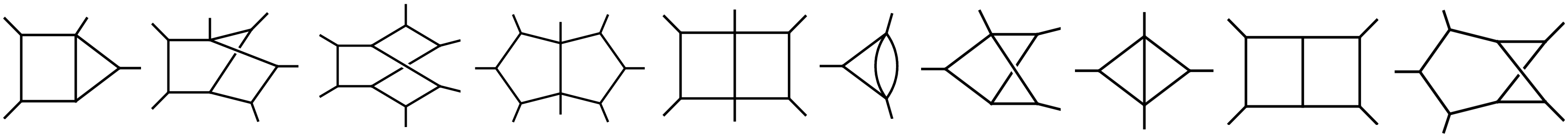
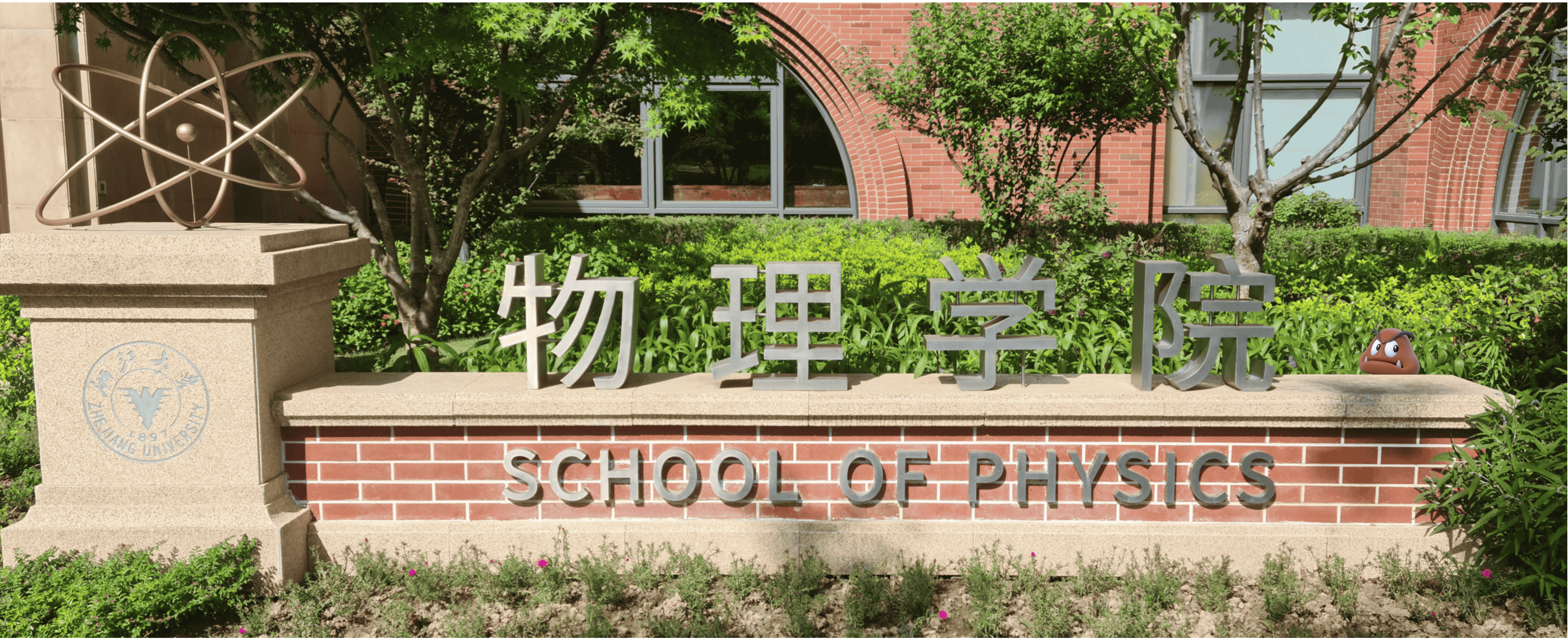
Full Classification of Feynman Integral Geometries at Two Loops





Zhejiang University

Hjalte Frellesvig

The spectrum of Feynman integral geometries at two loops

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Abstract: We provide a complete classification of the Feynman integral geometries at two-loop order in four-dimensional Quantum Field Theory. Concretely, we consider a basis of 79 Feynman integrals in the 't Hooft–Veltman scheme, i.e. with D-dimensional loop momenta and four-dimensional external momenta, and calculate their leading singularities via loop-by-loop Baikov for generic values of the masses and momenta. Aside from the Riemann sphere, we find that only elliptic curves, hyperelliptic curves of genus 2 and 3 as well as K3 surfaces occur. At an intermediate step, we also find a del Pezzo surface, a particular Fano variety known to be rationalizable, resulting in a hyperelliptic curve of genus 3. These geometries determine the space of functions relevant for Quantum Field Theories at two-loop order, including in the Standard Model.

Feynman integrals can be written as iterated integrals

$$f_i = \sum_j \epsilon^j f_i^{(j)} \qquad\qquad f_i^{(j)} = \int \sum_k A_{ik} f_k^{(j-1)}$$

But what are the integration surfaces?

