Higher-genus multiple zeta values

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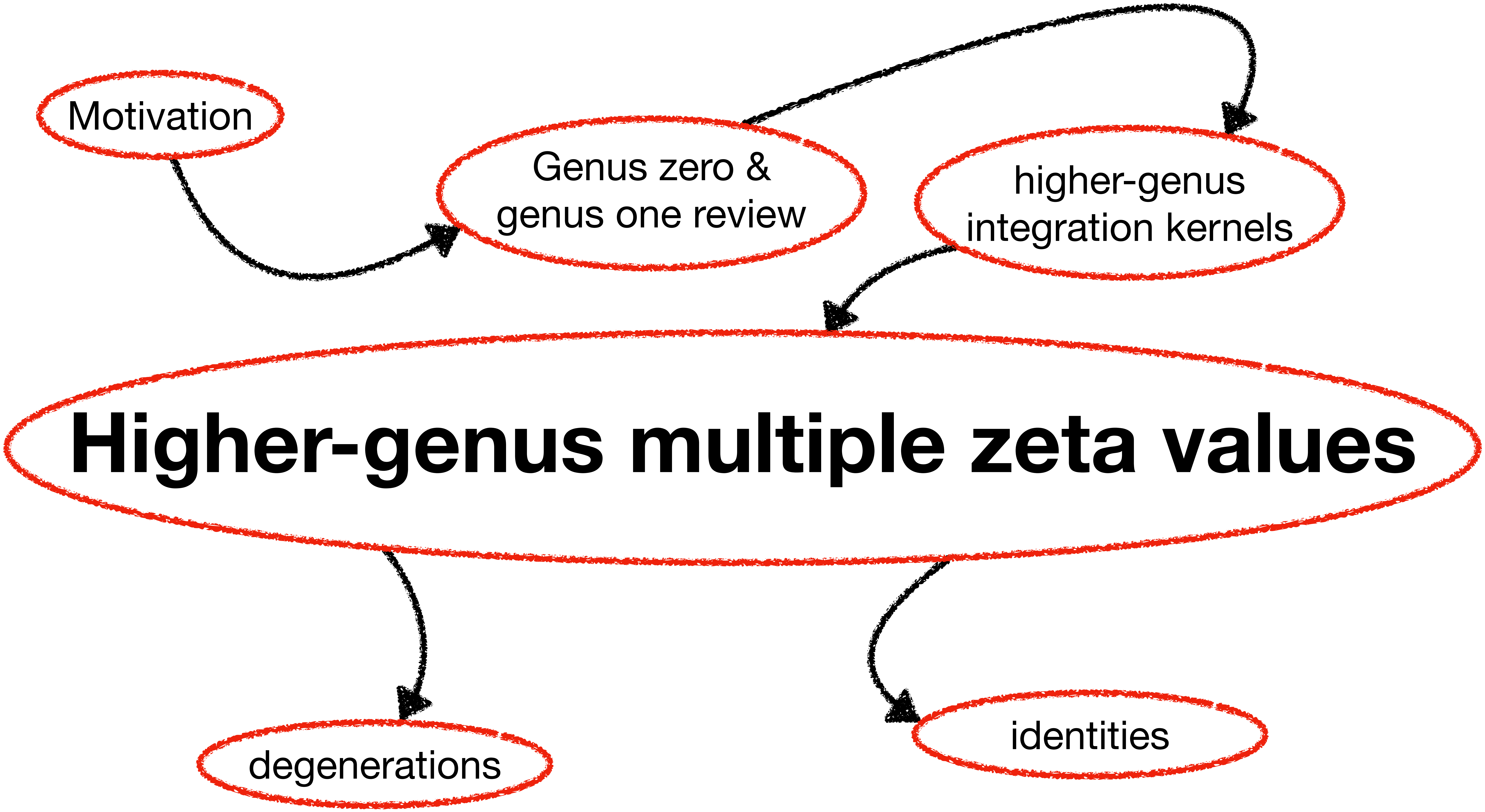
based on arXiv:2507.21765 with J. Broedel, E. Im, Z. Ji, Y. Moeckli


Motivation
Genus zero & genus one review
higher-genus integration kernels
Higher-genus multiple zeta values
degenerations
identities

Motivation

Higher-genus multiple zeta values

Higher-genus surfaces in theoretical physics

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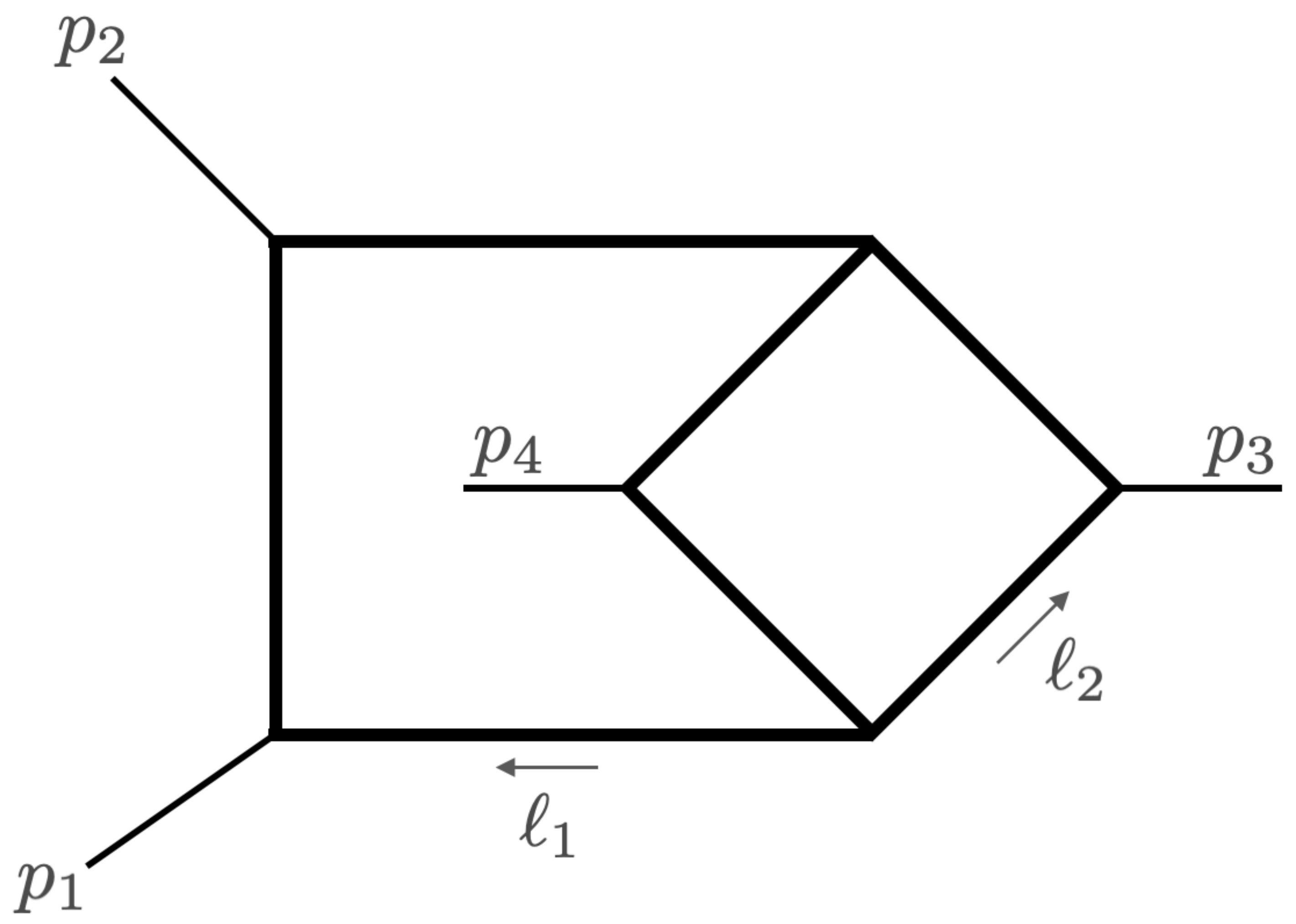

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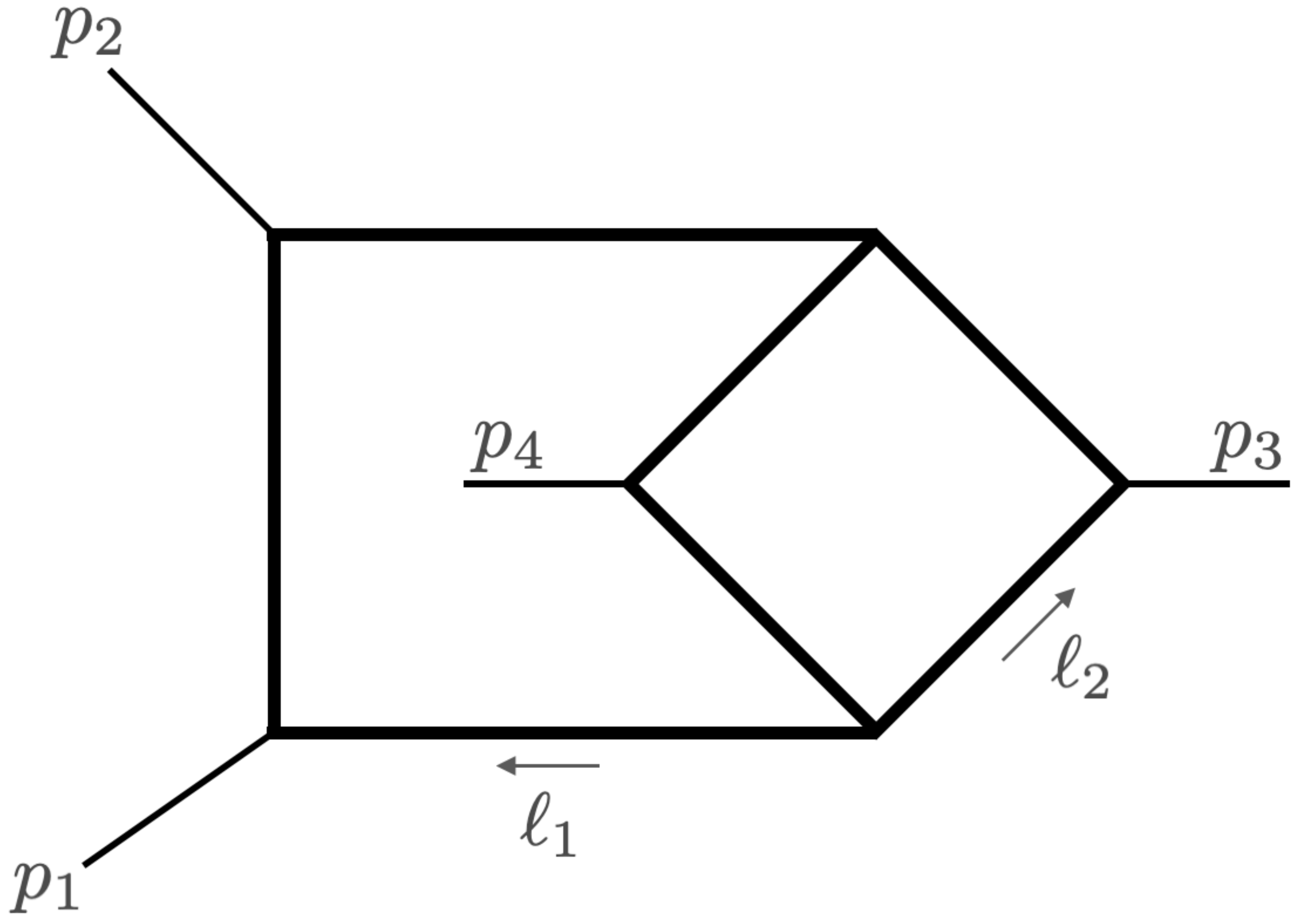
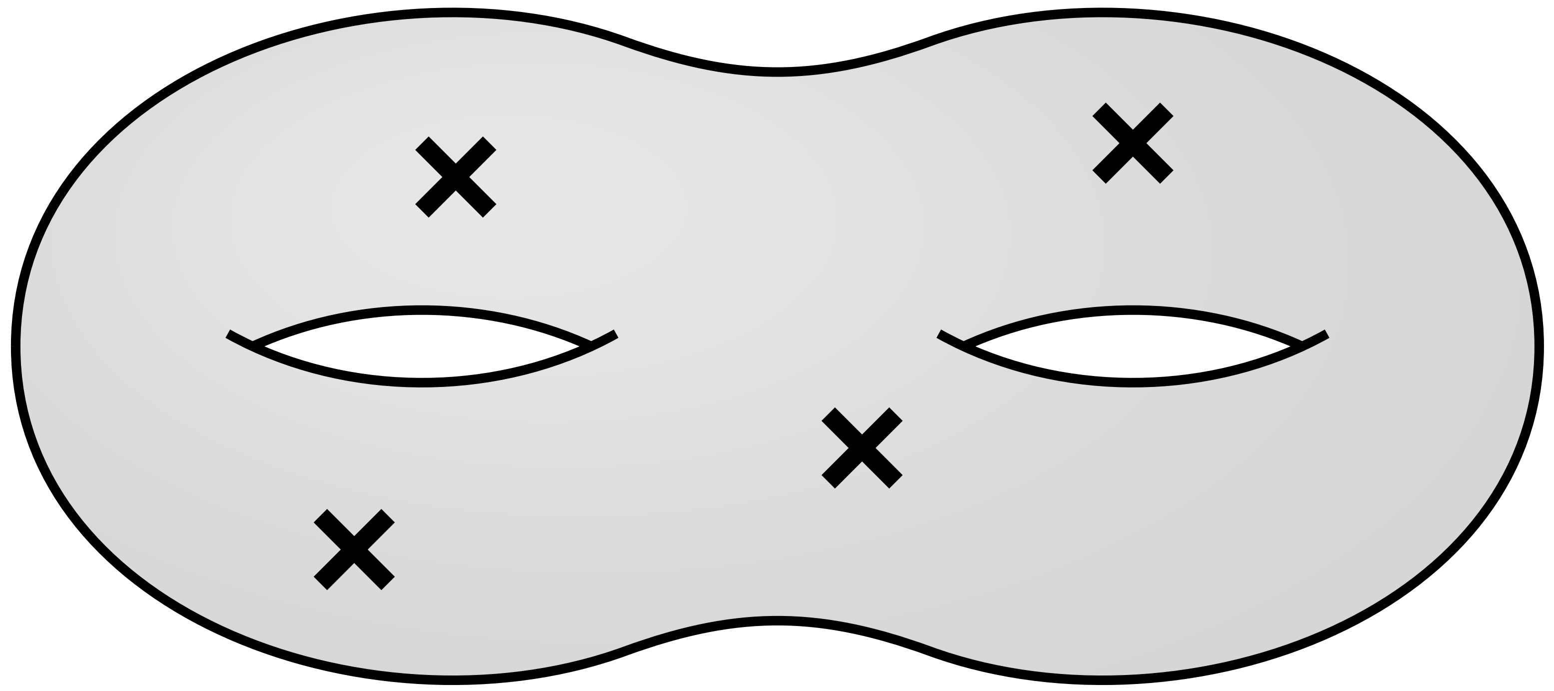


- **String theory:** amplitudes beyond genus one, studied e.g. in [D'Hoker, Schlotterer '25]



