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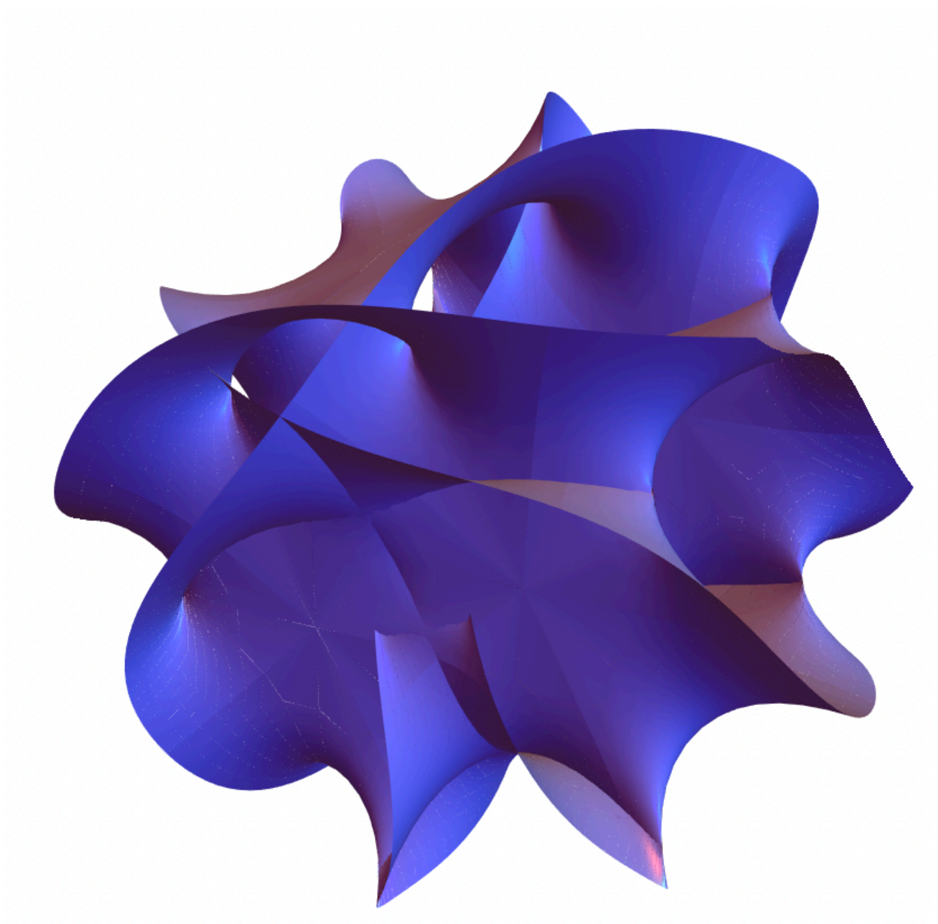
