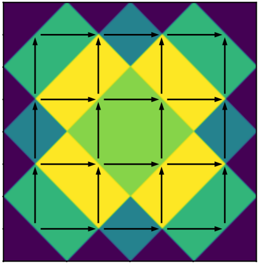

NuMeriQS Project C01

Lattice Gauge Theories: Combining Approaches with $\hat{H}$ and $\mathcal{L}$

Lena Funcke, Carsten Urbach

Gefördert durch
DFG Deutsche Forschungsgemeinschaft

⟨NuMeriQS⟩

UNIVERSITÄT BONN

Project C01: General Outline

Euclidean Monte Carlo

- well tested and developed
- access to equilibrium properties
- large systems accessible
- critical slowing down

Hamiltonian / QC

- currently being developed
- only small system sizes
- time evolution / non-equilibrium accessible

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Combine the strengths of both approaches!

- match both approaches non-perturbatively
- combine both approaches in a first application

Lattice Regularisation

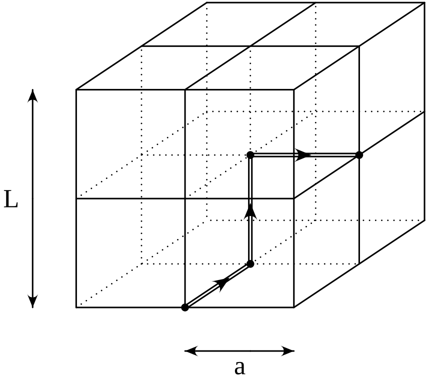
- quantum field theory requires regularisation
- lattice regularisation:

$\Rightarrow$ discretise space-time

  - hyper-cubic $L^3 \times T$-lattice with lattice spacing a

  $\Rightarrow$ momentum cut-off: $k_{\text{max}} \propto 1/a$

  - derivatives $\Rightarrow$ finite differences
  - integrals $\Rightarrow$ sums
  - gauge potentials A_μ in $G_{\mu\nu}$ $\Rightarrow$ link matrices U_μ ('')



Lattice Gauge Theories

Continuum Lagrange Density

$$\mathcal{L} = -\frac{1}{4}\left(G^a_{\mu\nu}\right)^2$$

- Field strength $G^a_{\mu\nu}$

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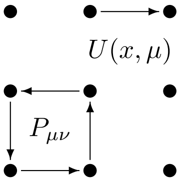
regularise / discretize

Lattice Lagrange Density

$$\mathcal{L}_{\mathsf{lat}} = -\frac{N}{g_0^2}\left(1 - \mathsf{Tr}\,\mathsf{Re}\,P_{\mu\nu}\right)$$

- Plaquette $P_{\mu\nu}$

Lattice Gauge Theories



Lattice Hamiltonian

$$\hat{H} = \frac{g^2}{2}\sum \hat{L}^2_{c,k} - \frac{1}{2g^2}\sum \mathrm{Tr}\,\mathrm{Re}\,\hat{P}_{kl}$$

Lattice Lagrange Density

$$\mathcal{L}_{\mathrm{lat}} = -\frac{N}{g_0^2}\left(1 - \mathrm{Tr}\,\mathrm{Re}\,P_{\mu\nu}\right)$$

- Plaquette $P_{\mu\nu}$

time continuum limit / Legendre Trafo

Hamiltonian Limit

Matching Lattice Lagrangian and Lattice Hamiltonian approach: $g(g_0)$

- starting in $4d$ Euclidean Space-Time:
  Hamiltonian should be obtained by taking the limit $a_t \to 0$
  [Creutz, PRD 15 (1977)]
- introduce anisotropy $\xi_0 = a_t/a_s$ and action

$$S_W = \frac{\beta}{\xi_0} \sum_{\mathbf{x},i} \mathrm{Re}\,(1 - P_{0i}(\mathbf{x})) + \beta\xi_0 \sum_{\mathbf{x},i>j} \mathrm{Re}\,(1 - P_{ij}(\mathbf{x}))$$

  with $\beta = N/g_0^2$. [Peardon, Morningstar, PRD 60, (1999)]
- which g-value corresponds to g_0 at $\xi_0 = 1$ and do observables match?