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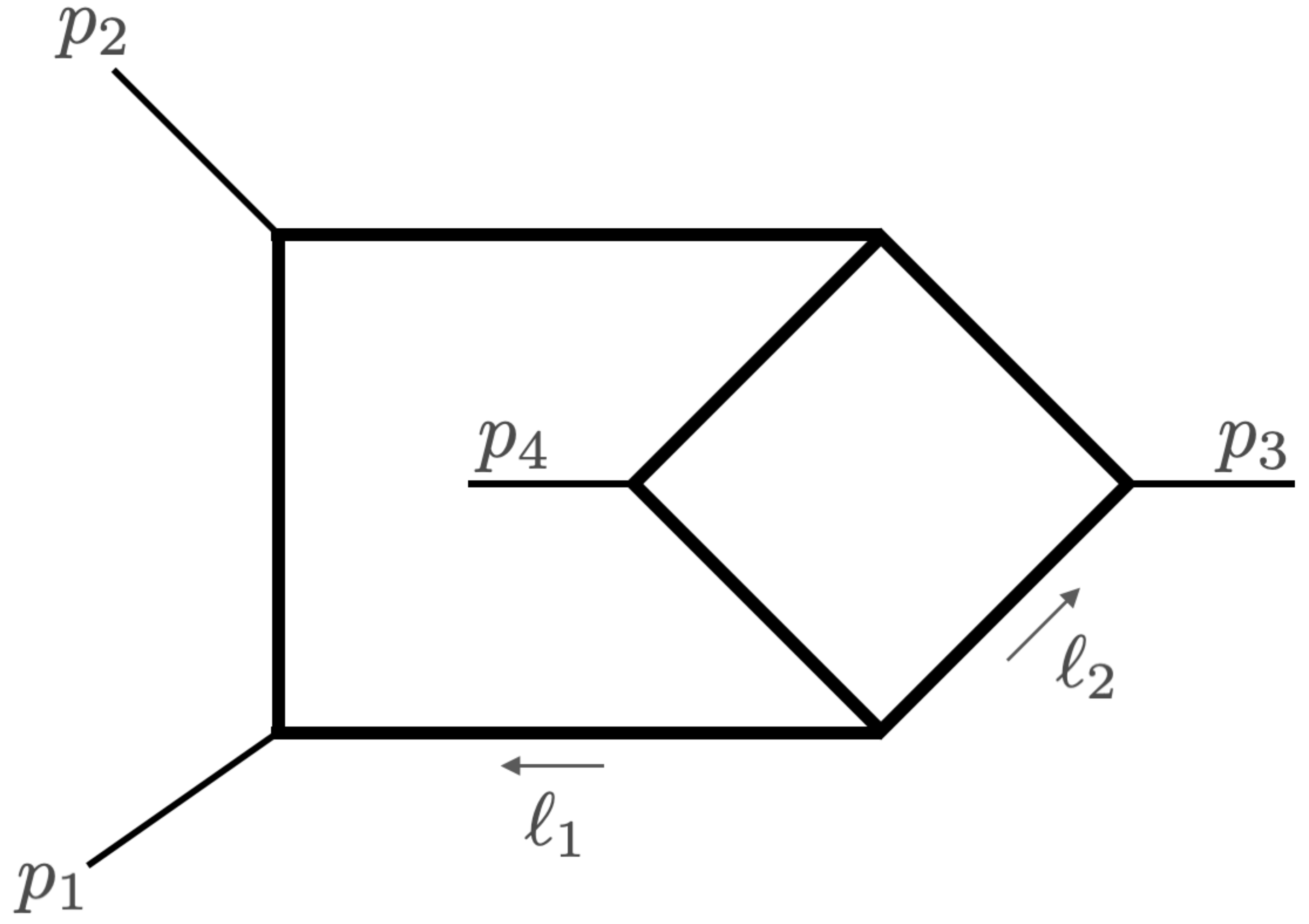

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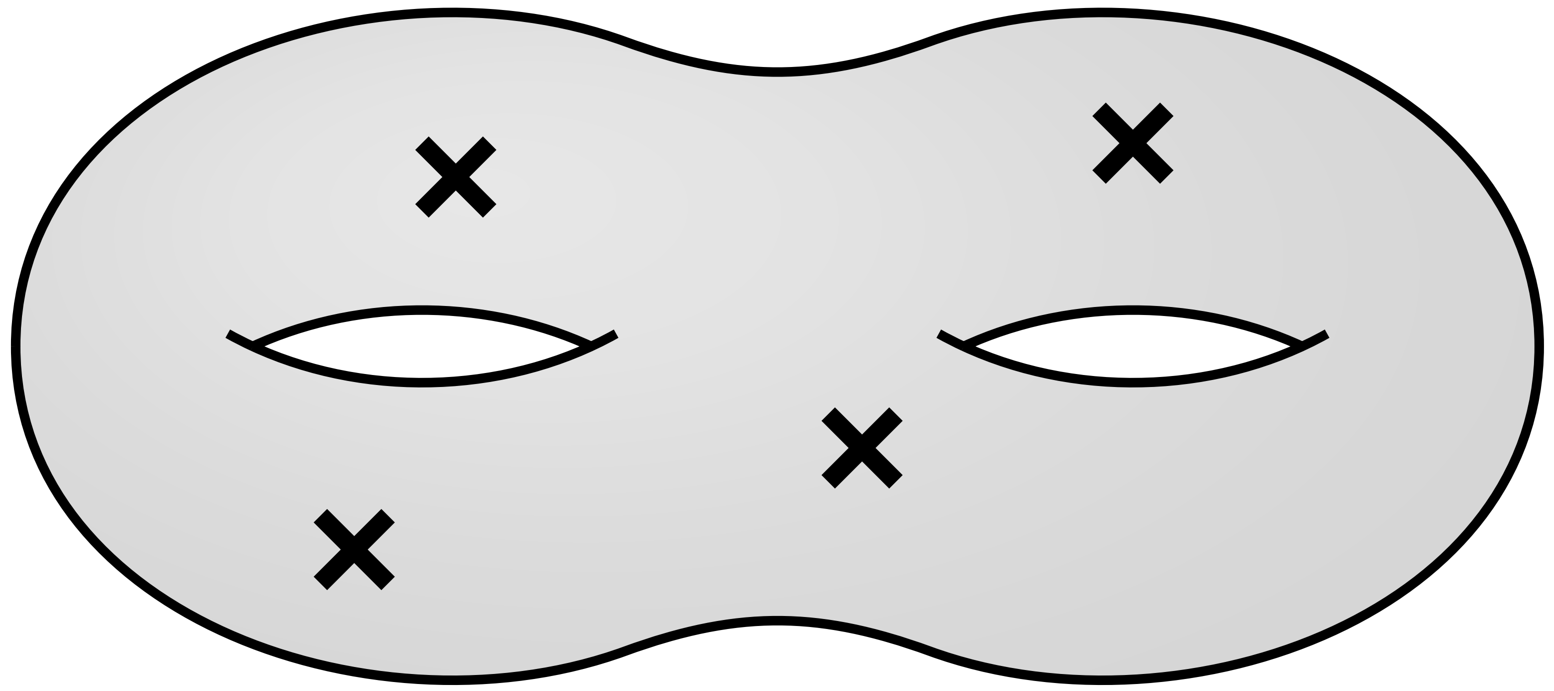

- **Maths:** integration of rational functions on any Riemann surface and their periods
  see e.g. [Enriquez, Zerbini '21&'22]

Motivation

Genus zero &
genus one review

Higher-genus multiple zeta values

Genus zero and one review

Genus zero and one review

Genus zero:

Goncharov polylogarithms:

$$G(a_1, \ldots, a_r \,|\, z) = \int_0^z \omega_{a_1}(t)\, G(a_2, \ldots, a_r \,|\, t), \text{ where } \omega_a(t) = \frac{dt}{t - a}$$

Genus zero and one review

Genus zero: $\left\{ \begin{array}{l} \text{Goncharov polylogarithms:} \\ G(a_1, \ldots, a_r | z) = \int_0^z \omega_{a_1}(t)\, G(a_2, \ldots, a_r | t), \text{ where } \omega_a(t) = \dfrac{dt}{t-a} \\ \\ \text{Multiple zeta values (MZVs):} \\ \zeta(n_1, \ldots, n_k) = \zeta_{0^{n_1-1}1\ldots0^{n_k-1}1} = G(\underbrace{0,\ldots,0}_{n_1-1},1,\ldots,\underbrace{0,\ldots,0}_{n_k-1},1\,|\,1), \end{array} \right.$

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Genus one:

Elliptic multiple polylogarithms (eMPLs):

$$\tilde\Gamma\left(\begin{smallmatrix} n_1 & \cdots & n_r \\ a_1 & \cdots & a_r\end{smallmatrix}\middle|\, z\right)=\int_0^z g^{(n_1)}(t,a_1)\,\tilde\Gamma\left(\begin{smallmatrix} n_2 & \cdots & n_r \\ a_2 & \cdots & a_r\end{smallmatrix}\middle|\, t\right),$$

where $F(z,\alpha\,|\,\tau)dz=\dfrac{\theta'(0\,|\,\tau)\theta(z+\alpha\,|\,\tau)}{\theta(z\,|\,\tau)\theta(\alpha\,|\,\tau)}dz=\dfrac{1}{\alpha}\sum_{n\geq 0}\alpha^n\,g^{(n)}(z\,|\,\tau)$ [Brown, Levin'13]

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Elliptic multiple zeta values (eMZVs): $\omega^{\mathrm{ell}}(n_1, \ldots, n_k) = \tilde{\Gamma}\left(\begin{smallmatrix} n_1 & \cdots & n_r \\ 0 & \cdots & 0 \end{smallmatrix}\middle| 1\right)$, [Enriquez'13]

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identities through double shuffle relations
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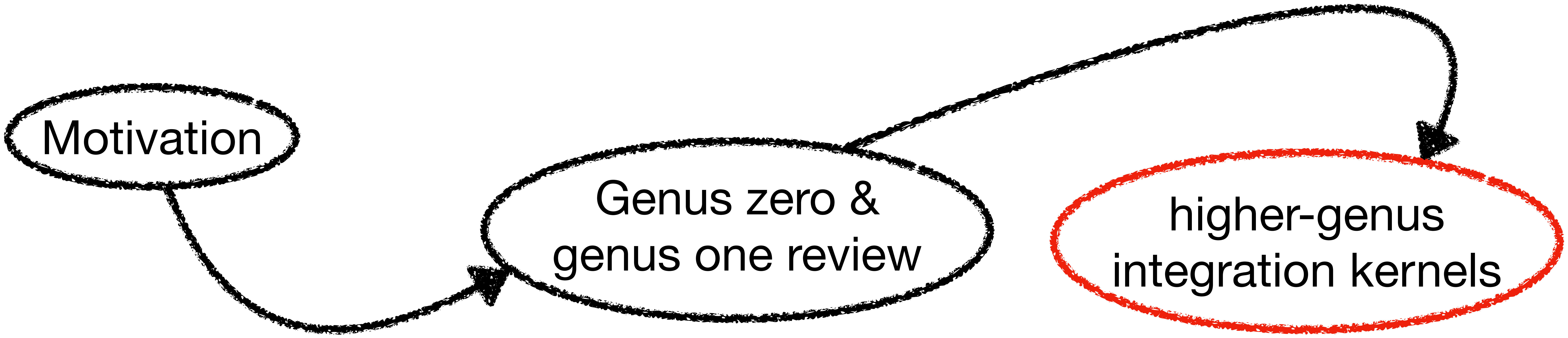
$$\begin{aligned} g^{(n_1)}(t-x)g^{(n_2)}(t) = &-(-1)^{n_1}g^{(n_1+n_2)}(x) \\ &+\sum_{j=0}^{n_2}\binom{n_1-1+j}{j}g^{(n_2-j)}(x)g^{(n_1+j)}(t-x) \\ &+\sum_{j=0}^{n_1}\binom{n_2-1+j}{j}(-1)^{n_1+j}g^{(n_1-j)}(x)g^{(n_2+j)}(t) \end{aligned}$$



Higher-genus multiple zeta values

Higher-genus integration kernels

- **Enriquez (meromorphic) kernels** from connection form [Enriquez'14]

$$K(z,x) = K_j(z,x)\, a_j = \sum_{r=0}^{\infty} \omega_{i_1\cdots i_r j}(z,x)\;\; b_{i_1}\cdots b_{i_r} a_j$$

with properties

- $K(z+\mathfrak{B}_i, x) = e^{b_i} K(z,x)$
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- functions in x
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- **Translation** between meromorphic and single-valued connection/kernels [D'Hoker, Enriquez, Schlotterer, Zerbini '25]

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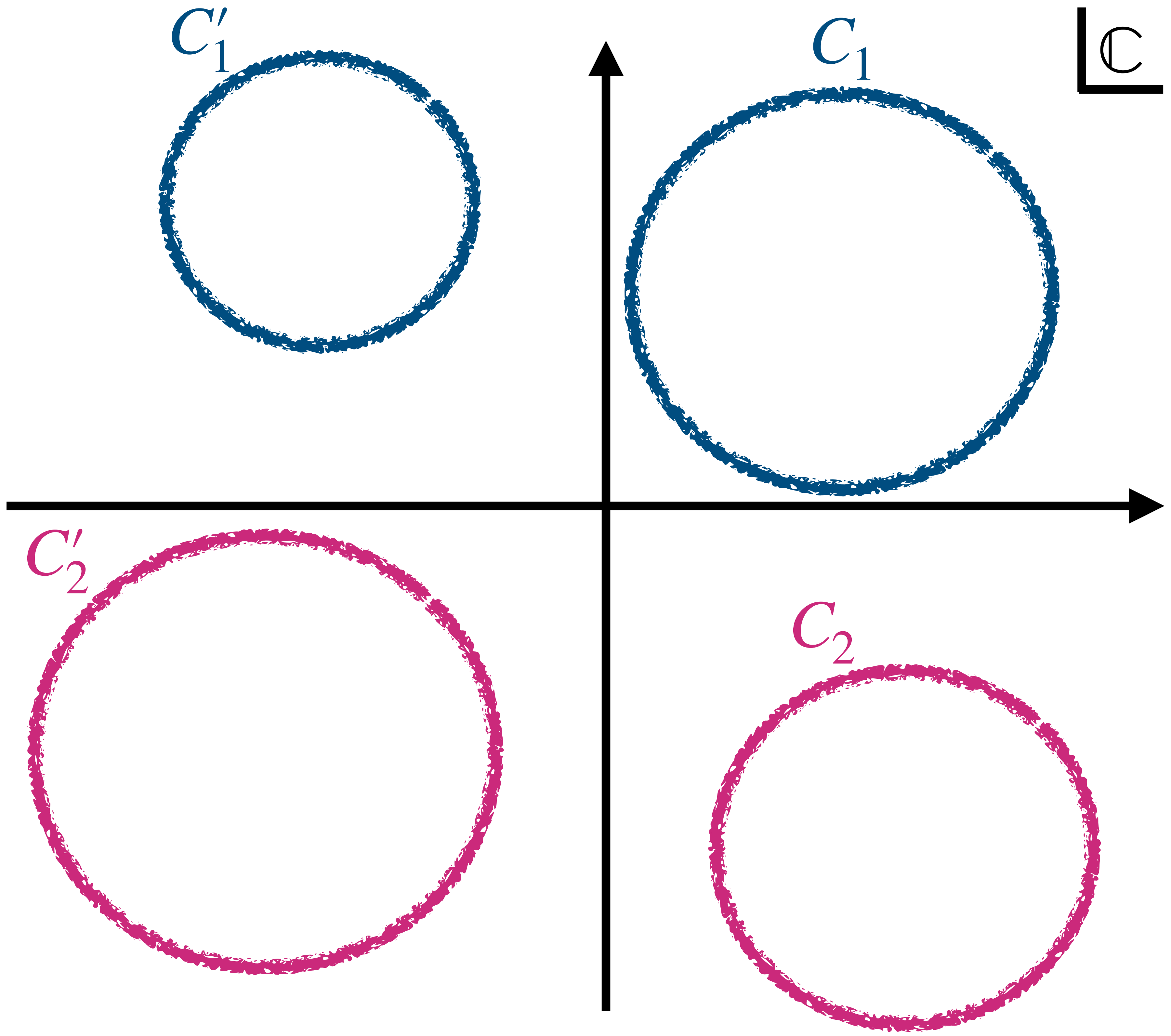
- **Fay identities** (quadratic 3-point identities of integration kernels) [D'Hoker, Schlotterer '24][KB, Broedel, Im, Lisitsyn, Moeckli '24]

- Can e.g. be computed through **convolution integrals** [D'Hoker, Schlotterer '25]

Higher-genus integration kernels

- Enriquez meromorphic kernels on the **Schottky cover** [KB, Broedel, Im, Lisitsyn, Zerbini '24]



