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U(1)-symmetric Matrix Product Operator and Symmetry-Resolved OTOCs

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NumeriQS Retreat

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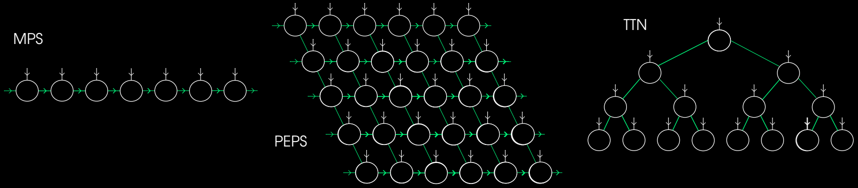
TNs as MB States and Operators

- TN compact representation of states and operators for many-body (MB) systems
- L-body system: reduce complexity

$$\sim O(2^L) \implies \sim poly(L)$$

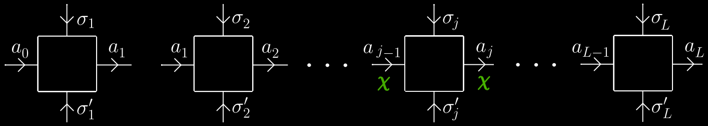
- efficient approximation states/operators with low entanglement
- TN based algorithms: ∘ unitary dynamics at long time scales;
  ∘ system-size larger than ED



Matrix Product Operator

- Operator acting on L-body system $\hat{O} : \mathcal{H} \to \mathcal{H}$
- 1D: matrix-product-operator (*MPO*),

$$\hat{O} = \sum_{\sigma_1,\dots,\sigma_L} \sum_{\sigma_1',\dots,\sigma_L'} \hat{O}^{\sigma_1,\sigma_1'} \dots \hat{O}^{\sigma_L,\sigma_L'} |\sigma_1,\dots,\sigma_L\rangle\langle\sigma_1',\dots,\sigma_L'|$$



- complexity: $\sim L\, \chi^2 d^2$

U(1) Symmetry

- action of $U(1)$: $\mathbb{V} = \bigoplus_n \mathbb{V}_n$ $\quad$ $\mathbb{V}_n$ sub-space
  - quantum charge $\quad n \in \mathbb{Z}$,
  - symmetry generator $\quad \hat{n} = \sum_n n \, \hat{\Pi}_n$, $\quad$ $\hat{\Pi}_n$ projector in $\mathbb{V}_n$

- $\hat{O} : \mathbb{V} \to \mathbb{V}$ charge-preserving $\quad \Leftrightarrow \quad \left[\hat{O}, \hat{n}\right] = 0$

$$\hat{O} = \bigoplus_n \hat{O}^{[n]} \qquad \hat{O}^{[n]} : \mathbb{V}_n \to \mathbb{V}_n$$

$$\begin{pmatrix} \hat{O}^{[1]} & 0 & \cdots & 0 \\ 0 & & \cdots & 0 \\ \vdots & \vdots & \hat{O}^{[n]} & \vdots \\ 0 & 0 & \cdots & \ddots \end{pmatrix}$$

e.g. spin L-body lattice with **total magnetization** conserved

$$\hat{S}^{z\ \mathrm{tot}} = \sum_{j=1}^{L} \hat{s}_j^z \qquad [\hat{H}, \hat{S}^{z\ \mathrm{tot}}] = 0$$

Operator Charge

- $\hat{\mathcal{O}} : \mathcal{H} \to \mathcal{H}$ maps states with charge n into states with n'

$$\hat{\mathcal{O}} = \sum_{i,j} O_{ij} |i, n\rangle\langle j, n'|$$

- *vectorized* operator $|\hat{\mathcal{O}}\rangle\rangle = \sum_{i,j} O_{ij} |i, n\rangle |j, n'\rangle$
- superoperator $\hat{\mathcal{Q}} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

$$\hat{\mathcal{Q}} \, |\hat{\mathcal{O}}\rangle\rangle = q_{\hat{\mathcal{O}}} \, |\hat{\mathcal{O}}\rangle\rangle,$$