A Non-perturbative Protocol

Keep spatial lattice spacing a_s fixed while taking $\xi_0 \to 0$

- use physical distance r_0/a_s to keep a_s fixed
- r_0/a_s so-called Sommer parameter defined as

$$r^2 F(r)|_{r=r_0} = c\,.$$

  with $F(r)$ the force between a static quark and anti-quark.

- r_0/a_s can be determined from the static potential $V(r)$

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1. start with some g_0-value
2. compute $r_{\text{iso}} = r_0/a_s(g_0, \xi_0 = 1)$
3. for every $\xi_0^i < 1$ tune g_0^i such that

$$r_0/a_s(g_0^i, \xi_0^i) \;=\; r_{\text{iso}}$$

4. compute other observable $O(g_0^i, \xi_0^i)$ and take the limit $\xi_0 \to 0$

Test-case: U(1) Lattice Gauge Theory

We consider compact U(1) LGT in $2+1$ dimensions as a first test-case

$\Rightarrow$ $U \in U(1)$ or $U_\mu \equiv e^{i\varphi_\mu}$

Pros:

- Similarities to QCD
- fast to simulate
- also accessible for QC / TN

Cons:

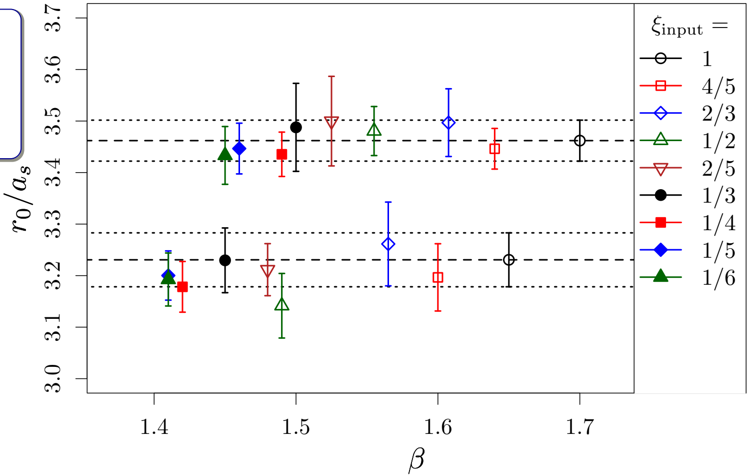
- no equivalent physical system
- no running coupling

Note: I'm omitting some renormalisation details in the following!