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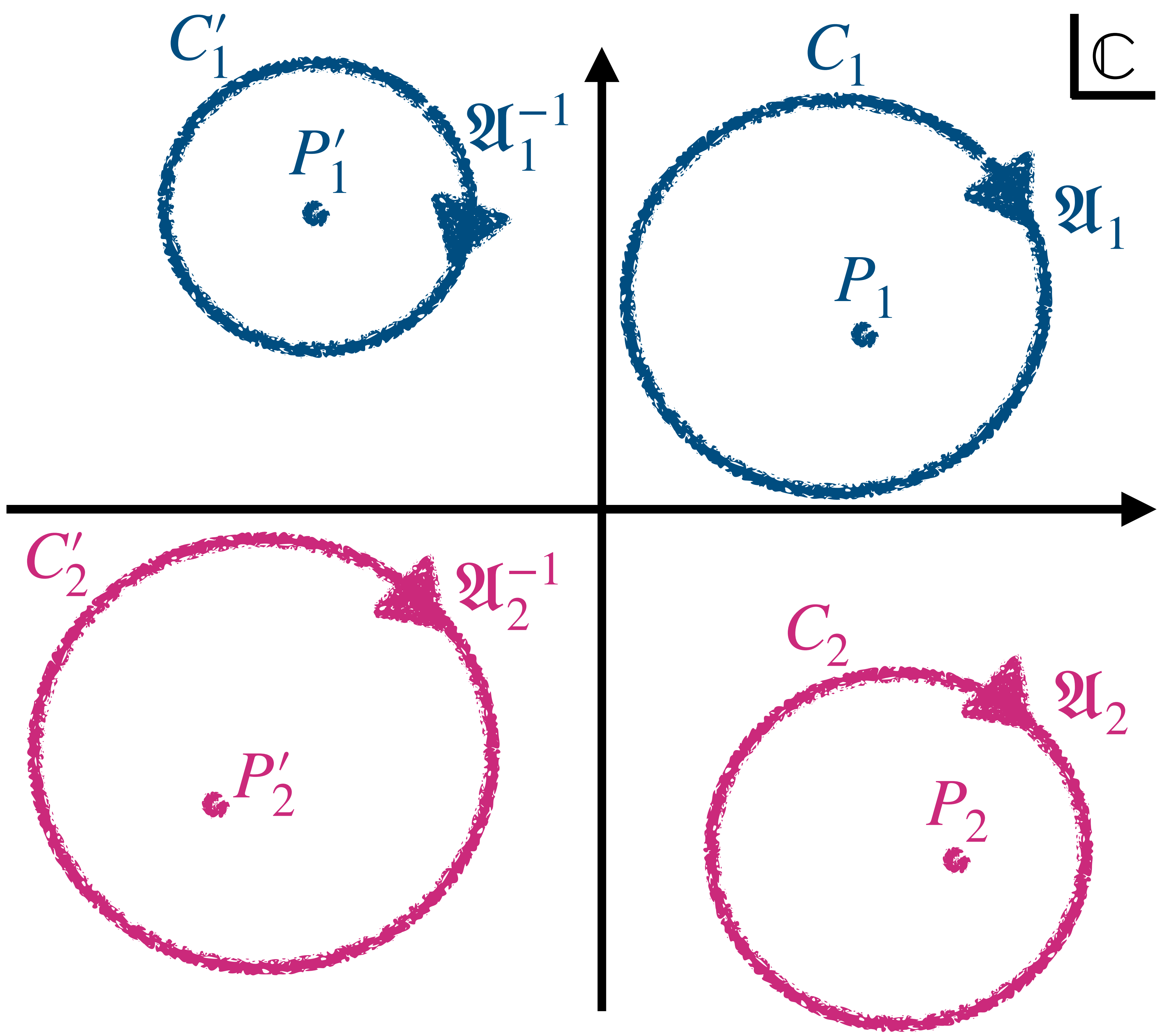

- σ_i: Möbius transformations ($i = 1,\ldots,h$); mapping outside of C_i to inside of C_i'
- P_i, P_i': fixed points of σ_i
- fundamental domain outside of the circles
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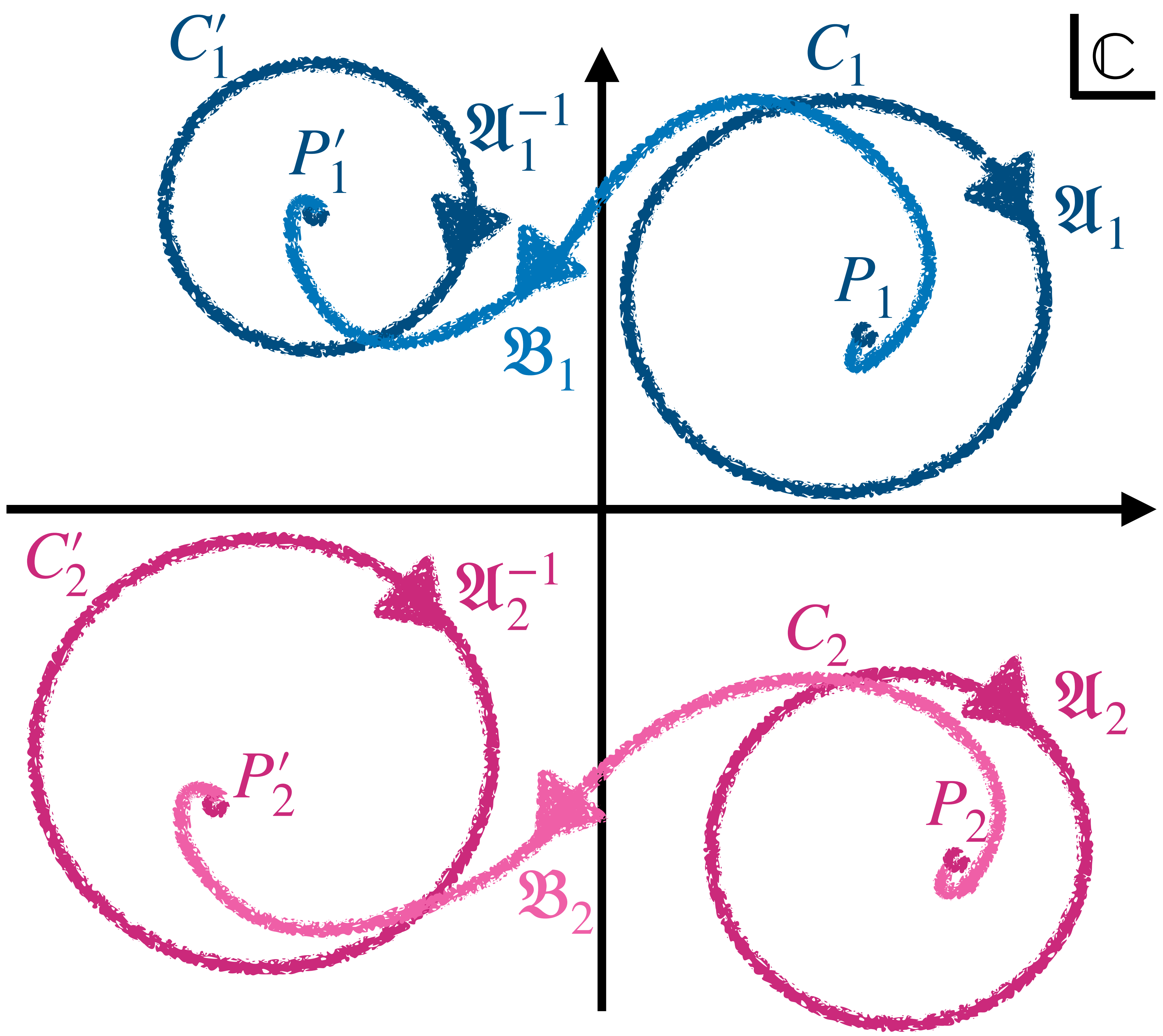

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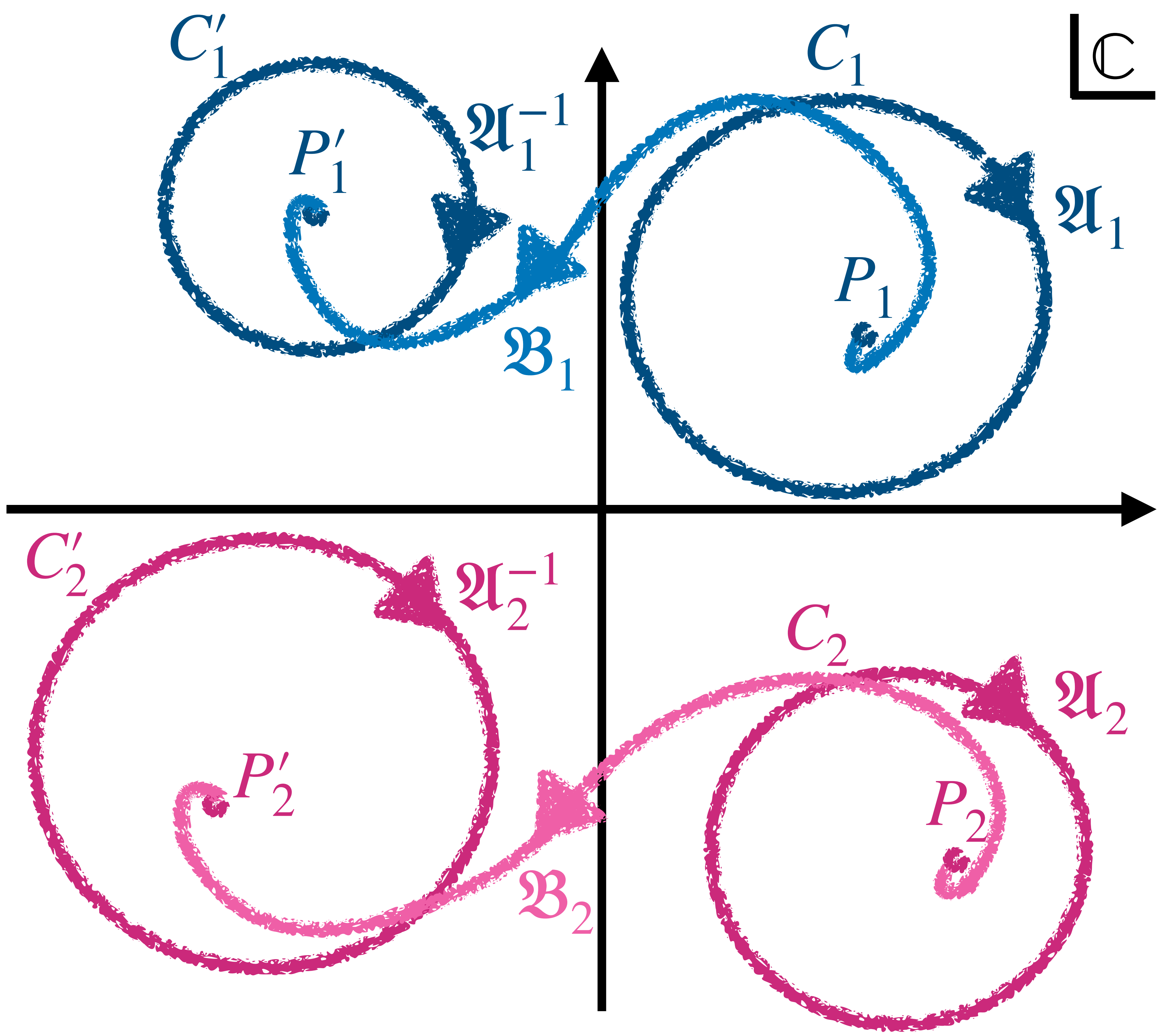

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holom. diffs.: $\omega_i(z) = \dfrac{1}{2\pi i} \displaystyle\sum_{\gamma \in G/\langle \sigma_i \rangle} \left(\frac{1}{z - \gamma P_i'} - \frac{1}{z - \gamma P_i} \right) dz$

period matrix: $\tau_{ij} = \displaystyle\int_{\mathfrak{B}_j} \omega_i = \frac{\delta_{ij}}{2\pi i} \log \lambda_i + \frac{1}{2\pi i} \sum_{\substack{\gamma \in \langle \sigma_j \rangle \backslash G / \langle \sigma_i \rangle \\ i = j: \gamma \neq \mathrm{id}}} \log \frac{(P_j' - \gamma P_i')(P_j - \gamma P_i)}{(P_j' - \gamma P_i)(P_j - \gamma P_i')}$

Higher-genus integration kernels

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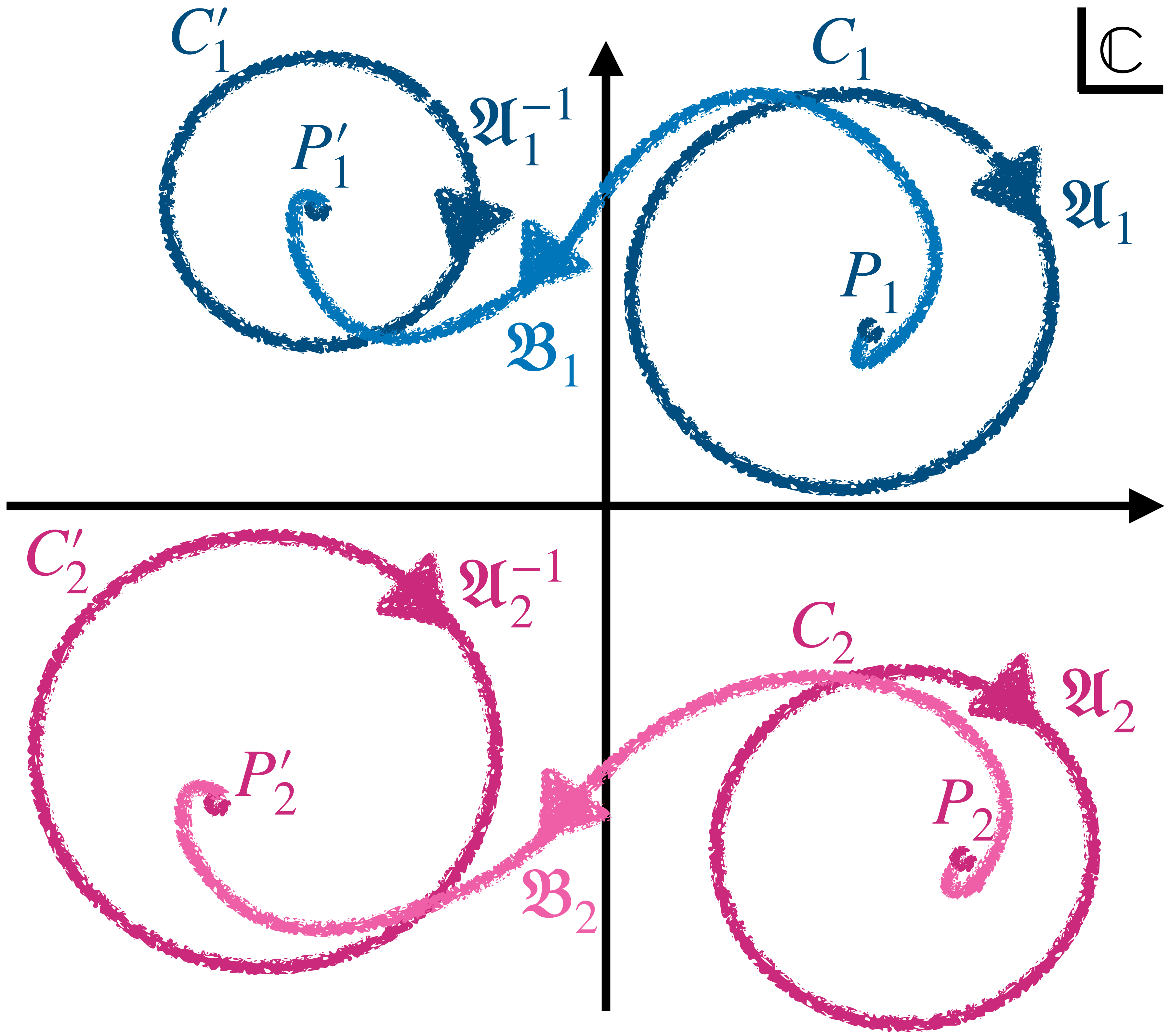

$$\omega_{i_1\cdots i_s j}(z,x) = \frac{1}{(-2\pi \mathring{i})} \sum_{\gamma \in G/\langle \sigma_j \rangle} \sum_{k=0}^{\delta_{j i_s} n_s} C\left(b_{i_1}^{n_1} \cdots b_{i_s}^{n_s - k}, \gamma\right) (-2\pi \mathring{i})^{1-k} g^{(k)}(\gamma^{-1} z, x \,|\, \tau_j)\, \omega(\gamma^{-1} z \,|\, \tau_j)$$

