

Tensor reduction for high-rank multi-loop integrals

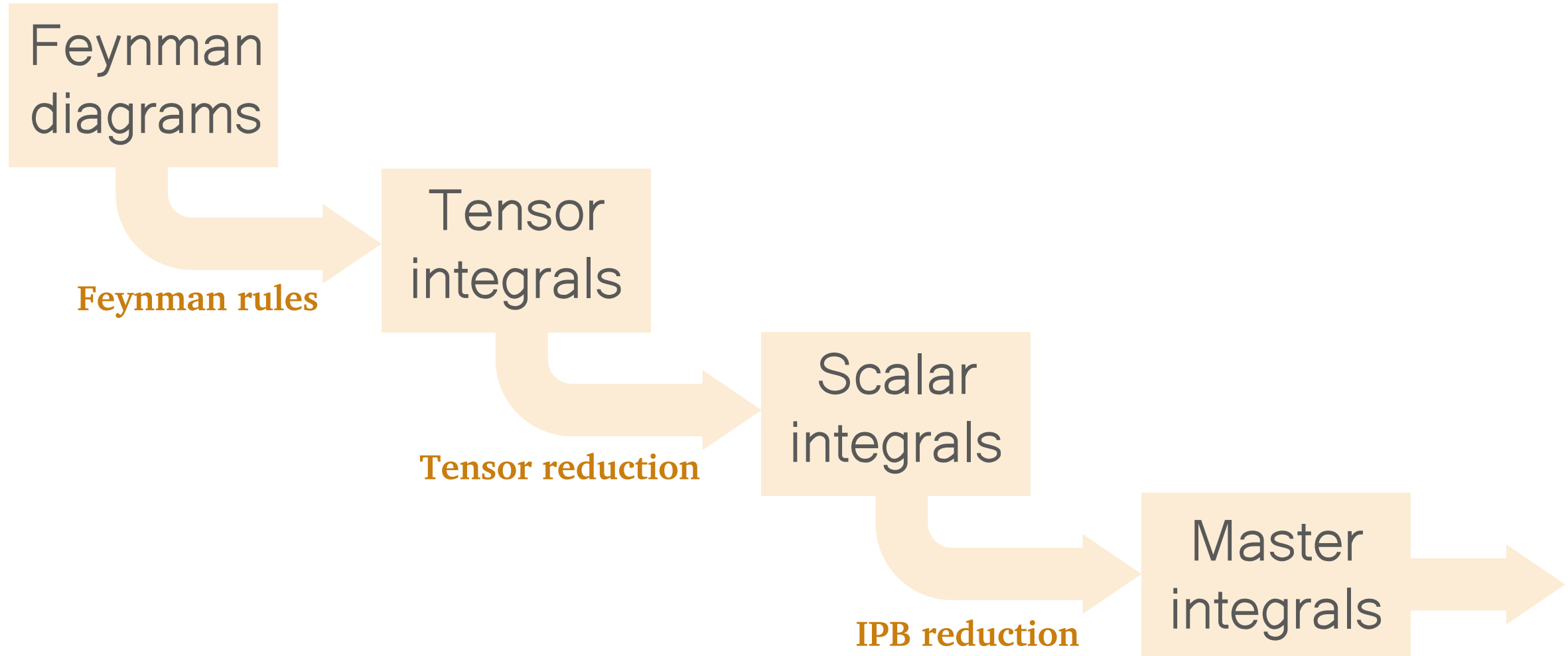
Bonn Fall HEP Meeting 2024:
Embracing Diversity in High Energy Physics



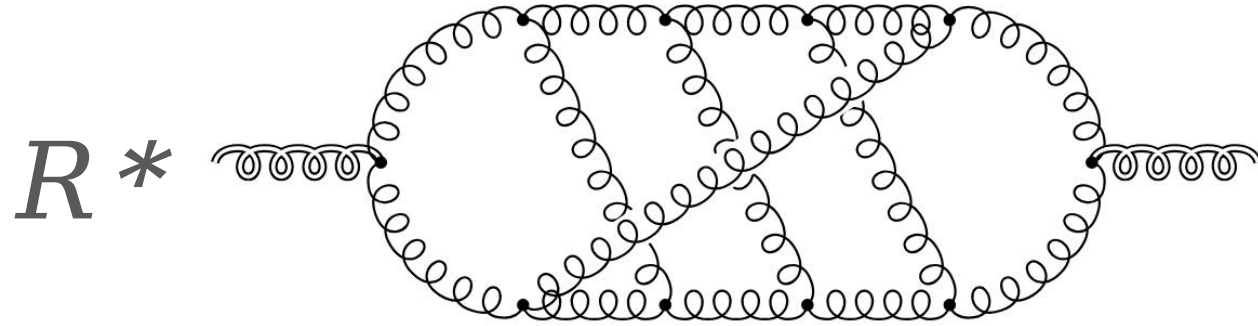
Sam Teale

In collaboration with F. Herzog, J. Goode, A. Kennedy, J. Vermaseren

Motivation



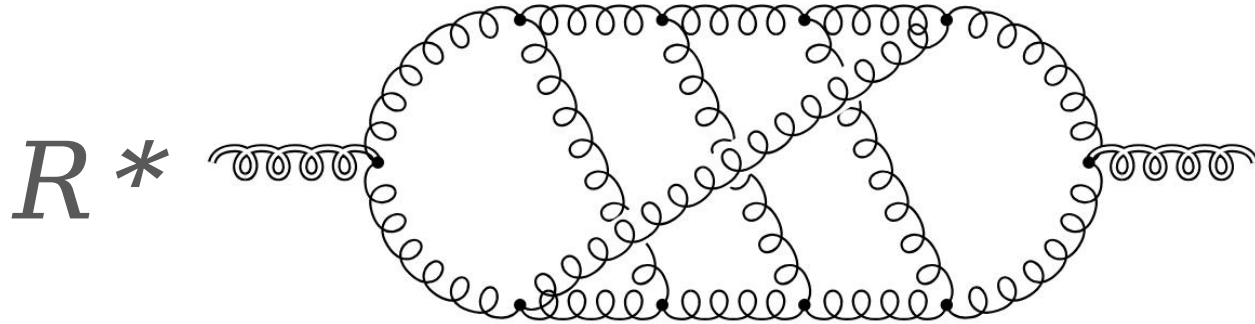
Motivation



[Herzog, Ruijl, Ueda, Vermaseren, Vogt 2017]

- R^* renormalization
- Dimensional regularization
- High rank!

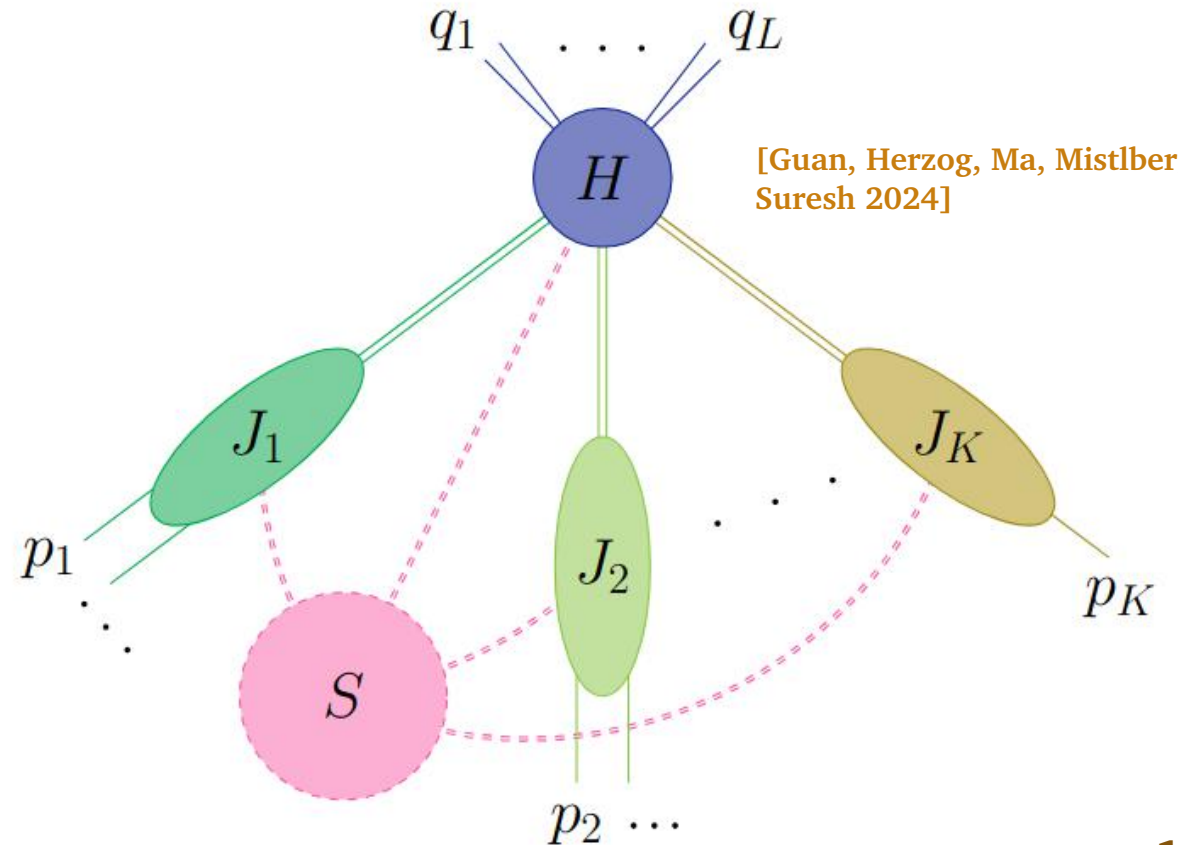
Motivation



[Herzog, Ruijl, Ueda, Vermaseren, Vogt 2017]

- Momentum space expansions
 - Method of regions
 - Small / large momentum expansions
 - Collinear expansions

- R^* renormalization
- Dimensional regularization
- High rank!



