Bethe Center for Theoretical Physics

⟨NuMeriQS⟩

UNIVERSITÄT BONN

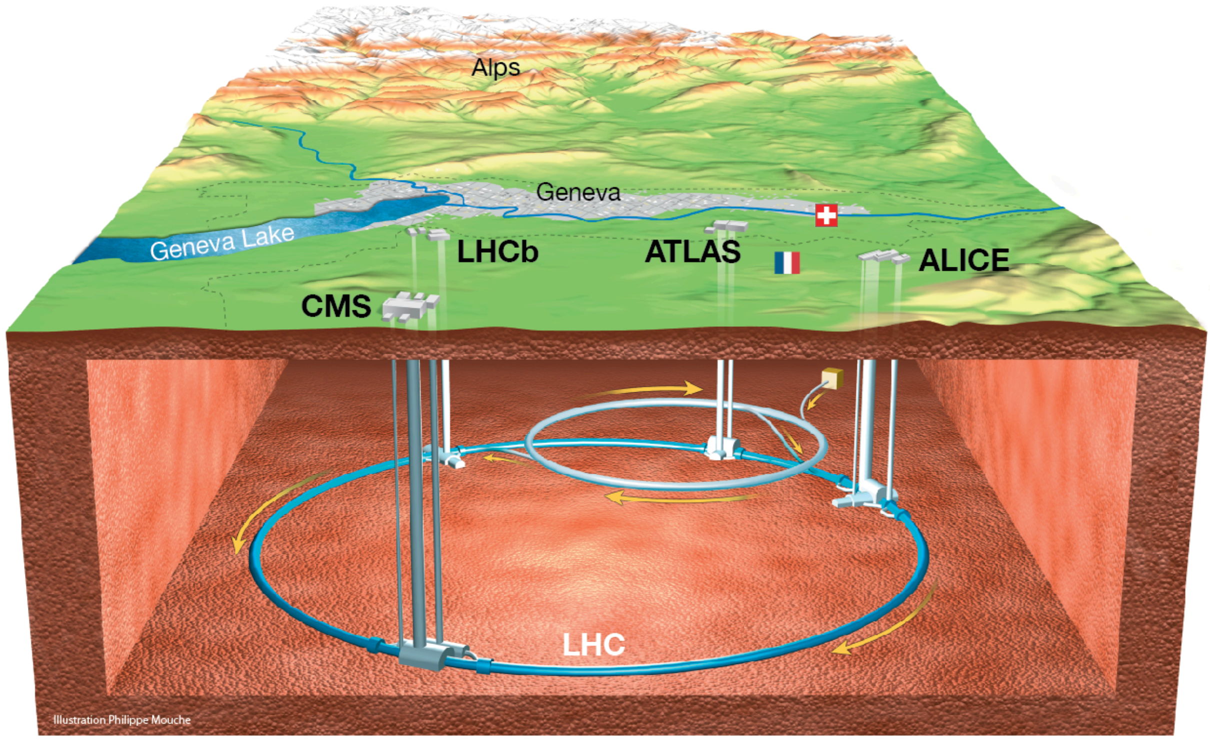
Project B04:
Precise Perturbative Computations from Quadrature Rules

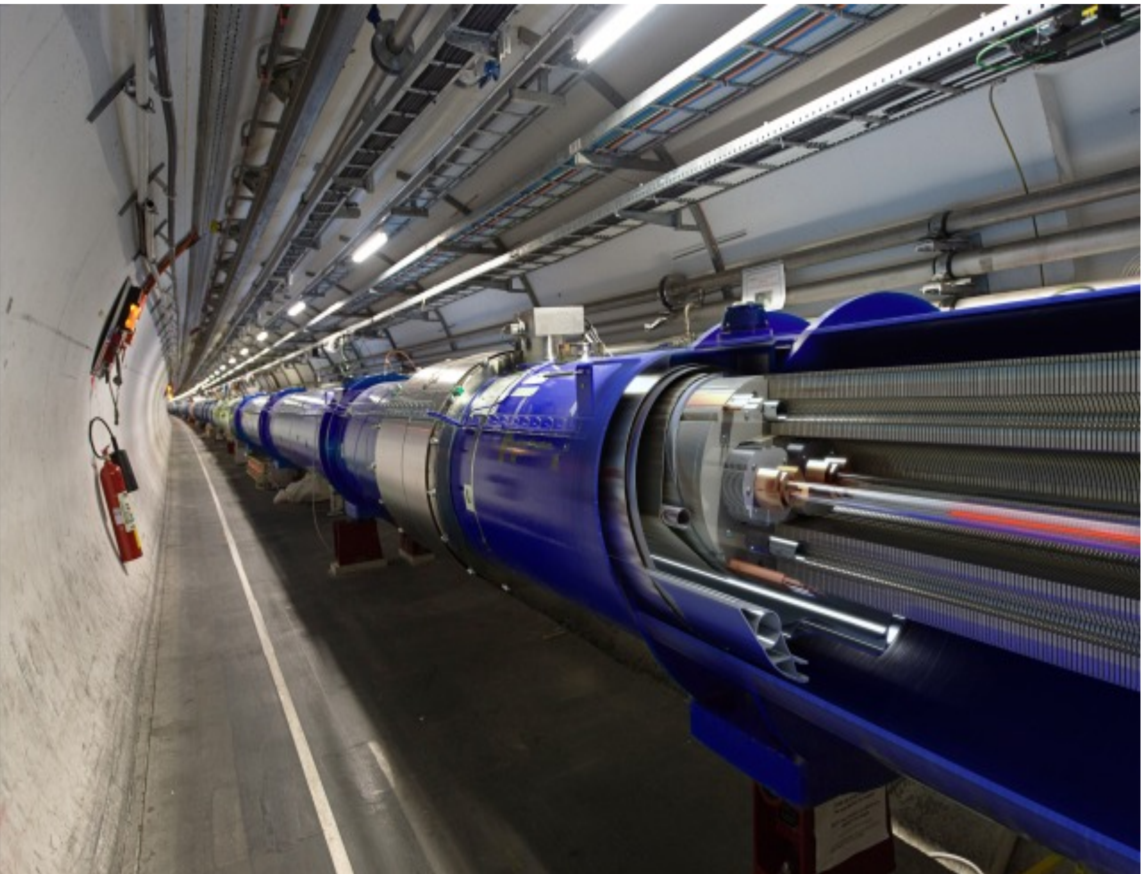
Claude Duhr

In collaboration with J. Dölz, B. Kovačić, C. Semper, A. Venkata

The need for precision ⟨NuMeriQS⟩

- Particle physics today is driven by the LHC.
- Collides protons at high energies.
  - ➡ Gives us access to physics at new energy scales!





- Some technical details:
  - ➡ Center-of-mass energy: 13.6 TeV
  - ➡ Circumference: 27 km.
  - ➡ 4 interaction points where the beams collide.

The need for precision

- Biggest success of the LHC in 2012: The discovery of the Higgs boson.
  - ➡ Are there other, unexpected, new particles?
- So far no signs of new physics.

