#\Sigma, & k = n. \end{cases}$$

Set of critical points

$\mathbb{C}_z$

Γ_3^+

Γ_3^-

σ_2

Γ_1^-

σ_1

Γ_1^+

σ_3

Γ_2^+

Γ_2^-

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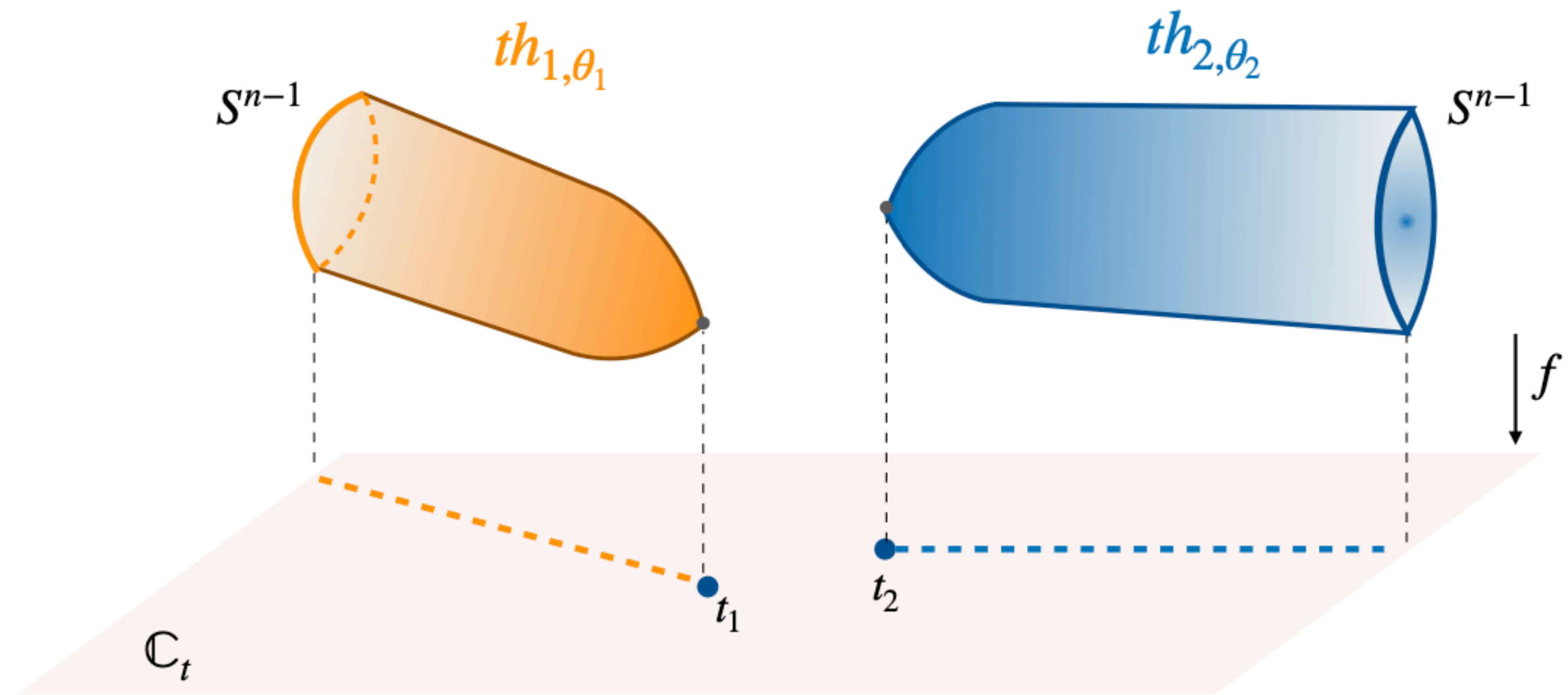*Analitically we should have to solve a differential equation: challenging!*

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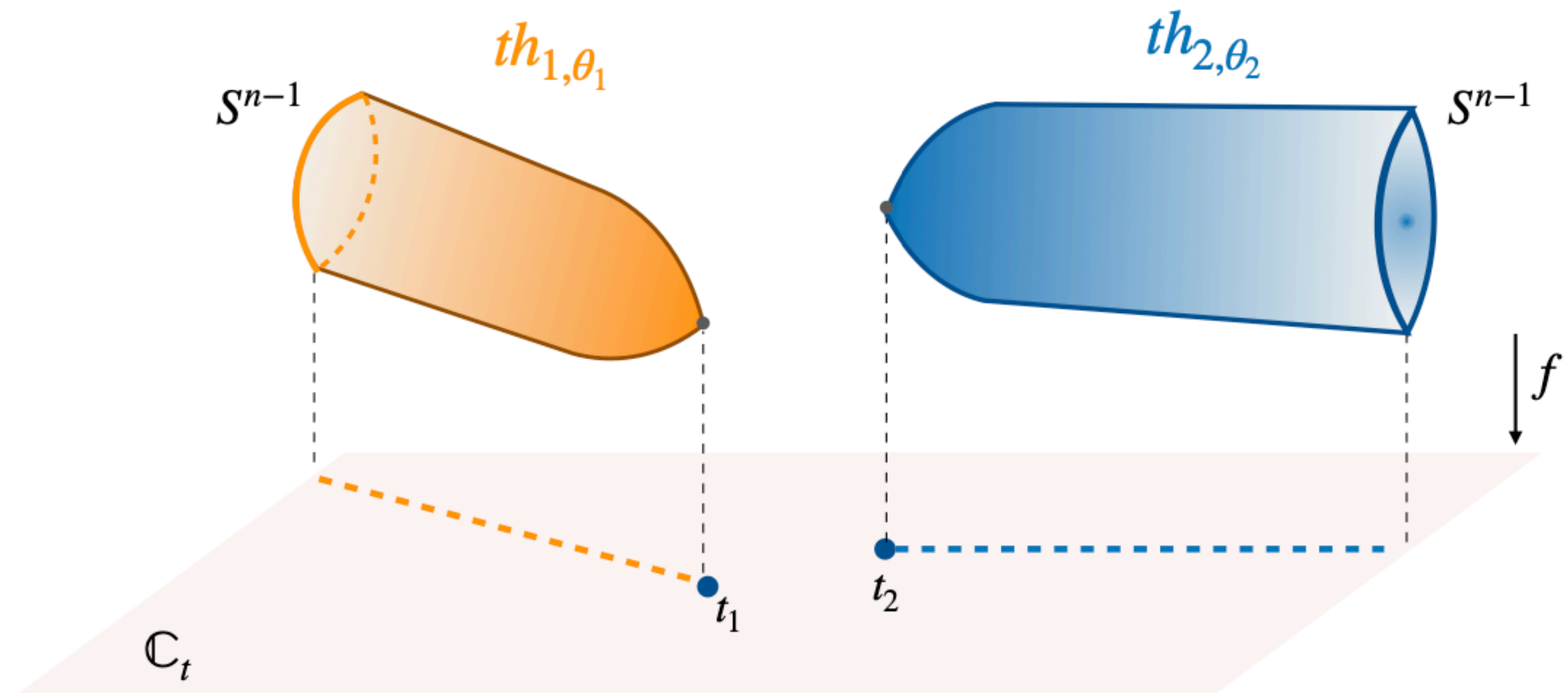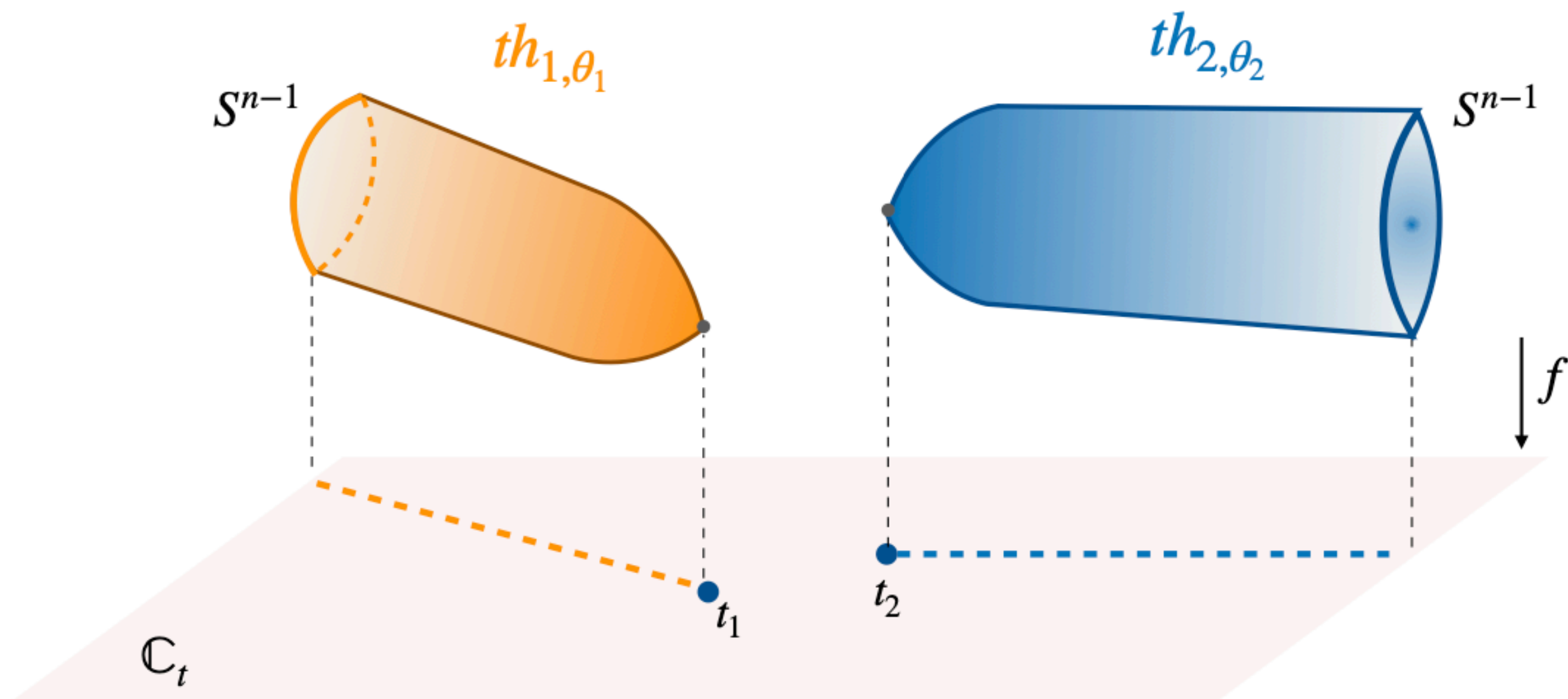*1. Consider $f : \mathbb{C}_z^n \mapsto \mathbb{C}_t$ as a fibration with fiber $f^{-1}(t)$*