[Guan, Herzog, Ma, Mistlberger, Suresh 2024]

Tensor reduction - Approaches

- Passarino-Veltmann [Passarino, Veltmann 1974]
 - Very general
 - High-rank gets expensive
- Parametric rep. [Tarasov, 1996] [Anastasiou, Glover, Oleari 2000]
 - Work in FP space
 - Required dimensional shift identities
 - Raises propagator powers
 - FaRe code - very general [Re Fiorentin 2015]
- Physical projectors [Peraro, Tancredi 2019]
 - 't Hooft-Veltman scheme **See Sara's talk!**
 - Effective for amplitude calculations
 - Physical constraints reduce basis
- Auxiliary vector [Feng, Li, Li; Hu, Li, Li; Feng, Li, Wang, Zhang; Feng, Hu, Li, Song]
 - Full analytic solution [Feng, Li, 2022]
 - So far only 2-loop for some topologies
- Unitarity [Ossola, Papadopoulos, Pittau; Forde; Giele, Kunszt, Melnikov; Berger, Bern, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre]
 - Circumvents the tensor reduction
 - So far only 2-loop [Ita; Abreu, Febres Cordero, Ita, Page, Zeng; Badger, Brønnum-Hansen, Hartanto, Peraro]
- Orbit-partition [Herzog, Ruijl, Ueda, Vermaseren, Vogt 2017] [Anastasiou, Karlen, Vicini, 2023]
 - In D dimensions [Goode, Herzog, Kennedy, ST, Vermaseren, 2024]
 - solved to rank 32 and arbitrary loop
 - Automated implementation - OPITeR

Motivating example

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma} A_1 + g^{\mu\rho}g^{\nu\sigma} A_2 + g^{\mu\sigma}g^{\nu\rho} A_3$$

D dimensional, Lorentz
invariant + vacuum

scalar integrals

short hand

$$I = \sum_i t_i A_i$$

Motivating example

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D dimensional, Lorentz
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scalar integrals

short hand

$$I = \sum_i t_i A_i$$

- Contract integral with each tensor in turn

$$\begin{bmatrix} I^{\mu}_{\mu}{}^{\rho}{}_{\rho} \\ I^{\mu}_{\rho\mu}{}^{\rho} \\ I^{\mu}_{\rho}{}^{\rho}{}_{\mu} \end{bmatrix} = \begin{bmatrix} D^2 & D & D \\ D & D^2 & D \\ D & D & D^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

Motivating example

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma} A_1 + g^{\mu\rho}g^{\nu\sigma} A_2 + g^{\mu\sigma}g^{\nu\rho} A_3$$

short hand

$$I = \sum_i t_i A_i$$

D dimensional, Lorentz invariant + vacuum

scalar integrals

- Contract integral with each tensor in turn
- And solve

$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \frac{1}{D(D+2)(D-1)} \begin{bmatrix} D+1 & -1 & -1 \\ -1 & D+1 & -1 \\ -1 & -1 & D+1 \end{bmatrix} \begin{bmatrix} I^{\mu}_{\mu}{}^{\rho}{}_{\rho} \\ I^{\mu}_{\rho\mu}{}^{\rho} \\ I^{\mu}_{\rho}{}^{\rho}{}_{\mu} \end{bmatrix}$$

Projectors

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma} A_1 + g^{\mu\rho}g^{\nu\sigma} A_2 + g^{\mu\sigma}g^{\nu\rho} A_3$$

- Define projectors such that:

$$P_i \cdot t_j = \delta_{ij}, \quad P_i \cdot I = A_i$$

short hand

$$I = \sum_i t_i A_i$$

Projectors

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- Define projectors such that:

$$P_i \cdot t_j = \delta_{ij}, \quad P_i \cdot I = A_i$$

- With an ansatz

$$P_1^{\mu\nu\rho\sigma} = g^{\mu\nu} g^{\rho\sigma} B_1 + g^{\mu\rho} g^{\nu\sigma} B_2 + g^{\mu\sigma} g^{\nu\rho} B_3$$

Projectors

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu} g^{\rho\sigma} A_1 + g^{\mu\rho} g^{\nu\sigma} A_2 + g^{\mu\sigma} g^{\nu\rho} A_3$$

short hand

$$I = \sum_i t_i A_i$$

- Define projectors such that:

$$P_i \cdot t_j = \delta_{ij}, \quad P_i \cdot I = A_i$$

- With an ansatz
- BUT should have the same **index symmetry** as the term $g^{\mu\nu} g^{\rho\sigma}$ so

$$P_1^{\mu\nu\rho\sigma} = g^{\mu\nu} g^{\rho\sigma} B_1 + (g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) B_2$$

orbit under the action of the stabilizer group

Solving the projector

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma} A_1 + g^{\mu\rho}g^{\nu\sigma} A_2 + g^{\mu\sigma}g^{\nu\rho} A_3$$

short hand

$$I = \sum_i t_i A_i$$

- Define projectors such that:

$$P_i \cdot t_j = \delta_{ij}, \quad P_i \cdot I = A_i$$

- Set up a **smaller** system to solve

$$\begin{bmatrix} P_1 \cdot t_1 \\ P_1 \cdot t_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} D^2 & 2D \\ D & D(D+1) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$B_1 = \frac{D+1}{D(D+2)(D-1)}$$

$$B_2 = \frac{1}{D(D+2)(D-1)}$$

Applying the projector

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma} A_1 + g^{\mu\rho}g^{\nu\sigma} A_2 + g^{\mu\sigma}g^{\nu\rho} A_3$$

short hand

$$I = \sum_i t_i A_i$$

$$A_1 = P_1 \cdot I = B_1 I^{\mu}_{\mu}{}^{\rho}_{\rho} + B_2 (I^{\mu}_{\rho\mu}{}^{\rho} + I^{\mu}_{\rho}{}^{\rho}_{\mu})$$

Applying the projector

$$I^{\mu\nu\rho\sigma} = g^{\mu\nu}g^{\rho\sigma} A_1 + g^{\mu\rho}g^{\nu\sigma} A_2 + g^{\mu\sigma}g^{\nu\rho} A_3$$

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$$A_1 = P_1 \cdot I = B_1 I^{\mu}_{\mu}{}^{\rho}_{\rho} + B_2 (I^{\mu}_{\rho\mu}{}^{\rho} + I^{\mu}_{\rho}{}^{\rho}_{\mu})$$

- Other projectors related by index symmetry

$$A_2 = P_2 \cdot I = B_1 I^{\mu}_{\rho\mu}{}^{\rho} + B_2 (I^{\mu}_{\mu}{}^{\rho}_{\rho} + I^{\mu}_{\rho}{}^{\rho}_{\mu})$$

$$A_3 = P_3 \cdot I = B_1 I^{\mu}_{\rho}{}^{\rho}_{\mu} + B_2 (I^{\mu}_{\rho\mu}{}^{\rho} + I^{\mu}_{\mu}{}^{\rho}_{\rho})$$

Scaling of the problem

$$I^{\mu_1 \dots \mu_N} = \sum_{\sigma \in S_2^N} g^{\mu_{\sigma(1)} \mu_{\sigma(2)}} \dots g^{\mu_{\sigma(N-1)} \mu_{\sigma(N)}} I_{\sigma(1) \dots \sigma(N)} = \sum_{\sigma \in S_2^N} g(\sigma) I(\sigma)$$


Set of independent permutations

- # tensor structures given by:

$$|S_2^N| = (N - 1)!!$$

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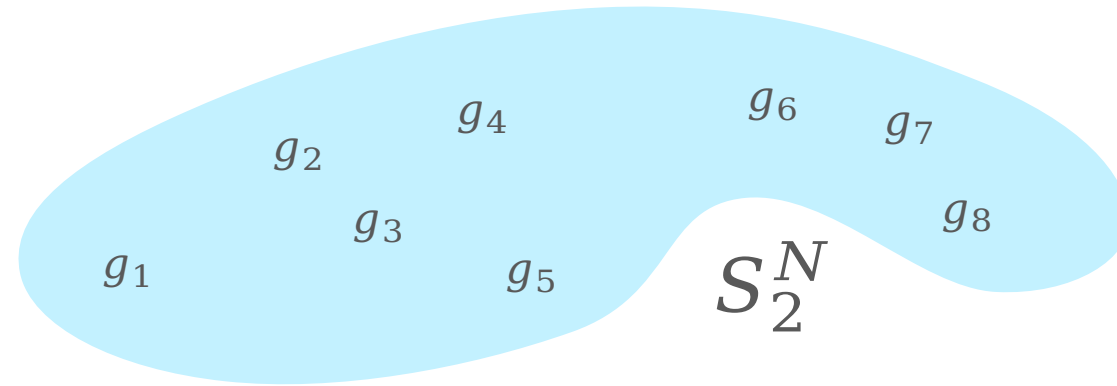
$$|S_2^N| = (N - 1)!!$$