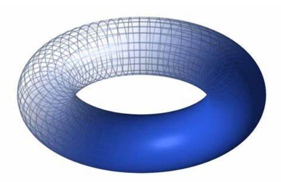
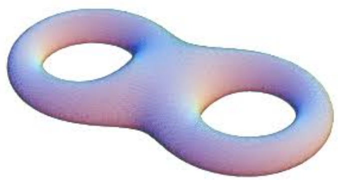
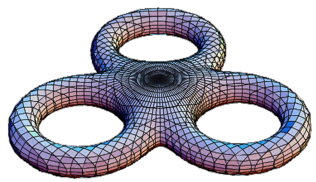
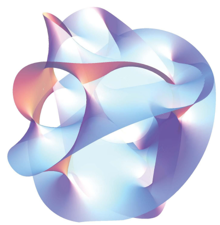
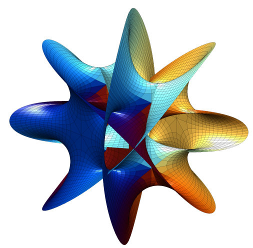
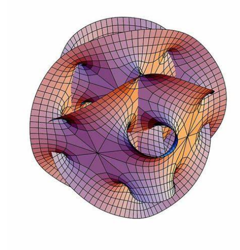
For many FIs it is just the complex plane / Riemann sphere

These are (more or less) the FIs that evaluate to polylogs

$$G(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{dz}{z - a_1} G(a_2, \ldots, a_n; z)$$

But there can also be more complicated surfaces:

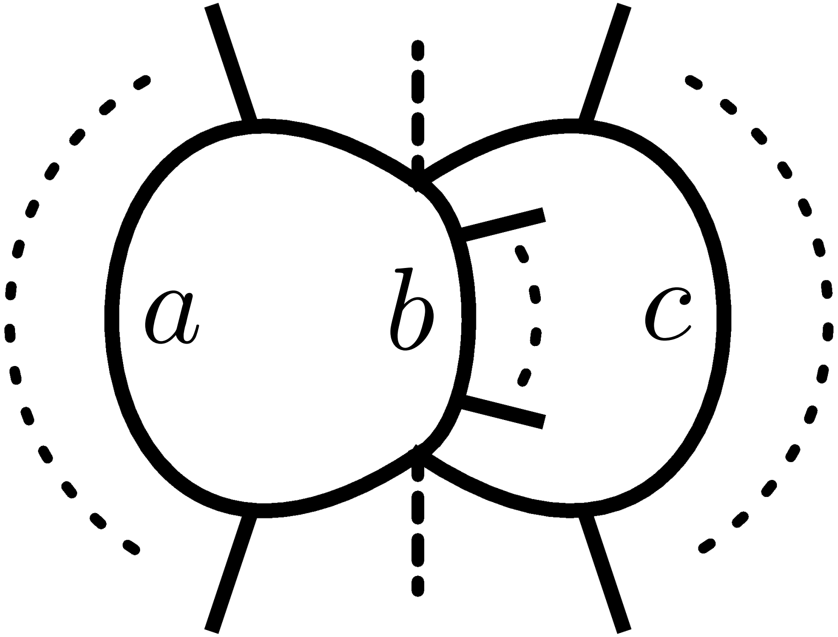
Elliptic curves, Hyper-elliptic curves, Calabi-Yau manifolds...

What else might lurk out there?

Let us investigate these geometries systematically

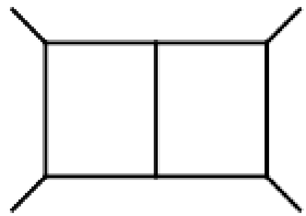
limiting ourselves to the case of two-loop integrals



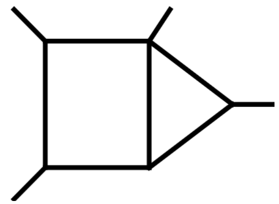
$$I^{\circ\circ}_{abc} \qquad a \geq c \geq b$$

Examples:

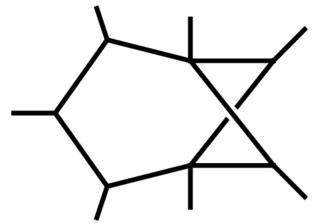
I^{**}_{313}



I^{*}_{312}



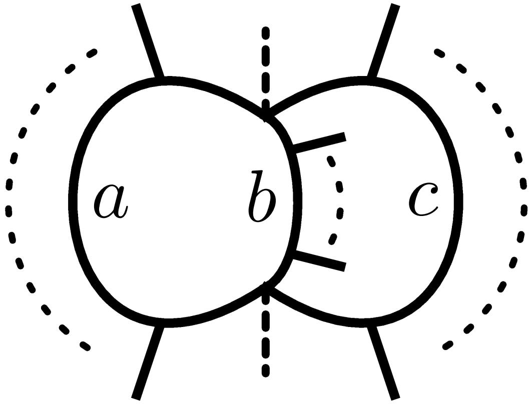
I_{422}



All masses (internal and external) and Mandelstams will be generic

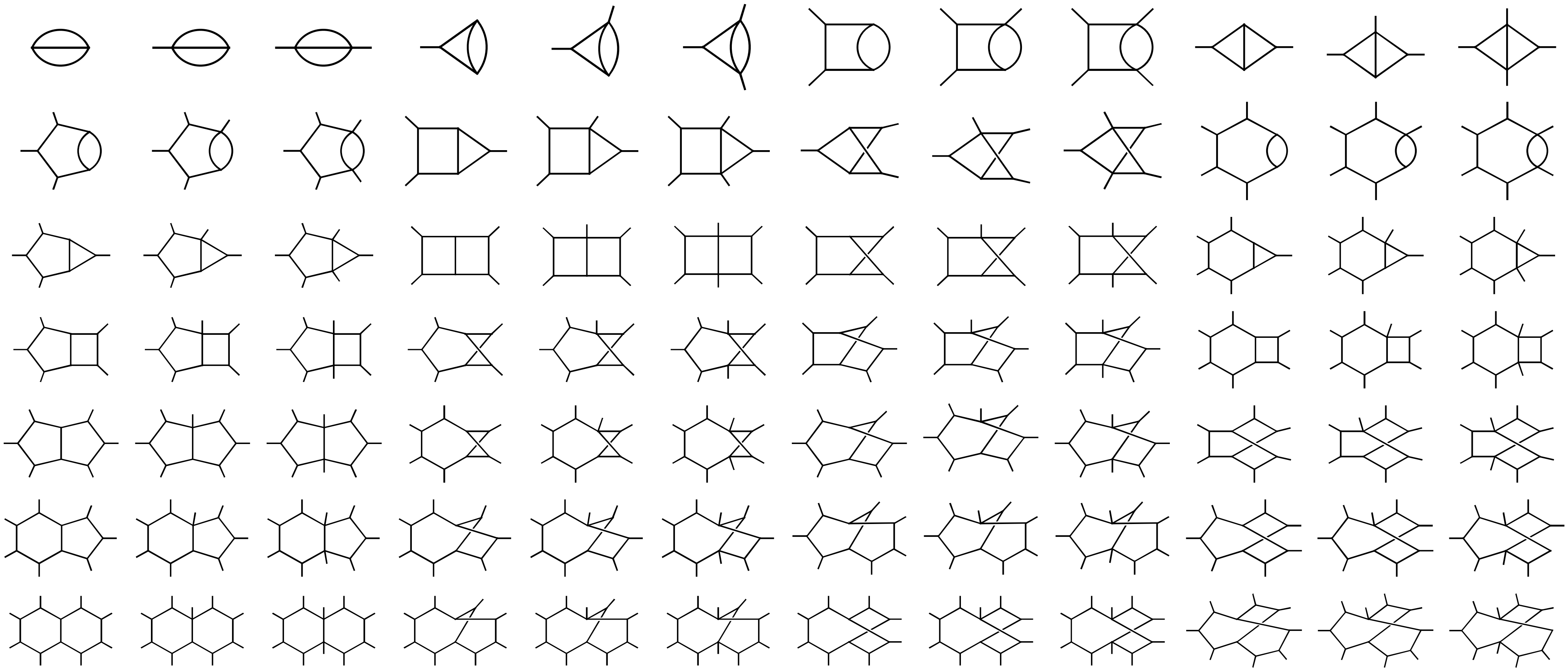
so we get an upper limit* for the complexity

* This limit is often saturated by the cases that have been computed in the literature

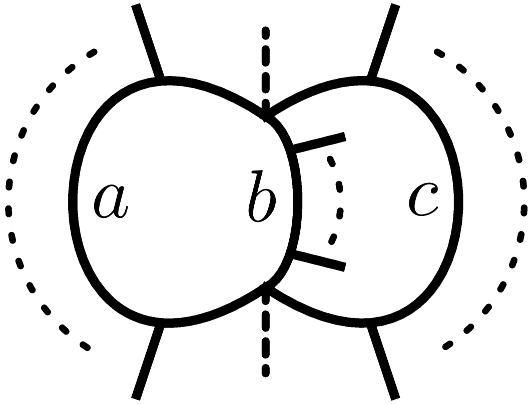


We work in the ‘t Hooft Veltman scheme where the external momenta are limited to four dimensions

$$5 \geq a \geq c \geq b \geq 1, \qquad 11 \geq a+b+c$$



There are 84 diagrams in total [Piotr Bargiela, Tong-Zhi Yang; 2024, 2025]