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2. Choose a non-critical value t_0 of f and construct a basis $\{\Delta_i\}$ for the middle homology of the variety $f(z) = t_0$
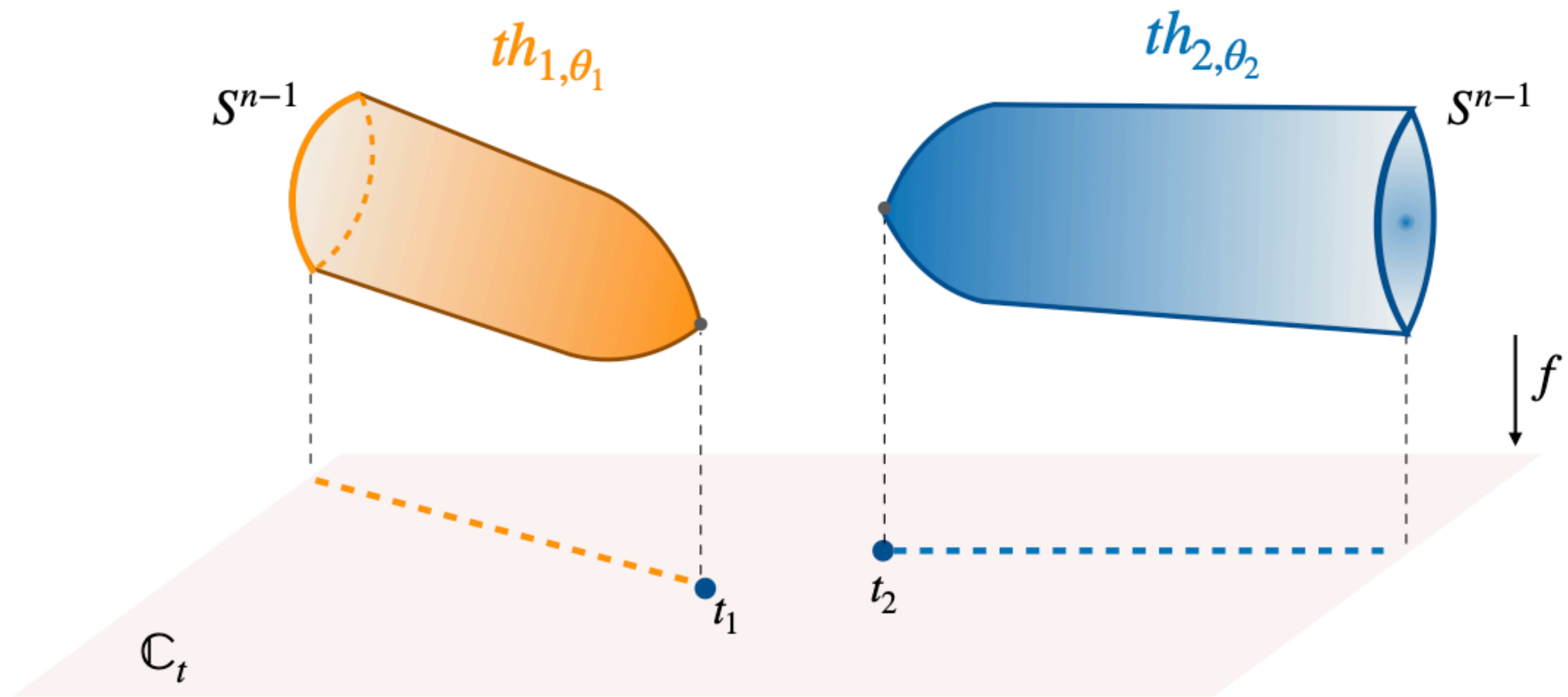
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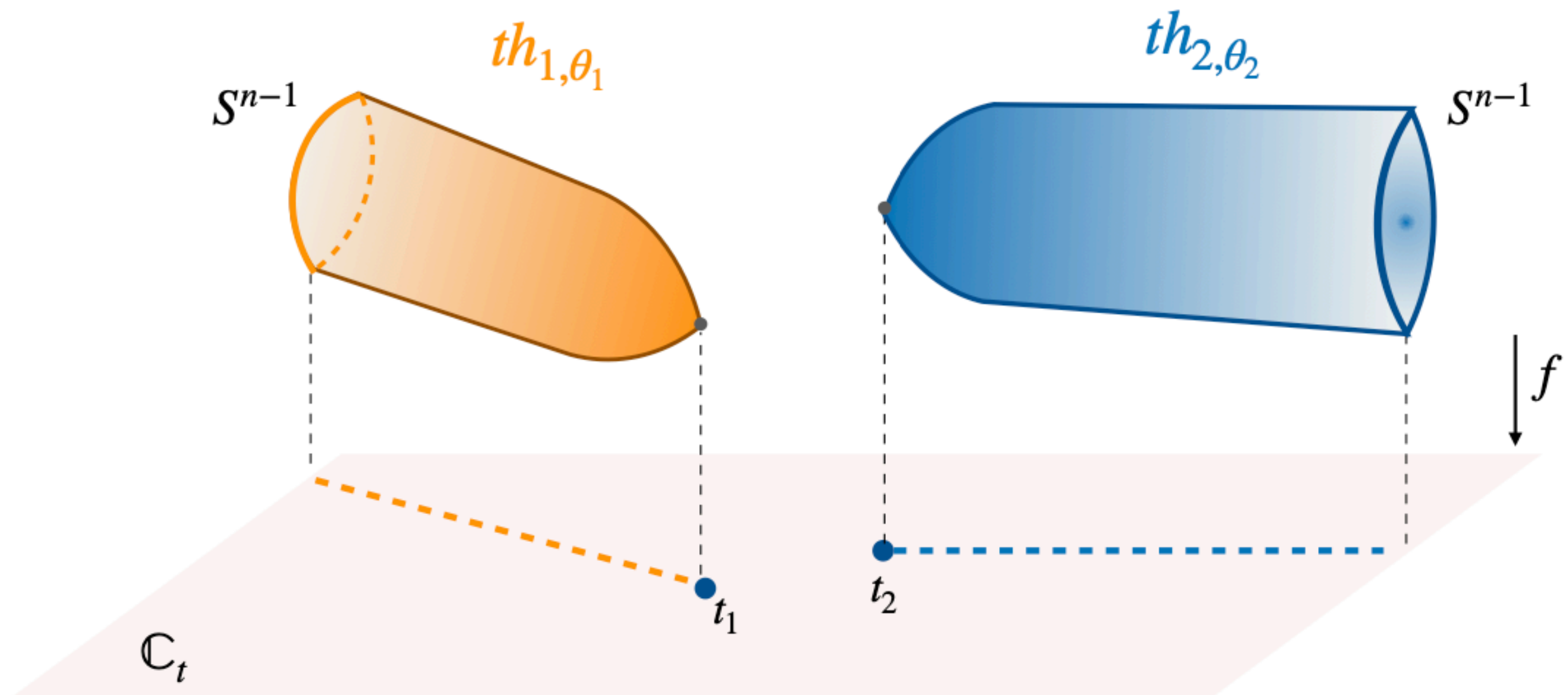