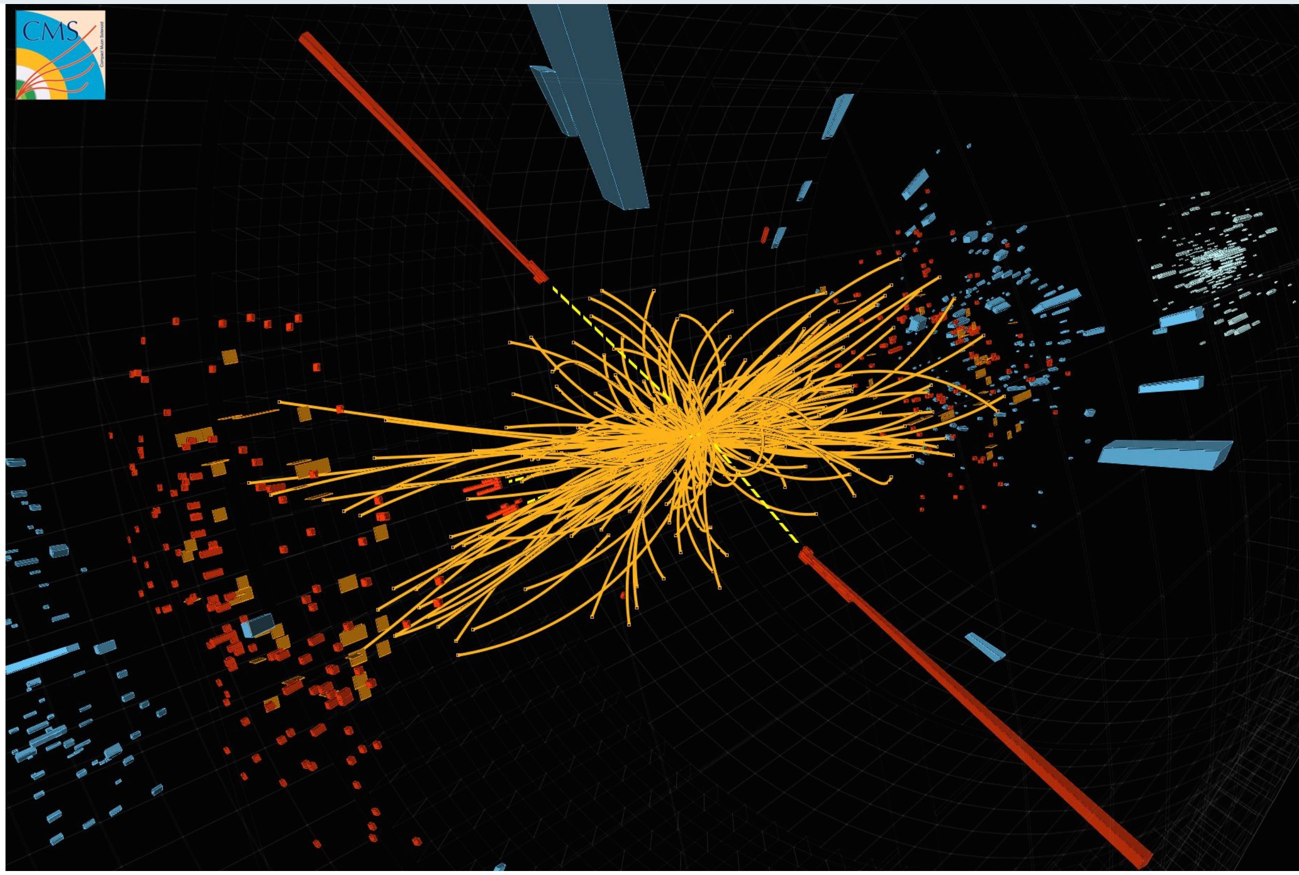
- In the absence of signals of new physics, we are entering a new era of precision physics!
  - ➡ Precise experimental measurements.
  - ➡ Precise theoretical predictions.

The need for precision ⟨NuMeriQS⟩

- In a quantum theory, we can predict probabilities.

  Proba $\sim |\mathcal{A}|^2$

  ➡ $\mathcal{A}$ = scattering amplitude
  = probability amplitude for a certain scattering to occur.



- In collider experiments, the observable is the (differential) cross section $d\sigma$.

  ➡ Corresponds (roughly) to the probability to find a certain final state in a certain region of a detector, normalised to the initial flux.

QCD factorisation

- The 'master formula' for LHC observables:

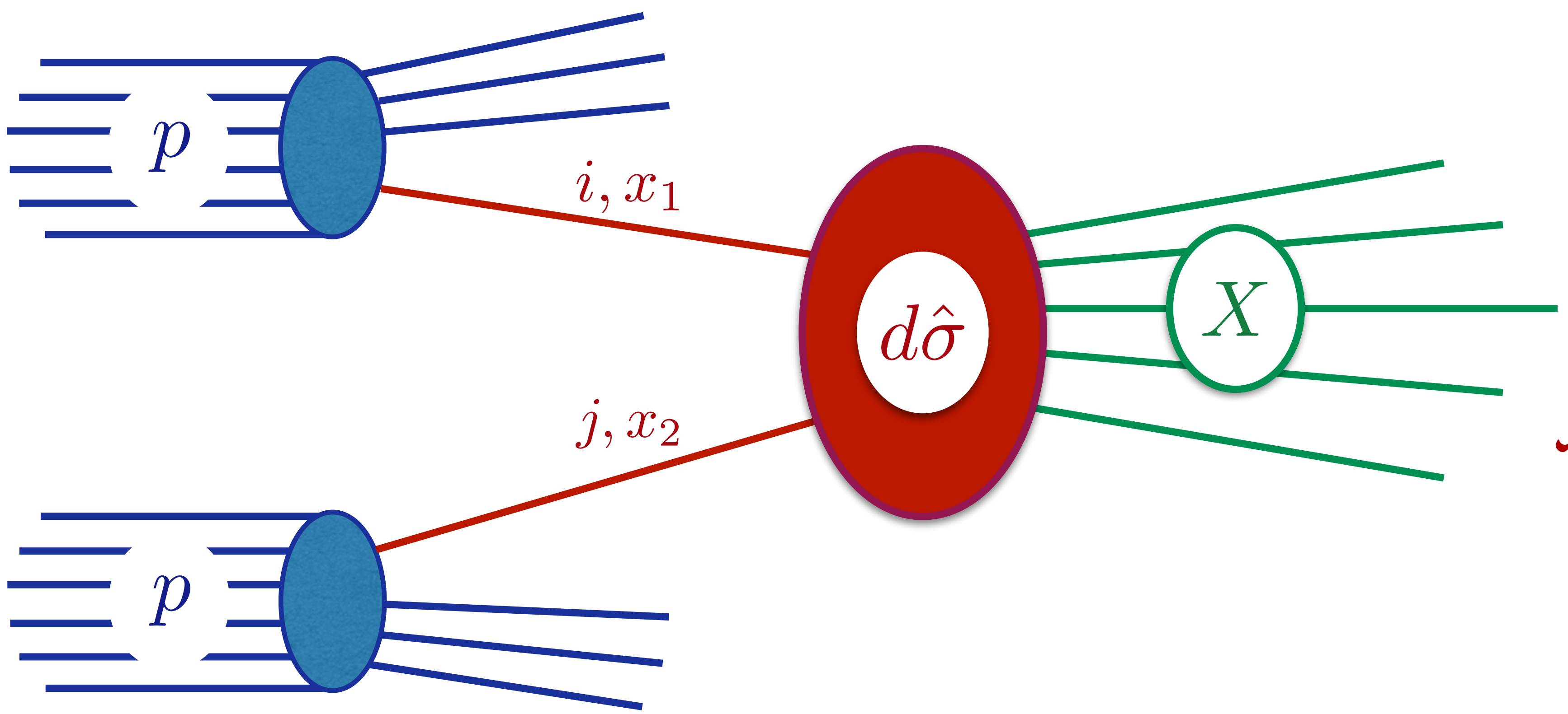
$$d\sigma(p\,p \rightarrow X) = \sum_{i,j} \int_0^1 dx_1\, dx_2\, f_i(x_1)\, f_j(x_2)\, d\hat{\sigma}(i\,j \rightarrow X)$$

Parton Distribution Functions

non-perturbative;
describe structure of the proton

Partonic cross section

computable in perturbation theory
as collisions between quarks and gluons



$$d\hat{\sigma} \sim \int dPS\, |\mathcal{A}|^2$$

$\mathcal{A}$ = scattering amplitude

The need for precision

- In general we do not know how to compute amplitudes exactly.