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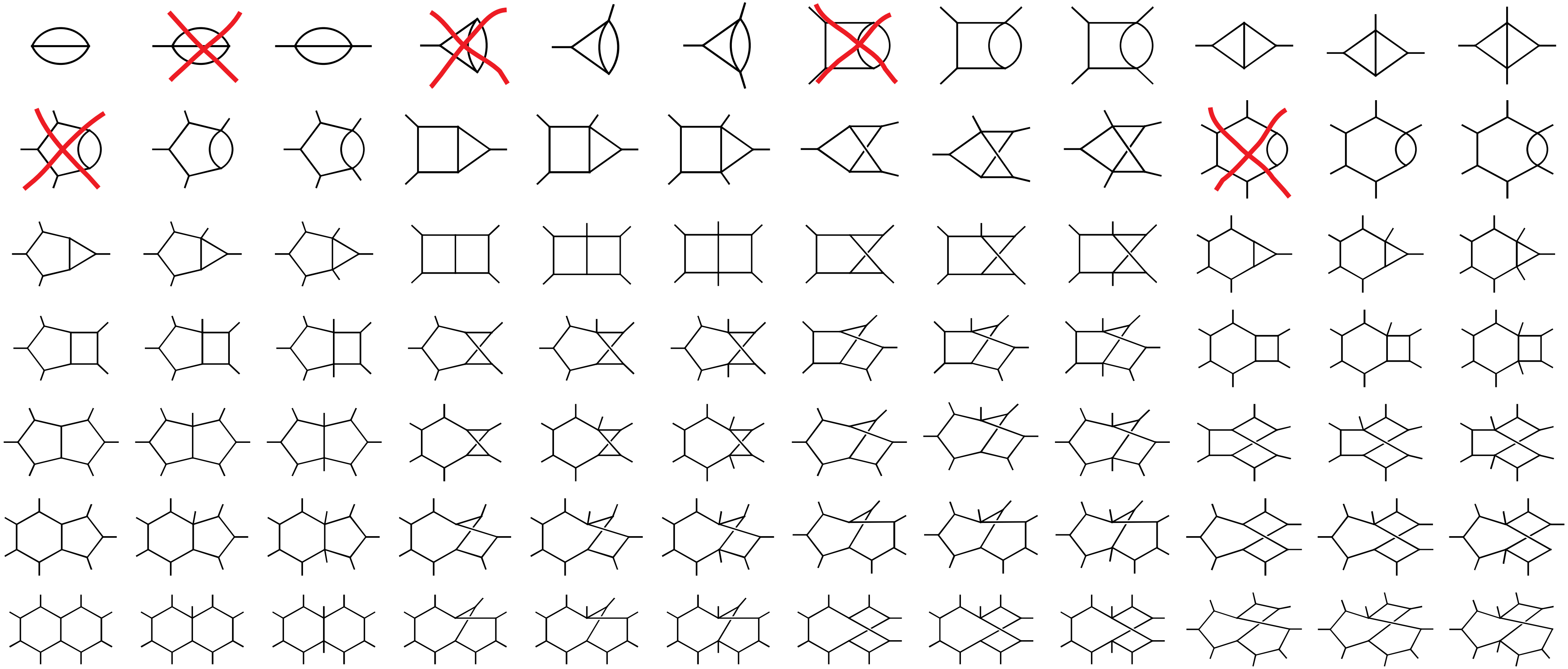


There are ~~84~~ diagrams in total
79

[Piotr Bargiela, Tong-Zhi Yang; 2024, 2025]

Methodology: The loop-by-loop Baikov representation

$$I = \mathcal{K} \int_{\mathcal{C}} \frac{N(x)\, d^n x}{x_1 \cdots x_{n_P}} \mathcal{B}_2(x)^{\beta_2} \mathcal{E}_1(x)^{\gamma_1} \mathcal{B}_1(x)^{\beta_1}$$

$\mathcal{B}_2\,,\ \mathcal{E}_1\,,\ \mathcal{B}_1$ are polynomials in x (Baikov polynomials)

$\beta_2\,,\ \gamma_1\,,\ \beta_1$ are irrational powers (become $\in \mathbb{Z}/2$ in integer dim)

E_i is the number of momenta external to loop nr. i

$$n = 2 + E_1 + E_2\,, \qquad n_{\text{ISP}} = n - n_P$$

If we cut all the propagators, we are left with an n_{ISP}-fold integral

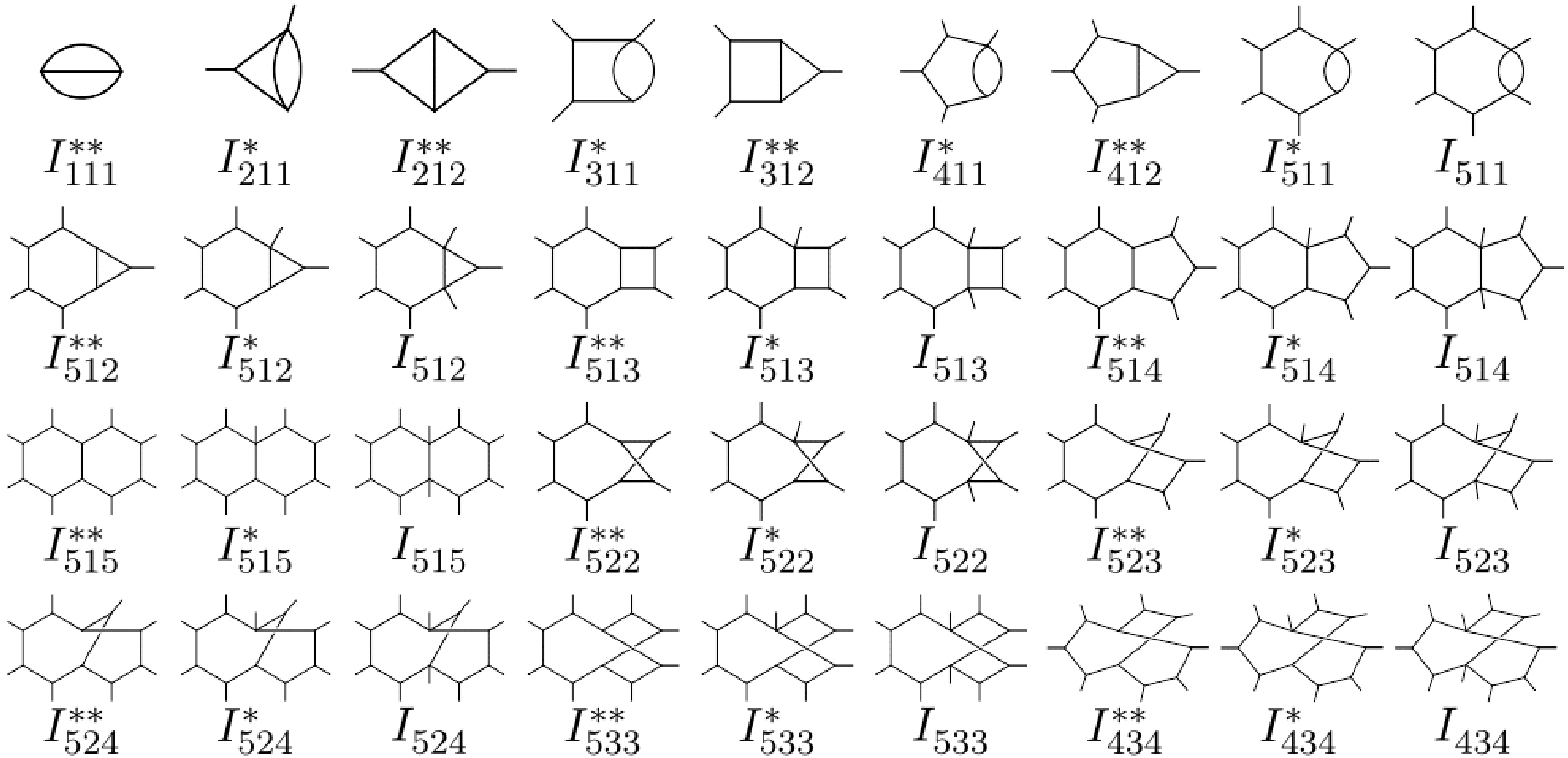
The remaining integral will indicate the geometry of the sector