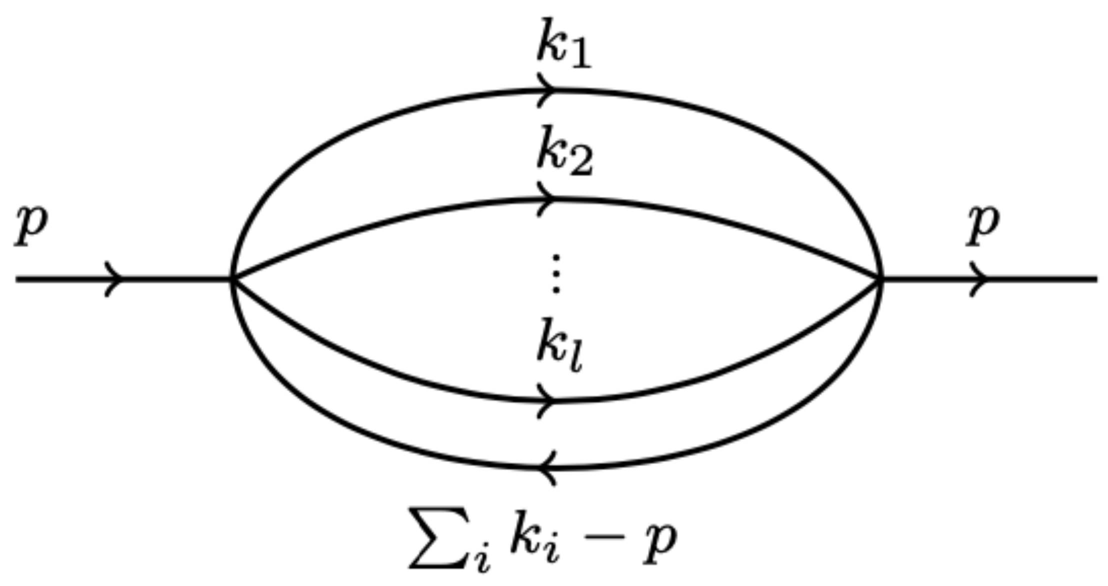
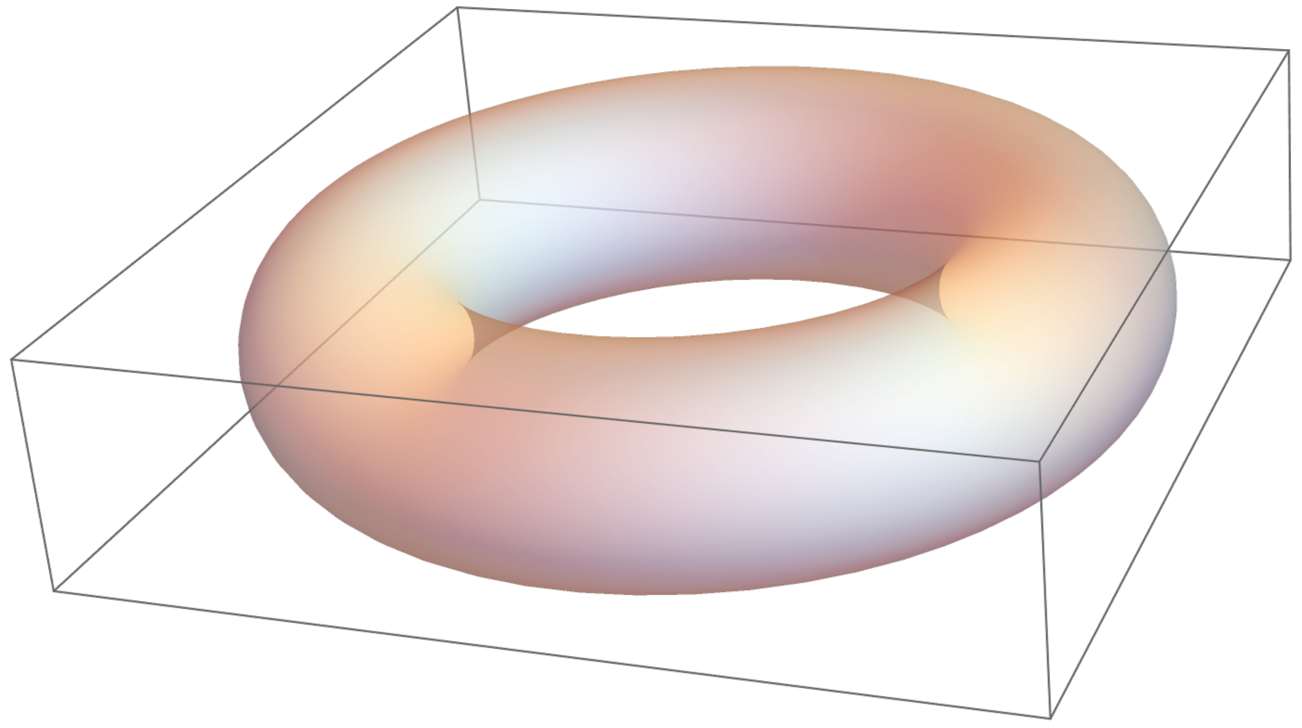