  ➡ Perturbation theory: $\alpha_s =$ coupling constant $\simeq 0.118$

$$\mathcal{A} = \mathcal{A}^{(0)} + \alpha_s\, \mathcal{A}^{(1)} + \alpha_s^2\, \mathcal{A}^{(2)} + \ldots$$

- Precision increases with the number of terms.

  ➡ How many terms needed?

The need for precision

- In general we do not know how to compute amplitudes exactly.
  - ➡ Perturbation theory: $\alpha_s =$ coupling constant $\simeq 0.118$

$$\mathcal{A} = \underbrace{\mathcal{A}^{(0)}}_{\text{LO}} + \underbrace{\alpha_s\,\mathcal{A}^{(1)}}_{\text{NLO } \sim 10\%} + \underbrace{\alpha_s^2\,\mathcal{A}^{(2)}}_{\text{NNLO } \sim 1\%} + \ldots$$

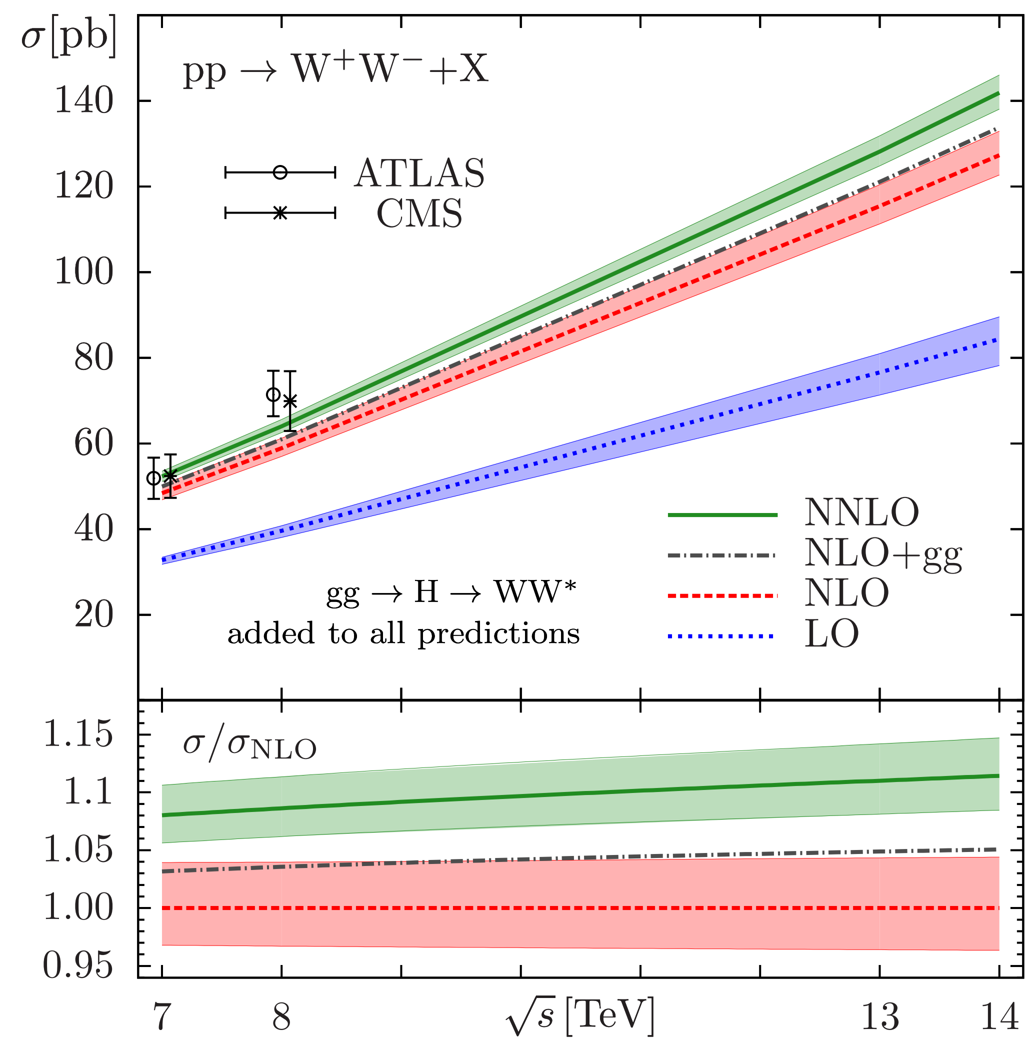
- Precision increases with the number of terms.
  - ➡ How many terms needed?
- To reach 1%, need next-to-next-to-leading order (NNLO) precision.
  - ➡ Is this needed?

The need for precision

- In general we do not know how to compute amplitudes exactly.
  - ➡ Perturbation theory: $\alpha_s =$ coupling constant $\simeq 0.118$

$$\mathcal{A} = \underbrace{\mathcal{A}^{(0)}}_{\text{LO}} + \underbrace{\alpha_s\, \mathcal{A}^{(1)}}_{\text{NLO} \sim 10\%} + \underbrace{\alpha_s^2\, \mathcal{A}^{(2)}}_{\text{NNLO} \sim 1\%} + \dots$$

- Precision increases with the number of terms.
  - ➡ How many terms needed?
- To reach 1%, need next-to-next-to-leading order (NNLO) precision.
  - ➡ Is this needed?