Higher-genus integration kernels

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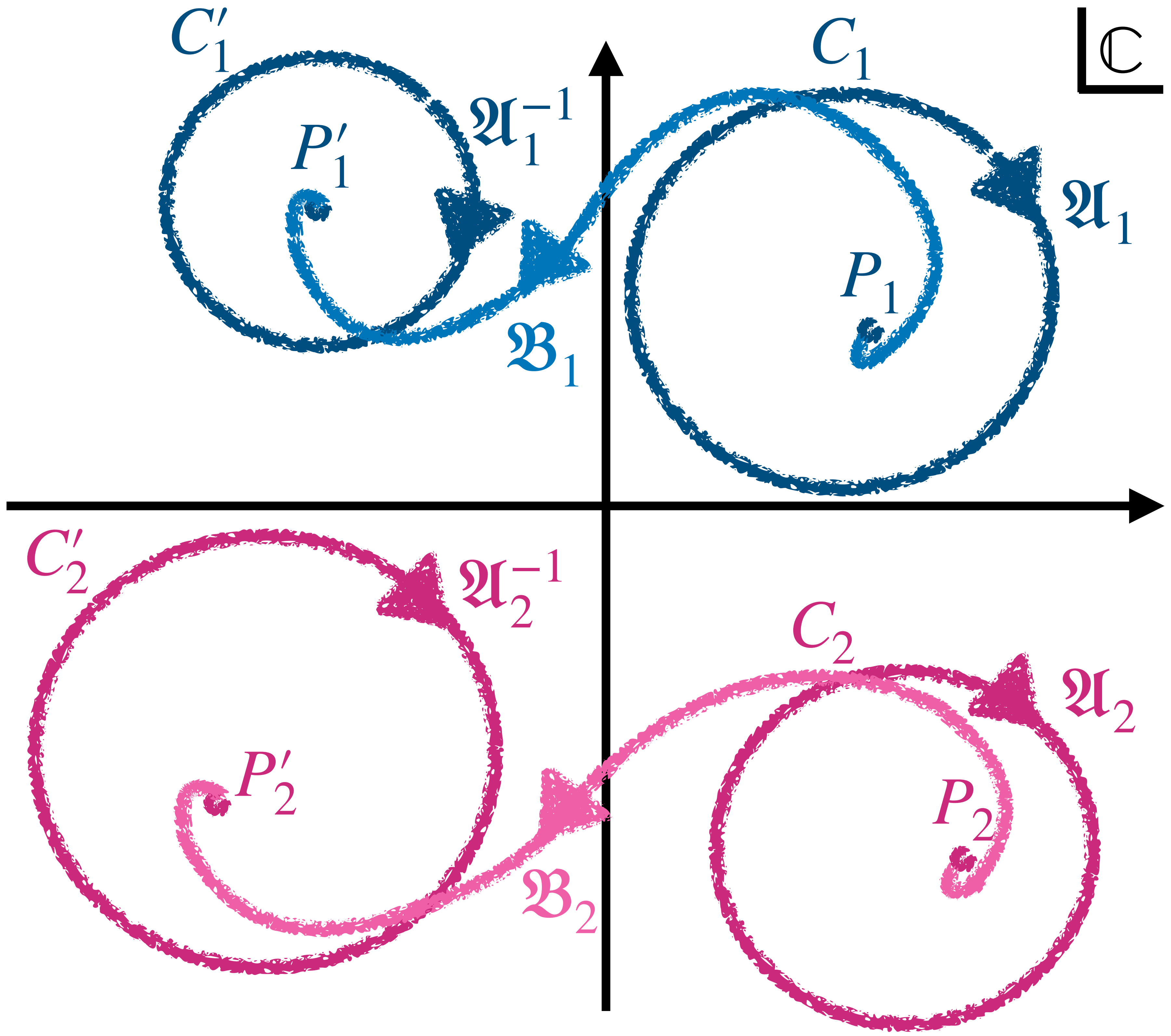
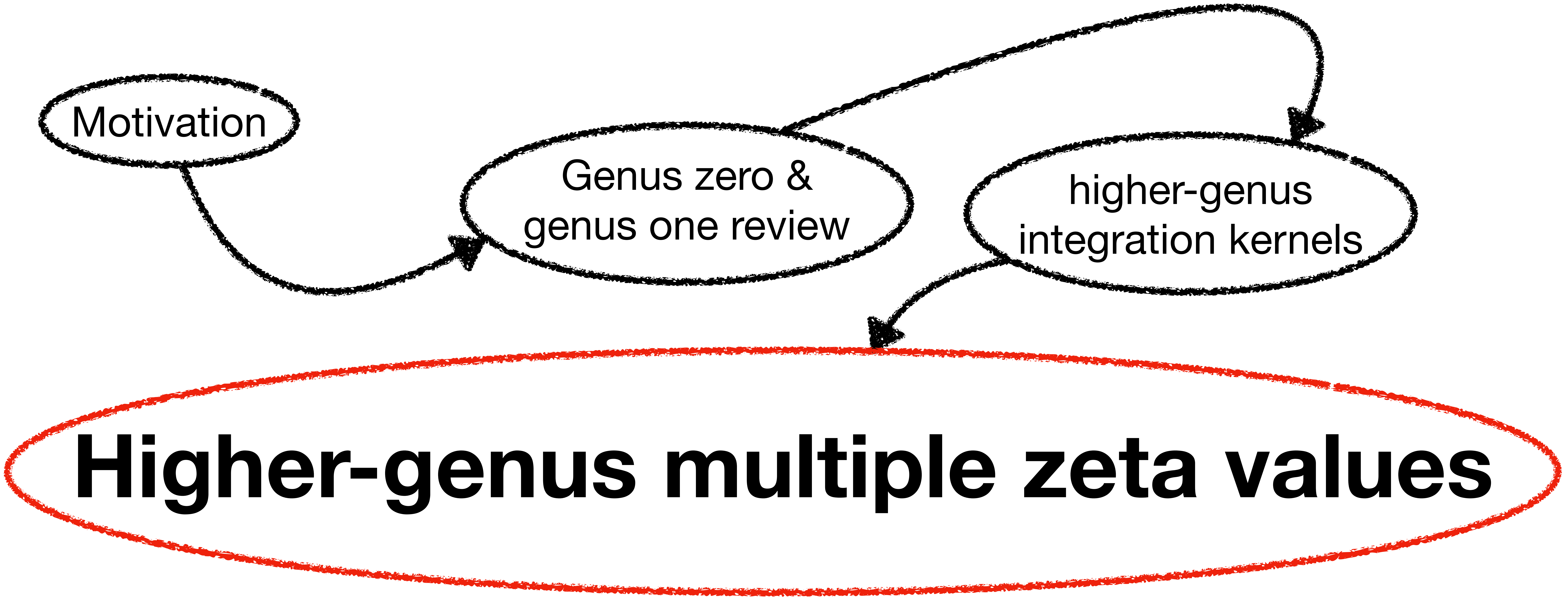


rational factors

$$\omega_{i_1\cdots i_s j}(z,x) = \frac{1}{(-2\pi \mathring{i})} \sum_{\gamma\in G/\langle\sigma_j\rangle} \sum_{k=0}^{\delta_{ji_s}n_s} C\big(b_{i_1}^{n_1}\cdots b_{i_s}^{n_s-k},\gamma\big)\,(-2\pi\mathring{i})^{1-k}\, g^{(k)}(\gamma^{-1}z,x\,|\,\tau_j)\,\omega(\gamma^{-1}z\,|\,\tau_j)$$

$\Longrightarrow$ expansion into **genus-one kernels**

$\Longrightarrow$ allows **numerical** evaluation and integration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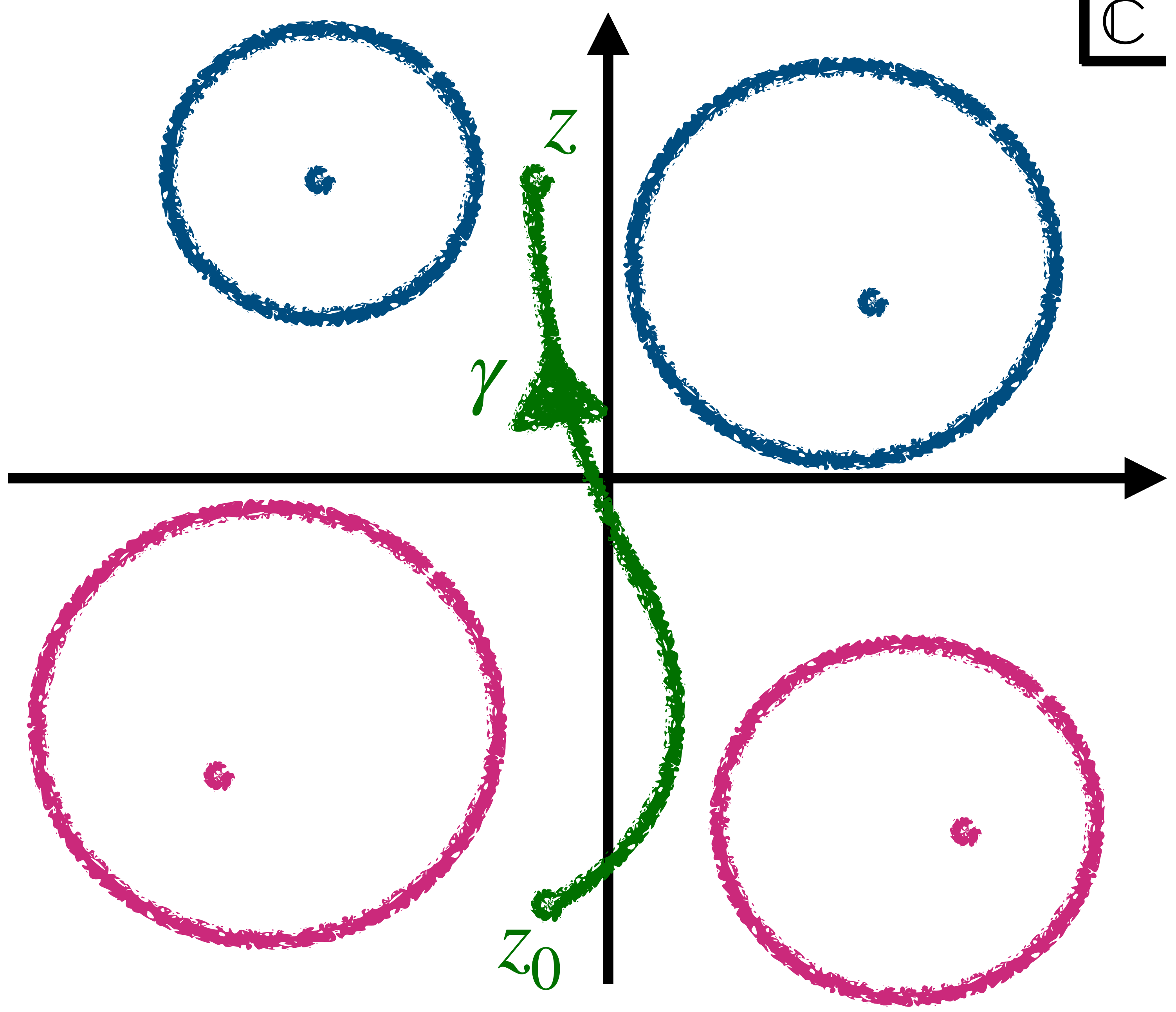
Higher-genus polylogarithms

Definition: $\tilde{\Gamma}\left(\begin{smallmatrix}\vec{\imath}_1 & \cdots & \vec{\imath}_k \\ x_1 & \cdots & x_k\end{smallmatrix}\middle| z_0, z\right) = \int_\gamma \omega_{\vec{\imath}_1} \circ \cdots \circ \omega_{\vec{\imath}_k} = \int_{t=z_0}^{z} \omega_{\vec{\imath}_1}(t, x_1)\, \tilde{\Gamma}\left(\begin{smallmatrix}\vec{\imath}_2 & \cdots & \vec{\imath}_k \\ x_2 & \cdots & x_k\end{smallmatrix}\middle| z_0, t\right)$

- Iterated integrals over Enriquez higher-genus integration kernels
- homotopy invariant integrals
- γ: path on surface from foot point z_0 to z