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Thimbles decomposition

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4. The trace of the vanishing cycle between t_0 and t_1 span the Thimble
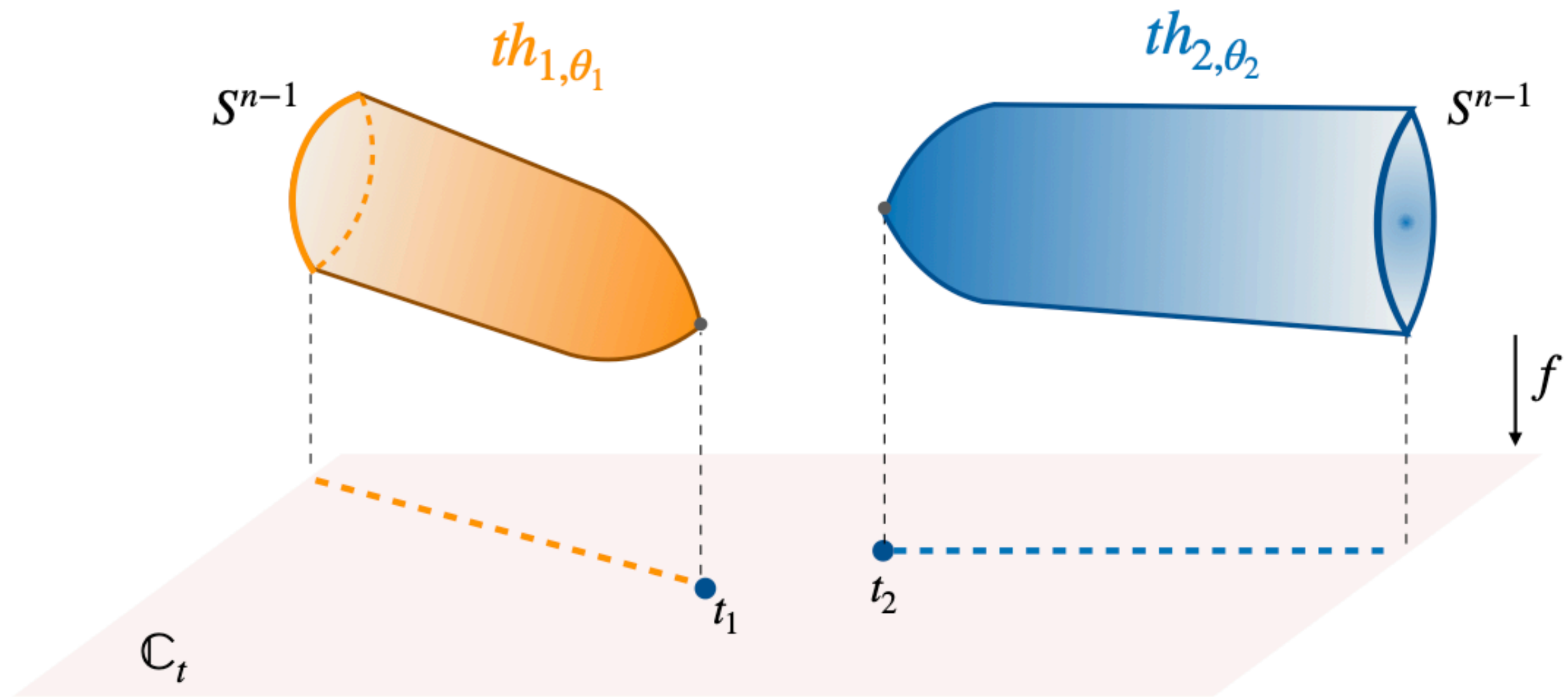
Remind we want the pairing to be defined for any value of
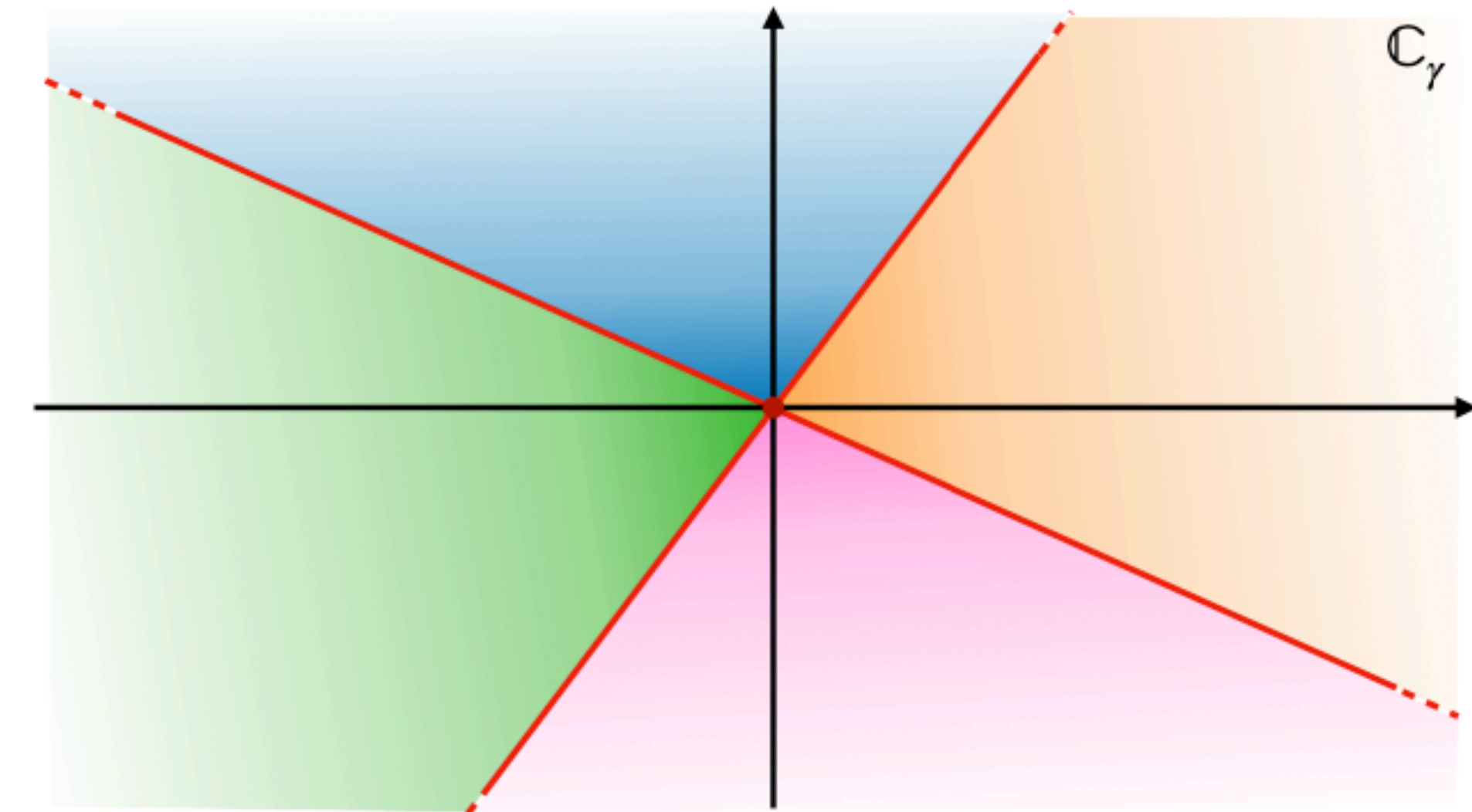*γ.*
Suppose then we have determine the basis of thimbles for
fixed generic γ. What happen for different γ ?

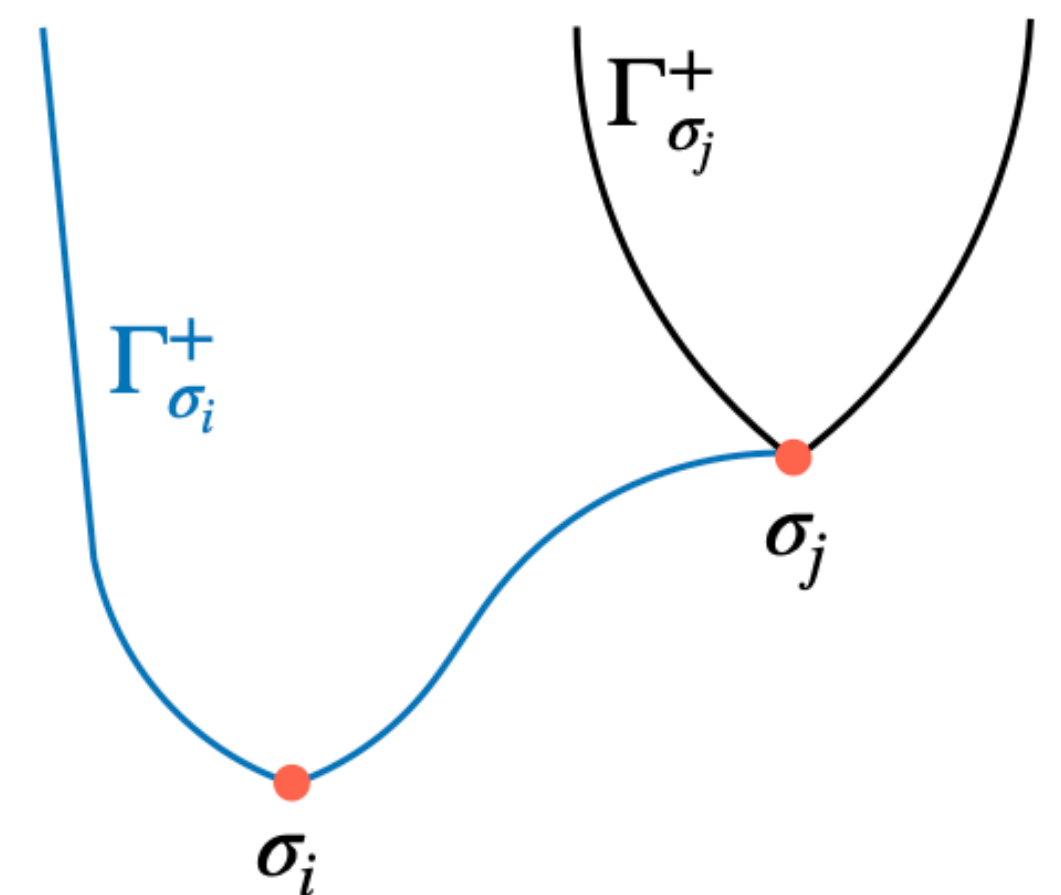*As γ varies it may happens that a thimble crosses more then*
one singular point:

$$l = \{\gamma \in \mathbb{C}^* \,|\, \mathrm{Im}(\gamma f(z))|_{\sigma_i} = \mathrm{Im}(\gamma f(z))|_{\sigma_j}\}$$ *Stokes' line*