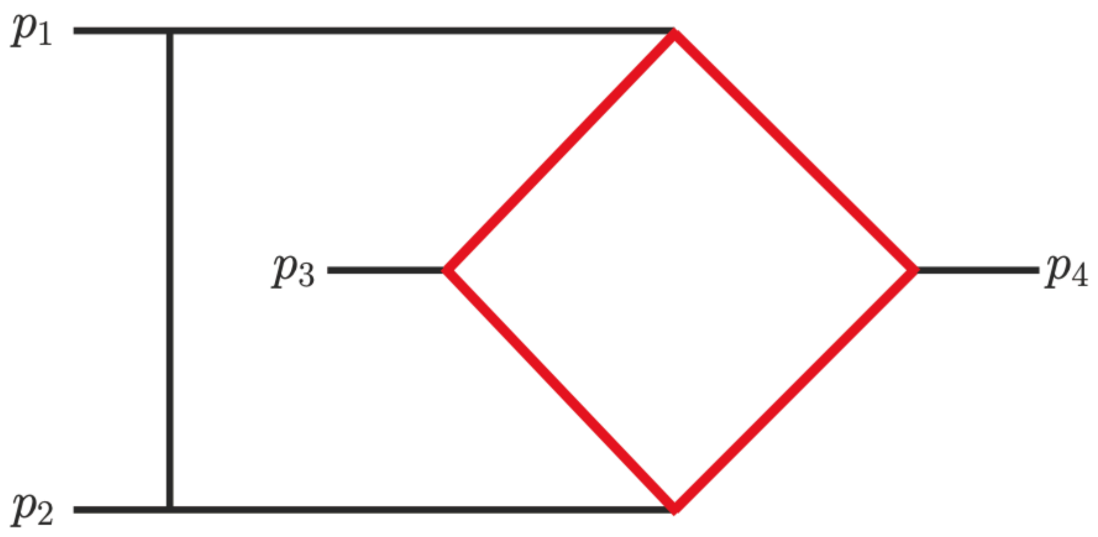
Geometry in Feynman integrals