We have to double-check it with *Picard-Fuchs operators*

i.e. a higher dim. differential operator that annihilates the integral

Let us start with the integrals with 0 ISPs

i.e. those where the maximal cut fixes all the degrees of freedom



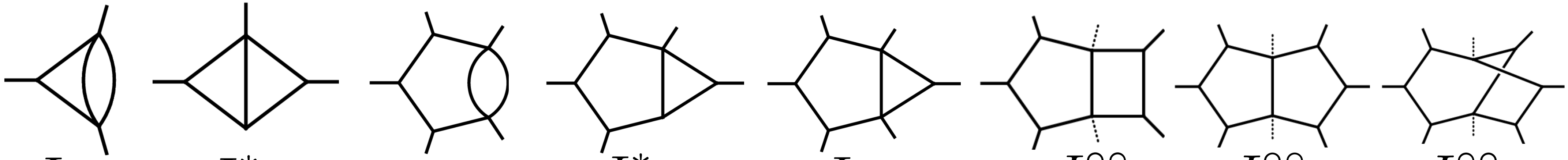
36 of the 79 diagrams. These are trivially polylogarithmic* **

* In this talk, such statements refer to the top-sector only

** *Polylogarithmic* means dlog-form and symbol, not actual polylogs

Then there are the integrals with 1 ISP

Type 1.1: $\int \frac{dz}{\sqrt{P_{[2]}(z)}}$ or $\int \frac{dz}{Q_{[2]}(z)\,\sqrt{P_{[2]}(z)}}$

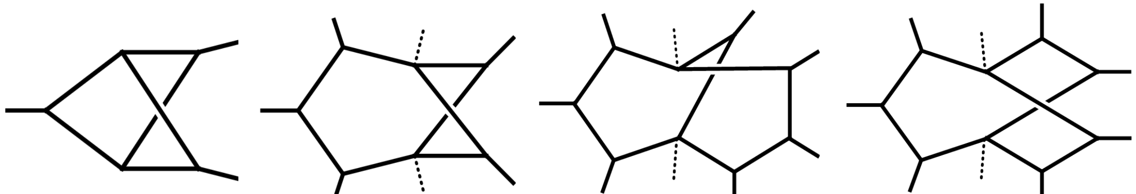


I_{211} I^*_{212} I_{411} I^*_{412} I_{412} $I^{\circ\circ}_{413}$ $I^{\circ\circ}_{414}$ $I^{\circ\circ}_{423}$

$P_{[2]} = (z-r_1)(z-r_2)$ $\qquad z = \frac{r_2(1+y)^2 - r_1(1-y)^2}{4y}$ $\qquad \frac{dz}{\sqrt{P_{[2]}(z)}} = \frac{-dy}{y}$

Type 1.1 are all **polylogarithmic**

Type 1.2: $\int \frac{dz}{\sqrt{P_{[4]}(z)}}$ or $\int \frac{dz}{Q_{[2]}(z)\,\sqrt{P_{[4]}(z)}}$

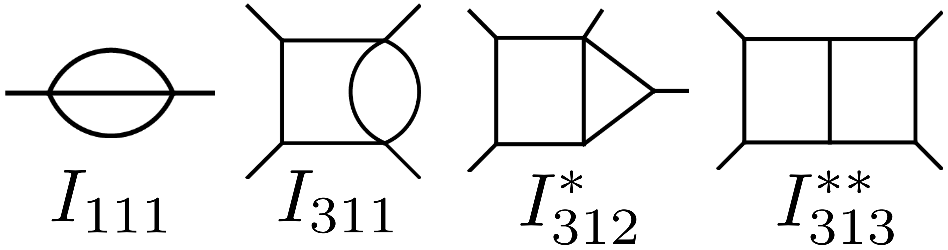


I^{**}_{222} $I^{\circ\circ}_{422}$ $I^{\circ\circ}_{424}$ $I^{\circ\circ}_{433}$