The number of Thimbles for such values of γ is less then for
generic γ

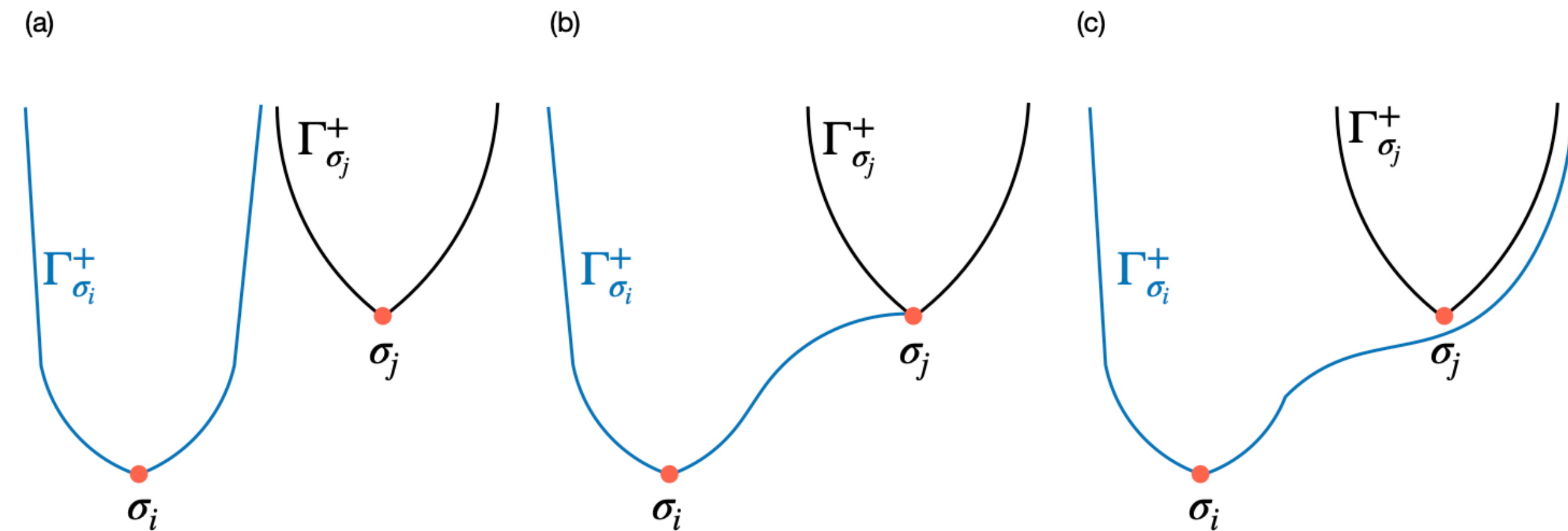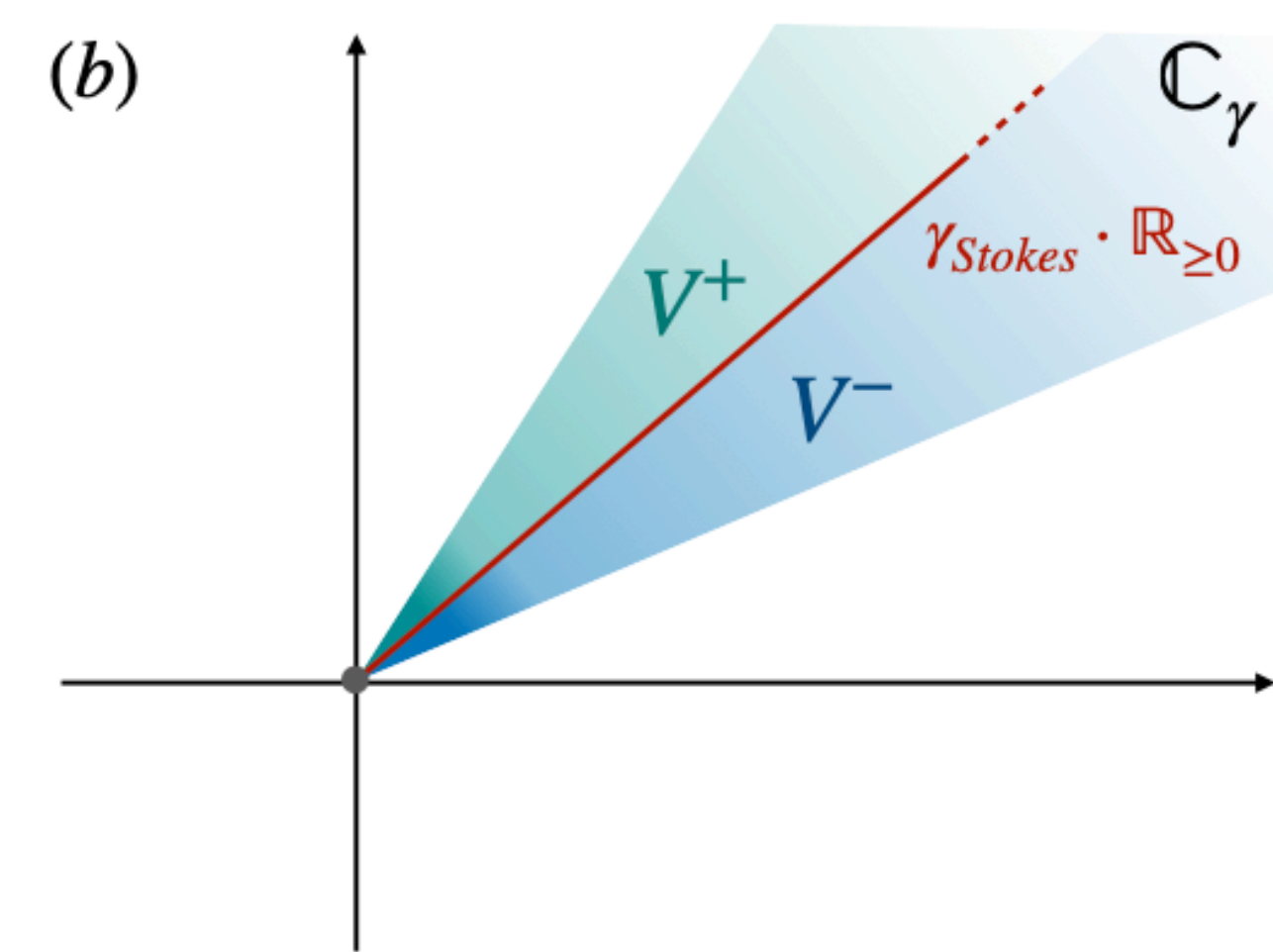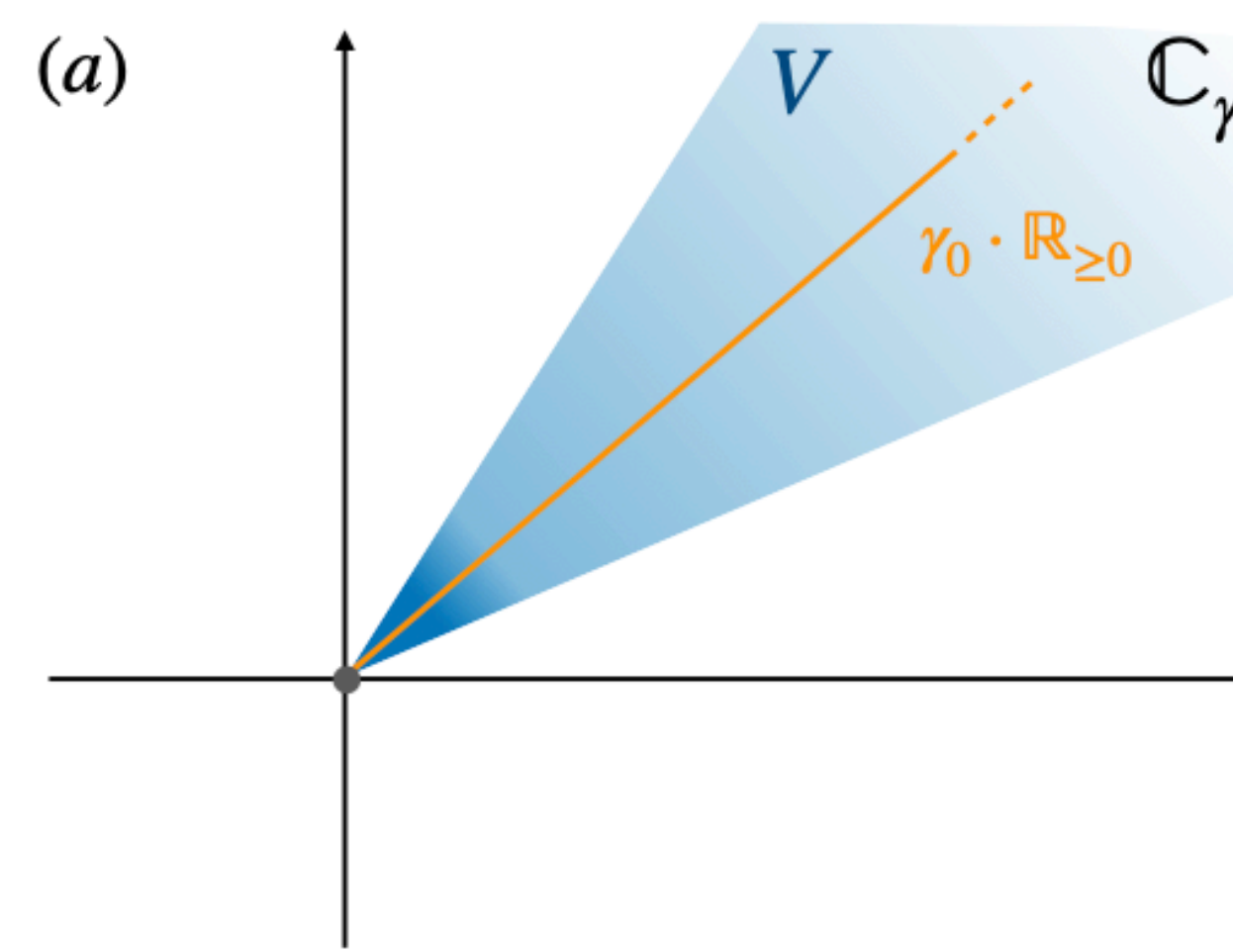*The space $\mathbb{C}_\gamma$ splits into regions separated by Stokes lines*

(b)

As we cross a Stokes ray, associated with a Stokes line between the critical points σ_i and σ_j the corresponding thimbles Γ_j^+ and Γ_i^+ undergo a discontinuous jump to the adjacent region of the form:

$$\begin{pmatrix} \Gamma_i^{+(1)} \\ \Gamma_j^{+(1)} \end{pmatrix} = \begin{pmatrix} 1 & \Delta_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Gamma_i^{+(0)} \\ \Gamma_j^{+(0)} \end{pmatrix}, \quad \text{for} \quad h_{\sigma_i} < h_{\sigma_j}$$

$$\underbrace{\phantom{\begin{pmatrix} 1 & \Delta_{ij} \\ 0 & 1 \end{pmatrix}}}_{T}$$