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$\hat{n}|n\rangle = n|n\rangle$

operator charge

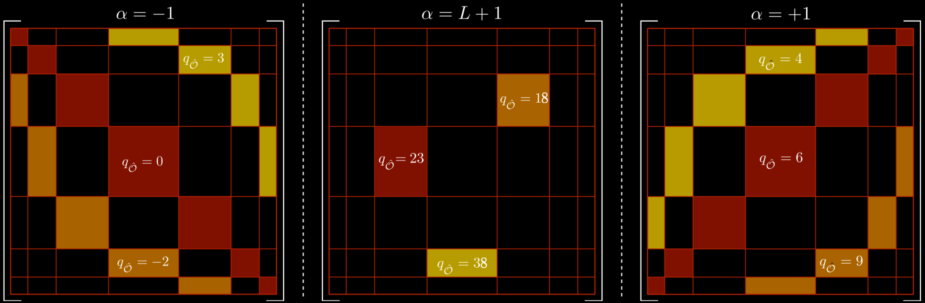
$$\hat{\mathcal{Q}}\, |\hat{\mathcal{O}}\rangle\rangle = q_{\hat{\mathcal{O}}}\, |\hat{\mathcal{O}}\rangle\rangle,$$

$$\mathcal{Q} = \hat{n} \otimes 1 + \alpha\, 1 \otimes \hat{n}^T$$

$$\underline{q_{\hat{\mathcal{O}}} = n + \alpha n'}$$

Sector Resolution

$$q_{\hat{\mathcal{O}}} = n + \alpha n'$$

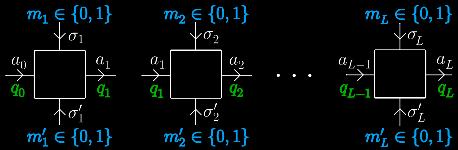


- $\alpha = -1$ block-diagonal operator
- $\alpha = (L+1)$ unique charge for each block, i.e. $q_{\hat{\mathcal{O}}} = n + (L+1)\, n'$;
- $\alpha = +1$ blocks anti-diagonal operators.

Invariance of TNs under U(1) symmetry

- *charge* or flow of conserved quantity

$$|\sigma_i\rangle \rightarrow |\sigma_i, m_i\rangle \qquad |\sigma'_i\rangle \rightarrow |\sigma'_i, m'_i\rangle \qquad |a_i\rangle \rightarrow |a_i, q_i\rangle$$



- local charge conservation: $q_i = \sum_{j=1}^{i} m_j + \alpha\ m'_j$
- boundary conditions: $q_0 = 0,\ q_L = q_{\hat{\mathcal{O}}},$

$$\Longrightarrow \text{operator charge conservation:} \quad q_{\hat{\mathcal{O}}} = \sum_{i=1}^{L} m_i + \alpha\ m'_i$$

OTOC

- Out-of-time-ordered correlator (OTOC): standard detector of quantum chaos

$$C_{j,j'}(t) = \frac{\mathrm{Tr}\left([\hat{O}_j, \hat{O}_{j'}(t)]^2\right)}{2\ \mathrm{Tr}\left(\mathbb{I}\right)}$$

- spin$-1/2$ system: $\hat{\sigma}^z_j(t) = \hat{U}(t)^\dagger \hat{\sigma}^z_j \hat{U}(t)$,

$$C_{j,j'}(t) = 1 - \frac{\mathrm{Tr}\left(\hat{\sigma}^z_{j'}(t)\ \hat{\sigma}^z_j\ \hat{\sigma}^z_{j'}(t)\ \hat{\sigma}^z_j\right)}{\mathrm{Tr}\left(\mathbb{I}\right)}$$

- U(1) symmetric systems: $\hat{\sigma}_j(t)$ is charge-preserving

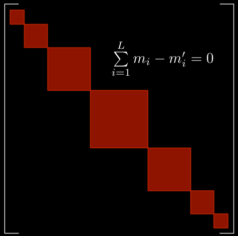
$\hat{U}(t) = e^{-i\ \hat{H}t}$

Projected OTOCs in U(1)-Symmetry Sector

charge-preserving MPO

$\alpha = -1$

$q_{\hat{\mathcal{O}}} = 0$

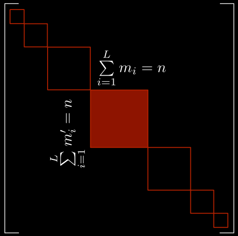


$$C_{j,j'}(t) = 1 - \frac{\mathrm{Tr}\left(\left(\hat{\sigma}^z_{j'}(t)\ \hat{\sigma}^z_j\right)^2\right)}{\mathcal{D}}$$