r_0-Matching in Practice

two starting β-values

- $\beta = 1/g_0^2 = 1.65$
- $\beta = 1/g_0^2 = 1.7$



r_0-Matching in Practice

two starting β-values

- $\beta = 1/g_0^2 = 1.65$
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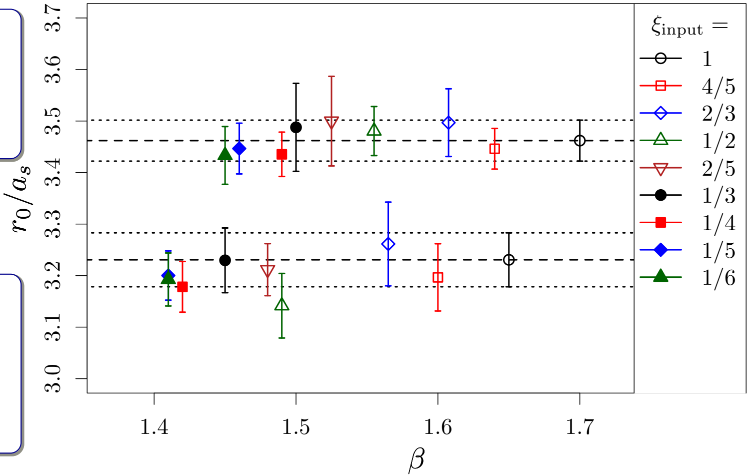
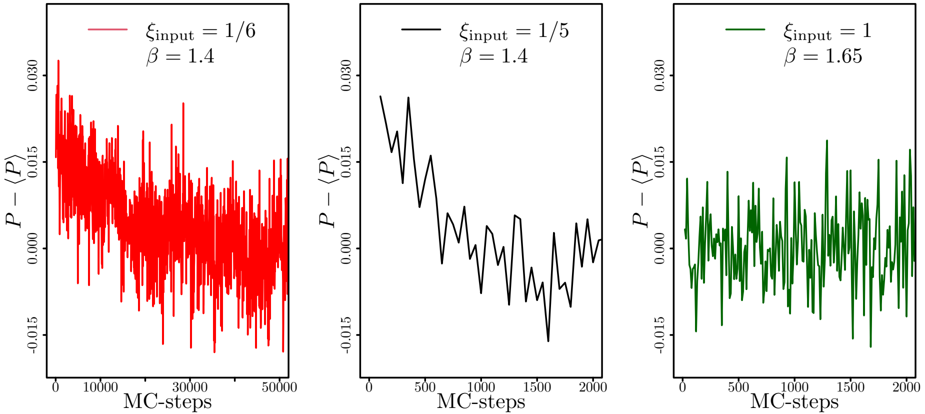
- matching within stat. errors
- at small ξ_0: critical slowing down



Critical Slowing Down in Practice

Severe slowing down with $\xi_0 \to 0$



The Finite Volume Issue

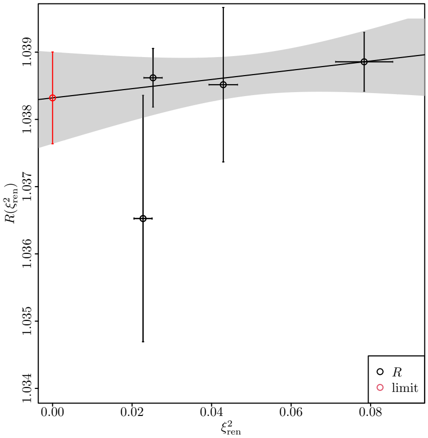
- Hamiltonian calculation: $L = 3$
- Lagrangian: $L = 16$ required

$\Rightarrow$ Repeat MC with $L = 3$

- define ratio

$$R = \frac{P(L = 16)}{P(L = 3)}$$

- and extrapolate $\xi \to 0$



Preliminary Comparison $\hat{H} \leftrightarrow \mathcal{L}$

- Hamiltonian results from exact diagonalisation
- agreement within 1.5σ or so...
- finite volume corrections essential
- currently still investigating extrapolations
- publication in preparation
  MSc work by **Christiane Groß**

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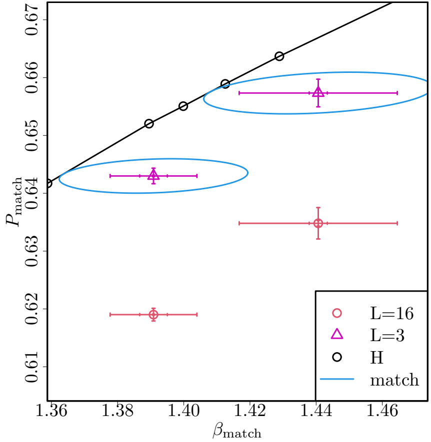
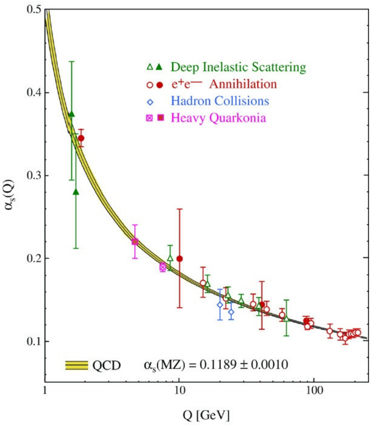