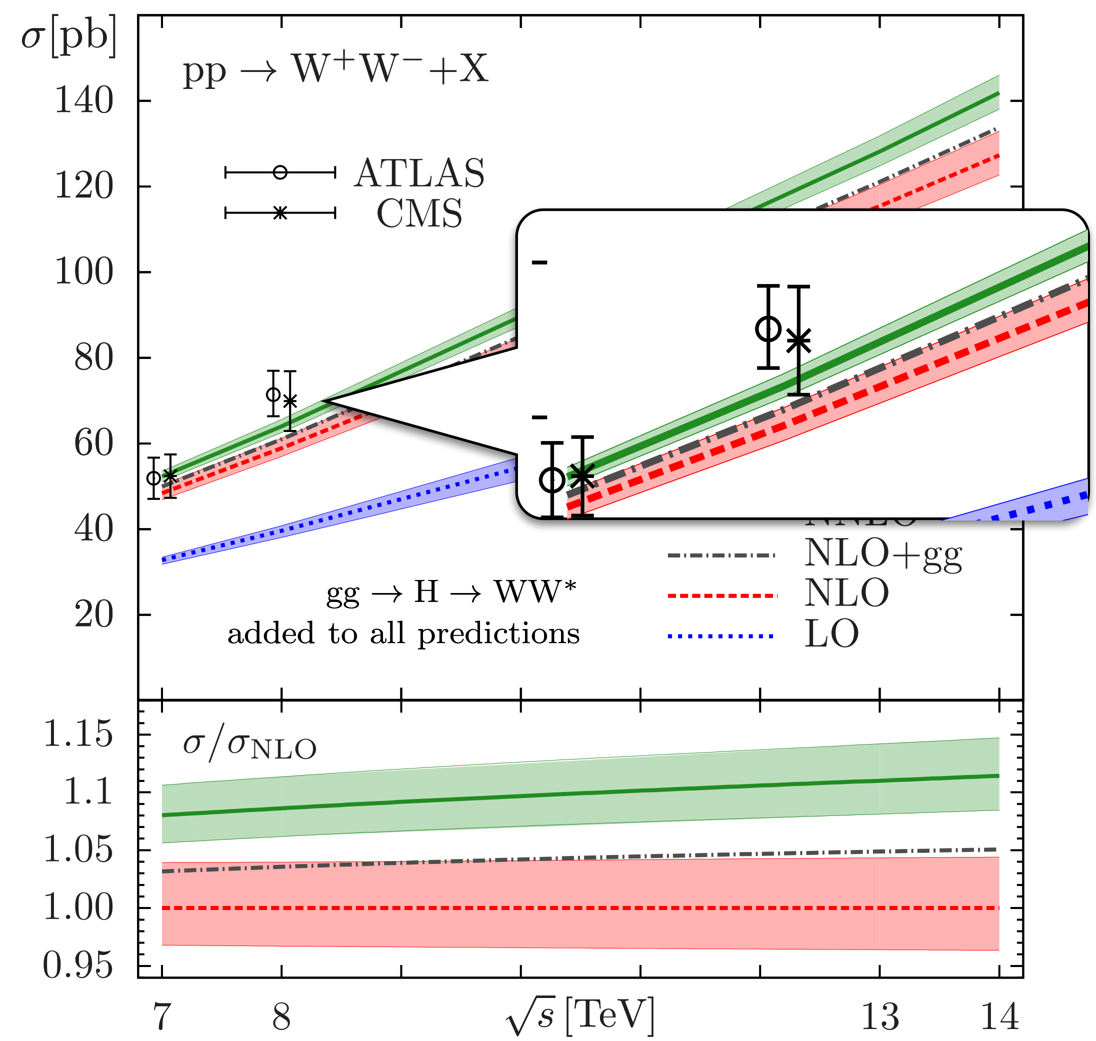
[Gehrmann, Grazzini, Kallweit, Maierhöfer, von Manteuffel, Pozzorini, Rathlev, Tancredi (2014)]

The need for precision

- In general we do not know how to compute amplitudes exactly.
  - ➡ Perturbation theory: $\alpha_s =$ coupling constant $\simeq 0.118$

$$\mathcal{A} = \underbrace{\mathcal{A}^{(0)}}_{\text{LO}} + \underbrace{\alpha_s\,\mathcal{A}^{(1)}}_{\text{NLO} \sim 10\%} + \underbrace{\alpha_s^2\,\mathcal{A}^{(2)}}_{\text{NNLO} \sim 1\%} + \dots$$

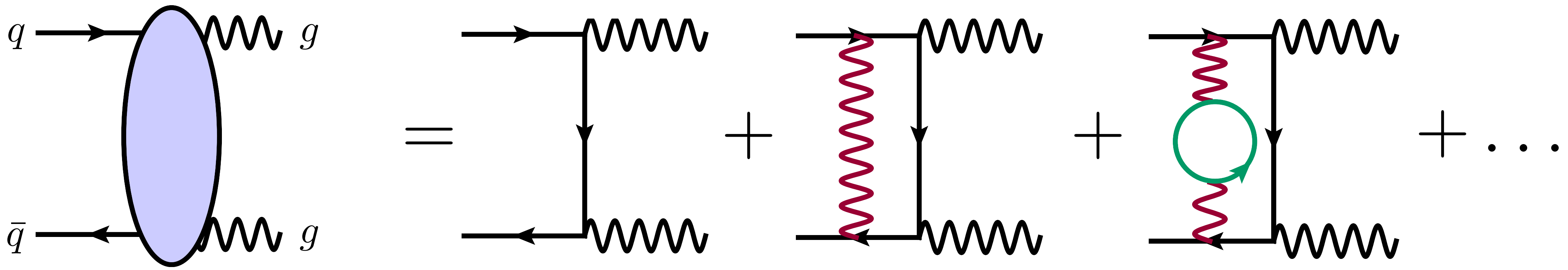
- Precision increases with the number of terms.
  - ➡ How many terms needed?
- To reach 1%, need next-to-next-to-leading order (NNLO) precision.
  - ➡ Is this needed?



[Gehrmann, Grazzini, Kallweit, Maierhöfer, von Manteuffel, Pozzorini, Rathlev, Tancredi (2014)]

Perturbation Theory

- $\mathcal{A}^{(L)}$ receives contributions from Feynman diagrams with L loops.

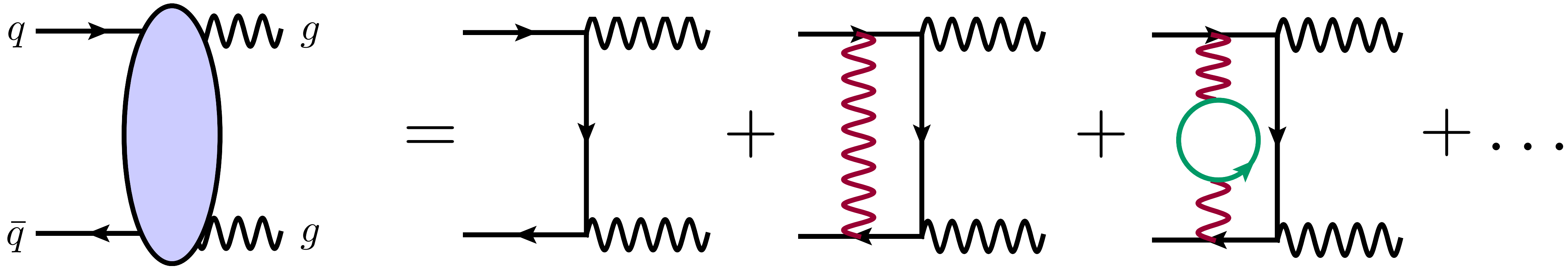


➡ Each diagram translates into an analytic formula.

Perturbation Theory

⟨NuMeriQS⟩

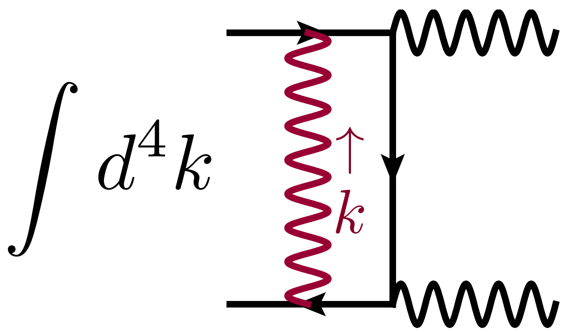
- $\mathcal{A}^{(L)}$ receives contributions from Feynman diagrams with L loops.