sector-projected MPO

$\alpha = L + 1$

$q_{\hat{\mathcal{O}}} = n + (L+1)n$

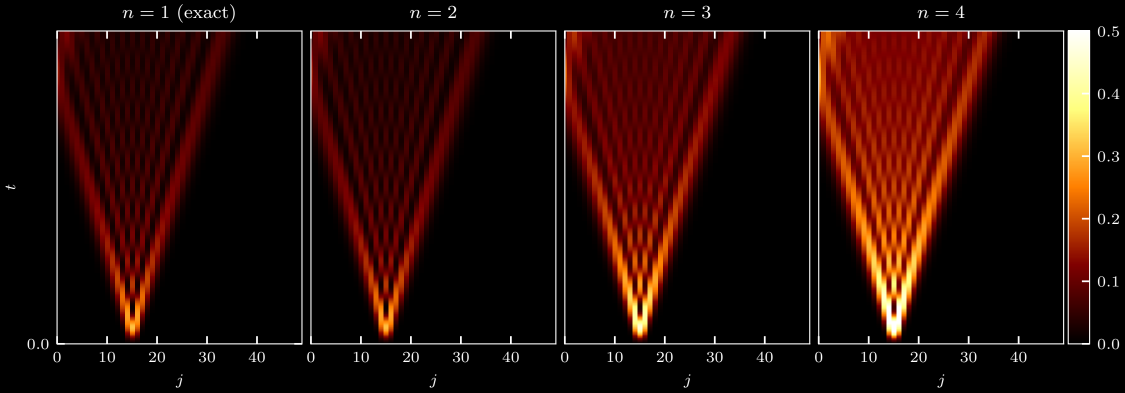


$$C_{j,j'}(t)^{[n]} = 1 - \frac{\mathrm{Tr}\left(\left(\hat{\sigma}^z_{j'}(t)^{[n]}\ \hat{\sigma}^{z\,[n]}_j\right)^2\right)}{\mathcal{D}_n}$$

$$\mathcal{D}_n = \mathrm{Tr}\left(\mathbb{I}^{[n]}\right), \qquad C_{j,j'}(t) = \sum_n \frac{\mathcal{D}_n}{\mathrm{Tr}\left(\mathbb{I}\right)} \cdot C_{j,j'}(t)^{[n]}$$

Projected OTOC

$$C_{j,j'}(t)^{[n]} = 1 - \mathrm{Tr}\left(\left(\hat{\sigma}^z_{15}(t)^{[n]}\ \hat{\sigma}^{z\,[n]}_i\right)^2\right)/\mathcal{D}_n$$

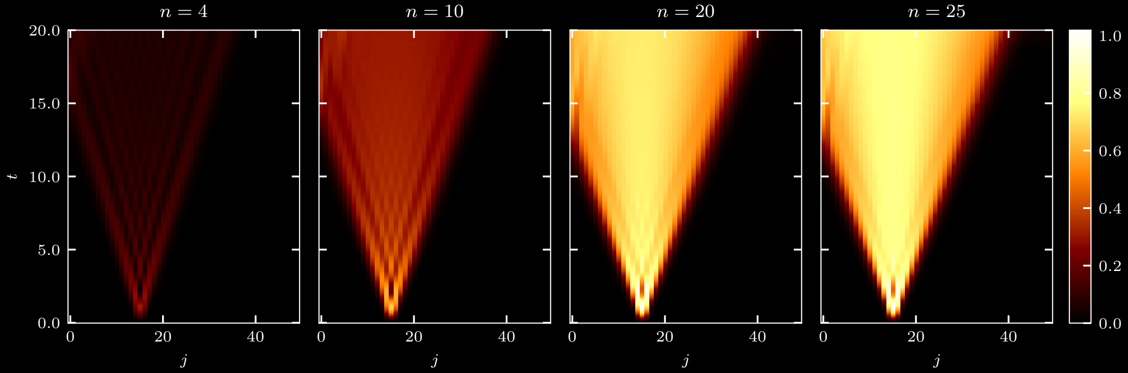


$L = 50$ spin chain, open boundary conditions, $H = \sum_{i=1}^{L} \frac{1}{2}\left(\sigma_i^+ \sigma_{i+1}^- + \sigma_i^- \sigma_{i+1}^+\right) + \sigma_i^z \sigma_{i+1}^z$.

Trotterized (4-th order) time-evolution operator with $dt = 0.005$.