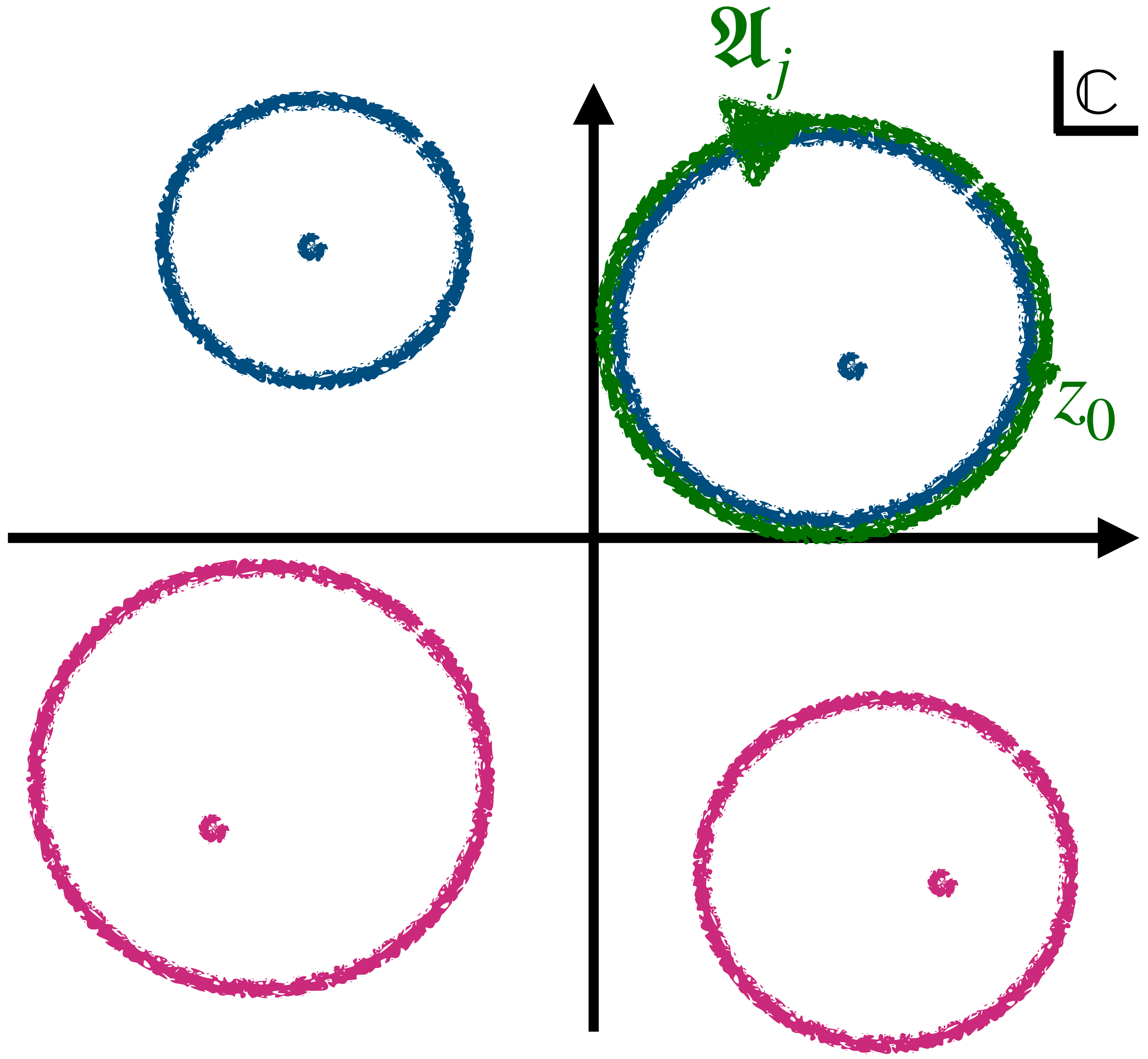
Higher-genus multiple zeta values

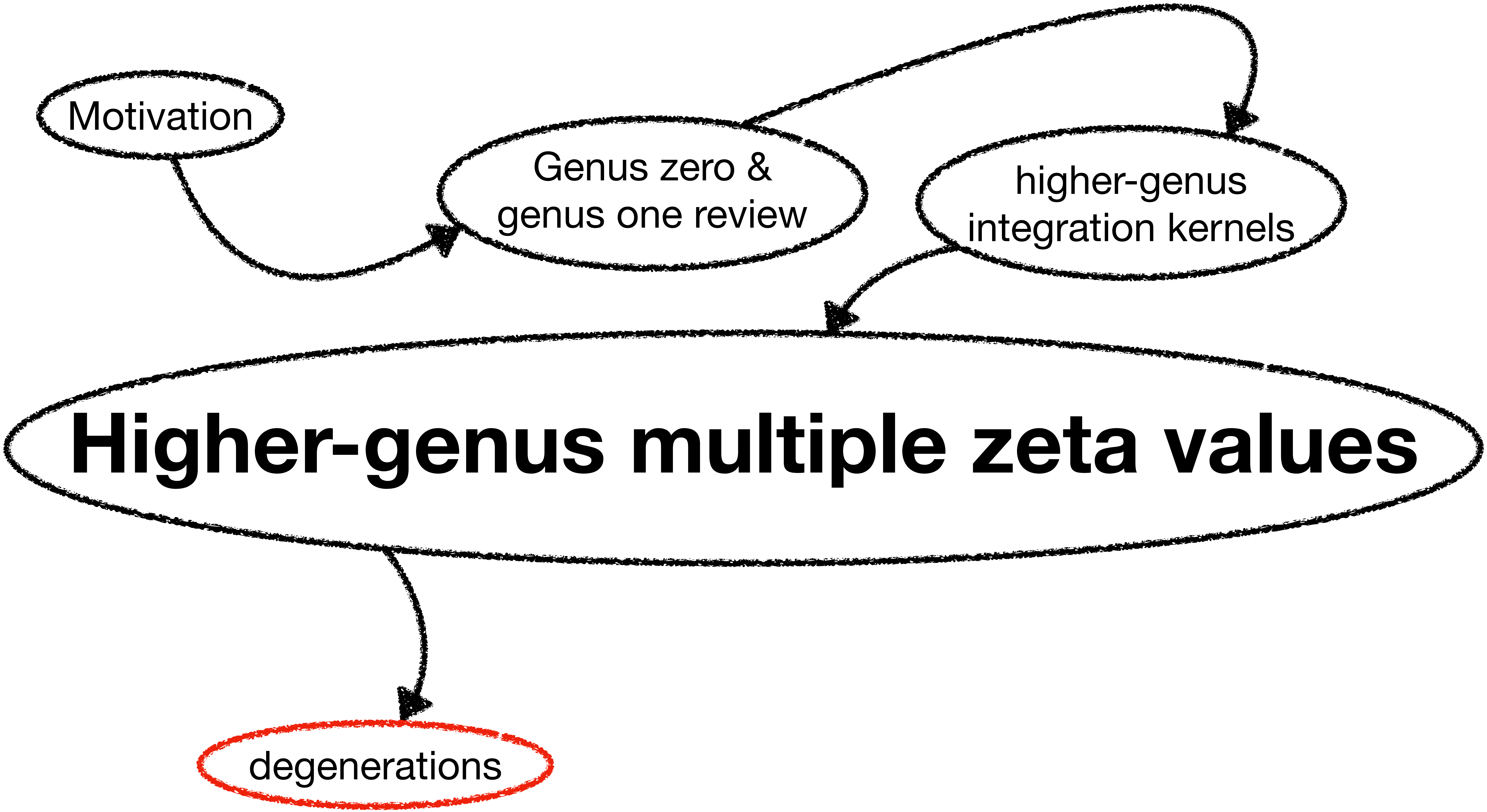
Definition: $^{[h]}\zeta_{\mathfrak{A}_j}(\vec{\imath}_1, \ldots, \vec{\imath}_k) = \int_{\mathfrak{A}_j} \omega_{\vec{\imath}_1} \circ \cdots \circ \omega_{\vec{\imath}_k} = \tilde{\Gamma}\left(\begin{matrix} \vec{\imath}_1 & \cdots & \vec{\imath}_k \\ z_0 & \cdots & z_0 \end{matrix} \middle| z_0, z_0 + \mathfrak{A}_j\right)$

integration over
the j-th $\mathfrak{A}$-cycle

z_0 is a point on $\mathfrak{A}_j$

- functions of the geometry of the surface

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degenerations

Degenerations of hgMZVs

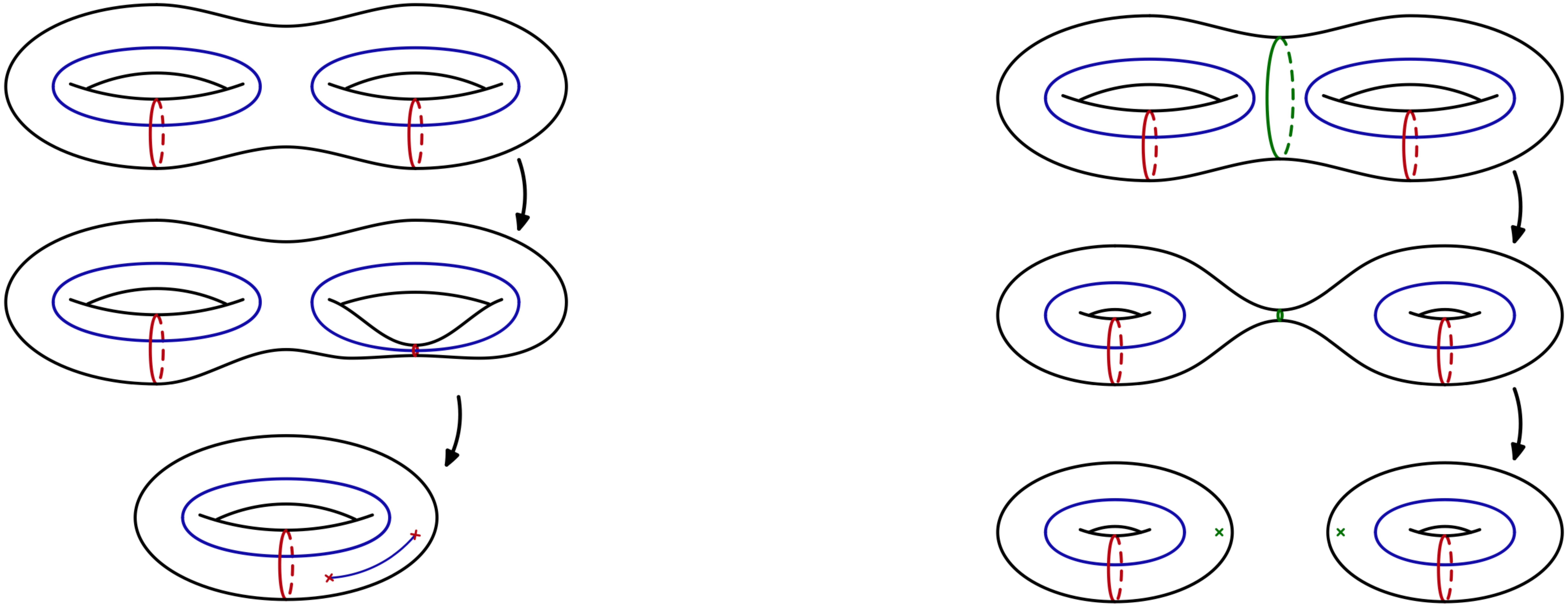
How do hgMZVs behave under degenerations of the Riemann surface?

Can we recover eMZVs and hgMZVs from degenerating hgMZVs?

Degenerations of hgMZVs

Can study the usual two types of degenerations:

Non-separating degeneration **Separating degeneration**



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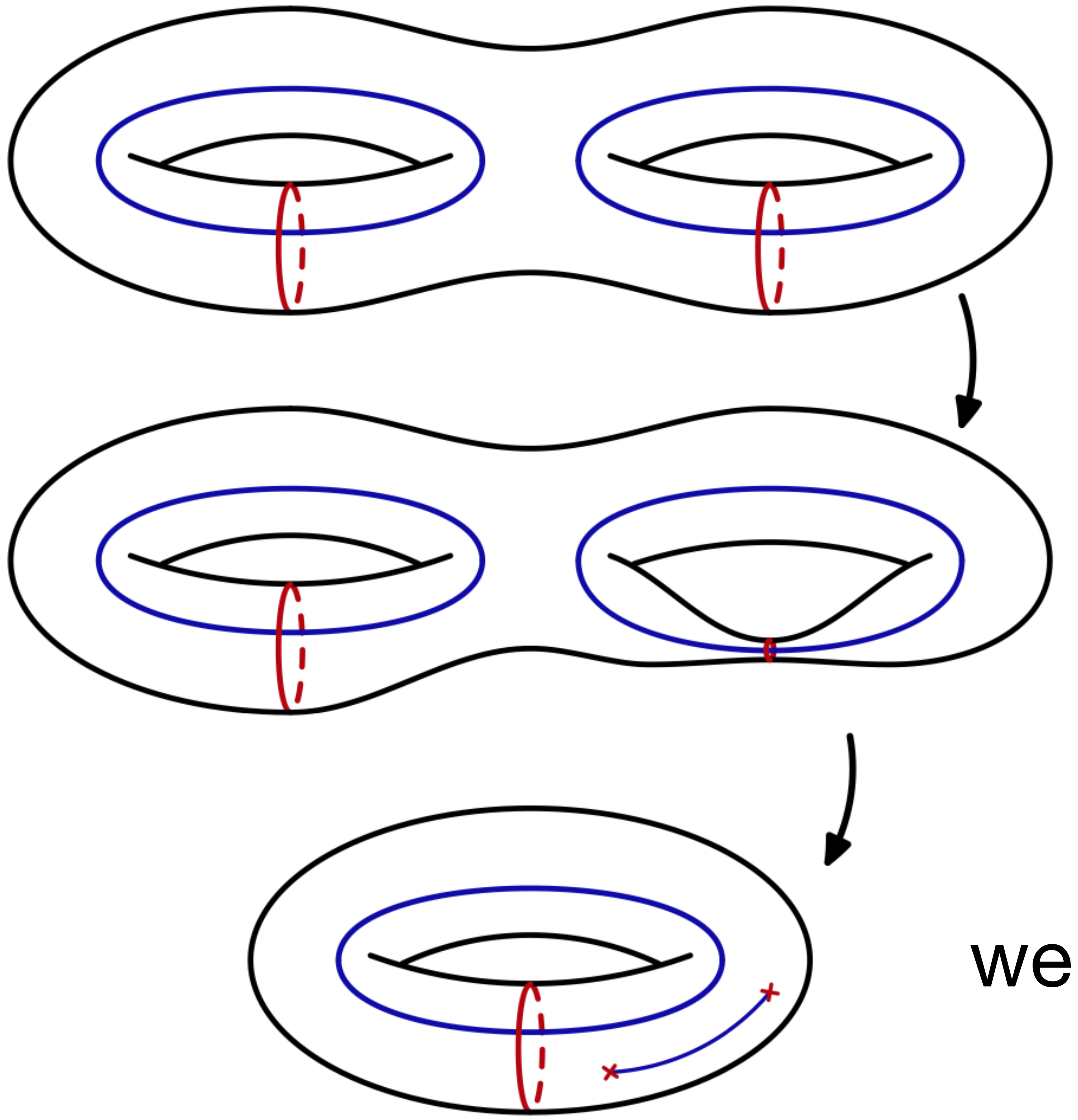
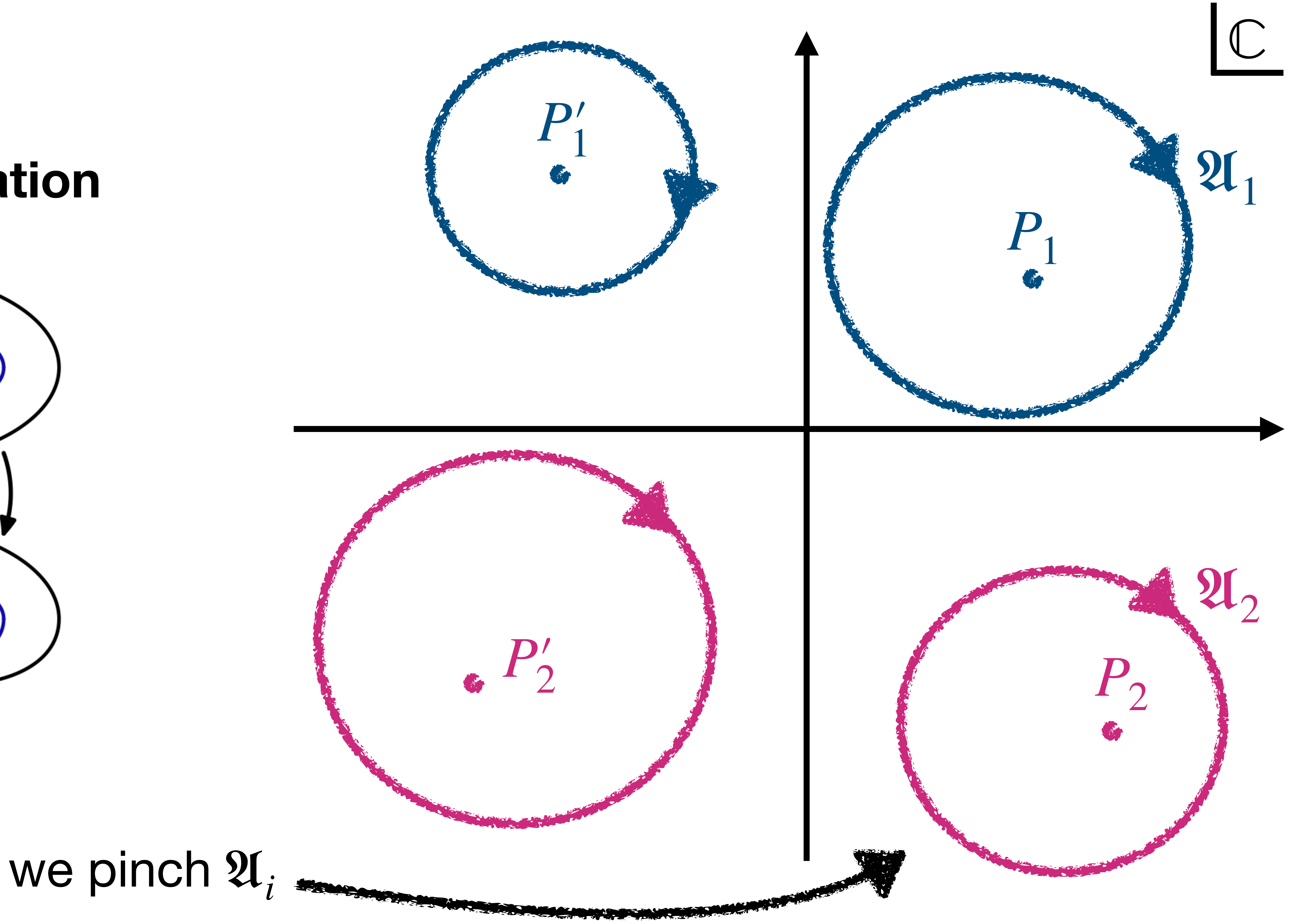
Non-separating degeneration