➡ Each diagram translates into an analytic formula.

- Quantum mechanics: We have to sum over all unobserved quantum numbers.

➡ Integrate over loop momentum k.



➡ 1 loop: usually doable.

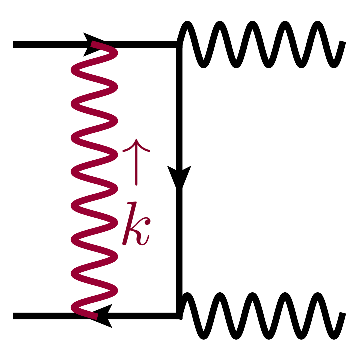
➡ 2 loops: some $2 \to 2$ or 3.

➡ 3 loops: some $2 \to 1$ or 2.

Regularisation

- Feynman integrals typically diverge, and need to be regulated.
- Most common scheme: Dimensional regularisation.
  - ➡ Perform computation in arbitrary dimensions D.
  - ➡ Take the limit $D \to 4$ at the very end.
  - ➡ Divergences show up as poles in $\dfrac{1}{D-4}$.
  - ➡ Result is a Laurent series:

$\int d^D k$



$$= \frac{a_{-2}}{\epsilon^2} + \frac{a_{-1}}{\epsilon} + a_0 + a_1\,\epsilon + \ldots \qquad D = 4 - \epsilon$$

  - ➡ The a_i are functions of momenta and masses of the particles.

Feynman integrals

Analytic

Numerical

Feynman integrals

Analytic

- If we can obtain analytic expressions for the Laurent coefficients, the problem is solved!

Numerical

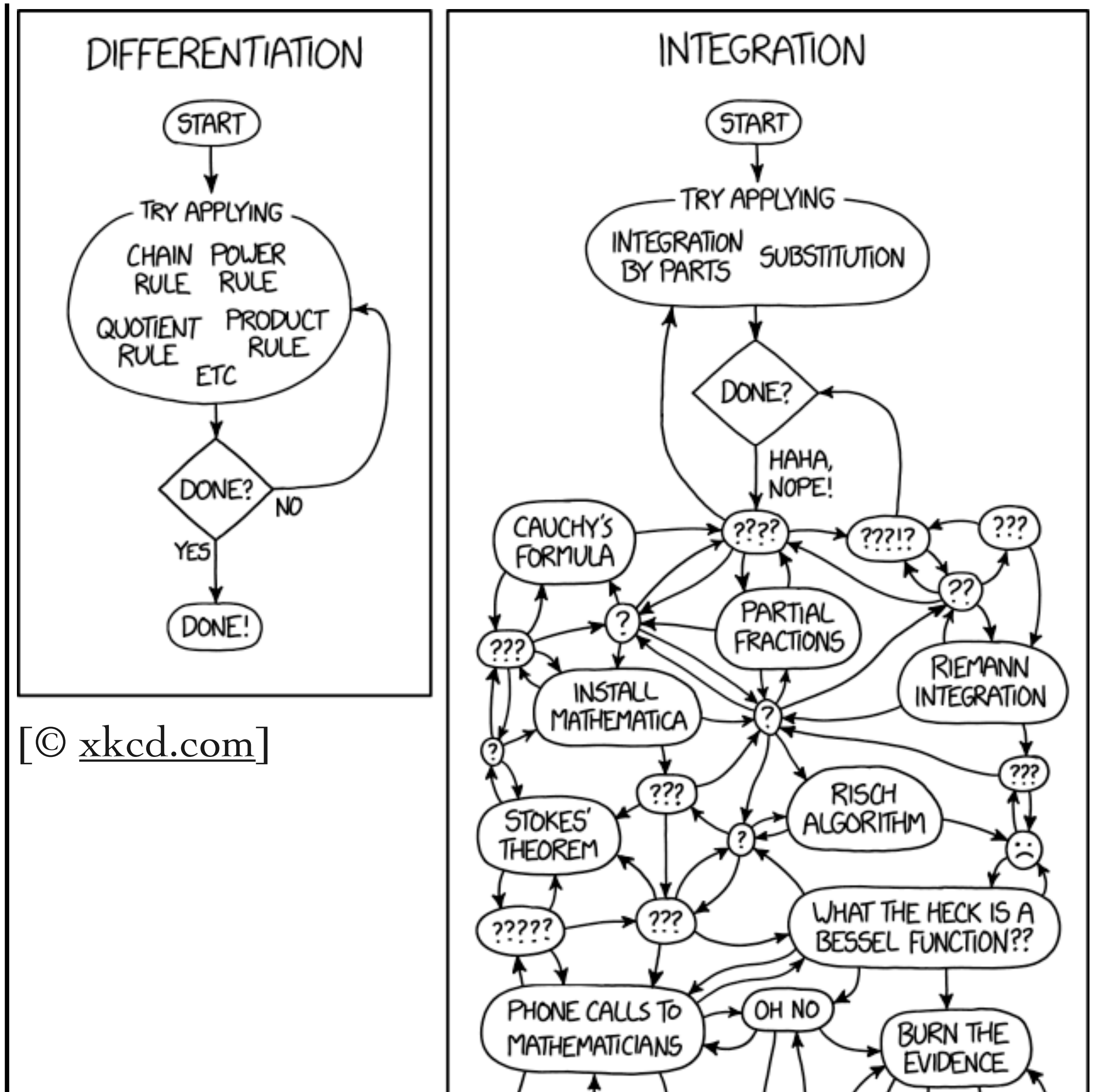
Feynman integrals

Analytic

- If we can obtain analytic expressions for the Laurent coefficients, the problem is solved!

- Feynman integrals must be complicated transcendental functions, depending on many variables.

- Mathematically: Periods of (very complicated, singular) algebraic varieties.



[© xkcd.com]

Feynman integrals

Analytic

- If we can obtain analytic expressions for the Laurent coefficients, the problem is solved!
- Feynman integrals must be complicated transcendental functions, depending on many variables.
- Mathematically: Periods of (very complicated, singular) algebraic varieties.

Numerical

- Advantage: form of functions and number of variables irrelevant.

Feynman integrals

Analytic

- If we can obtain analytic expressions for the Laurent coefficients, the problem is solved!
- Feynman integrals must be complicated transcendental functions, depending on many variables.
- Mathematically: Periods of (very complicated, singular) algebraic varieties.

Numerical

- Advantage: form of functions and number of variables irrelevant.
- Problem: Divergences and regularisation!
- There are algorithms that provide integrands for the Laurent coefficients.
- These algorithms are typically not very efficient.

Project B04

- Goal of Project B04: Development of novel purely numerical approaches for Feynman integrals!

- Important questions we will address:

  ➡ Can one identify a 'basis' of Feynman integrals such that most of the complicated integrals are finite?

  ➡ Can we develop an efficient numerical algorithm to integrate those integrals?

Feynman integrals

$$\int_{\mathbb{R}^D} d^D k \frac{\mathcal{N}}{D_1 D_2 \cdots D_p}$$ Polynomial

Propagator: $D_i = (k - q_i)^2 - m_i^2$ Real positive number (mass)

Scalar product: $v^2 = v_0^2 - (v_1^2 + v_2^2 + \ldots + v_{D-1}^2)$

➡ Singularities lie on quadrics: hyperbolas and ellipsoids.