Where $\qquad \Delta_{ij} = (\pm 1)\Delta_i \circ \Delta_j.$

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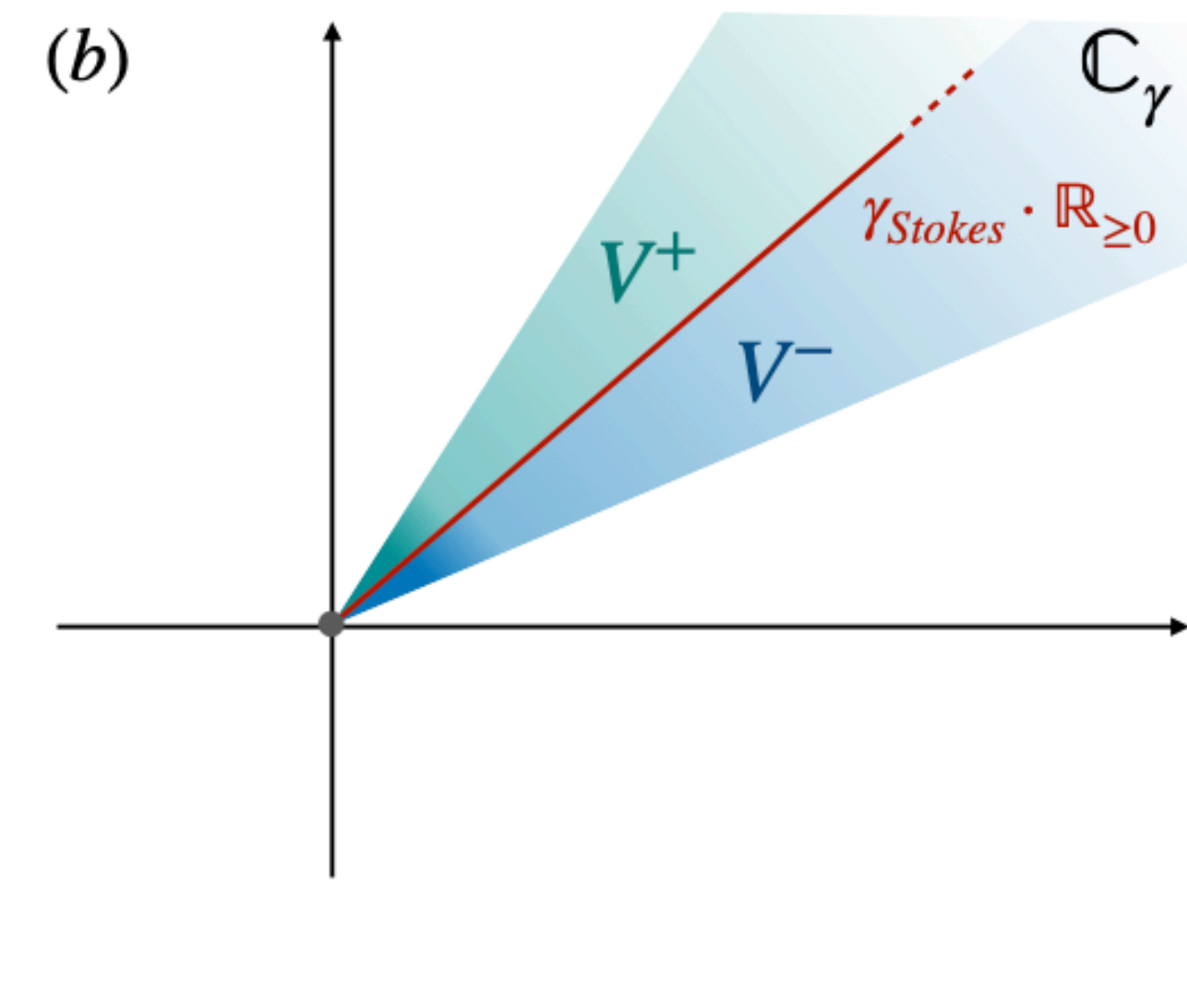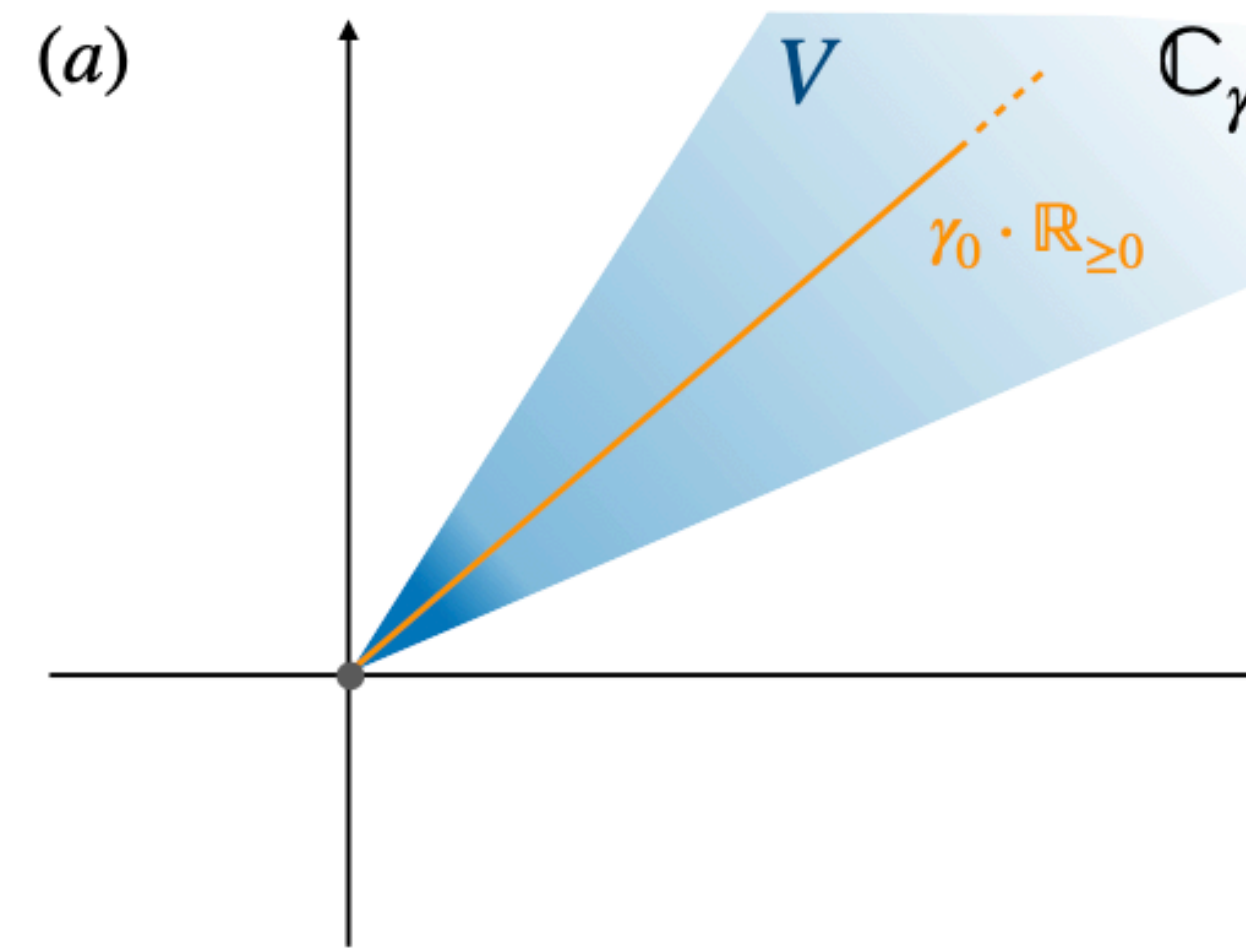*Where* $\qquad \Delta_{ij} = (\pm 1)\Delta_i \circ \Delta_j .$ $\longrightarrow$ *Intersection among the corresponding vanishing cycles*
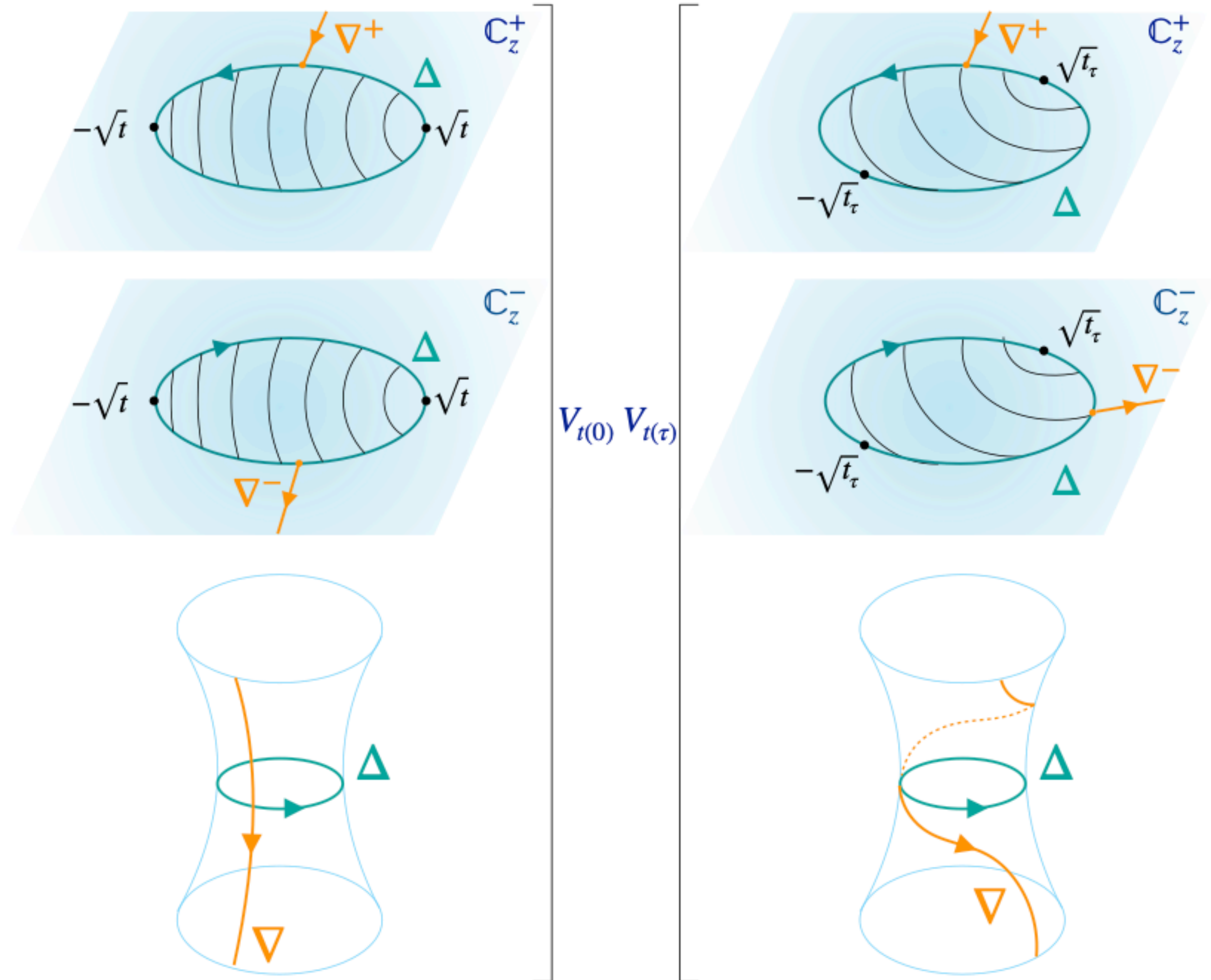
Picard-Lefshetz theorem:

$$h_\tau(a) = a + (-1)^{n(n+1)/2}(a \circ \Delta)\Delta$$

Representation of the monodromy acting on vanishing cycles

Let consider $f = \mathscr{P}_\ell \in \mathbb{C}[z]$ *and extend it to* $\overline{\mathscr{P}}_\ell : \mathbb{P}^1 \to \mathbb{P}^1$.
We want to compute the Hypercohomology of the complex:

$$(\Omega^\bullet_{\mathbb{P}^1, p}, \nabla) : 0 \to \mathscr{O}_{\mathbb{P}^1}(*p) \xrightarrow{\nabla} \Omega^1_{\mathbb{P}^1}(*p) \to 0$$