N	2	4	6	8	10	12	14	16	18	20
$ S_2^N $	1	3	15	105	945	10,395	135,135	2,027,025	34,459,425	654,729,075

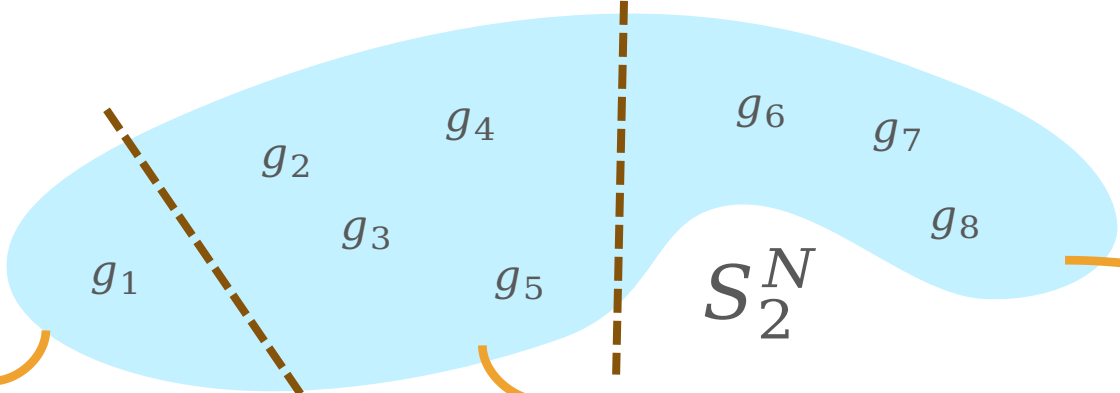
The orbit-partition formula

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

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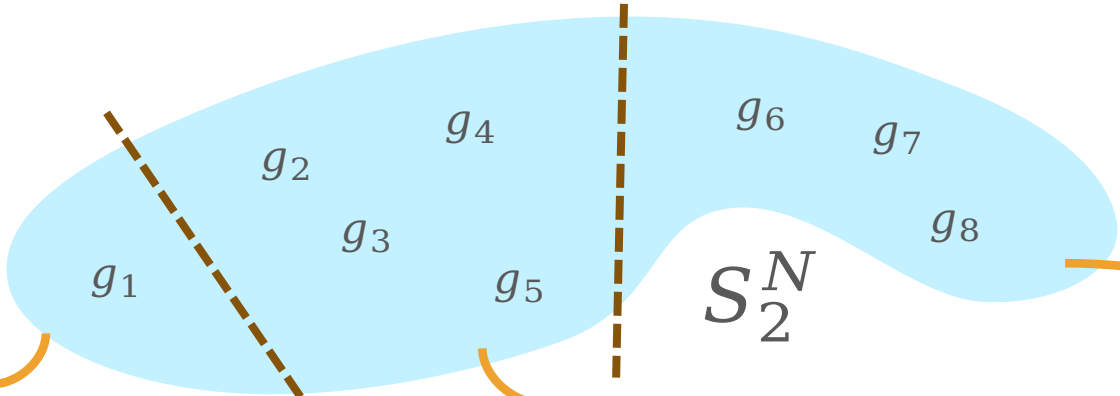


The orbit-partition formula



$$P(g_1) = c_1(g_1) + c_2(g_2 + g_3 + g_4 + g_5) + c_3(g_6 + g_7 + g_8)$$

The orbit-partition formula

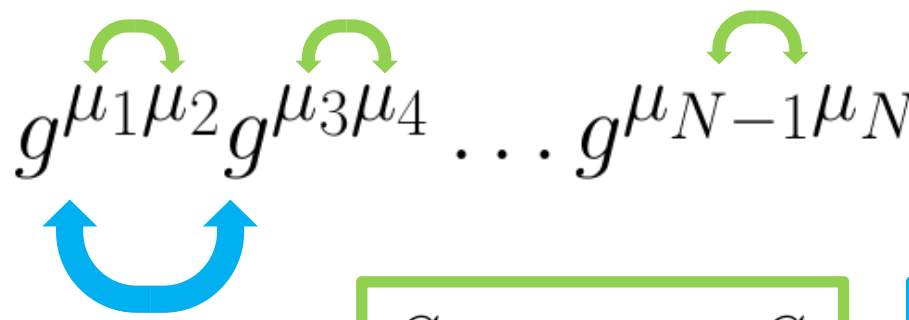
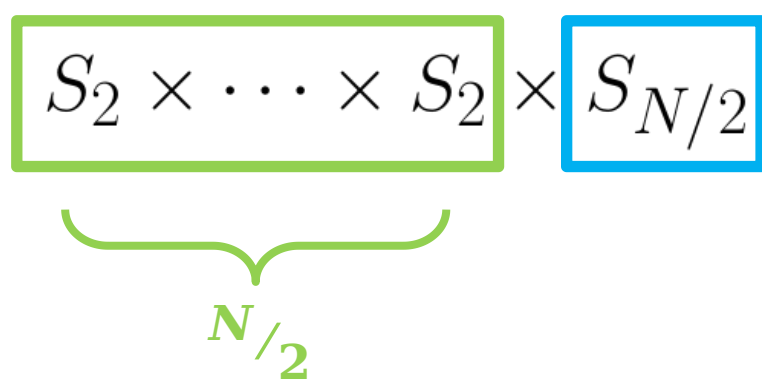


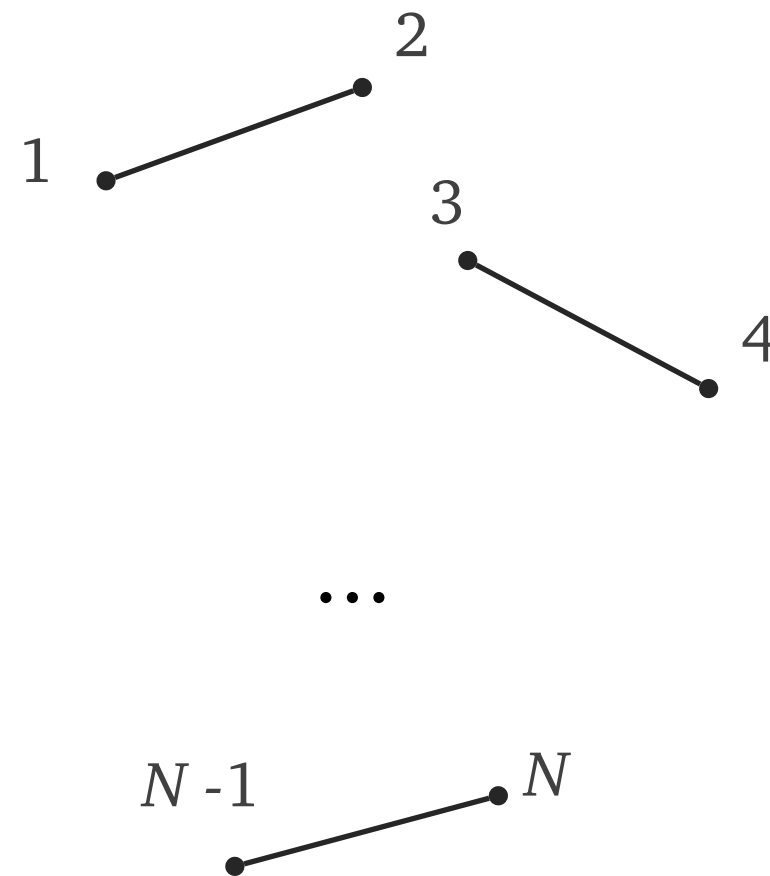
$$P(g_1) = c_1(g_1) + c_2(g_2 + g_3 + g_4 + g_5) + c_3(g_6 + g_7 + g_8)$$

$$P(\sigma) = \sum_k c_k \sum_{\sigma' \in C_k(\sigma)} g(\sigma')$$

- k labels orbit
- $C_k(\sigma)$ a set that builds the orbits

Finding orbits

$$g^{\mu_1\mu_2} g^{\mu_3\mu_4} \dots g^{\mu_{N-1}\mu_N}$$

$$\underbrace{S_2 \times \dots \times S_2}_{N/2} \times S_{N/2}$$


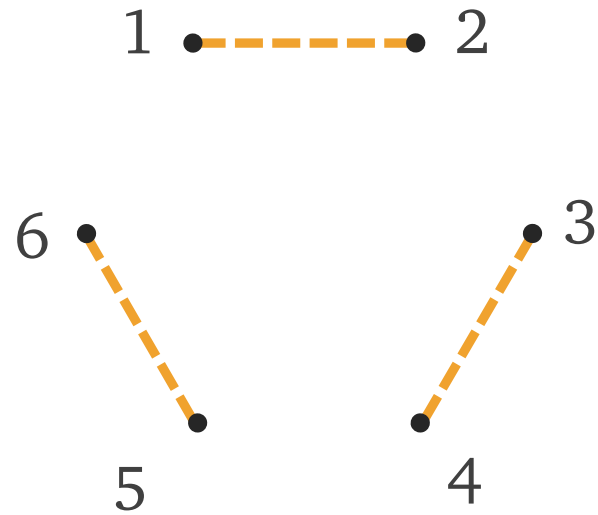


Finding orbits

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

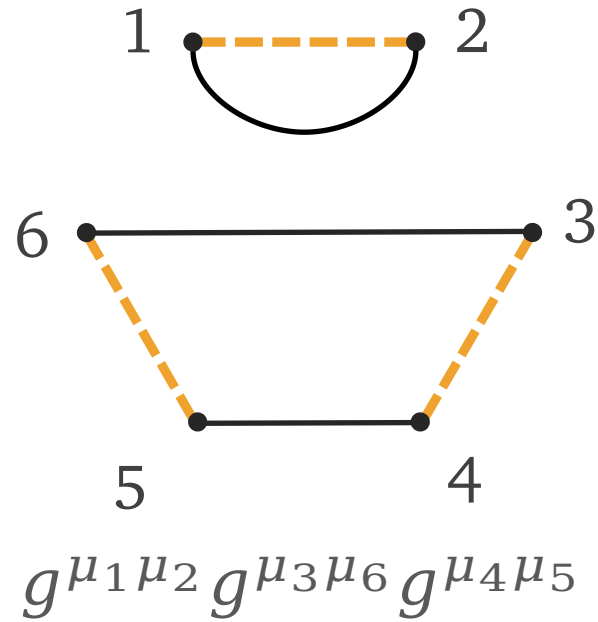
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- Projector for $g^{\mu_1\mu_2} g^{\mu_3\mu_4} g^{\mu_5\mu_6}$