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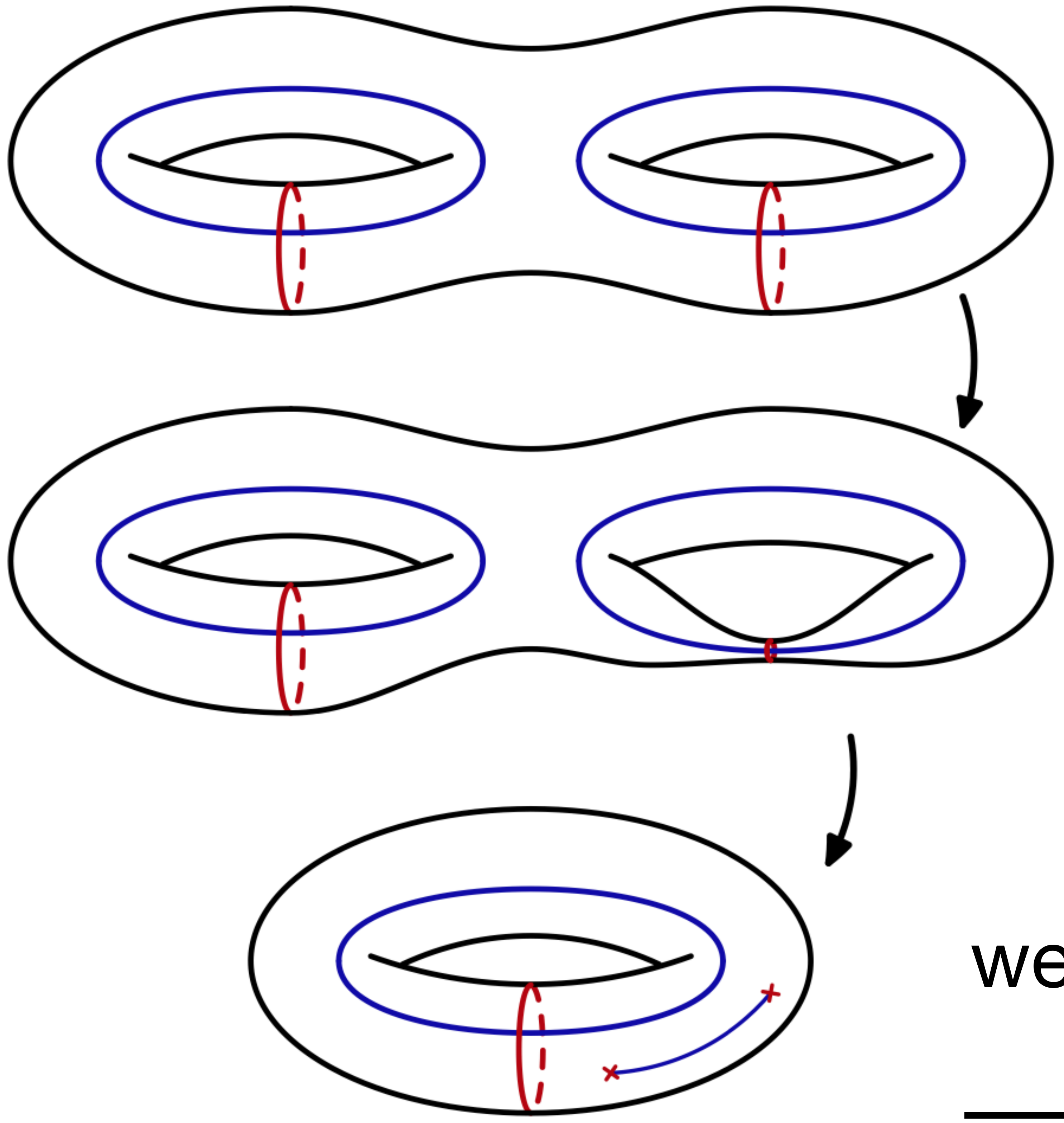
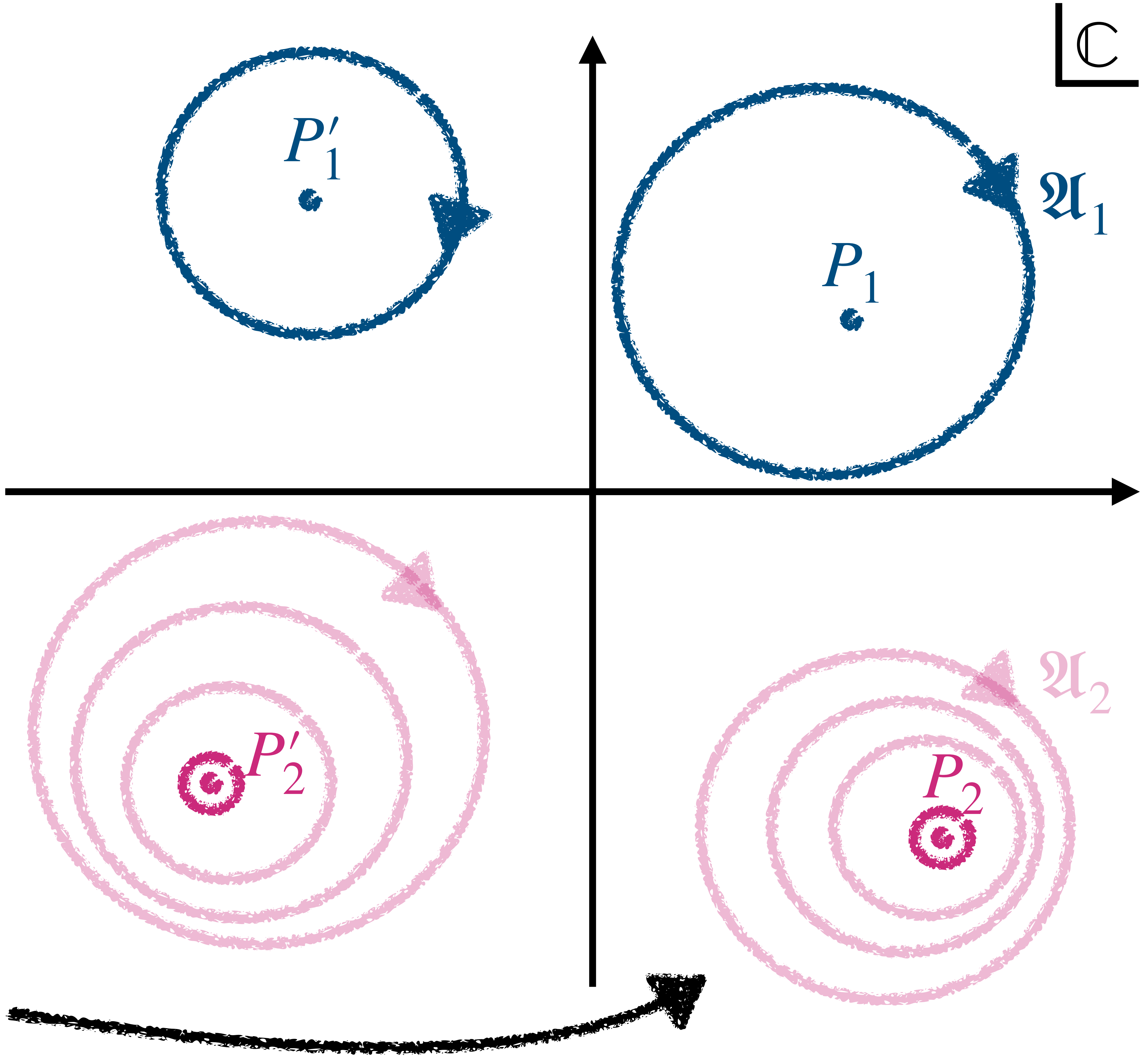




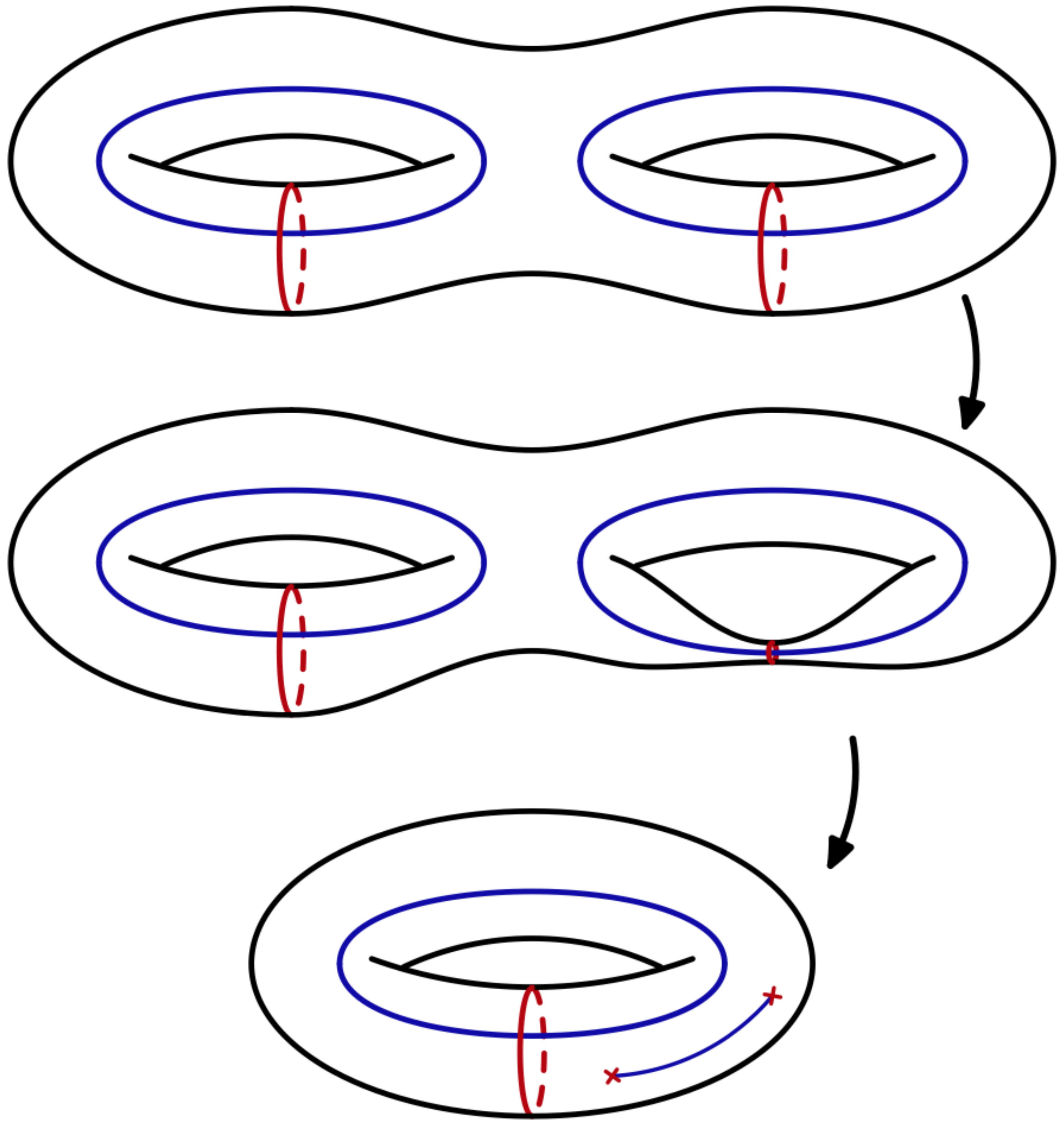
we pinch $\mathfrak{A}_i$

$\Longrightarrow$ σ_i becomes singular and decouples from other generators

$\Longrightarrow$ ${}^{[h]}K_{j\neq i} \xrightarrow{\mathfrak{A}_i \to 0} {}^{[h-1]}K_{j\neq i}$

Degenerations of hgMZVs

Non-separating degeneration



$$j \neq i\colon\ {}^{[h]}\omega_{i_1\cdots i_r j} \xrightarrow{\mathfrak{A}_i \to 0} \begin{cases} {}^{[h-1]}\omega_{i_1\cdots i_r j}, & i_k \neq i \\ 0, & \text{else} \end{cases}$$

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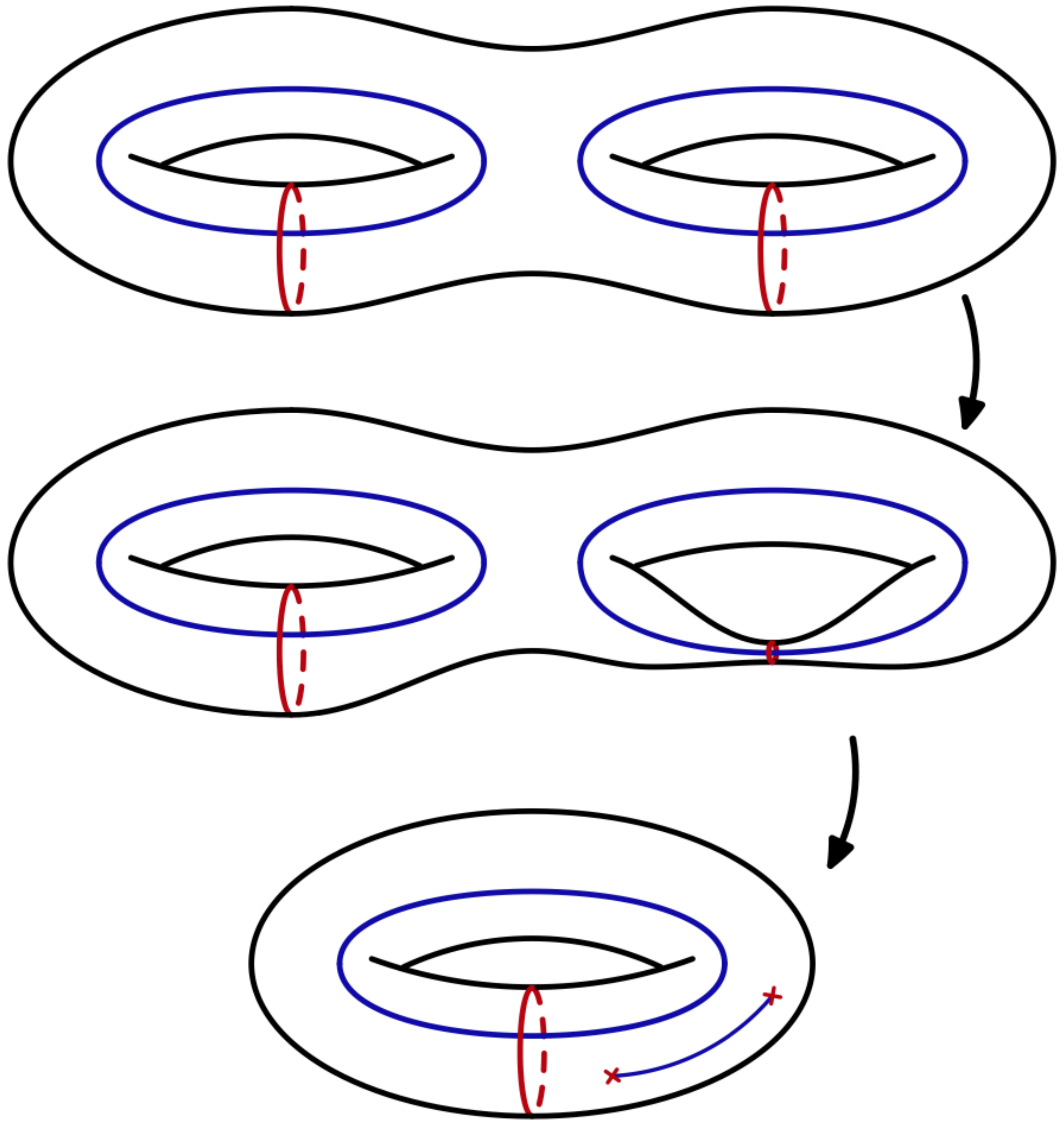