An Application: The Running Coupling

- running coupling α_{ren} depends on energy / length scale
- example α_s in QCD
- by matching to perturbation theory, access to dynamically generated scale Λ
- requires $g_0 \ll 1$



An Application: The Running Coupling

- running coupling α_{ren} depends on energy / length scale
- example α_s in QCD
- by matching to perturbation theory, access to dynamically generated scale Λ
- requires $g_0 \ll 1$
- can be determined via step scaling $s \in \mathbb{R}_+$

$$\sigma_s(\alpha_{\text{ren}}(r)) = \alpha_{\text{ren}}(s \cdot r)$$

[Lüscher, Weisz, Wolff, NPB 359, (1991)]

$$\begin{array}{l} \alpha_{\text{ren}}(r_0, g_0) \rightarrow \alpha_{\text{ren}}(sr_0, g_0) \\ \qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad \alpha_{\text{ren}}(r_1, g_1) \rightarrow \alpha_{\text{ren}}(sr_1, g_1) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad \alpha_{\text{ren}}(r_2, g_2) \ldots \end{array}$$

$U(1)$ Gauge Theories

$$\hat{H} = \frac{g^2}{2}\sum \hat{L}_k^2 - \frac{1}{2g^2}\sum \mathrm{Tr}\,\mathrm{Re}\,\hat{P}_{kl}$$

- classically, one parametrises a $U(1)$ object as

$$U(\varphi) \;=\; e^{i\varphi}$$

- φ becomes quantum number labeling states $|\varphi\rangle \in \mathcal{H}$

$$\hat{\varphi}|\varphi\rangle \;=\; \varphi|\varphi\rangle$$

- canonical momentum operator for $\hat{\varphi}$ reads ($\hat{L} \equiv \hat{p}_\varphi$)

$$\hat{L} = \;-\mathrm{i}\frac{\partial}{\partial\varphi}\,, \qquad [\hat{L},\hat{U}] = \hat{U}\,, \qquad \hat{U} = e^{i\hat{\varphi}}$$

- equivalent to the commutator $[\hat{\varphi}, \hat{L}]$

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$+$ Gauss Law

$$\sum_\mu (\hat{L}_{x,k} - \hat{L}_{x-\hat{k},k})|\psi\rangle = 0$$

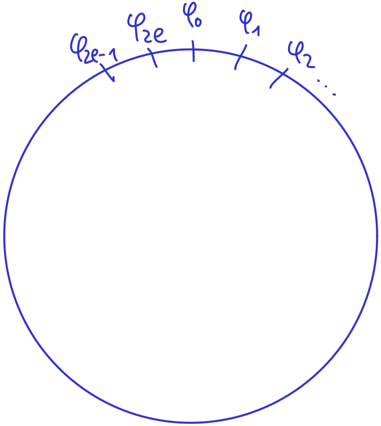
$\Rightarrow$ physical states

Discretised $U(1)$ Gauge Theory

- need to discretise gauge dofs
- interestingly, one can find discrete operators $\hat{L}$ and $\hat{U}$ exactly fulfilling $[\hat{L}, \hat{U}] = \hat{U}$
- but in a basis where $\hat{L}$ is diagonal

$$\hat{L} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 2 & 0 & \\ 0 & 0 & 3 & \\ \vdots & & & \ddots \end{pmatrix}, \hat{U} = \begin{pmatrix} 0 & -1 & 0 & \\ 0 & 0 & -1 & \\ \vdots & & & \\ 0 & 0 & & \dots 0 \end{pmatrix}$$

- angular momentum interpretation!