Sara Maggio | Fall Bonn HEP Meeting 2024: Embracing Diversity in High Energy Physics | 07/10/2024

Outline

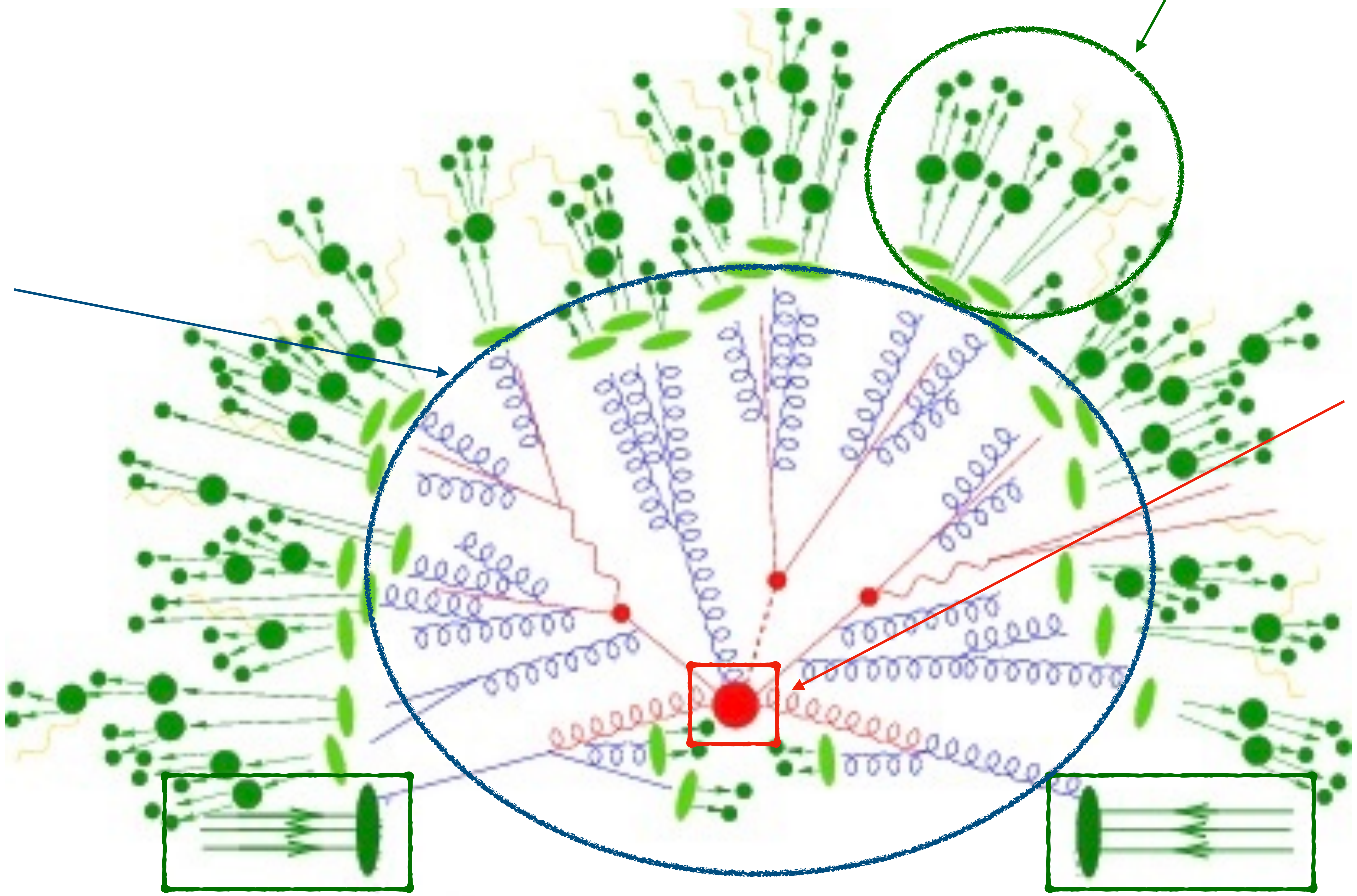
- Motivation
- Modern techniques
- Examples
- Conclusions

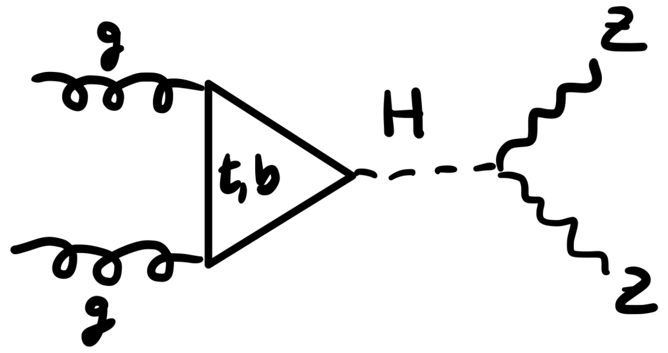
Motivation

Why do we need to compute Feynman integrals?

- The LHC will give us new data for the next 2 decades
- Precision measurements in the Higgs sector are fundamental for testing the Standard Model or for finding New Physics
- This requires as accurate as possible theoretical predictions

What happens at the LHC?





[Sherpa's artistic view of protons' collisions]

Modern techniques for solving Feynman integrals

Family of Feynman integrals

↓ IBPs

Master Integrals

↓ Differential equations method

$$d\vec{I} = A\vec{I}$$

↓ Gauge transformation

$$\vec{J} = U\vec{I}$$

↓ epsilon-form [Henn, '13]

$$d\vec{J} = \epsilon\tilde{A}\vec{J} \longrightarrow \vec{J} = P\exp[\epsilon\int_{\mathcal{C}}\tilde{A}(\vec{x'})]\vec{J}_0$$

Integration By Part identities

- Identities among integrals in a family
- The most general way to perform a reduction to *master integrals*