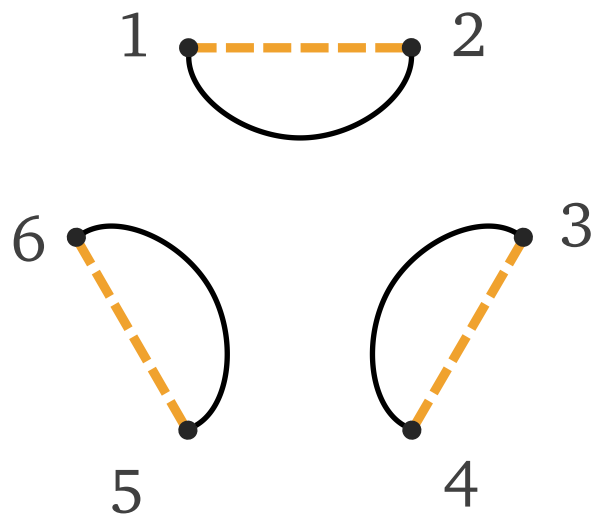
Finding orbits

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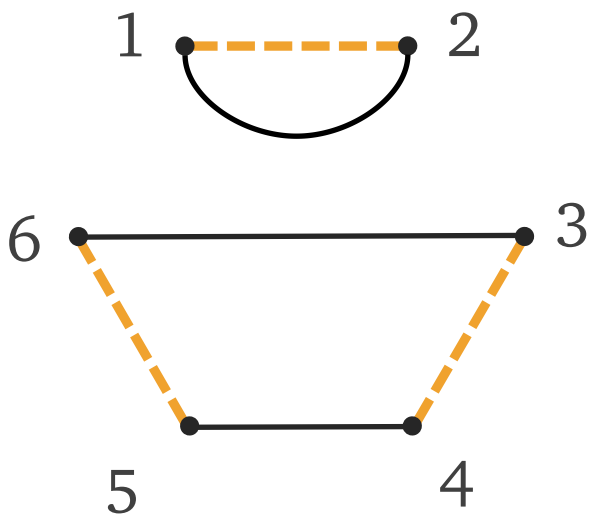


Finding orbits

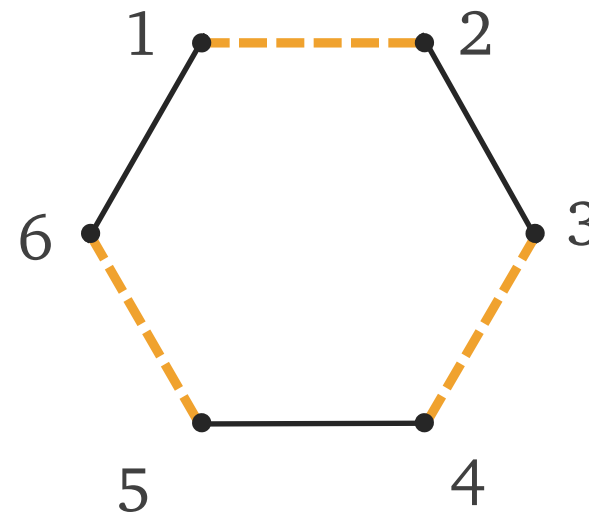
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$$g^{\mu_1\mu_2} g^{\mu_3\mu_4} g^{\mu_5\mu_6}$$



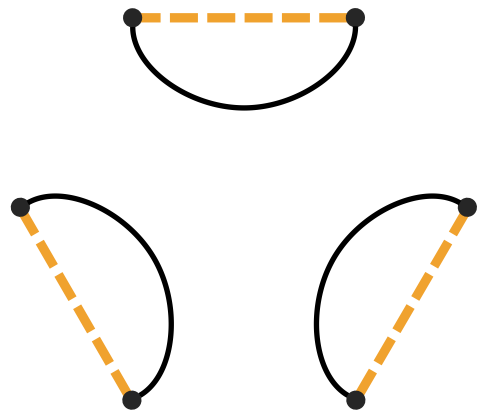
$$g^{\mu_1\mu_2} g^{\mu_3\mu_6} g^{\mu_4\mu_5}$$



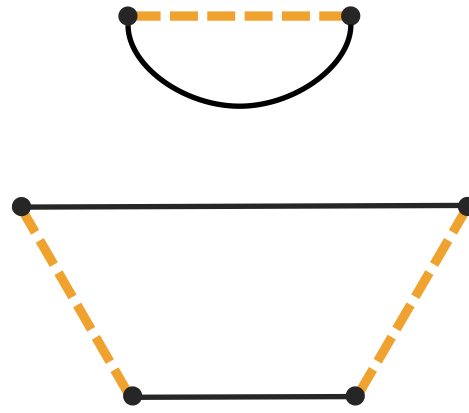
$$g^{\mu_1\mu_6} g^{\mu_2\mu_3} g^{\mu_4\mu_5}$$

Finding orbits

- Projector for $g^{\mu_1\mu_2} g^{\mu_3\mu_4} g^{\mu_5\mu_6}$

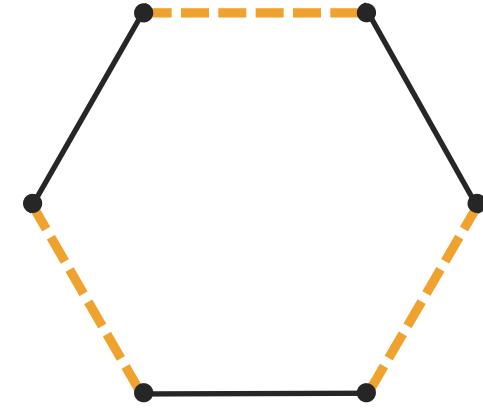


$$g^{\mu_1\mu_2} g^{\mu_3\mu_4} g^{\mu_5\mu_6}$$



$$g^{\mu_1\mu_2} g^{\mu_3\mu_6} g^{\mu_4\mu_5}$$

+ 5 terms



$$g^{\mu_1\mu_6} g^{\mu_2\mu_3} g^{\mu_4\mu_5}$$

+ 7 terms

15 =	1	+	6	+	8
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Finding orbits

- Orbits \leftrightarrow Diagrams
- Diagrams characterised by *cycle structure*

$$P(\sigma) = \sum_{\lambda \vdash N/2} c_\lambda T_\lambda(\sigma)$$

diagrams = integer partitions of $N/2$

Finding orbits

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N	2	4	6	8	10	12	14	16	18	20
$ S_2^N $	1	3	15	105	945	10,395	135,135	2,027,025	34,459,425	654,729,075
$p(N/2)$	1	2	3	5	7	11	15	22	30	42

Finding orbits

- We calculated the coefficients rank 32

N	Graph	c_λ
$N = 2$		$c_1 = \frac{1}{D}$
$N = 4$		$c_2 = \frac{1}{D(D+2)(D-1)}$
		$c_{11} = \frac{D+1}{D(D+2)(D-1)}$
$N = 6$		$c_3 = \frac{2}{D(D+2)(D-1)(D+4)(D-2)}$
		$c_{21} = -\frac{1}{D(D-1)(D-2)(D+4)}$
		$c_{111} = \frac{D^2+3D-2}{D(D+2)(D-1)(D+4)(D-2)}$
$N = 8$		$c_4 = \frac{-(5D+6)}{D(D-1)(D-2)(D-3)(D+6)(D+4)(D+2)(D+1)}$
		$c_{31} = \frac{2}{(D-1)(D-2)(D+2)(D-3)(D+6)(D+1)}$
		$c_{22} = \frac{D^2+5D+18}{D(D-1)(D-2)(D-3)(D+6)(D+4)(D+2)(D+1)}$
		$c_{211} = \frac{-(D^3+6D^2+3D-6)}{D(D-1)(D-2)(D-3)(D+6)(D+4)(D+2)(D+1)}$
		$c_{1111} = \frac{(D+3)(D^2+6D+1)}{D(D+4)(D+2)(D-1)(D-3)(D+6)(D+1)}$

- Ready to apply to integrals

Applying to Integrands

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

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- Vacuum projectors
- Coefficients to rank 32

Applying to Integrands

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

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- Vacuum projectors
- Coefficients to rank 32
- Symmetry of the integrand?



Fully symmetric example

$$I_1^{\mu_1 \dots \mu_6} = \int d^D k \, k^{\mu_1} k^{\mu_2} k^{\mu_3} k^{\mu_4} k^{\mu_5} k^{\mu_6} (\dots)$$

- Naively 15 terms!