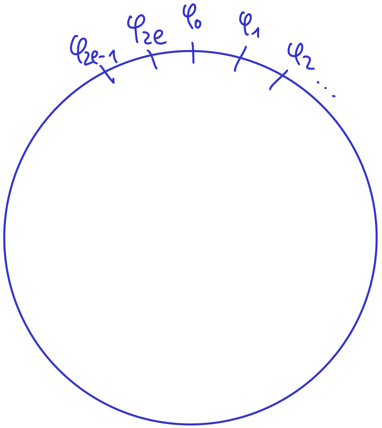
Approximation / Truncation

- discretisation requires finite l approximation
- replace $\mathrm{U}(1)$ by $\mathbb{Z}_{2l+1}$

  [Haase et al., Quantum 5, 393 (2021)]

- $\hat{U}$ not unitary, but commutation relations exact
- $\hat{L}$ diagonal
- at small g_0: magnetic basis favourable ($\hat{U}$ diagonal)

  [Haase et al., Quantum 5, (2021); Kaplan and Stryker, PRD 102, (2020); Paulson et al., PRX Quantum 2, (2021)]



Step Scaling in $2+1$d Compact U(1) Gauge Theory

- compact pure U(1) gauge theory shares confinement with QCD
- however, theory is trivial, no renormalisation of the coupling

$\Rightarrow$ proof-of-concept calculation

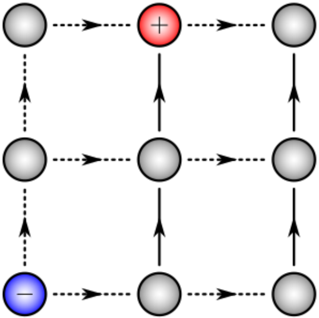
- we employ exact diagonalisation (ED) and variational quantum eigensolver (VQE)
- and use $r^2F(r)$ as a proxy for the renormalised coupling

  [Crippa et int., Funcke, et int., CU, arXiv:2404.17545]

- $L=3$ with periodic open conditions

Results: Step Scaling at fixed $l = 1$

- example for fixed truncation $l = 1$
- electrical basis only
- compare ED and VQE
- $r_1 = 1, r_2 = \sqrt{5}$
- from $\beta = 1.4$ to the perturbative regime
- at large β



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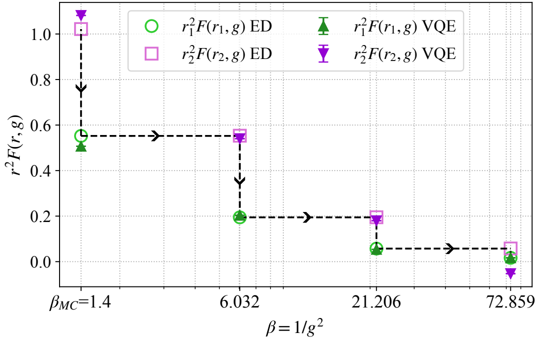
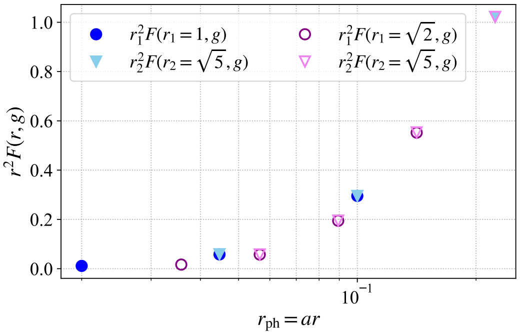


[Crippa et int., Funcke, et int., CU, arXiv:2404.17545]

Results: Running Coupling

- assume the scale has been set at $\beta = 1.4$
- e.g. 0.1 fm
- this would be determined via Monte Carlo
- now follow the step scaling to β_{max}
- reasonable agreement for different r-pairs
- residual lattice artefacts visible