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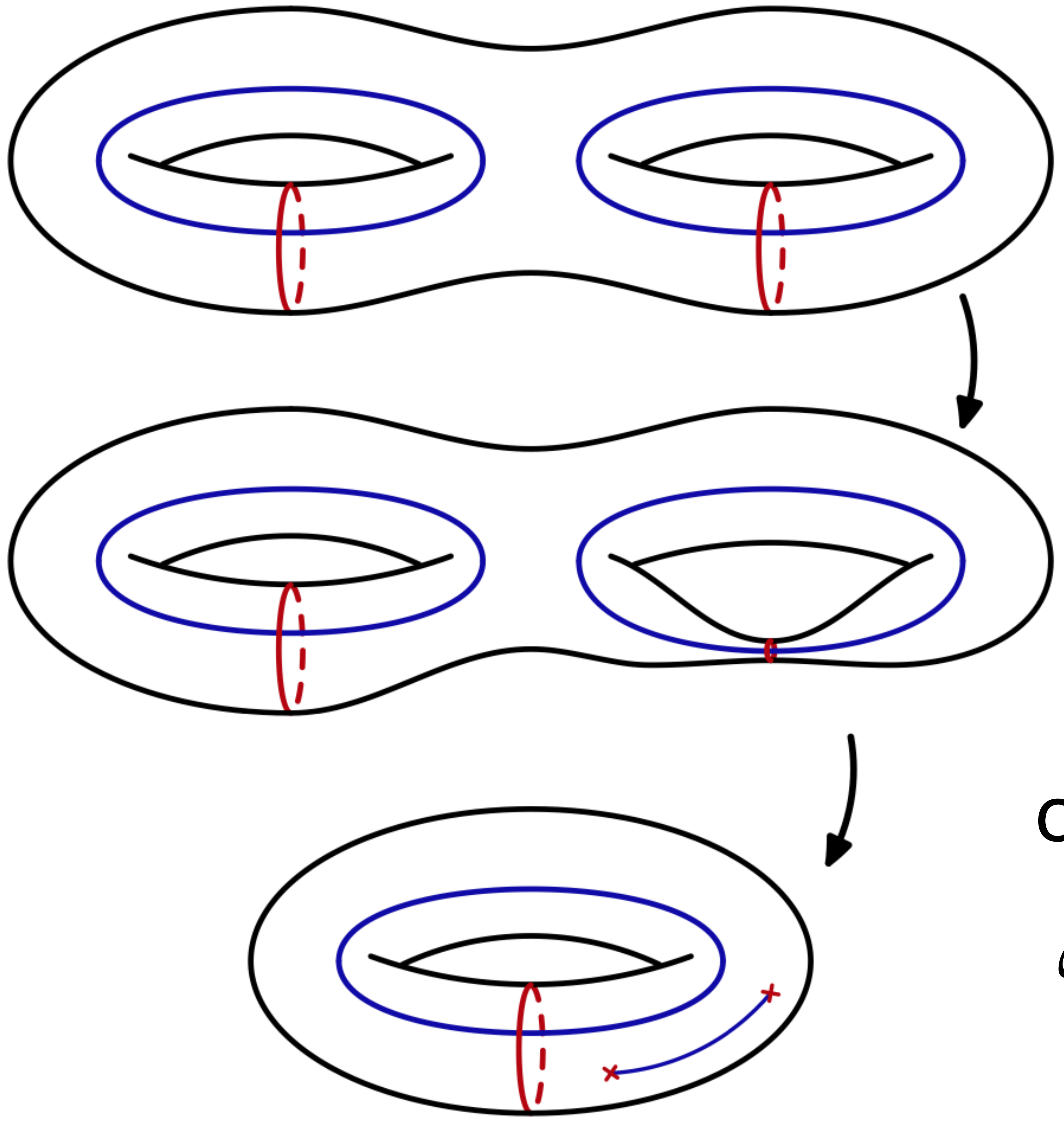
$\Longrightarrow$ hgMZVs **reduce to a genus lower** or vanish ($j_l \neq i$):

$${}^{[h]}\zeta_{\mathfrak{A}_j}(\vec{i}_1 j_1, \ldots, \vec{i}_k j_k) \xrightarrow{\mathfrak{A}_i \to 0} \begin{cases} {}^{[h-1]}\zeta_{\mathfrak{A}_j}(\vec{i}_1 j_1, \ldots, \vec{i}_k j_k), & i \notin \vec{i}_l \\ 0, & \text{else} \end{cases}$$

Degenerations of hgMZVs

Non-separating degeneration



$$j \neq i: \ {}^{[h]}\omega_{i_1\cdots i_r j} \xrightarrow{\mathfrak{A}_i \to 0} \begin{cases} {}^{[h-1]}\omega_{i_1\cdots i_r j}, & i_k \neq i \\ 0, & \text{else} \end{cases}$$

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can **recover all eMZVs** from hgMZVs through degeneration:

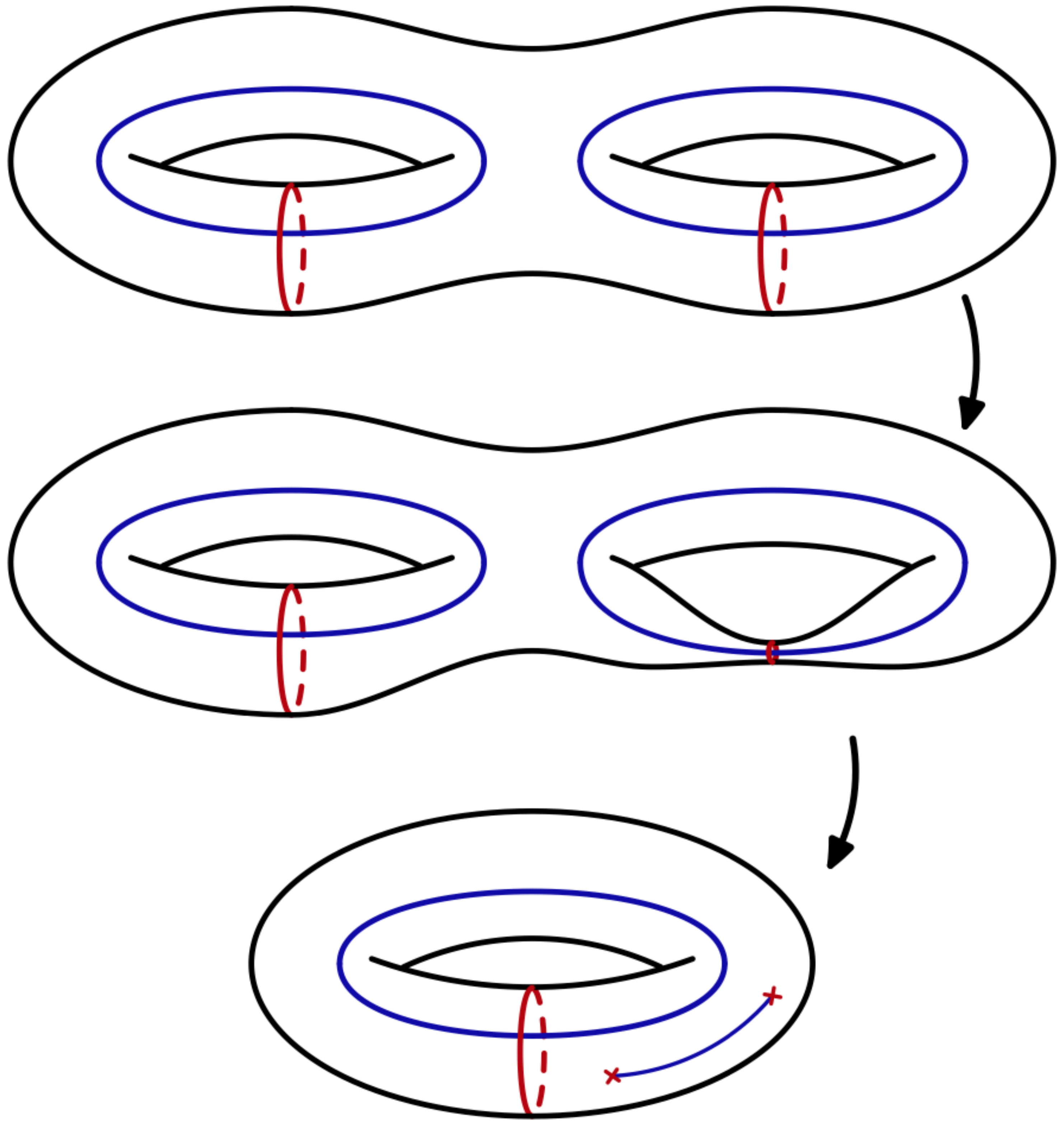
$$\zeta_{\mathfrak{A}_j}(\vec{i}_1, \ldots, \vec{i}_k) \xrightarrow{\{\mathfrak{A}_i \to 0\}_{i \neq j}} (-2\pi \mathring{\imath})^{-\sum_{m=1}^{k} |\vec{i}_m|} \delta_{\vec{i}_1, \ldots, \vec{i}_k, j} \ \omega^{\text{ell}}(|\vec{i}_1|, \ldots, |\vec{i}_k| \,|\, \tau_j)$$

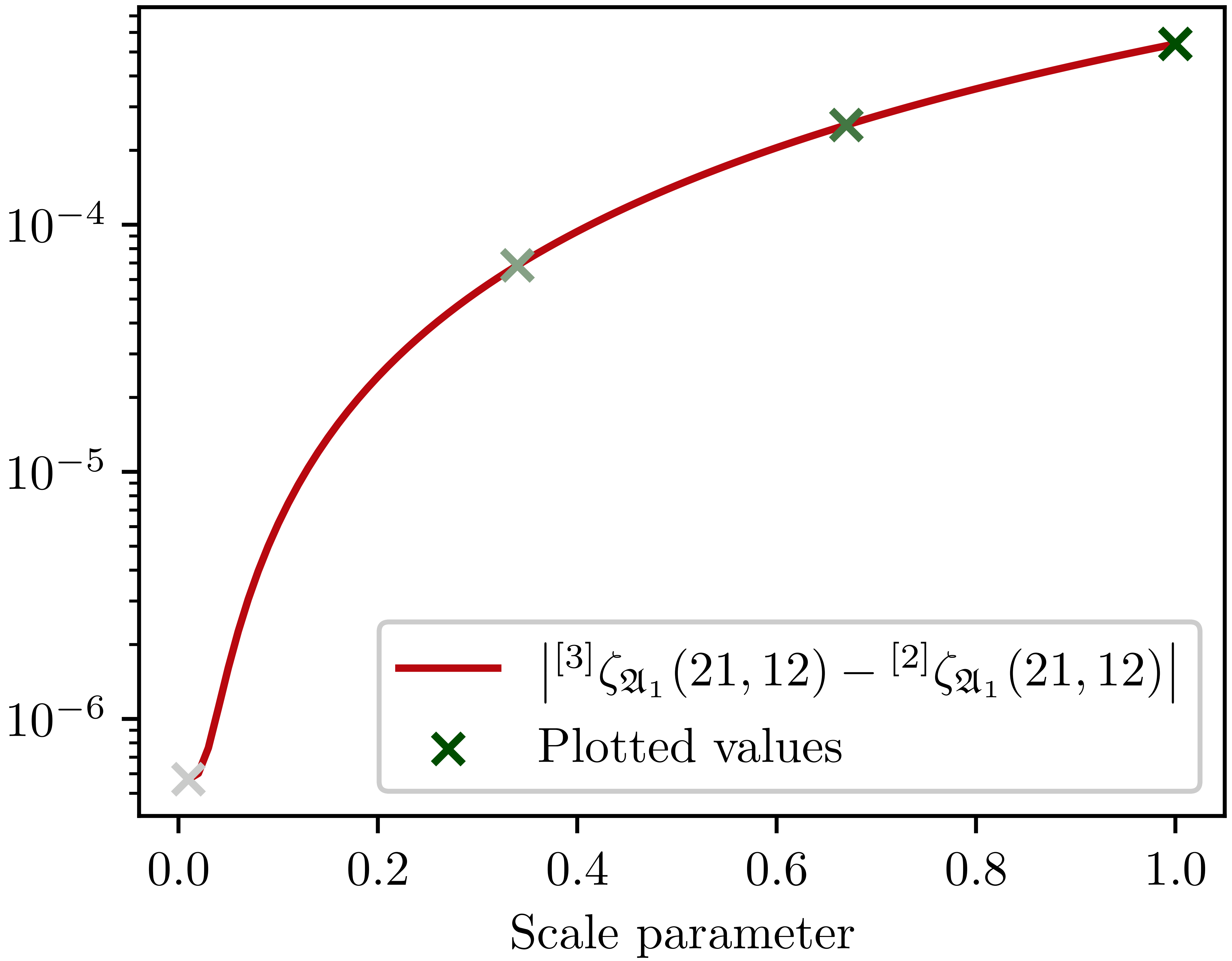
elliptic modulus of the j-th subcover

(case $j = i$ is more involved)