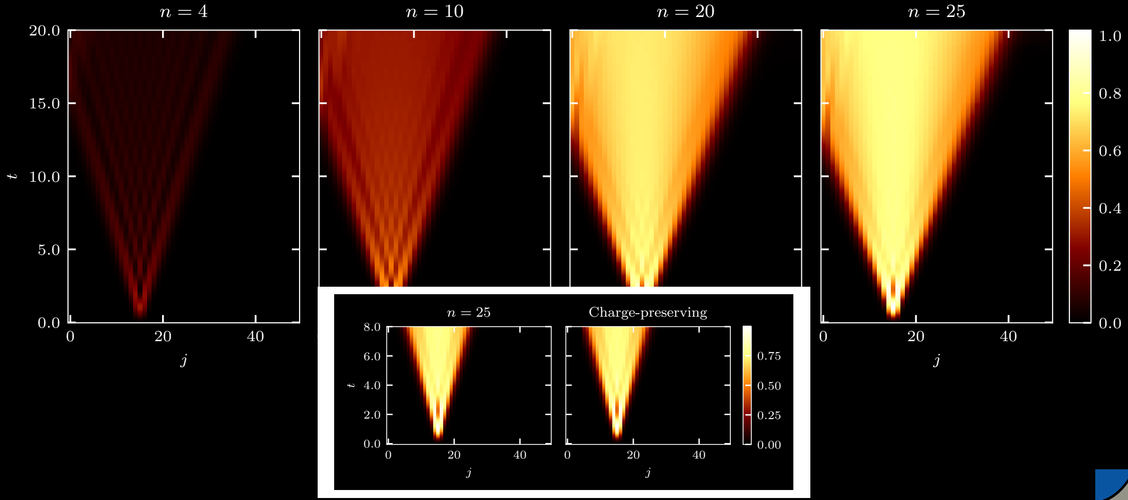
Projected OTOC

$$C_{j,j'}(t)^{[n]} = 1 - \mathrm{Tr}\left(\left(\hat{\sigma}^z_{15}(t)^{[n]}\ \hat{\sigma}^{z\,[n]}_j\right)^2\right)/\mathcal{D}_n$$

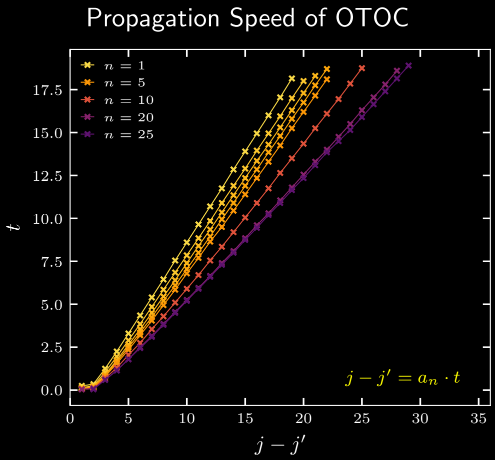


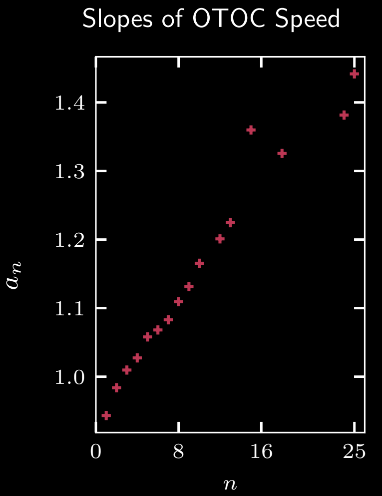
Projected OTOC

$$C_{j,j'}(t)^{[n]} = 1 - \mathrm{Tr}\left(\left(\hat{\sigma}^z_{15}(t)^{[n]}\ \hat{\sigma}^{z\,[n]}_j\right)^2\right)/\mathcal{D}_n$$



Projected OTOC Speed





Summary and Outlook

- Implementing symmetry in MPO

  - Computations of global quatities in symmetry-sector
    - reduction of computational cost $\longrightarrow$ blockwise operations
    - dynamics at large times for large system-size

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    - significant information lies on the biggest sector

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- Implementing symmetry in MPO

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    - significant information lies on the biggest sector

Outlook

- study operator entanglement entropy with symmetry-resolution
- implement symmetry in higher D lattice

Thank you!