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Results: Running Coupling

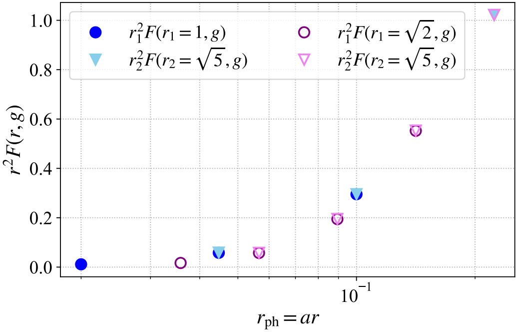
- assume the scale has been set at

Non-trivial dependence on r?

- in $2+1$d: g_0^2 has dimension of a mass
- define $\tilde{g}^2 = g_0^2/\mu$ with $[\mu]$ mass
- in terms of $\tilde{g}^2$, β-function becomes linear

$\Rightarrow$ dimensionless r^2F becomes non-trivial dependence on physical r

- residual lattice artefacts visible



[Crippa et int., Funcke, et int., CU, arXiv:2404.17545]

Currently Ongoing: Adding Matter

- $\mathrm{U}(1)$ in $2+1$d with matter fields:

$\Rightarrow$ non-trivial β-function

- requires effort on Hamiltonian as well as on Lagrangian side

- ongoing work by Emil Rosanowski and Alessio Negro

Additional Terms in $\hat{H}$

- mass term

$$\hat{H}_m \propto m\hat{\phi}_x^\dagger\hat{\phi}_x$$

- kinetic term

$$\hat{H}_{\mathrm{kin}} \propto \hat{\phi}_x^\dagger\hat{U}_{x,k}\hat{\phi}_{x+\hat{k}}$$

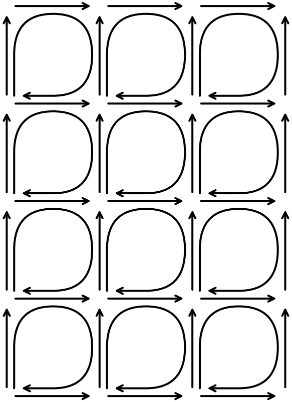
- possibly Wilson term

$$\hat{H}_W \propto a\phi^\dagger\nabla_k^\star\nabla_k\phi$$

Currently Ongoing: Magnetic Basis SU(2) Gauge Fields

- apply canonical transformation
- local string/loop formulation
- with additional *helper fields*
- plus smart way to represent Tr(P)
- promising **preliminary** results

$\rightarrow$ poster by Timo Jakobs



[[Mathur, Rathor, PRD 107 (2023)]]

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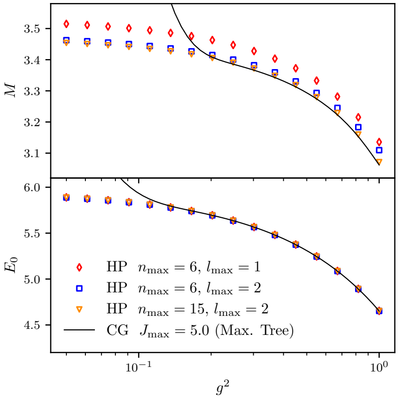
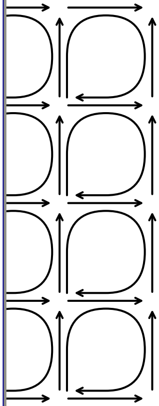
$\rightarrow$ poster by

Preliminary





Summary

- matching $\hat{H}$ and $\mathcal{L}$ in compact U(1) LGT
- steps scaling in compact U(1) LGT
- next steps are
  - including matter fields
  - non-Abelian gauge groups
- thanks to:
  A. Crippa, L. Funcke, C. Groß, T. Jakobs, A. Negro, S. Romiti, E. Rosanowski, and more
  and for your attention!