- The integral over k_0 can be done using residues. [Feynman; Catani, Rodrigo, …]
  - ➡ Integrand more complicated.
  - ➡ Fewer integrations left.
  - ➡ Can use MC methods to do remaining integrations (if integral converges).

Feynman integrals

- Important observation: After k_0 integration, all singularities lie on ellipsoids.

  [Capatti, Hirschi, Kermanschah, Pelloni, Ruijl]

  ➡ Ellipsoids are compact.

- There are quadrature methods that can be applied to such integrals!

  ➡ Possible advantage: Exponential convergence!

- Goal of B04:

  ➡ Develop and implement a quadrature algorithm for Feynman integrals!

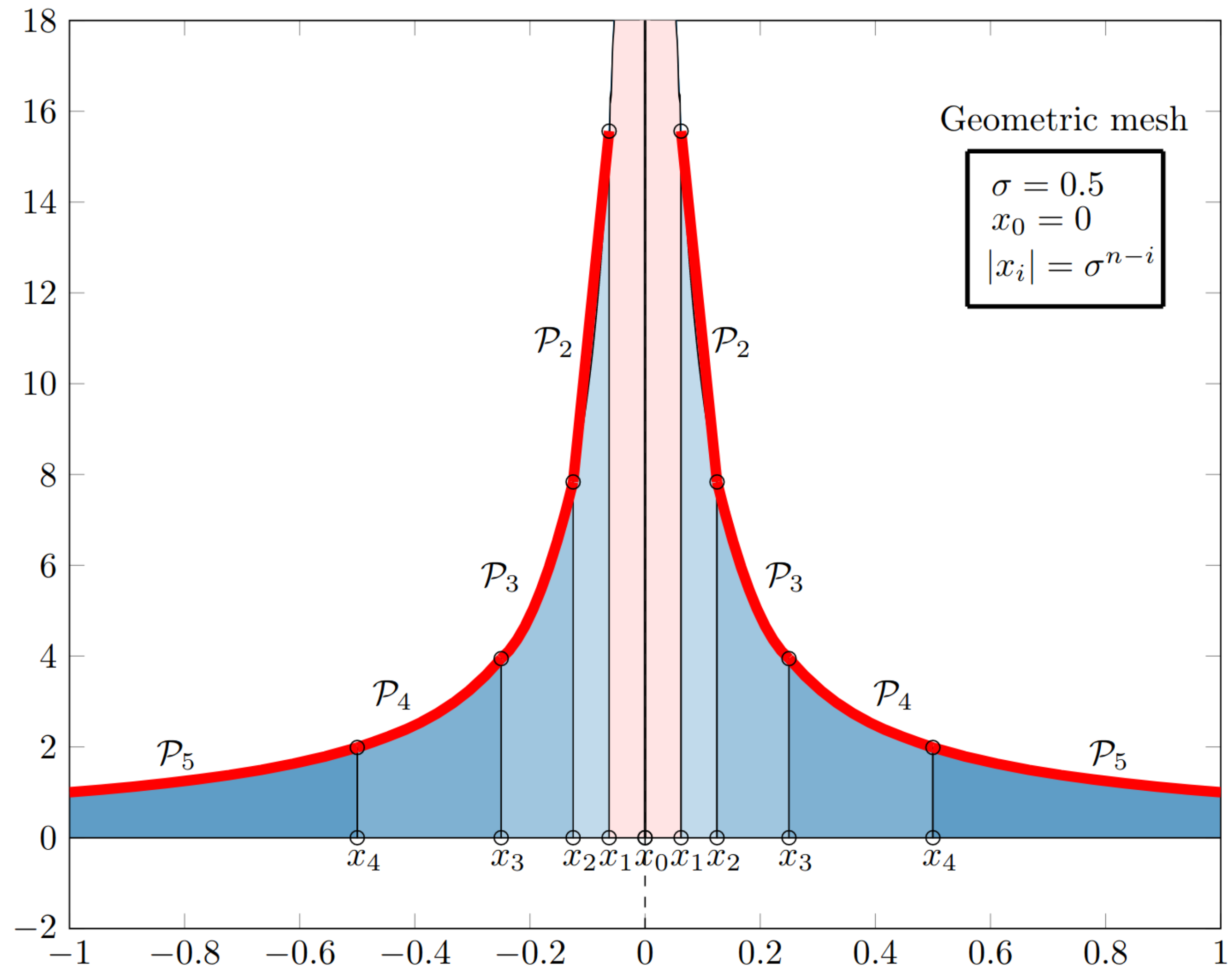
1D hp-quadrature

⟨NuMeriQS⟩

- hp interpolation in 1D

➡ Using a geometric mesh, domain is subdivided such that all domain elements not bordering the singularity have (exponential) p-scaling.

➡ Elements bordering singularity have general h-scaling, which is exponential with respect to mesh size n.