Type 1.2 are all **elliptic**

Integrals with 1 ISP (continued)

Type 1.3: $$\int \frac{dz}{\sqrt{Q_{[2]}(z)}\sqrt{P_{[2]}(z)}}$$



Both combining the roots, and rationalizing, gives an **elliptic** curve

But they are different. "isogenic, not isomorphic"

[**HF**, Vergu, Volk, von Hippel; 2021]

Rationalizing gives the right result, but either way we get the right geometry

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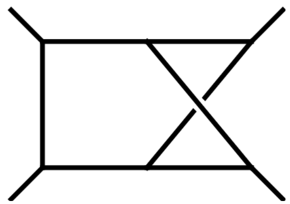
Type 1.4: $$\int \frac{dz}{\sqrt{Q_{[2]}(z)}\sqrt{P_{[4]}(z)}}$$

I^{**}_{322}



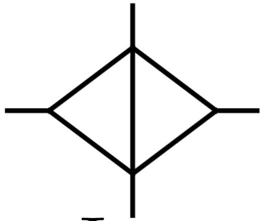
Combining the roots gives deg. 6 corresponding to hyper-elliptic with genus 2

Rationalizing $Q_{[2]}$ gives deg. 8 corresponding to hyper-elliptic with genus 3

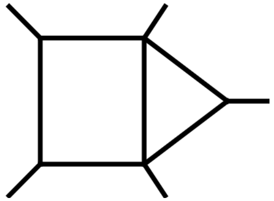
but there is a genus drop back down to **genus 2**

[Marzucca, McLeod, Page, Pögel, Weinzierl; 2023]

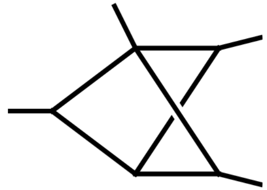
Integrals with 2 ISPs:



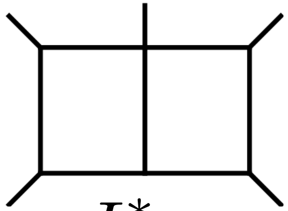
I_{212}



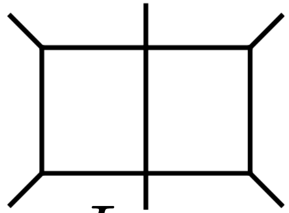
I_{312}



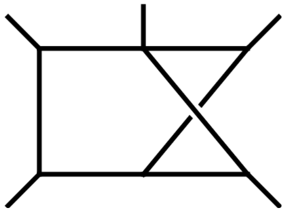
I^{*}_{222}



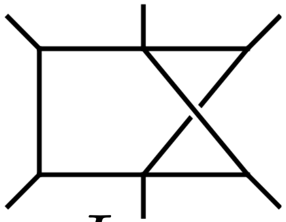
I^{*}_{313}



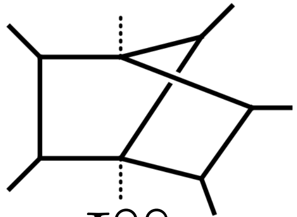
I_{313}



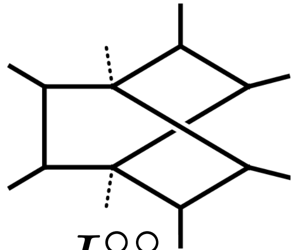
I^{*}_{322}



I_{322}

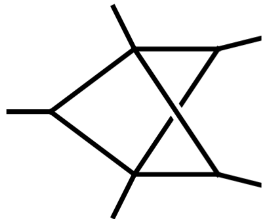


$I^{\circ\circ}_{323}$



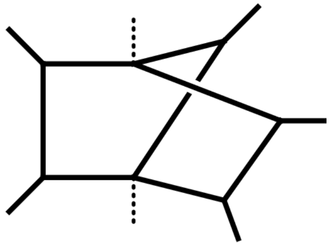
$I^{\circ\circ}_{333}$

Integral with 3 ISPs:

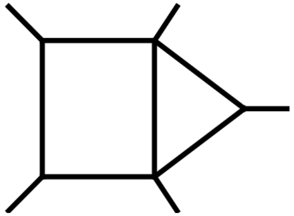


I_{222}

We investigate them one by one



$$I_{323}^{\circ\circ}\Big|_{\substack{d=6\\ \text{max cut}}} = \int \frac{d^2 z}{\sqrt{P_{[2]}(z_1, z_2)}}$$



$$I_{312}\Big|_{\text{max cut}} = \int \frac{d^2 z}{Q_{[2]}(z_1, z_2)\sqrt{R_{[2]}(z_1, z_2)}}$$

$$P_{[2]} = \sum_{i,j=0}^{i+j=2} a_{ij} z_1^i z_2^j$$

Rationalizing in z_2 gives $\frac{d^2 z}{\sqrt{P_{[2]}(z_1, z_2)}} = \frac{-dz_1 dy}{\sqrt{a_{02}}\, y}$