Fully symmetric example

$$I_1^{\mu_1 \dots \mu_6} = \int d^D k \, k^{\mu_1} k^{\mu_2} k^{\mu_3} k^{\mu_4} k^{\mu_5} k^{\mu_6} (\dots)$$

- Naively 15 terms!
- Only one invariant tensor

e.g. $\delta^{\mu_1 \mu_2 \mu_3 \mu_4} = g^{\mu_1 \mu_2} g^{\mu_3 \mu_4} + g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} + g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}$

$$I_1^{\mu_1 \dots \mu_N} = \int d^D k \, k^{\mu_1} \dots k^{\mu_N} (\dots) = \frac{\delta^{\mu_1 \dots \mu_N}}{C(N)} \int d^D k (k^2)^{N/2} (\dots)$$

- More loops?

$$C(N) = 2^{N/2} \frac{\Gamma((D+N)/2)}{\Gamma(D/2)}$$

Integrand symmetry

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

[Goode, Herzog, ST, in preparation]

$$I_2^{\mu_1 \dots \mu_{N_1}; \nu_1 \dots \nu_{N_2}} = \int d^D k_1 d^D k_2 k_1^{\mu_1} k_1^{\mu_2} k_2^{\nu_1} k_2^{\nu_2} k_2^{\nu_3} k_2^{\nu_4} (\dots)$$

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- Contract integrand with symmetric tensor $\delta^{\mu_1 \mu_2 \nu_1 \nu_2 \nu_3 \nu_4}$

$$\delta^{k_1 k_1 k_2 k_2 k_2 k_2} = \underbrace{3(k_1 \cdot k_1)(k_2 \cdot k_2)^2}_{\delta_1^{k_1 k_1 k_2 k_2 k_2 k_2}} + \underbrace{12(k_1 \cdot k_2)^2(k_2 \cdot k_2)}_{\delta_2^{k_1 k_1 k_2 k_2 k_2 k_2}}$$

Integrand symmetry

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- recover δ_1 and δ_2 by :
 - Replace dot product with metric
 - Replace k_1 with μ_i
 - Replace k_2 with ν_i

Integrand symmetry

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

[Goode, Herzog, ST, in preparation]

$$I_2^{\mu_1 \dots \mu_{N_1}; \nu_1 \dots \nu_{N_2}} = \int d^D k_1 d^D k_2 k_1^{\mu_1} k_1^{\mu_2} k_2^{\nu_1} k_2^{\nu_2} k_2^{\nu_3} k_2^{\nu_4} (\dots)$$

$$\delta_1^{\mu_1 \mu_2 \nu_1 \nu_2 \nu_3 \nu_4} = \delta^{\mu_1 \mu_2} \delta^{\nu_1 \nu_2 \nu_3 \nu_4}$$

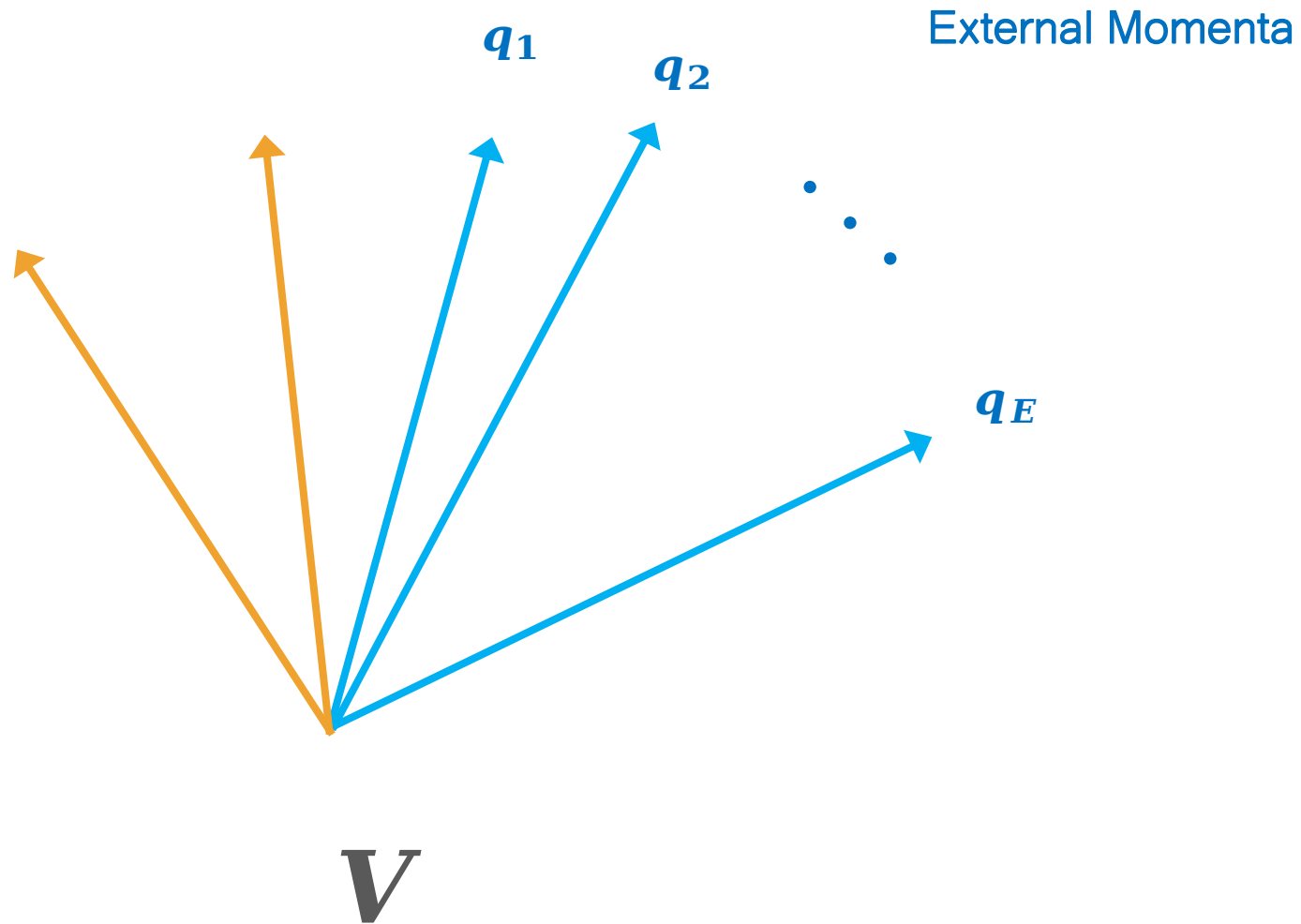
- Left with the very nice form

$$I_2^{\mu_1 \mu_2; \nu_1 \dots \nu_4} = \delta^{\mu_1 \mu_2} \delta^{\nu_1 \nu_2 \nu_3 \nu_4} \int d^D k_1 d^D k_2 P^{k_1 k_1 k_2 k_2 k_2 k_2} (\dots) \\ + \delta_2^{\mu_1 \mu_2 \nu_1 \nu_2 \nu_3 \nu_4} \int d^D k_1 d^D k_2 P^{k_1 k_2 k_1 k_2 k_2 k_2} (\dots)$$

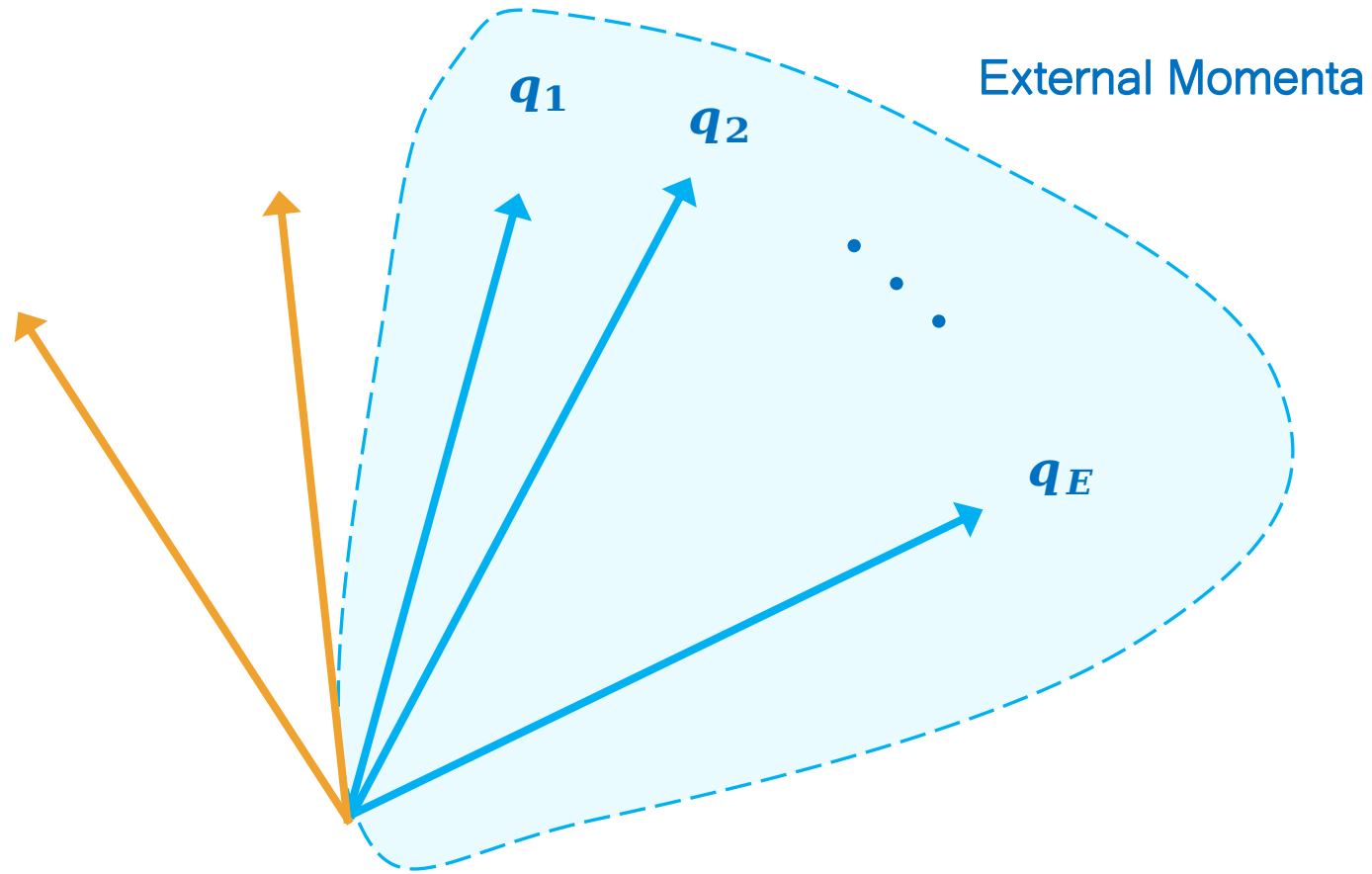
- Beyond vacuum?