Symmetry Resolved OpEE

- Bipartition of the system into A and B, i.e. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
- $\hat{Q}$: generator of $U(1)$ symmetry, $\hat{Q} = \hat{Q}_A \otimes 1_B + 1_A \otimes \hat{Q}_B$
- Schmidt decomposition of operator $\hat{\mathcal{O}}$:

$$\frac{\hat{\mathcal{O}}}{\sqrt{\mathrm{Tr}(\hat{\mathcal{O}}^\dagger \hat{\mathcal{O}})}} = \sum_{q_A} \sum_j \lambda_j^{(q_A)} \, \hat{\mathcal{O}}_{A,j}^{(q_A)} \otimes \hat{\mathcal{O}}_{B,j}^{(q_{\hat{\mathcal{O}}} - q_A)}$$

with $\left[\hat{Q}_A, \hat{\mathcal{O}}_{A,j}^{(q_A)}\right]_\alpha = q_A \hat{\mathcal{O}}_{A,j}^{(q_A)} \qquad \left[\hat{Q}_B, \hat{\mathcal{O}}_{B,j}^{(q_B)}\right]_\alpha = q_B \, \hat{\mathcal{O}}_{B,j}^{(q_B)} \, \delta_{q_B,(q_{\hat{\mathcal{O}}} - q_A)}$.[a]

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[a] α-deformed commutator $[\hat{A}, \hat{B}]_\alpha = \hat{A} + \alpha \hat{B}$

Symmetry Resolved OpEE

Operator Entanglement Entropy (OpEE) : indicator of the operator complexity

$$S(\hat{\mathcal{O}}) = \sum_{q_A} p(q_A) S_{q_A}(\hat{\mathcal{O}}) + \sum_{q_A} -p(q_A) \log\left(p(q_A)\right)$$

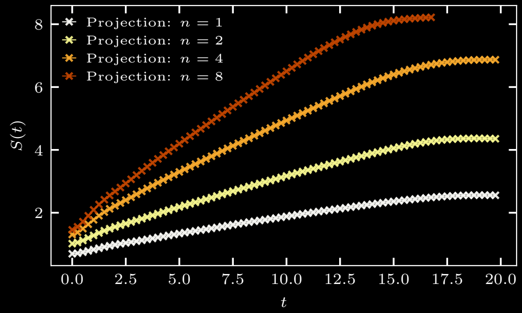
$$S_{q_A}(\hat{\mathcal{O}}) = -\sum_j \left(\frac{(\lambda_j^{(q_A)})^2}{p(q_A)} \right) \log \left(\frac{(\lambda_j^{(q_A)})^2}{p(q_A)} \right)$$

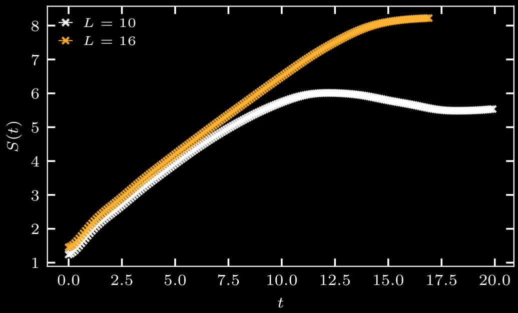
$$p(q_A) = \sum_j (\lambda_j^{(q_A)})^2$$

interplay between entanglement of a state and symmetries

[2] Rath A, Vitale V, Murciano S, Votto M, Dubail J, Kueng R, Branciard C, Calabrese P and Vermersch B 2023, *Entanglement barrier and its symmetry resolution: theory and experiment*, PRX Quantum 4 010318

Symmetry Resolved OpEE





Heisenberg chain with $L = 16$ Trotterised (4-th order) time-evolution operator until time $t = 20$. $S(\hat{U}(t)^{[n]})$ for the sectors $n = 1, 2, 4, 8$.

Heisenberg chain with $L = 10$ and $L = 16$ Trotterised (4-th order) time-evolution operator until time $t = 20$. $S(\hat{U}(t)^{[n]})$ for the biggest sector $n = 5$ and $n = 8$.

Reduction of the computational cost

Projected OTOCs in U(1)-Symmetry Sector

$$\epsilon_n = 1 - \mathrm{Tr}\left(\left(U^{(n)}_{\mathrm{exact}}\right)^\dagger U^{(n)}_{\mathrm{MPO}}\right)/\mathcal{D}_n$$