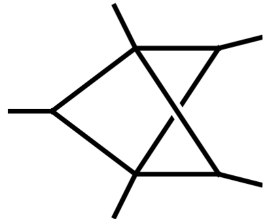
Taking the residue in y (and integrating z_1) makes $I_{323}^{\circ\circ}$ a constant revealing it to be **polylogarithmic**

Rationalizing $R_{[2]}$ makes $Q_{[2]}$ of the form $Q_{[2]} = \sum_i^2 \sum_{j=0}^{i+j\leq 4} b_{ij} y_1^i y_2^j$

Lastly taking a residue in y_1 reveals an **elliptic** curve

The lesson: It it not just what is under the root that determines the geometry

The tardigrades



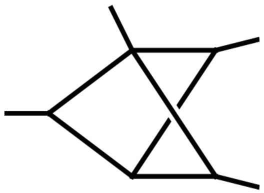
$$I_{222}\big|_{\text{max cut}} = \int \frac{d^3z}{Q_{[4]}(z_1,z_2,z_3)\sqrt{P_{[4]}(z_1,z_2,z_3)}}$$

A non-trivial variable change allows us to eliminate one variable

$$\int \frac{d^2y}{\sqrt{R_{[6;4]}(y_1,y_2)}}$$ (deg. 6 in total, but deg. 4 in the individual vars.)

This characterizes a **K3** manifold

Also studied in e.g. [Doran, Harder, Vanhove, Pichon-Pharabod, 2024]



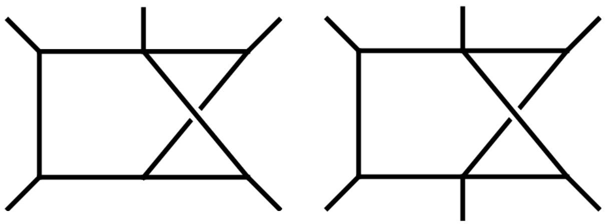
$$I^*_{222}\big|_{\text{max cut}} = \int \frac{d^2z}{\sqrt{S_{[4]}(z_1,z_2)}\sqrt{T_{[2]}(z_1,z_2)}}$$

A var. change brings this to the form $\int \frac{d^2y}{\sqrt{\mathcal{S}_{[4;2]}(y_1,y_2)}\sqrt{\mathcal{T}_{[2]}(y_1,y_2)}}$

Could we combine the roots, we would get the K3, but we can't...

In the end the DHVP analysis reveals it as **K3**

Five and six-point crossed box



$$I^{\circ}_{322}\Big|_{\text{max cut}} = \int \frac{d^2 z}{Q_{[4]}(z_1, z_2)\sqrt{P_{[2]}(z_1, z_2)}}$$

Rationalizing $P_{[2]}$ and taking a residue, gives

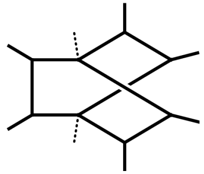
$$I^{\circ}_{322}\Big|_{\text{max cut}} = \int \frac{dz}{\sqrt{A_{[2]}(z)}\sqrt{B_{[4]}(z) + C_{[3]}(z)\sqrt{A_{[2]}(z)}}}$$

Then rationalizing $A_{[2]}$ gives $\quad I^{\circ}_{322}\Big|_{\text{max cut}} = \int \frac{x\, dx}{\sqrt{R_{[8]}(x)}}$

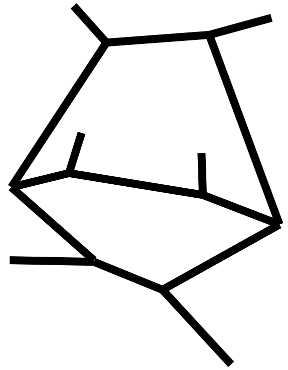
This corresponds to a hyper-elliptic curve with **genus 3**

This time there is no extra involution, so

I_{322} and I^{*}_{322} have genus 3 while I^{**}_{322} had genus 2

One of the highlights: $I^{\circ\circ}_{333}$ 

The Goomba



$$I^{\circ\circ}_{333}\Big|^{(d=6)}_{\text{max cut}} = \int \frac{d^2 z}{\sqrt{P_{[4;4]}(z_1, z_2)}}$$

This defines a "Del Pezzo surface of degree 2"