Beyond vacuum

[Goode, Herzog, ST, in preparation]

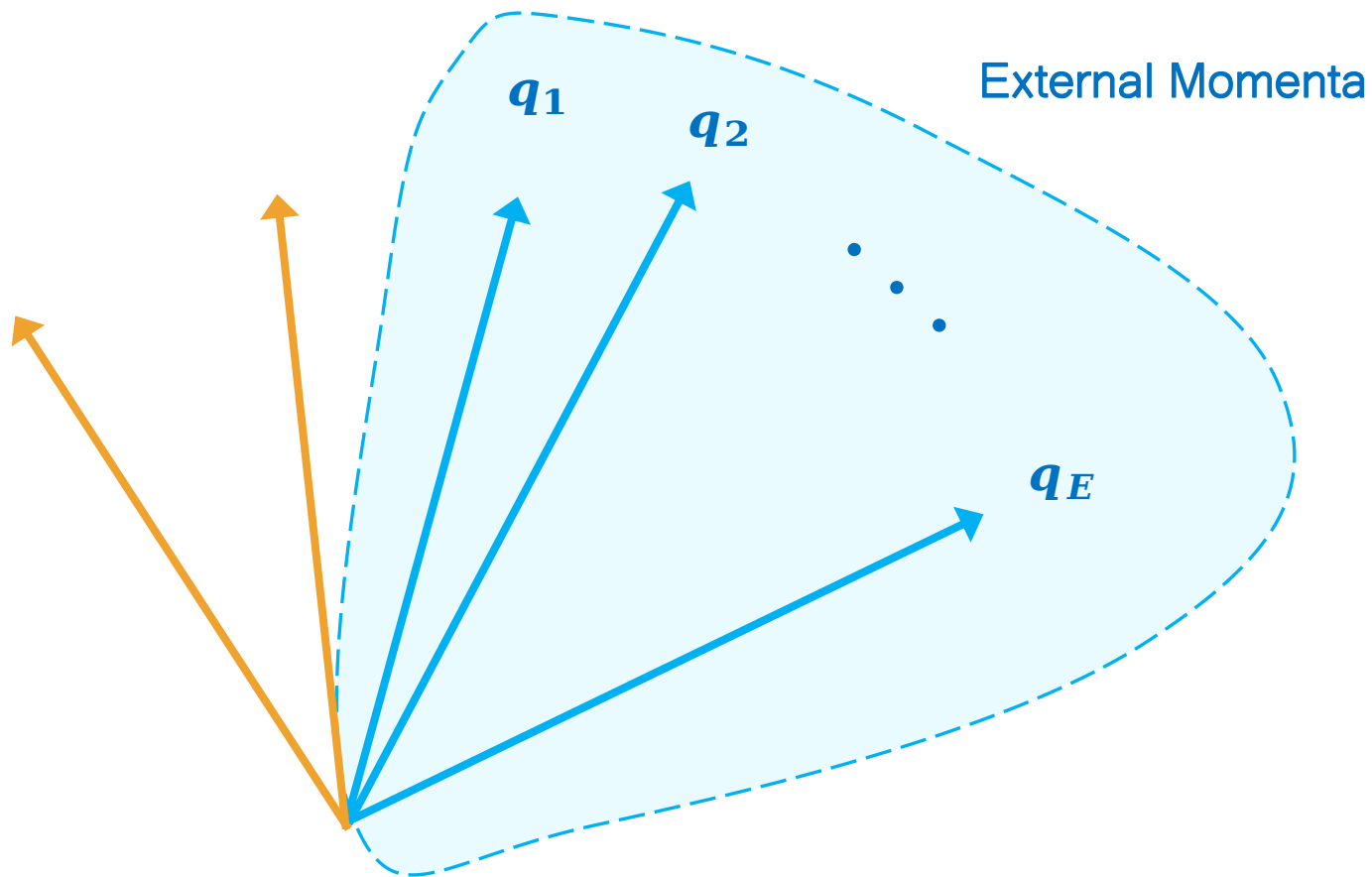


Beyond vacuum



$$\mathbf{V} = \mathbf{V}_{\perp} \oplus \mathbf{V}_{\parallel}$$

Beyond vacuum



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Van Neerven-Vermaseren basis

[van Neerven, Vermaseren 1984]

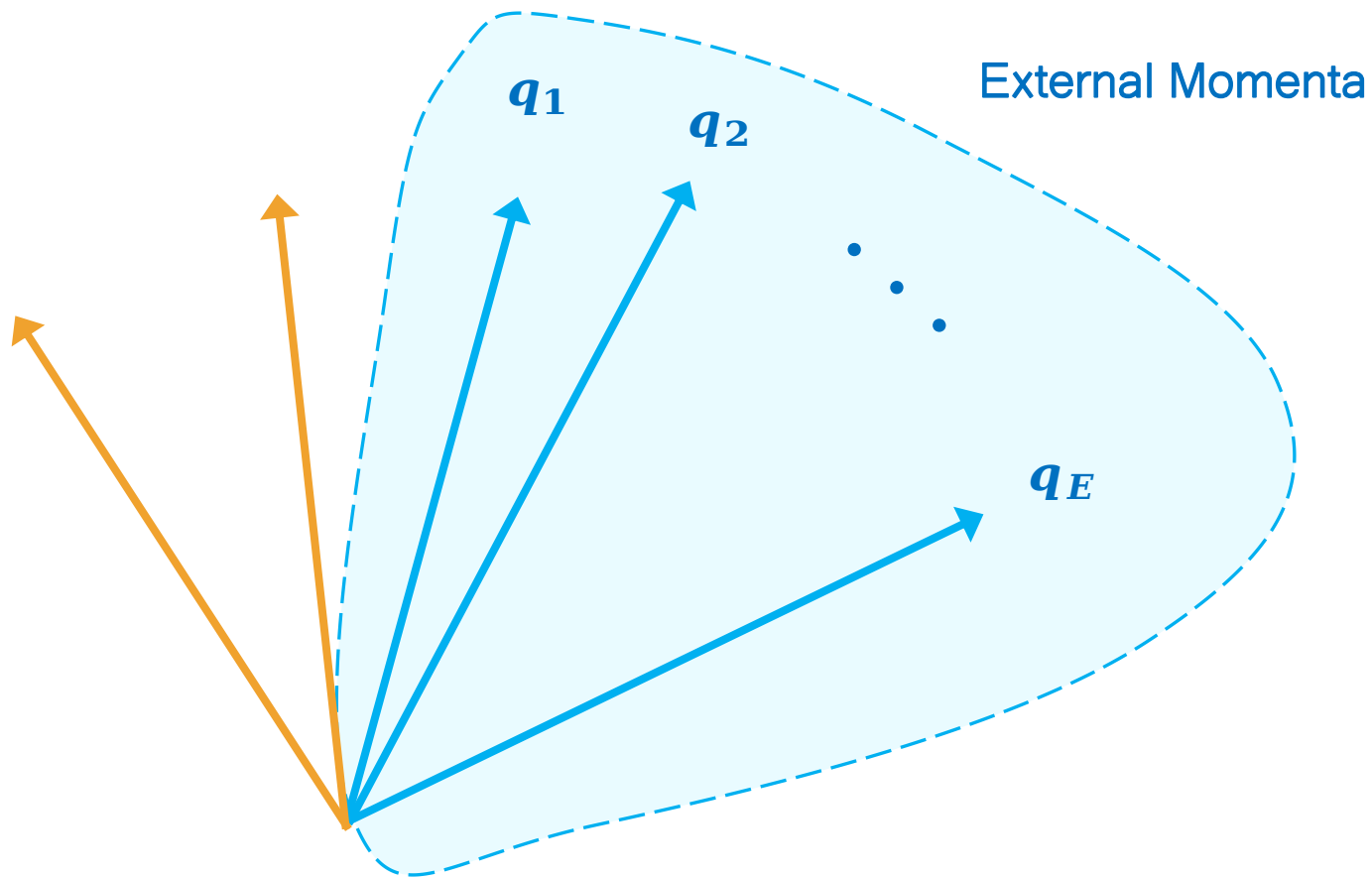
[Ellis, Kunstz, Melnikov, Zanderighi 2012]

$$g^{\mu\nu} = g_{\perp}^{\mu\nu} + g_{\parallel}^{\mu\nu}$$

$$g_{\perp}^{\mu\nu} = g^{\mu\nu} - \sum_{i,j} p_i^{\mu} H_{ij} p_j^{\nu},$$

$$q_i \cdot q_j = G_{ij}, \quad H_{ij} = (G^{-1})_{ij}$$

Beyond vacuum



$$\mathbf{V} = \mathbf{V}_{\perp} \oplus \mathbf{V}_{\parallel}$$

A similar method was used by

[Anastasiou, Karlen, Vicini, 2023]