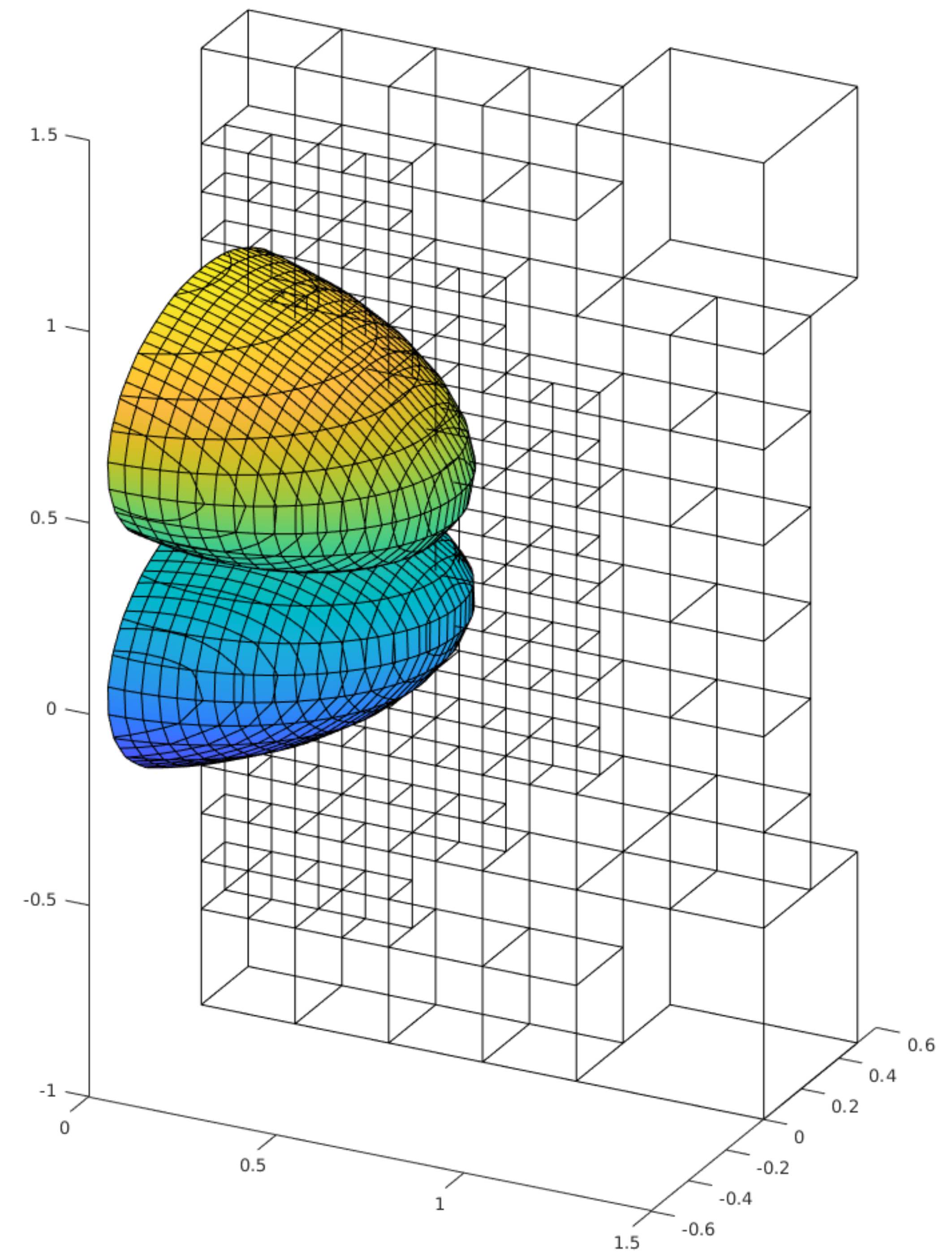


[Slide by B. Kovačić]

hp-quadrature

- hp-quadrature for ellipsoidal singularities

➡ Domain is partitioned such that the diameter of each element is proportional to distance from singularity.



➡ Employ quadratures of orders scaling with respect to the volume of each element.

➡ Regularize singular elements using Duffy transformations.

[Slide by B. Kovačić]

Project B04

- Numerical methods can only be used if integral converges.

$$\int_{\mathbb{R}^D} d^D k \frac{\mathcal{N}}{D_1 D_2 \cdots D_p}$$

- It is possible to choose the numerator polynomials so that many of the most complicated integrals are finite! [Gambuti, Kosower, Novichkov, Tancredi]

- Goal of B04:

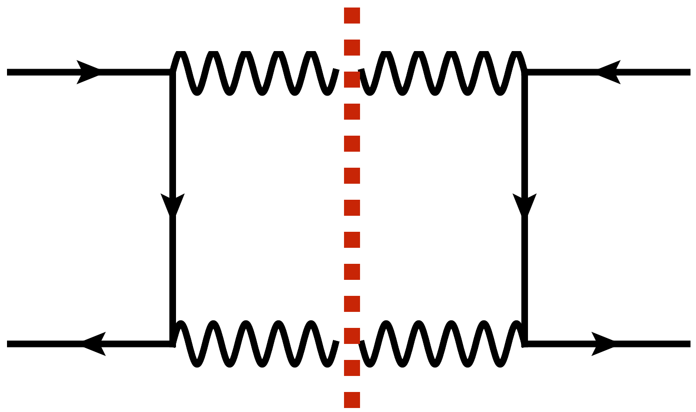
  ➡ Develop and implement an algorithm to render most complicated (basis) integrals finite.

Conclusion

- Work on Project B04 has started!
    - ➡ We have implemented a general algorithm to perform the residue integration.
    - ➡ We have identified a simple example of a finite integrals, and we are implementing a quadrature algorithm.
    - ➡ We are implementing the algorithm to choose the numerators.
- Final goal: divide et impera!
    - ➡ Find a basis of integrals where:
        - divergent integrals are 'easy': can be done analytically.
        - 'complicated' integrals are finite: needs efficient numerical algorithms.

Perturbation Theory

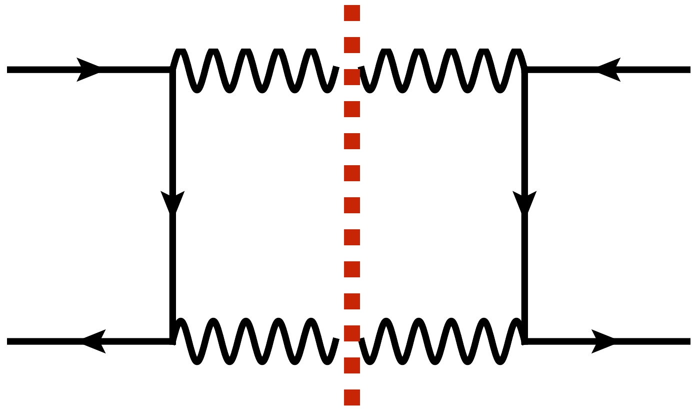
- Leading order (LO):



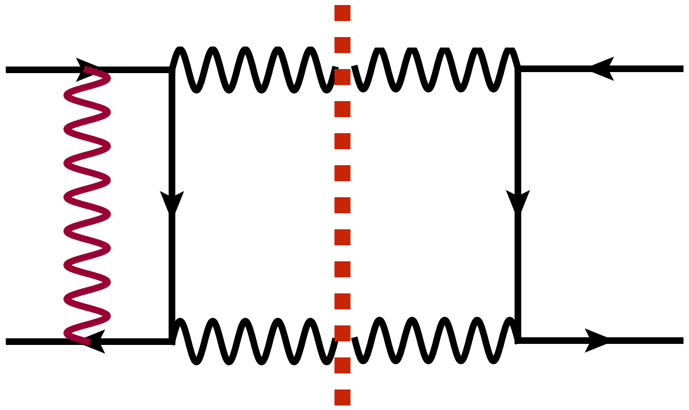
Perturbation Theory

⟨NuMeriQS⟩

- Leading order (LO):



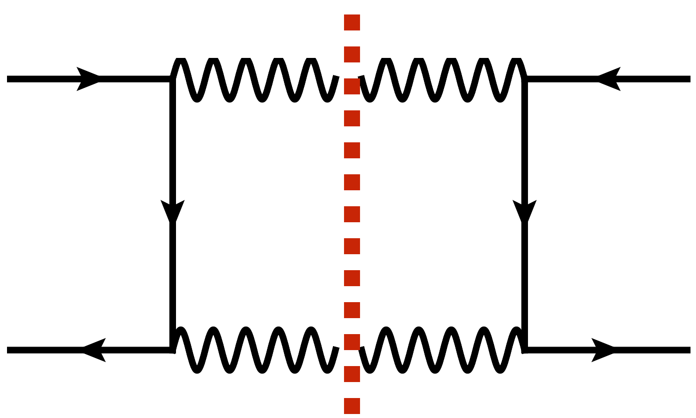
- Next-to-LO (NLO):



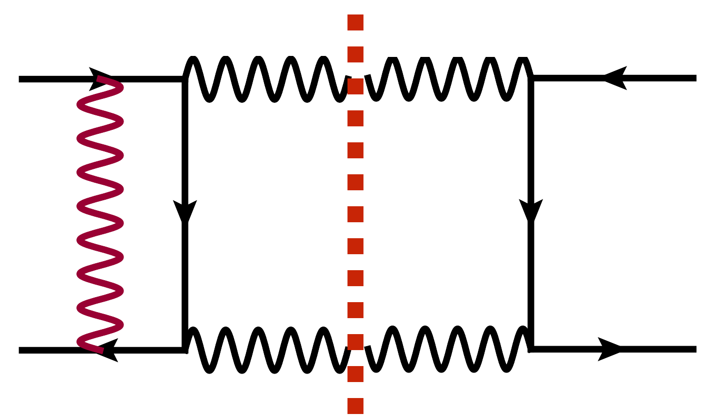
Perturbation Theory

⟨NuMeriQS⟩

- Leading order (LO):