First we projectivize: $$y^2 = \sum_{ij}^{i+j+k=4} a_{ij} z_1^i z_2^j z_0^k$$

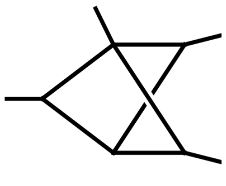
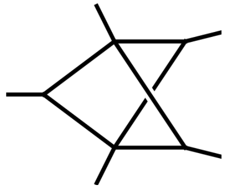
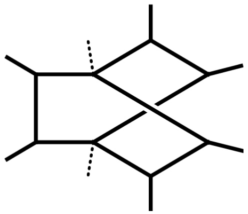
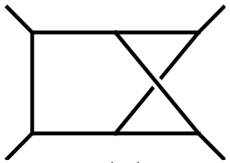
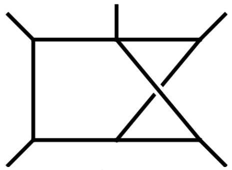
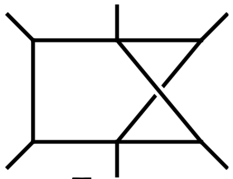
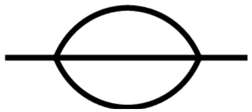
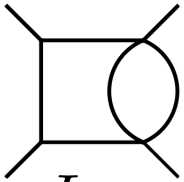
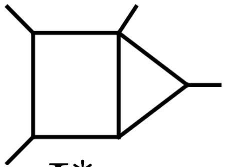
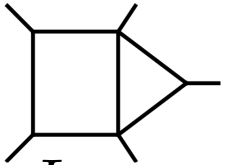
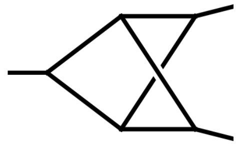
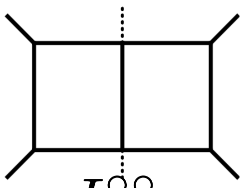
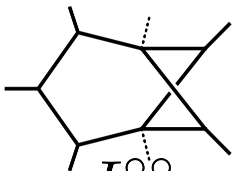
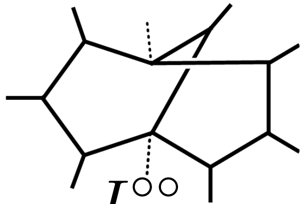
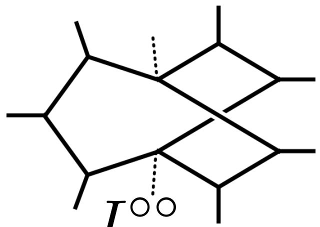
Then we do the var. change: $$z_n = \sum_{ijk}^{i+j+k=3} q_{nijk} \alpha^i \beta^j \gamma^k \qquad y = \sum_{ijk}^{i+j+k=6} r_{ijk} \alpha^i \beta^j \gamma^k$$

[J. Schicho, *Elementary theory of Del Pezzo Surfaces;* 2005]

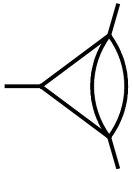
After 'deprojectivizing': $$I^{\circ\circ}_{333}\Big|_{\text{max cut}} = \int \frac{d\alpha\, d\beta}{P_{[6]}(\alpha, \beta)}$$

This integrand describes a surface with algebraic **genus 3**

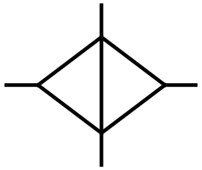
Summary (preliminary)

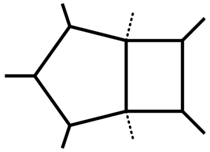
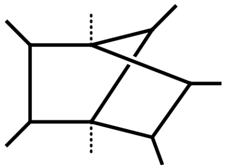
K3 surfaces:	 I^{*}_{222}  I_{222} (Del Pezzo g3) $I^{\circ\circ}_{333}$ 
Hyper-elliptic:	Genus 2:  I^{**}_{322} Genus 3:  I^{*}_{322}  I_{322}
Elliptic:	 I_{111}  I_{311}  I^{*}_{312}  I_{312}  I^{**}_{222}  $I^{\circ\circ}_{313}$  $I^{\circ\circ}_{422}$  $I^{\circ\circ}_{424}$  $I^{\circ\circ}_{433}$
Polylogarithmic:	Everything else

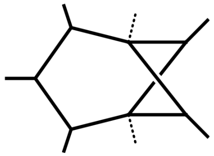
Other surprises

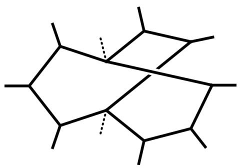
I_{211}  is polylog but seems to have $\mathcal{L}_4 = \mathcal{L}_2\mathcal{L}_1\mathcal{L}_1$

The $\mathcal{L}_2$ factorizes after the same variable change you do to take the maximal cut!

I_{212}  seems to be polylog. More investigation is needed

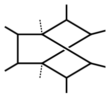
$I^{\circ\circ}_{413}$  and $I^{\circ\circ}_{323}$  are both polylog

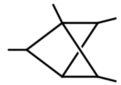
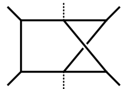
$I^{\circ\circ}_{422}$  is elliptic, but (probably) polylog to $\mathcal{O}(\varepsilon^0)$

$I^{\circ\circ}_{434}$  Cannot be (straightforwardly) parametrized with loop-by-loop Baikov. (Standard is fine)

Perspectives

There were no huge surprises showing up at two-loop

The biggest highlight was that Del Pezzo surface for the Goomba $I_{333}^{\circ\circ}$  giving the genus three

Other highlights were the new K3 in I_{222}^{*} 
and the hyper-elliptics in the three $I_{322}^{\circ\circ}$ 

Which of these will be needed for pheno?
(Les Houches Wishlist)

How about three-loop?

What are the *true* rules for combining roots?

Perspectives

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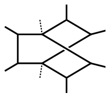
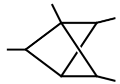
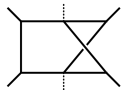
The biggest highlight was that Del Pezzo surface for the Goomba $I_{333}^{\circ\circ}$  giving the genus three

Other highlights were the new K3 in I_{222}^{*} 
and the hyper-elliptics in the three $I_{322}^{\circ\circ}$ 

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What are the *true* rules for combining roots?

Thank you for listening!

Hjalte Frellesvig