Van Neerven-Vermaseren basis

[van Neerven, Vermaseren 1984]

[Ellis, Kunszt, Melnikov, Zanderighi 2012]

$$g^{\mu\nu} = g_{\perp}^{\mu\nu} + g_{\parallel}^{\mu\nu}$$

$$g_{\perp}^{\mu\nu} = g^{\mu\nu} - \sum_{i,j} p_i^{\mu} H_{ij} p_j^{\nu},$$

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Beyond vacuum

- Decompose momenta

$$p_i^\mu = (p_i)_\perp^\mu + (p_i)_\parallel^\mu$$

Beyond vacuum

- Decompose momenta

$$p_i^\mu = (p_i)_\perp^\mu + (p_i)_\parallel^\mu$$

- Introduce dual vectors r_i^μ $q_i \cdot r_j = \delta_{ij}$

- Decompose all the loop momenta p_i^μ

$$p_i^\mu = (p_i)_\perp^\mu + \sum_{j=1}^E p_i \cdot q_j r_j^\mu$$

Beyond vacuum

- Use our vacuum projectors on transverse integrals with:

$$D \rightarrow D - E, \quad g^{\mu\nu} \rightarrow g_{\perp}^{\mu\nu}$$

Beyond vacuum

- Use our vacuum projectors on transverse integrals with:

$$D \rightarrow D - E, \quad g^{\mu\nu} \rightarrow g_{\perp}^{\mu\nu}$$

- Convert back after integration

$$r_i^{\mu} = \sum_{j=1}^E H_{ij} q_j^{\mu}$$

$$H_{ij} = \frac{\delta_{\parallel} \begin{matrix} q_1 \dots q_{i-1} & q_{i+1} \dots q_E \\ q_1 \dots q_{j-1} & q_{j+1} \dots q_E \end{matrix}}{\Delta(q_1 \dots q_E)} (-1)^{i+j}$$

Generalised
Kronecker delta

new !

$$\Delta(q_1 \dots q_E) = \delta_{q_1 \dots q_E}^{q_1 \dots q_E}$$

Fermion lines

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

2408.05137

- Pure Lorentz
- Beyond vacuum
- fermions????

Fermion lines

- Pure Lorentz
- Beyond vacuum
- fermions????

Fermionic projectors

- construct Projectors including γ matrices

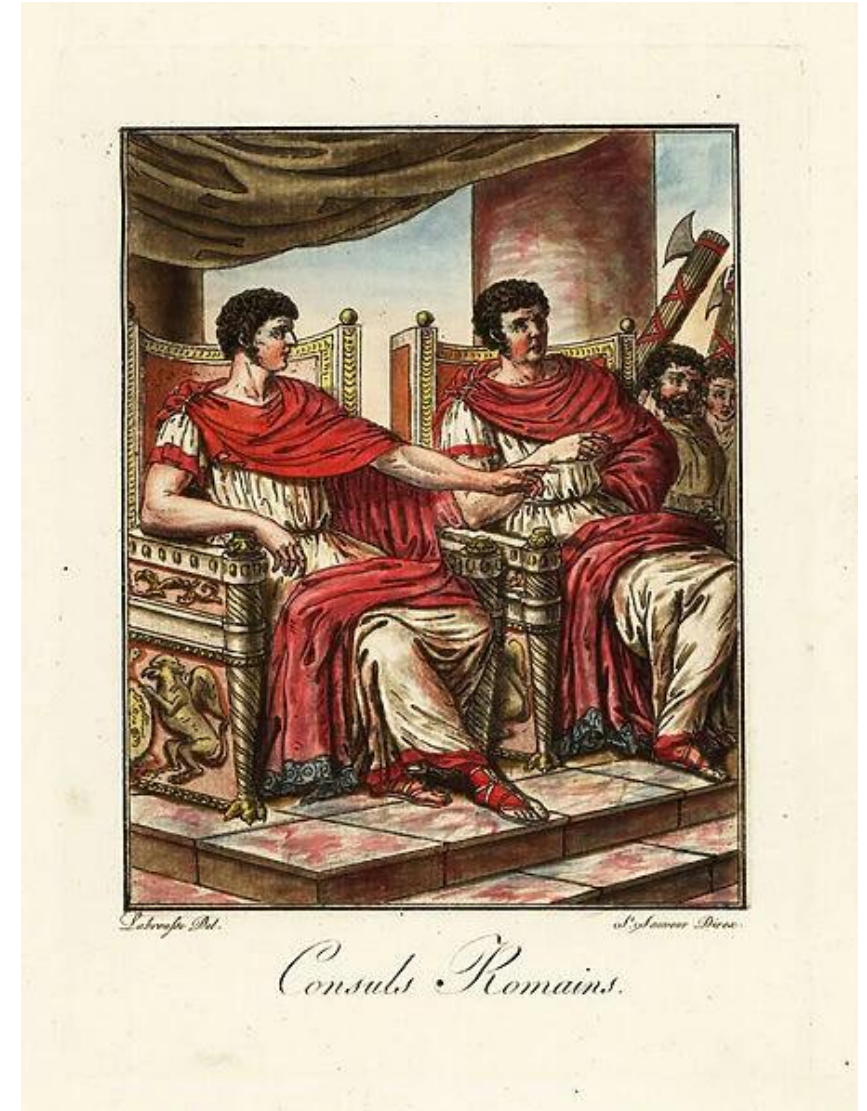
γ -factorisation

- Factorise the γ structure out of the integral

$$\int d^D k \dots \not{k} \dots = \gamma_\mu \int d^D k \dots k^\mu \dots$$

OPITER

Orbit Partition Improved
Tensor Reduction



OPITeR

Orbit Partition Improved Tensor Reduction

- Tensor reduction up to rank 20
- Arbitrary number of spinor indices
- Arbitrary loop
- Up to 8 external momenta
- Options for exploiting integrand symmetry
- Packaged as a **Form** procedure

OPITeR examples

$$\int d^D p_1 d^D p_2 p_1^{\mu_1} p_1^{\mu_2} p_2^{\nu_1} p_2^{\nu_2} p_2^{\nu_3} p_2^{\nu_4} (\dots)$$

- Define the integral

```
*-----  
#include opiter.frm  
*-----  
Local integral=ext () *loop (p1, p2)  
                *p1 (mu1) *p1 (mu2) *p2 (nu1)  
                *p2 (nu2) *p2 (nu3) *p2 (nu4) ;
```


OPITeR examples

$$\int d^D p_1 d^D p_2 p_1^{\mu_1} p_1^{\mu_2} p_2^{\nu_1} p_2^{\nu_2} p_2^{\nu_3} p_2^{\nu_4} (\dots)$$

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```

- Call the `opiter` procedure

```
#call opiter
```

OPITeR examples

[Goode, Herzog, ST, in preparation]

- `opiter/opitersettings.dat`
- Reduction mode

`tenormode 1`

no symmetry improvement

`tenormode 2`

with integrand symmetries

- Output basis

`tensorbasis 1`

g_{\perp} and dual momenta

`tensorbasis 2`

g and external momenta

OPITeR examples

- `tensormode 2`

```
integral =
```

```
+ sym(ind1(mu1, mu2) * ind2(MMu1, MMu2)) * sym(ind1(nu1, nu2, nu3, nu4) * ind2(MMu3, MMu4, MMu5, MMu6)) * ext*loop(p1, p2) * d_(MMu1, MMu2) * d_(MMu3, MMu6) * d_(MMu4, MMu5) * (  
  + p1.p1*p2.p2^2*rat(3*D + 9, D^4 + 5*D^3 + 2*D^2 - 8*D)  
  + p1.p2^2*p2.p2*rat(-12, D^4 + 5*D^3 + 2*D^2 - 8*D)  
)  
  
+ sym(ind1(mu1, mu2) * ind2(MMu1, MMu2)) * sym(ind1(nu1, nu2, nu3, nu4) * ind2(MMu3, MMu4, MMu5, MMu6)) * ext*loop(p1, p2) * d_(MMu1, MMu6) * d_(MMu2, MMu4) * d_(MMu3, MMu5) * (  
  + p1.p1*p2.p2^2*rat(-12, D^4 + 5*D^3 + 2*D^2 - 8*D)  
  + p1.p2^2*p2.p2*rat(12, D^3 + 5*D^2 + 2*D - 8)  
) ;
```

OPITeR examples

- `tensormode 2`

```
integral =
```

```
+ sym(ind1(mu1, mu2) * ind2(MMu1, MMu2)) * sym(ind1(nu1, nu2, nu3, nu4) * ind2(MMu3, MMu4, MMu5, MMu6)) * ext*loop(p1, p2) * d_(MMu1, MMu2) * d_(MMu3, MMu6) * d_(MMu4, MMu5) * (  
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+ p1.p2^2*p2.p2*rat(-12, D^4 + 5*D^3 + 2*D^2 - 8*D)  
)
```

```
+ sym(ind1(mu1, mu2) * ind2(MMu1, MMu2)) * sym(ind1(nu1, nu2, nu3, nu4) * ind2(MMu3, MMu4, MMu5, MMu6)) * ext*loop(p1, p2) * d_(MMu1, MMu6) * d_(MMu2, MMu4) * d_(MMu3, MMu5) * (  
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+ p1.p2^2*p2.p2*rat(12, D^4 + 5*D^3 + 2*D^2 - 8*D)  
)
```