With respect to the differential:

$$\nabla \equiv (\gamma^{-1} d + d\mathscr{P}_\ell \wedge)$$

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Skipping the details, the final result is

$$H^0_{dR}(\mathbb{C}, d\mathscr{P}_l) \cong 0$$

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This result depends on the global Jacobian ring

$$J_{\mathscr{P}_\ell} \cong \mathbb{C}^{\ell-1}$$

that is NOT sensitive to the coalescence of critical points! If one point as multiplicity m, by considering the middle extension of the connexion we get

$$H^1_{dR}(\mathbb{C}, d\mathscr{P}_\ell)_{(m)} \cong \mathbb{C}^{\ell-m}$$

$$P(\gamma) = \int_\Gamma e^{-\gamma(z^4 + bz^2 + cz + d)} dz$$

Describing the Grand-canonical partition function of gauge Skyrme models for nuclear matter
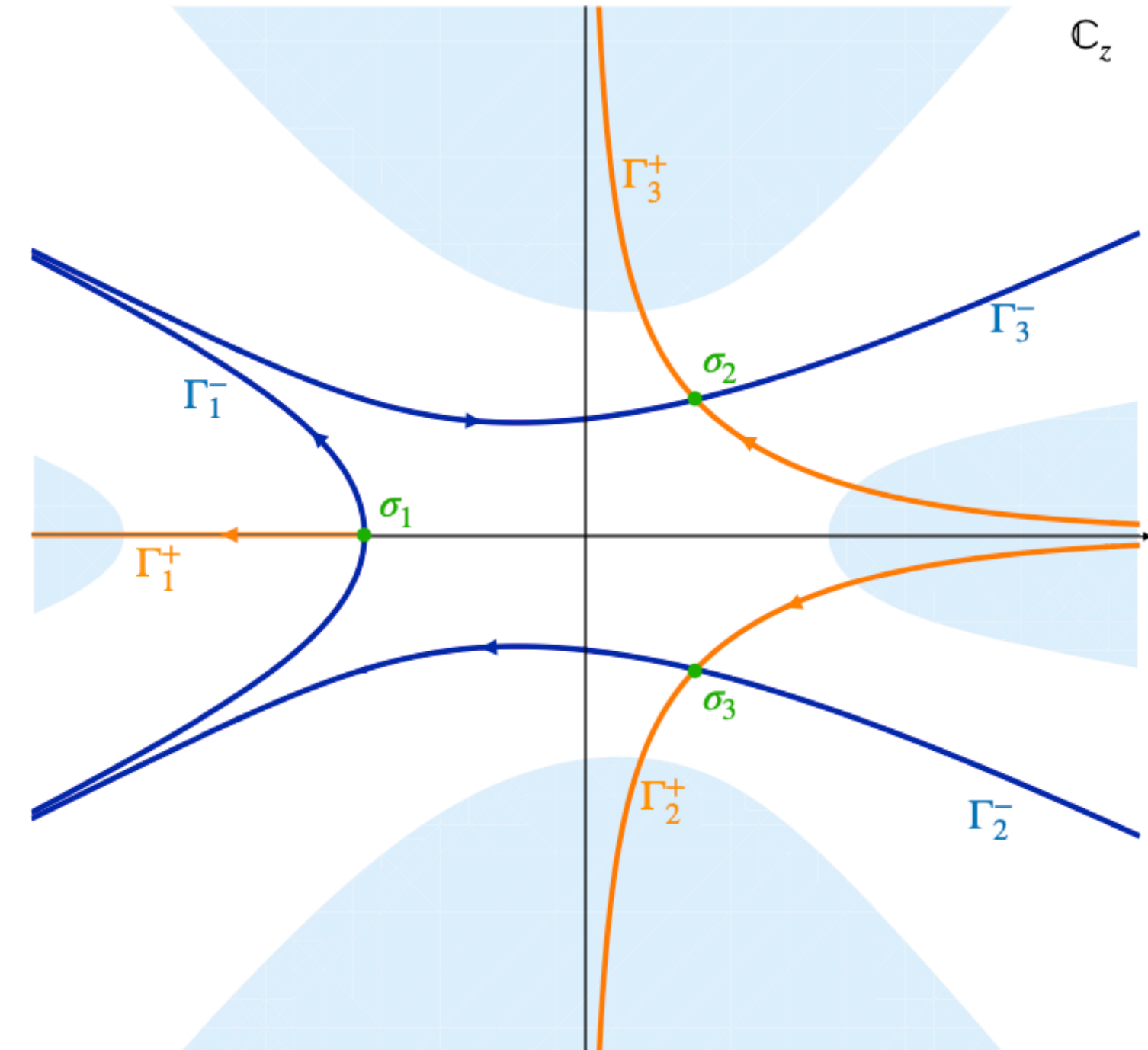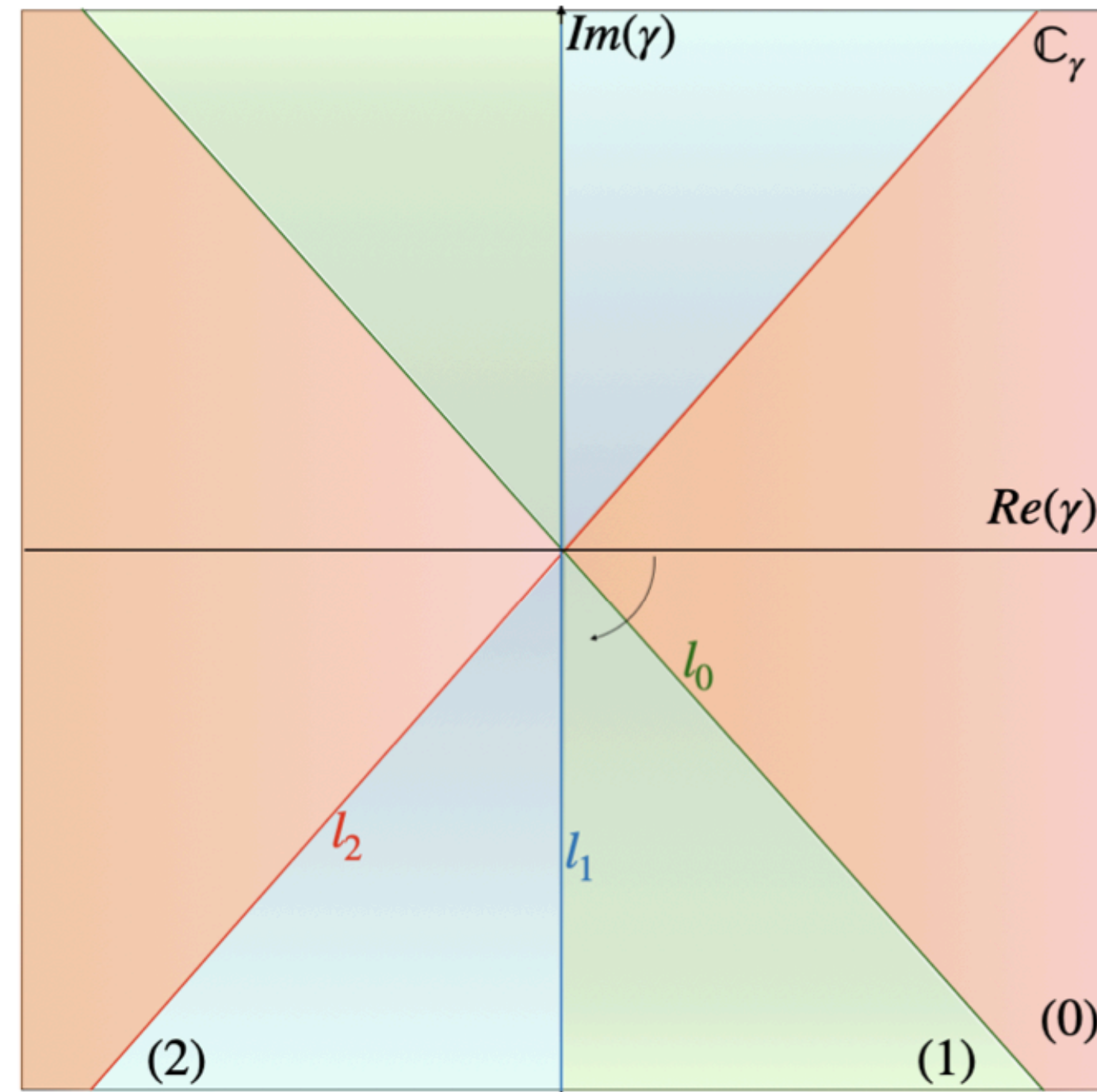
The singular locus is the set

$$\Sigma = \{ z \in \mathbb{C} \,|\, f'(z) = 0 \} = \{ \sigma_i(\Delta) \}$$

Where $\Delta = 8b^2 + 27c^2$

$$\Delta \equiv \begin{cases} > 0 & 1 \text{ real and } 2 \text{ complex conjugate solutions,} \\ < 0 & 3 \text{ real different solutions,} \\ = 0 & 3 \text{ real solutions with at least a multiple root.} \end{cases}$$

$$l_0 : Re(\gamma) = -\frac{11}{16}\sqrt{\frac{3}{2}}\text{Im}(\gamma), \qquad where \quad \text{Im}(\gamma\text{f}(z))\big|_{\sigma_1} = \text{Im}(\gamma\text{f}(z))\big|_{\sigma_2},$$

$$l_1 : Re(\gamma) = 0, \qquad where \quad \text{Im}(\gamma\text{f}(z))\big|_{\sigma_2} = \text{Im}(\gamma\text{f}(z))\big|_{\sigma_3},$$

$$l_2 : Re(\gamma) = \frac{11}{16}\sqrt{\frac{3}{2}}\text{Im}(\gamma), \qquad where \quad \text{Im}(\gamma\text{f}(z))\big|_{\sigma_1} = \text{Im}(\gamma\text{f}(z))\big|_{\sigma_3}.$$