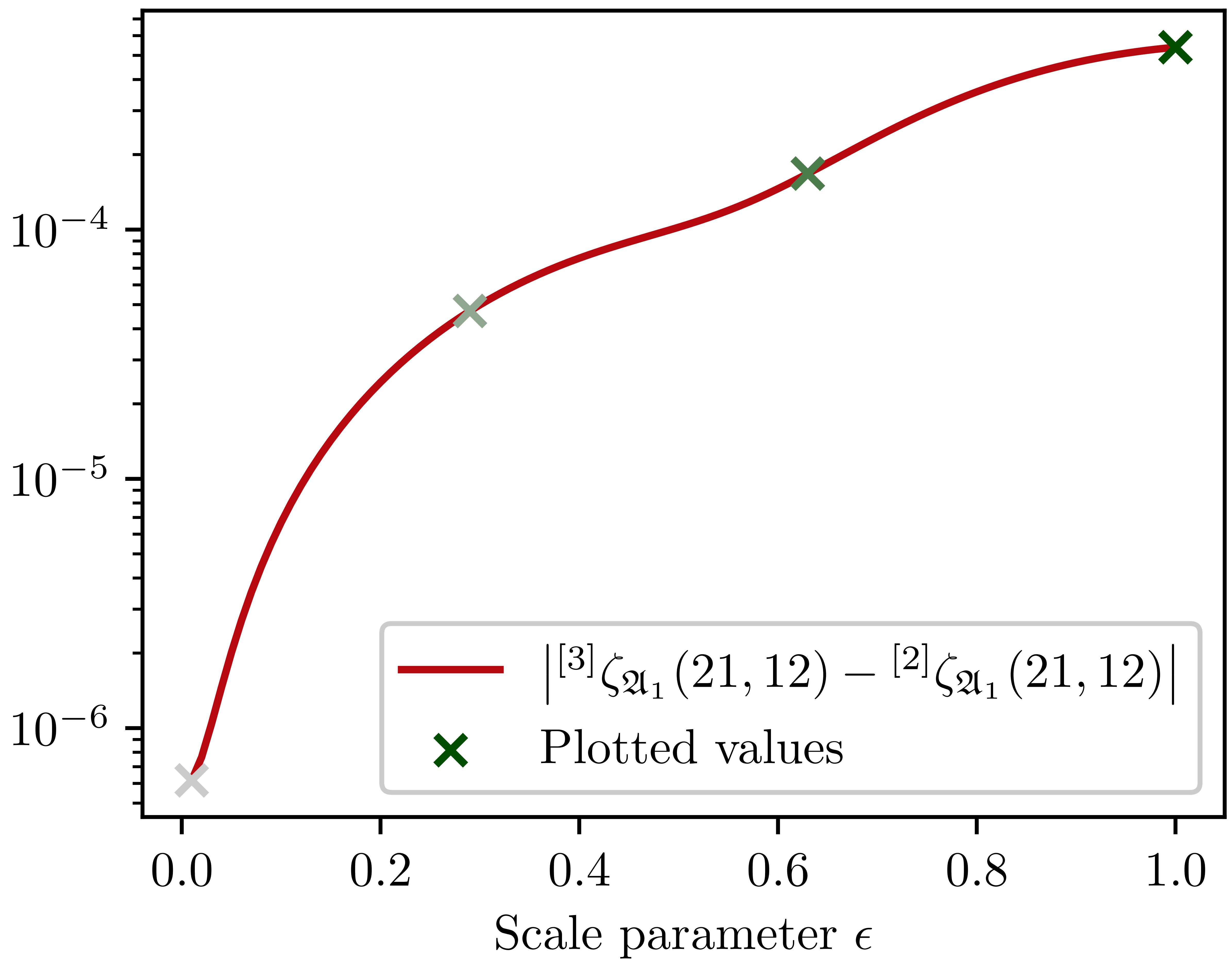
Degenerations of hgMZVs

Non-separating degeneration

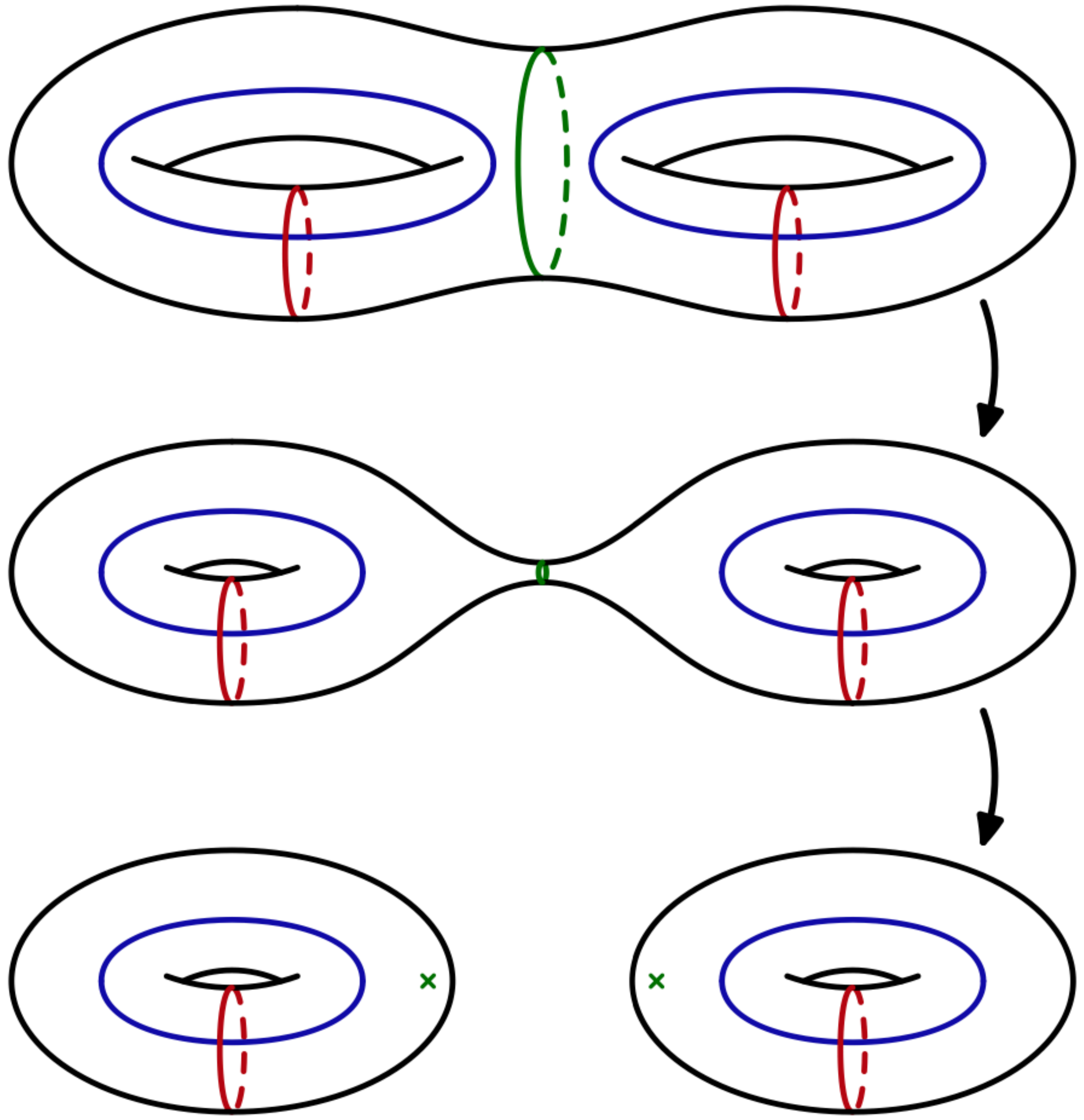


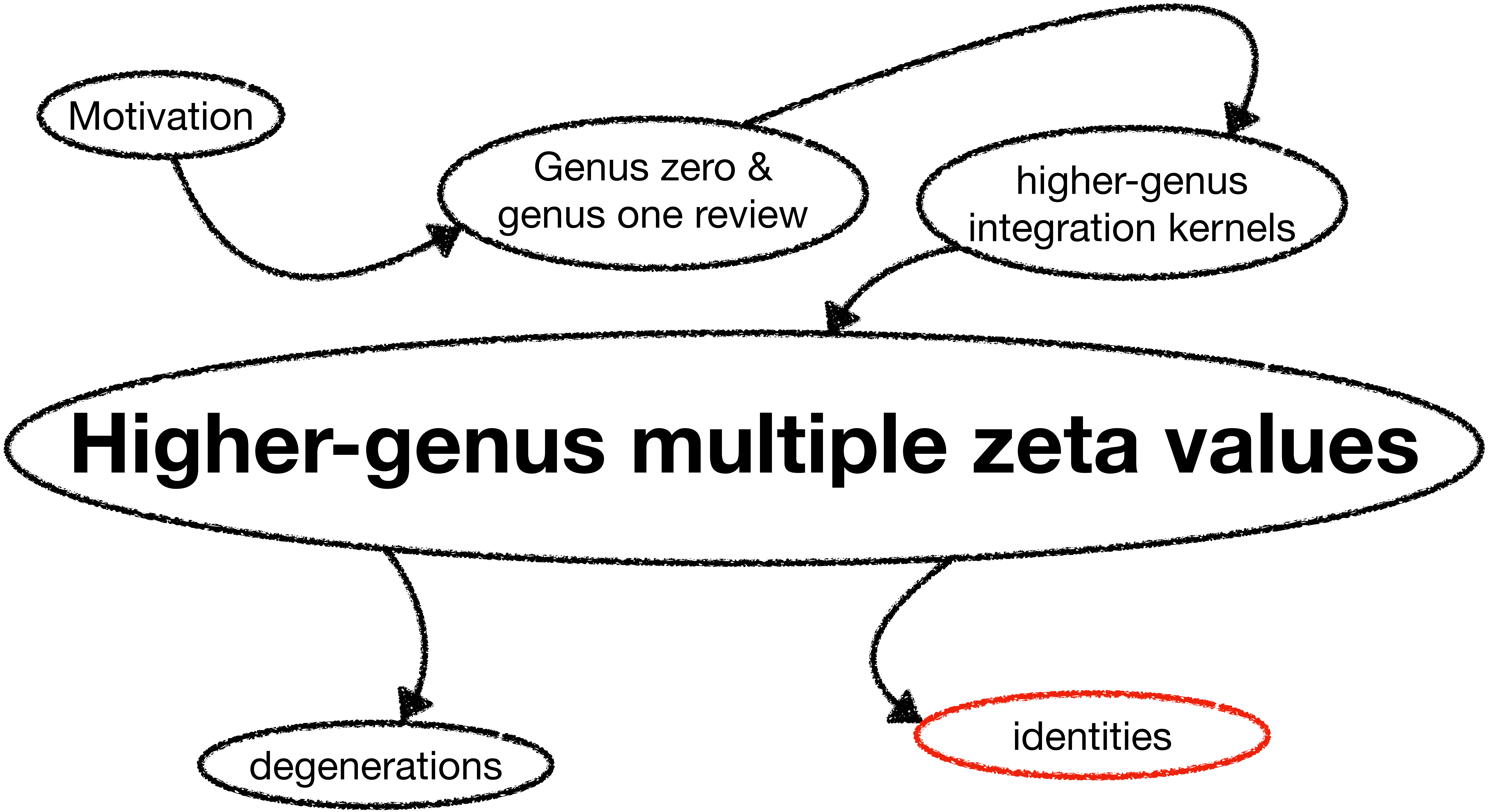


Degenerations of hgMZVs



Separating degeneration




Motivation
Genus zero &
genus one review
higher-genus
integration kernels
Higher-genus multiple zeta values
degenerations
identities

Relations between hgMZVs

- Fay relations

- Cycle exchange

- Alternating identity (hyperelliptic)

- eMZV $\Longrightarrow$ hgMZV identities*

** Conjectured*

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higher-genus Fay-like identities: [D'Hoker, Schlotterer '24][KB, Broedel, Im, Lisitsyn, Moeckli '24]

$$\omega_{i_1\cdots i_r k}(z,x)\,\omega_{p_1\cdots p_s p p'}(y,z)\Big|_{p'=p} = \omega_{i_1\cdots i_r k}(z,x)\,\omega_{p_1\cdots p_s p p'}(y,x)\Big|_{p'=p}$$
$$-\sum_{l=0}^{r-1}\sum_{m=0}^{s}(-1)^{m-s}\omega_{(i_1\cdots i_l \sqcup\!\sqcup p_s\cdots p_{m+1}) j i_{l+1}\cdots i_r k}(z,x)\,\omega_{p_1\cdots p_m j}(y,x)$$
$$-\sum_{m=0}^{s}(-1)^{m-s}\omega_{(i_1\cdots i_r \sqcup\!\sqcup p_s\cdots p_{m+1}) jk}(z,y)\,\omega_{p_1\cdots p_m j}(y,x)$$
$$-\sum_{l=0}^{r}\sum_{m=0}^{s}(-1)^{m-s}\omega_{(i_1\cdots i_l \sqcup\!\sqcup p_s\cdots p_{m+1}) j}(z,y)\,\omega_{p_1\cdots p_m j i_{l+1}\cdots i_r k}(y,x)$$

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can be used to derive hgMPL and hgMZV relations
e.g. for genus two:

$$0 = {}^{[2]}\zeta_{\mathfrak{A}_1}(21^2,1) + {}^{[2]}\zeta_{\mathfrak{A}_1}(2^21,2) + {}^{[2]}\zeta_{\mathfrak{A}_1}(2^2,21) + {}^{[2]}\zeta_{\mathfrak{A}_1}(21,1^2)$$

(more complicated cases can involve a lot more terms)

* Conjectured

Relations between hgMZVs

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Relations between hgMZVs

can use convolution formulation from [D'Hoker, Schlotterer '25]

- Fay relations

- Cycle exchange

$$\zeta_{\mathfrak{A}_i}(j_1 \cdots j_r kl, mn) = \int_{t_1 \in \mathfrak{A}_i} \omega_{j_1 \cdots j_r kl}(t_1, z_0) \int_{t_2 = z_0}^{t_1} \omega_{mn}(t_2, z_0)$$

- Alternating identity (hyperelliptic)

- eMZV $\Longrightarrow$ hgMZV identities*

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Relations between hgMZVs

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- Alternating identity (hyperelliptic)
- eMZV $\Longrightarrow$ hgMZV identities*

** Conjectured*

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$\Longrightarrow$ choosing $i \neq m$ and carefully exchanging integrations leads to a $\mathfrak{A}_m$-hgMZV

$$\zeta_{\mathfrak{A}_i}(j_1\cdots j_r kl, mn) = \zeta_{\mathfrak{A}_m}(n, ij_1\cdots j_r kl) + \delta_{ij_1\cdots j_r k}\frac{\mathrm{B}_{r+1}}{(r+1)!}\zeta_{\mathfrak{A}_m}(n, kl) + \sum_{p=1}^{r}\frac{\mathrm{B}_p}{p!}\delta_{ij_1\cdots j_p}\,\zeta_{\mathfrak{A}_m}(n, ij_{p+1}\cdots j_r kl)$$

(for $k\neq l,\ i\neq m,\ m\neq n$)

- Alternating identity (hyperelliptic)

- eMZV $\Longrightarrow$ hgMZV identities*

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Relations between hgMZVs

can use convolution formulation from [D'Hoker, Schlotterer '25]

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$$\omega_{mn}(t_2) \quad (m \neq n)$$

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$$+ \sum_{p=1}^{r} \frac{\mathrm{B}_p}{p!}\delta_{ij_1\cdots j_p}\, \zeta_{\mathfrak{A}_m}(n, ij_{p+1}\cdots j_r kl)$$

$i \neq m$

(for $k \neq l,\ i \neq m,\ m \neq n$)

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hyperelliptic curve: $y^2 = P_n(x)$

$\hookrightarrow$ additional symmetry (hyperelliptic involution)

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hyperelliptic curve: $y^2 = P_n(x)$

additional symmetry (hyperelliptic involution)

can be used to derive new relations: $(i \neq j)$

$$\zeta_{\mathfrak{A}_j}(ji, ij) = \zeta_{\mathfrak{A}_j}(i, jij)$$

$$\zeta_{\mathfrak{A}_j}(jiji, ij) = \zeta_{\mathfrak{A}_j}(iji, jij) = \zeta_{\mathfrak{A}_j}(ji, ijij) = \zeta_{\mathfrak{A}_j}(i, jijij)$$

$$\cdots$$

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$$\cdots$$

Note: all compact genus-two Riemann surfaces are hyperelliptic!

numerically: does not hold for hgMZVs of non-hyperelliptic curves!

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numerically: eMZV identities translate into hgMZV identities

e.g.

$$\omega^{\text{ell}}(0,5) = \omega^{\text{ell}}(2,3) \quad \rightsquigarrow \quad \zeta_{\mathfrak{A}_j}(j, j^6) = \zeta_{\mathfrak{A}_j}(j^3, j^4)$$

$$\omega^{\text{ell}}(1,2) = -2\,\zeta_2\,\omega^{\text{ell}}(1,0) - \omega^{\text{ell}}(0,3) \quad \rightsquigarrow \quad \zeta_{\mathfrak{A}_j}(j^2, j^3) = -\frac{2\,\zeta_2}{(-2\pi \mathring{\imath})^2}\zeta_{\mathfrak{A}_j}(j^2, j) - \zeta_{\mathfrak{A}_j}(j, j^4)$$

$$\dots$$

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$$\ldots$$

this would imply that all eMZV structures carry over to hgMZVs!

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$$\ldots$$

this would imply that all eMZV structures carry over to hgMZVs!

possible way to investigate:
higher-genus associator relations [Gonzales '20]

* *Conjectured*

Relations between hgMZVs

- Fay relations $$0 = {}^{[2]}\zeta_{\mathfrak{A}_1}(21^2,1) + {}^{[2]}\zeta_{\mathfrak{A}_1}(2^21,2) + {}^{[2]}\zeta_{\mathfrak{A}_1}(2^2,21) + {}^{[2]}\zeta_{\mathfrak{A}_1}(21,1^2)$$

- Cycle exchange $$\zeta_{\mathfrak{A}_1}(12,23) = \zeta_{\mathfrak{A}_2}(3,112) - \frac{1}{2}\zeta_{\mathfrak{A}_2}(3,12)$$

- Alternating identity (hyperelliptic) $${}^{[2]}\zeta_{\mathfrak{A}_j}(ji,ij) = {}^{[2]}\zeta_{\mathfrak{A}_j}(i,jij), \quad i \neq j$$

- eMZV $\Longrightarrow$ hgMZV identities* $$\omega^{\mathrm{ell}}(0,5) = \omega^{\mathrm{ell}}(2,3) \quad \Longrightarrow \quad \zeta_{\mathfrak{A}_j}(j,j^6) = \zeta_{\mathfrak{A}_j}(j^3,j^4)$$

* *Conjectured*

Summary

☑ Definition of $\mathfrak{A}$-cycle hgMZVs

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☑ Degenerations of hgMZVs

- **Separating** and **non-separating degenerations** studied

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Summary

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☑ hgMZV relations

- **Structures of eMZVs carry over to hgMZVs**
  <u>Conjecture:</u> Any eMZV identity translates into a hgMZV identity
  (Elliptic associator structures [Enriquez'10] generalize to higher-genera [Gonzales '20])
- **New Structures appear**
  Cycle exchange identity: e.g.
  $\zeta_{\mathfrak{A}_1}(12{,}21) = \zeta_{\mathfrak{A}_2}(1{,}112) + \frac{1}{2}\zeta_{\mathfrak{A}_2}(1{,}12)$
- **Certain identities only present for hyperelliptic surfaces**
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  ${}^{[2]}\zeta_{\mathfrak{A}_j}(ji, ij) = {}^{[2]}\zeta_{\mathfrak{A}_j}(i, jij), \quad (i \neq j)$

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- ☑ Degenerations of hgMZVs
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- ☑ Regularization of hgMZVs
  - see [KB, Broedel, Im, Ji, Moeckli '25]

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Outlook

- modular properties of hgMZVs
- $\mathfrak{B}$-cycle hgMZVs
- algebra structures underlying hgMZVs
- higher-genus associator relations (cf. e.g. [Gonzales '20])
- relation to higher-genus iterated Eisenstein integrals?
- application to physics

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Thank you!