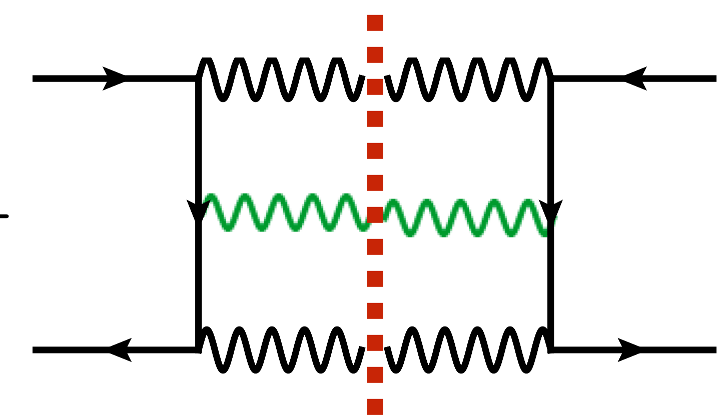


- Next-to-LO (NLO):



Virtual

+

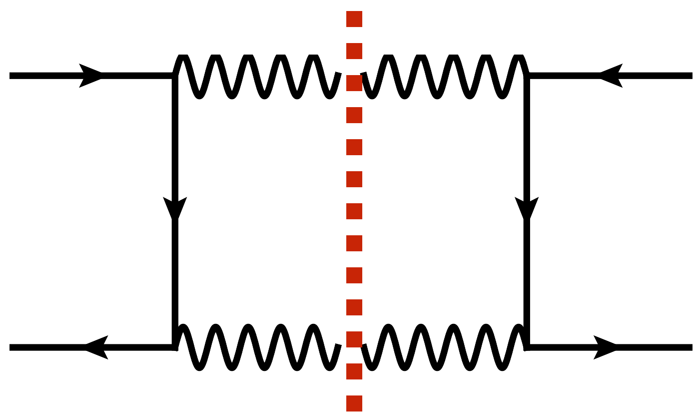


Real

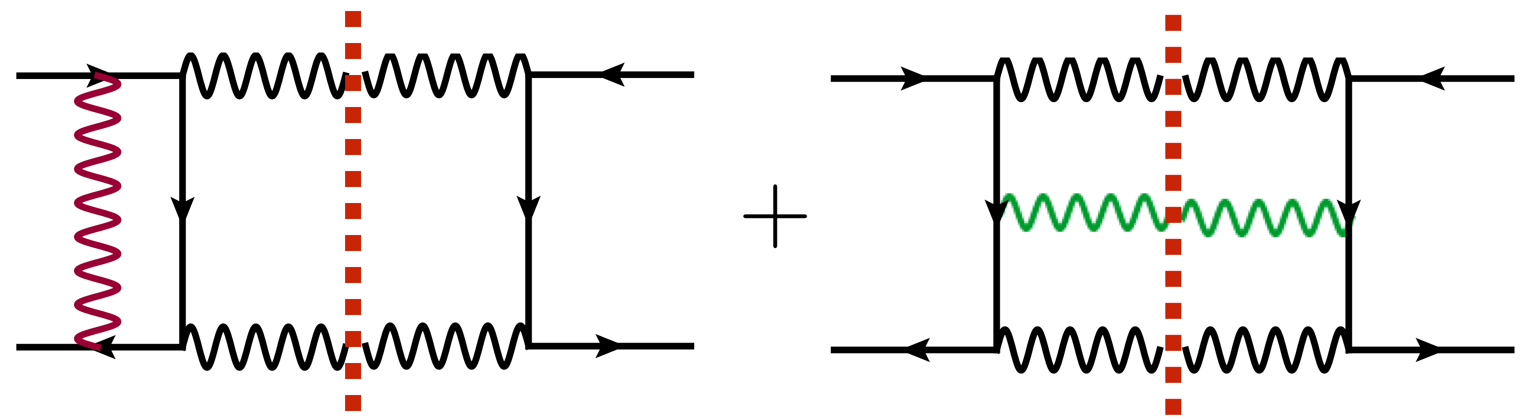
Individually divergent, but sum is finite.

Perturbation Theory

- Leading order (LO):



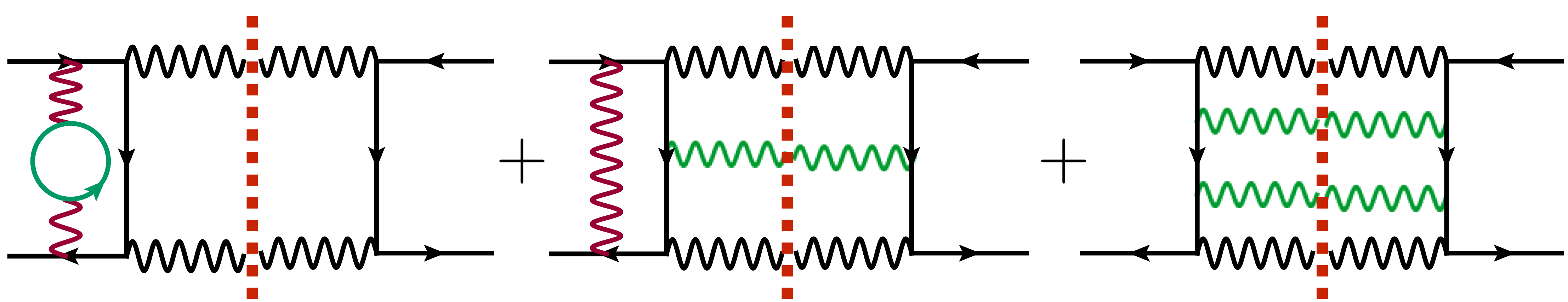
- Next-to-LO (NLO):



Virtual Real

Individually divergent, but sum is finite.

- Next-to-next-to-LO (NNLO):



Perturbation Theory

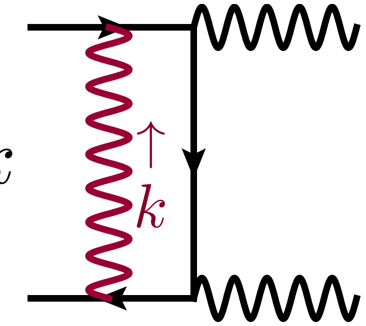
- Loop integrations:
  - ➡ 1 loop: usually doable.
  - ➡ 2 loops: typically $2 \to 2$ and $2 \to 3$ with massless particles.
  - ➡ 3 loops: $2 \to 1$ and first $2 \to 2$ with massless particles.

- Combing real and virtual corrections:
  - ➡ NLO: usually doable.
  - ➡ NNLO: typically $2 \to 2$ and first $2 \to 3$.
  - ➡ N3LO: $2 \to 1$.

Perturbation Theory

- Loop integrations:

$$\int d^4k$$



  - ➡ 1 loop: usually doable.
  - ➡ 2 loops: typically $2 \to 2$ and $2 \to 3$ with massless particles.
  - ➡ 3 loops: $2 \to 1$ and first $2 \to 2$ with massless particles.

- Combing real and virtual corrections:
  - ➡ NLO: usually doable.
  - ➡ NNLO: typically $2 \to 2$ and first $2 \to 3$.
  - ➡ N3LO: $2 \to 1$.