Fix $\gamma \notin l_i$. Consider the preimage of the level set $f(z) = t$

$$f^{-1}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix}$$

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Vanishing cycles

$$\Delta_1 = \{z_3\} - \{z_4\} \quad and \quad \Delta_2 = \Delta_3 = \{z_1\} - \{z_4\}$$

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Monodromy matrices

$$M_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad and \quad M_2 = M_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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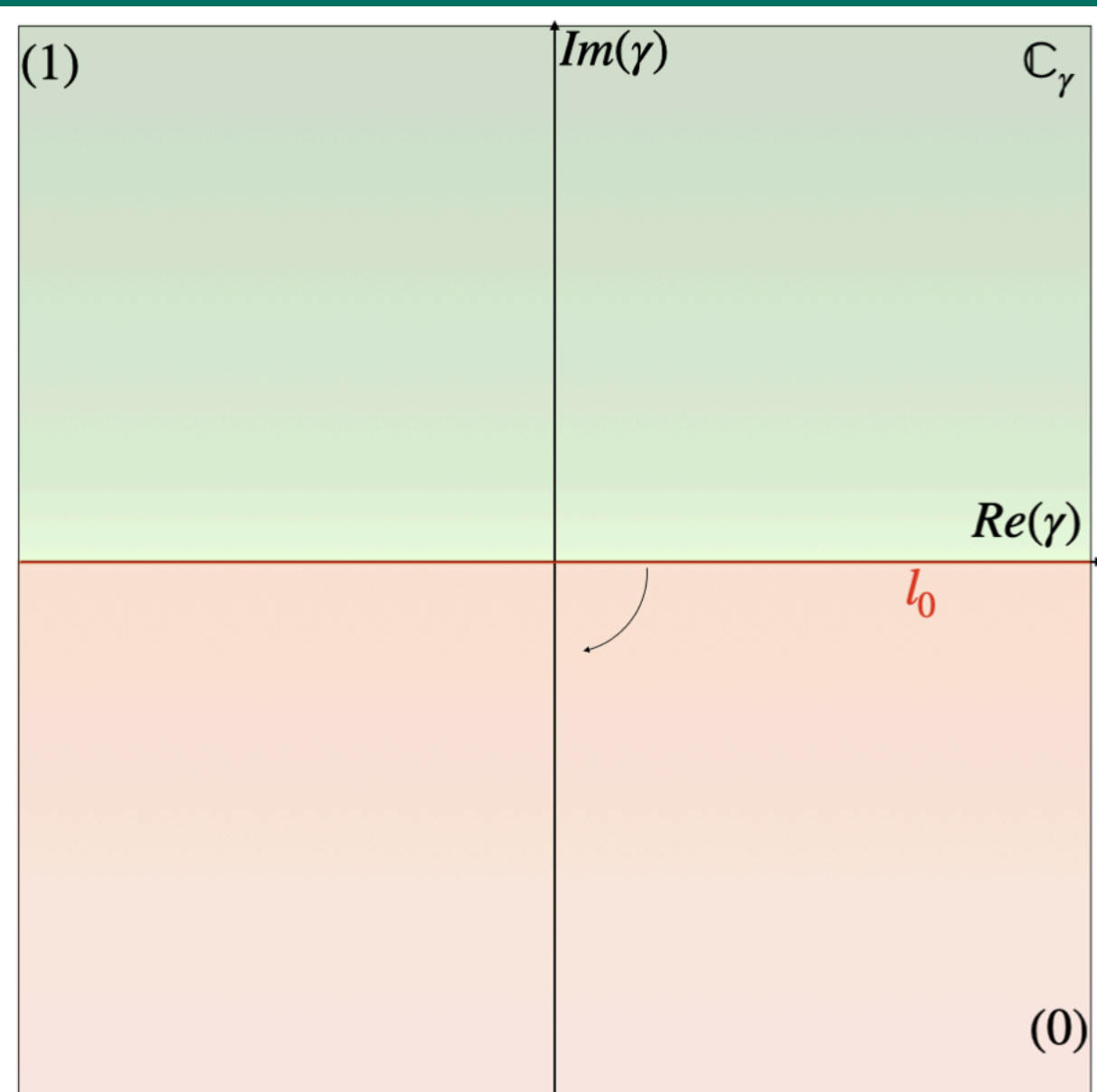$$f^{-1}(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix}$$

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Jump matrices

$$T^{(0)} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad T^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \qquad T^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
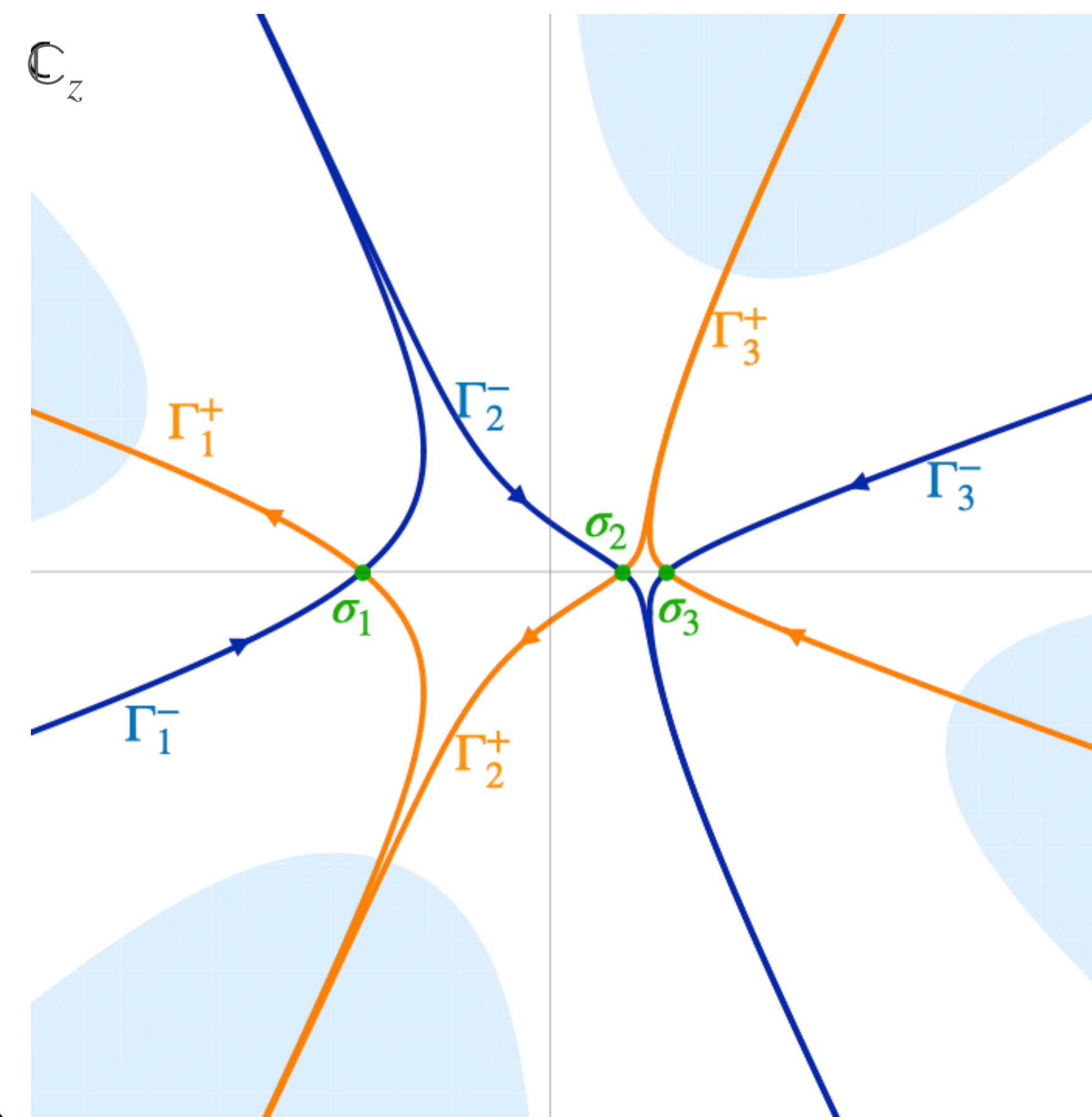
Stoke's line: $\mathrm{Im}(\gamma) = 0$
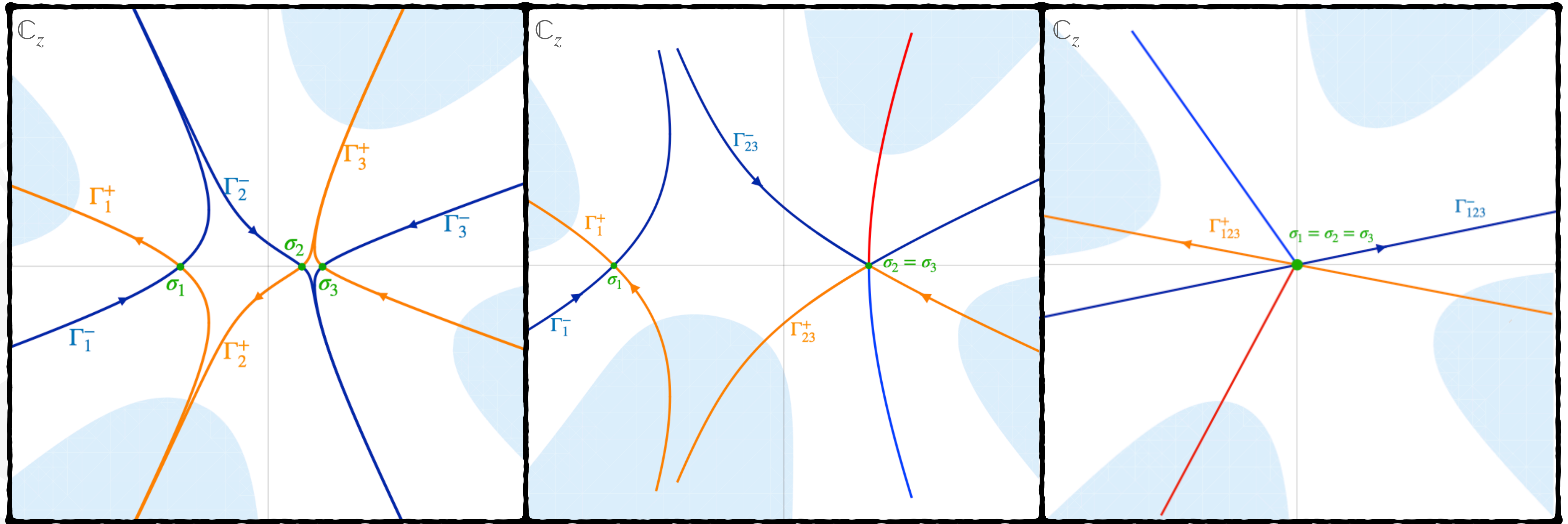
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$$H_1(X, D_N, \mathbb{Z}) = span\{\Gamma_1^+, \Gamma_2^+, \Gamma_3^+\} \cong \mathbb{Z}^3$$
$$H_1(X, D_N, \mathbb{Z})^\vee = span\{\Gamma_1^-, \Gamma_2^-, \Gamma_3^-\} \cong \mathbb{Z}^3$$