Invariant tensor

Scalar integral

```
+ p1.p1*p2.p2^2*rat(-12, D^4 + 5*D^3 + 2*D^2 - 8*D)  
+ p1.p2^2*p2.p2*rat(12, D^4 + 5*D^3 + 2*D^2 - 8*D)  
);
```

- `tensor mode 1` or after `#call symmetrise`

```
integral =
+ ext*loop(p1,p2)*d_(mu1,mu2)*d_(nu1,nu2)*d_(nu3,nu4) * ( p1.p1*p2.p2^2
*rat(D + 3,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(-4,D^4 + 5*
D^3 + 2*D^2 - 8*D) )
+ ext*loop(p1,p2)*d_(mu1,mu2)*d_(nu1,nu3)*d_(nu2,nu4) * ( p1.p1*p2.p2^2
*rat(D + 3,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(-4,D^4 + 5*
D^3 + 2*D^2 - 8*D) )
+ ext*loop(p1,p2)*d_(mu1,mu2)*d_(nu1,nu4)*d_(nu2,nu3) * ( p1.p1*p2.p2^2
*rat(D + 3,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(-4,D^4 + 5*
D^3 + 2*D^2 - 8*D) )
+ ext*loop(p1,p2)*d_(mu1,nu1)*d_(mu2,nu2)*d_(nu3,nu4) * ( p1.p1*p2.p2^2
*rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
+ 2*D - 8) )
+ ext*loop(p1,p2)*d_(mu1,nu1)*d_(mu2,nu3)*d_(nu2,nu4) * ( p1.p1*p2.p2^2
*rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
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*rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
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*rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
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+ ext*loop(p1,p2)*d_(mu1,nu2)*d_(mu2,nu3)*d_(nu1,nu4) * ( p1.p1*p2.p2^2
*rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
+ 2*D - 8) )
+ ext*loop(p1,p2)*d_(mu1,nu2)*d_(mu2,nu4)*d_(nu1,nu3) * ( p1.p1*p2.p2^2
*rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
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+ ext*loop(p1,p2)*d_(mu1,nu4)*d_(mu2,nu1)*d_(nu2,nu3) * ( p1.p1*p2.p2^2
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+ 2*D - 8) )
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*rat(-1,D^4 + 5*D^3 + 2*D^2 - 8*D) + p1.p2^2*p2.p2*rat(1,D^3 + 5*D^2
+ 2*D - 8) );
```

Summary

Projectors

- Orbit-partition \rightarrow rank 32 projectors
- Exploiting integrand symmetry

OPITeR

- Fully automated reduction
- Soon to be published!

Summary

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Thank you for
listening



Bonus slides

Fermion lines

- Adding gamma matrices the tensors

- Work in anti symmetric basis $\Gamma^{\mu_1 \dots \mu_p} = \gamma^{[\mu_1} \dots \gamma^{\mu_p]}$

Fermion lines

- Adding gamma matrices the tensors

- Work in anti symmetric basis $\Gamma^{\mu_1 \dots \mu_p} = \gamma^{[\mu_1} \dots \gamma^{\mu_p]}$

$$\text{tr} (\Gamma_{\mu_1 \dots \mu_a} \Gamma^{\nu_1 \dots \nu_b}) = \delta_{ab} \text{tr}(\mathbb{I}) \delta_{\mu_1 \dots \mu_a}^{\nu_1 \dots \nu_b} (-1)^{a(a-1)/2}$$

Fermion lines

- Adding gamma matrices the tensors

- Work in anti symmetric basis $\Gamma^{\mu_1 \dots \mu_p} = \gamma^{[\mu_1} \dots \gamma^{\mu_p]}$

$$\text{tr}(\Gamma_{\mu_1 \dots \mu_a} \Gamma^{\nu_1 \dots \nu_b}) = \delta_{ab} \text{tr}(\mathbb{I}) \delta_{\mu_1 \dots \mu_a}^{\nu_1 \dots \nu_b} (-1)^{a(a-1)/2}$$

- Efficiently transform into basis

$$\gamma^{\mu_1} \dots \gamma^{\mu_n} = \sum_{k=0}^n \sum_{\pi \in \Sigma_n^k} \text{sgn}(\pi) \Gamma^{\mu_{\pi(1)} \dots \mu_{\pi(k)}} \text{tr}(\gamma^{\mu_{\pi(k+1)}} \dots \gamma^{\mu_{\pi(n)}})$$

new !

Σ_n^k shuffles the first k indices with the remaining $n - k$ indices over the two tensors

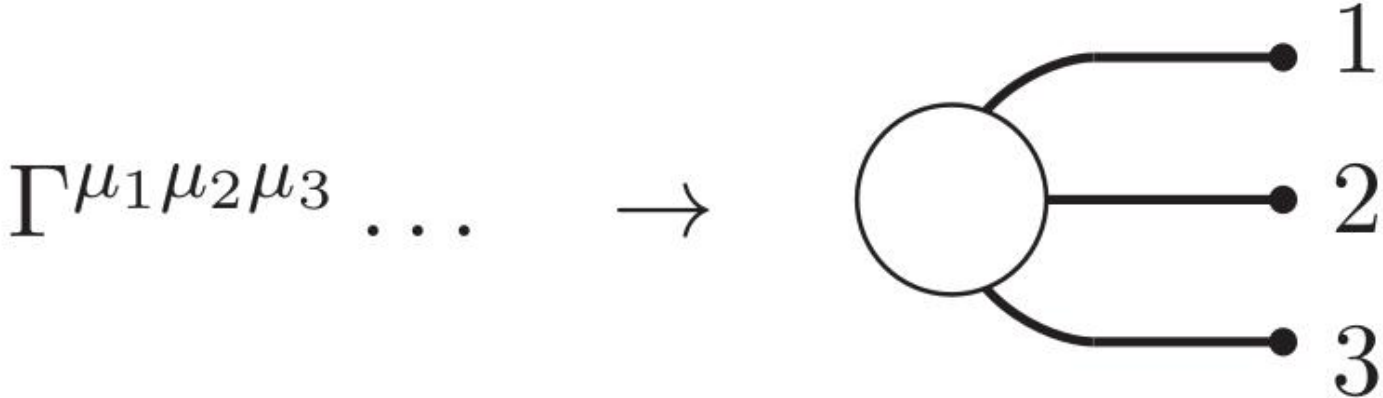
Fermion lines

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

- Add new tensors
- Constrain the basis
- Create projector ansatz as before

Fermion lines

- Add new tensors
- Constrain the basis
- Create projector ansatz as before
- Need extra diagram elements



Fermion lines



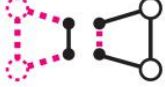




[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]





N	n_γ	Graph	$c_{\lambda_g, \lambda_\gamma}$
$N = 1$	1		$c_{(), ()} = \frac{1}{D \operatorname{tr}(\mathbb{1})}$
$N = 2$	0		$c_{(1), ()} = \frac{1}{D \operatorname{tr}(\mathbb{1})}$
	2		$c_{(), ()} = \frac{1}{2D(D-1) \operatorname{tr}(\mathbb{1})}$
$N = 3$	1		$c_{(1), ()} = \frac{-1}{D(D+2)(D-1) \operatorname{tr}(\mathbb{1})}$,
			$c_{(), (1)} = \frac{D+1}{D(D+2)(D-1) \operatorname{tr}(\mathbb{1})}$
	3		$c_{(), ()} = \frac{1}{D(D-2)(D-1) \operatorname{tr}(\mathbb{1})}$

$N = 4$	0		$c_{(1,1), ()} = \frac{D+1}{D(D+2)(D-1) \operatorname{tr}(\mathbb{1})}$,
			$c_{(2), ()} = \frac{-1}{D(D+2)(D-1) \operatorname{tr}(\mathbb{1})}$
	2		$c_{(1), ()} = \frac{1}{D(D-1)(D-2)(D+2) \operatorname{tr}(\mathbb{1})}$,
			$c_{(), (1)} = \frac{1}{(D-1)(D-2)(D+2) \operatorname{tr}(\mathbb{1})}$
4		$c_{(), ()} = \frac{1}{D(D-1)(D-2)(D-3) \operatorname{tr}(\mathbb{1})}$	

Fermion lines

[Goode, Herzog, Kennedy, ST, Vermaseren, 2024]

\vec{n}	k	Graph	c_k
(1, 1, 1, 1)	1		$\frac{D^3+3D^2-3D-3}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4) \text{tr}(\mathbb{1})^2}$
	2		$\frac{-D^2-2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4) \text{tr}(\mathbb{1})^2}$
	3		$\frac{3D+5}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4) \text{tr}(\mathbb{1})^2}$
	4		$\frac{-D^2-3D-3}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4) \text{tr}(\mathbb{1})^2}$
	5		$\frac{-D^2-3D-3}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4) \text{tr}(\mathbb{1})^2}$
	6		$\frac{2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4) \text{tr}(\mathbb{1})^2}$
	7		$\frac{2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4) \text{tr}(\mathbb{1})^2}$

(2, 0, 0, 2)	8		$\frac{-D^2-2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4) \text{tr}(\mathbb{1})^2}$
	9		$\frac{2D+1}{D(D-1)(D-2)(D-3)(D+2)(D+1)(D+4) \text{tr}(\mathbb{1})^2}$
(0, 2, 2, 0)	10		$\frac{-1}{(D-1)(D-2)(D-3)(D+2)(D+1) \text{tr}(\mathbb{1})^2}$
	11		$\frac{1}{D(D-1)(D-2)(D-3)(D+2)(D+1) \text{tr}(\mathbb{1})^2}$

Workflow

[Goode, Herzog, ST, in preparation]