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$$H_1(X, D_N, \mathbb{Z})^\vee = span\{\Gamma_{123}^-\} \cong \mathbb{Z}$$

Let us consider now

$$f(z_1, \ldots, z_n) = log \mathscr{B}(z_1, \ldots, z_n)$$

f is defined on $X = \mathbb{C}^n \backslash \{\mathscr{B} = 0\}$ $\longrightarrow$ $\overline{X} = \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$

We classify the added divisor according to th behavior of $\overline{\mathscr{B}}$*, the extension of* $\mathscr{B}$ *over* $\overline{X}$*:*

$$\overline{X} - X = D_h \cup D_v \cup D_{log}$$

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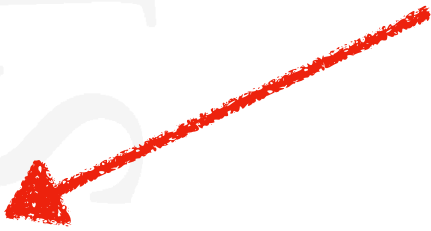
Horizontal divisor: $\overline{\mathscr{B}}$ *si finite*

Let us consider now

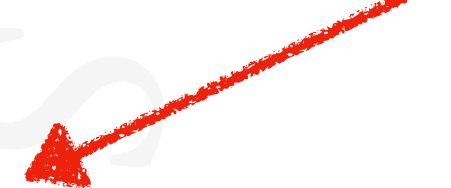$$f(z_1, \ldots, z_n) = \log \mathscr{B}(z_1, \ldots, z_n)$$

f is defined on $X = \mathbb{C}^n \backslash \{\mathscr{B} = 0\}$ $\longrightarrow$ $\overline{X} = \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$

We classify the added divisor according to th behavior of $\overline{\mathscr{B}}$, *the extension of* $\mathscr{B}$ *over* $\overline{X}$:

$$\overline{X} - X = D_h \cup D_v \cup D_{log}$$

Horizontal divisor: $\overline{\mathscr{B}}$ *si finite*

Vertical divisor(at infinity): $\overline{\mathscr{B}}$
has poles

Let us consider now

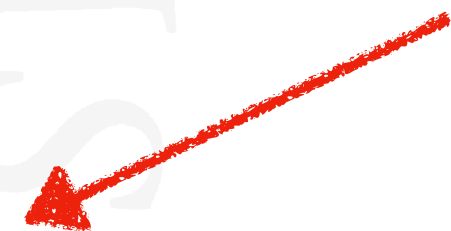$$f(z_1, \ldots, z_n) = log\mathscr{B}(z_1, \ldots, z_n)$$

f is defined on $X = \mathbb{C}^n \backslash \{\mathscr{B} = 0\}$ ⟶ $\overline{X} = \mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}$

We classify the added divisor according to th behavior of $\overline{\overline{\mathscr{B}}}$*, the extension of* $\mathscr{B}$ *over* $\overline{X}$*:*
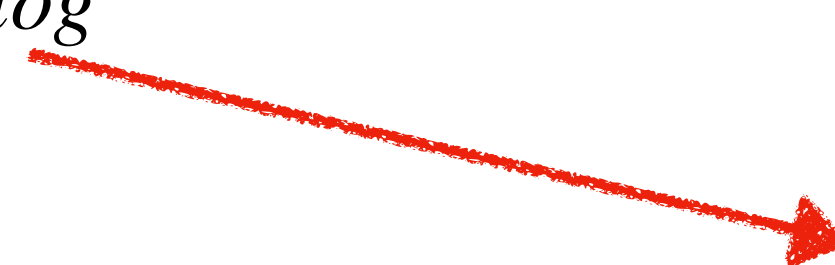
$$\overline{X} - X = D_h \cup D_v \cup D_{log}$$

Horizontal divisor: $\overline{\overline{\mathscr{B}}}$ *si finite*

Vertical divisor(at infinity): $\overline{\overline{\mathscr{B}}}$ *has poles*

Logarithmic divisor: $d\overline{\overline{\mathscr{B}}}$ *has logarithmic poles*

Consider the local system.
$$\mathscr{L}_{\alpha,\gamma}(U) = \ker \nabla|_U, \qquad U \subset X \qquad \text{with } \nabla = d + \gamma\alpha$$

The global Betti cohomology is defined as

$$H^\bullet(X,\alpha) \cong H^\bullet\left(\widetilde{X}, D_v^{\mathbb{R}+} \cup D_{log}^{\mathbb{R}-}, \Pi_*(\mathscr{L}_{\alpha,\gamma})\right)$$

We get:

$$H^\bullet(X,\alpha) \cong \begin{cases} H^\bullet\left(\tilde{X}, D_\infty^{\mathbb{R}}, \Pi_*(\mathscr{L}_{\alpha,\gamma})\right), & Re(\gamma) > 0, \\ H^\bullet\left(\tilde{X}, D_{\mathscr{B}}^{\mathbb{R}}, \Pi_*(\mathscr{L}_{\alpha,\gamma})\right), & Re(\gamma) < 0. \end{cases}$$

Consider the integral

$$\mathscr{I} = \int_\Gamma \frac{dx \wedge dy}{\left[y^2 + x(x-1)(x-\lambda)\right]^\gamma} = \int_\Gamma e^{-\gamma \log\left[y^2 + x(x-1)(x-\lambda)\right]} dx \wedge dy = \int_\Gamma e^{-\gamma \log \mathscr{B}(x,y;\lambda)} dx \wedge dy$$

We extend $\mathscr{B}$ *to* $\mathbb{P}^2$: $\qquad \overline{\mathscr{B}}(x, y, \eta; \lambda) = y^2\eta - x(x-\eta)(x-\eta\lambda)$

And define the close form

$$d \log \overline{\mathscr{B}} = \frac{2\eta y \, dy + [y^2 + x^2 + x\lambda(x-2\eta)]d\eta + [-3x^2 - \eta^2\lambda + 2x\eta(1+\lambda)]dx}{y^2\eta - x(x-\eta)(x-\eta\lambda)}$$

Analyzing the behavior of $\overline{\mathscr{B}}$ we get

$$D_h = D_v = \varnothing \qquad\qquad D_{log} = D_{\overline{\mathscr{B}}} \cup D_\infty$$

With:

$$D_{\overline{\mathscr{B}}} = \overline{\mathscr{E}}_\lambda = \{[x:y:\eta] \in \mathbb{P}^2 \,|\, \overline{\mathscr{B}} = 0\}$$

$$D_\infty = \mathbb{P}^1 = \{[x:y:0] \in \mathbb{P}^2\}$$

Intersecting at $D_{\overline{\mathscr{B}}} \cap D_\infty = [0:1:0]$

The computation of Betti cohomology gives:

$$H^\bullet_{Betti,glob,\gamma}(X,\alpha)(\widetilde{X}, S^3, \Pi_*(\mathscr{L}_{\alpha,\gamma})) \cong 0 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus 0 \oplus \mathbb{C}^2$$

$$H^\bullet_{Betti,glob,\gamma}(X,\alpha)(\widetilde{X}, D^R_{\overline{\mathscr{B}}}, \Pi_*(\mathscr{L}_{\alpha,\gamma})) \cong 0 \oplus 0 \oplus \mathbb{C}^2 \oplus \mathbb{C}^4 \oplus 0$$

- The identification of exponential integrals as periods of twisted de Rham cohomology and Betti homology over complex manifolds, allows to accomodate in the same framework a wide range of physically relevant integrals

- The analysis of the wall crossing structure allows to analytic continue the master integral decomposition in the parameter γ and carefully avoid Stokes' phenomena for a sharp counting of the co-homology dimension

Outlooks

- *Concrete application of this formalism to Feynman integrals in different representation and multiple variables (working in progress with Angius, Cacciatori, Mastrolia and Noja)*

- *Analysis of multi parameter dependence (Complex structure moduli of algebraic varieties = kinematic physical variables)*

- *Application to conformal correlators: string amplitudes*

Thank you