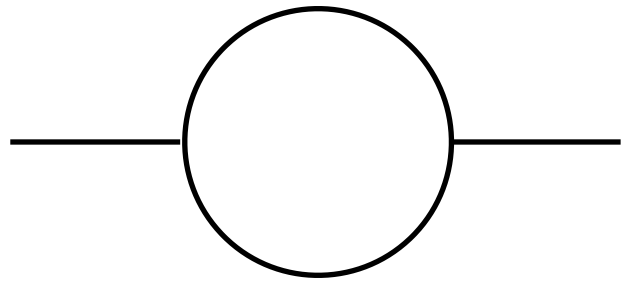
Considering an integral family: $I(b_1, ..., b_\tau, -a_1, ..., -a_\sigma) = \int \prod_{l=1}^{L} \frac{d^D k_l}{\pi^{D/2}} \frac{S_1^{a_1} \cdots S_\sigma^{a_\sigma}}{D_1^{b_1} \cdots D_\tau^{b_\tau}}$

Gauss theorem in D dimensions: $\int \prod_{l=1}^{L} \frac{d^D k_l}{\pi^{D/2}} \frac{\partial}{\partial k_j^\mu} \left[v^\mu \frac{S_1^{a_1} \cdots S_\sigma^{a_\sigma}}{D_1^{b_1} \cdots D_\tau^{b_\tau}} \right] = 0$

Example: 1 loop tadpole $I(n) = \int \prod_{l=1}^{L} \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2 + m^2)^n}$

IBPs: $\int \prod_{l=1}^{L} \frac{d^D k}{\pi^{D/2}} \frac{\partial}{\partial k^\mu} \left[k^\mu \frac{1}{(k^2 + m^2)^n} \right] = 0 \longrightarrow I(n+1) = -\frac{D-2n}{2nm^2} I(n)$

Differential equation method

- Scalar Feynman integrals depend on the masses and $s_{ij} = (p_i + p_j)^2 = 2p_i \cdot p_j$
- Taking derivatives of one integral, we get another integral in the family that can be reduced to master integrals using IBPs
- Example 1-loop bubble: 

$$B(p^2, m^2) = \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2 + m^2)((k-p)^2 + m^2)} = (m^2)^{\frac{D-4}{2}} B\left(\frac{p^2}{m^2}\right) \overset{m^2=1}{=} B(p^2)$$

$$\frac{\partial}{\partial p^2} B(p^2) = \frac{1}{2p^2} p_\mu \frac{\partial}{\partial p_\mu} \int \frac{d^D k}{\pi^{D/2}} \frac{1}{(k^2+1)((k-p)^2+1)} = \frac{1}{2p^2} p_\mu \frac{\partial}{\partial p_\mu} \int \frac{d^D k}{\pi^{D/2}} \frac{1}{D_1 D_2}$$

$$p_\mu \frac{\partial}{\partial p_\mu} \frac{1}{D_1} = 0$$

$$p_\mu \frac{\partial}{\partial p_\mu} \frac{1}{D_2} = \frac{1}{D_2^2}(2k \cdot p - 2p^2) = \frac{1}{D_2^2}(D_1 - D_2 - p^2)$$

All together: $\frac{\partial}{\partial p^2} I(1,1) = \frac{1}{2p^2}\Big[I(0,2) - I(1,1)\Big] - \frac{1}{2} I(1,2)$

From the IBPs: $I(0,2) = -\frac{(D-2)}{2m^2} I(1,0)$

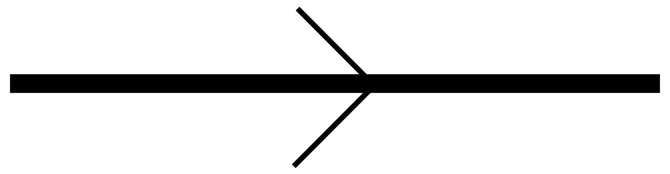
$$I(1,2) = -\frac{(D-2)}{2m^2(p^2+4m^2)} I(1,0) - \frac{(D-3)}{p^2+4m^2} I(1,1)$$

We finally get: $\frac{d}{dp^2} I(1,1) = \frac{1}{2}\Big(\frac{D-3}{p^2+4} - \frac{1}{p^2}\Big) I(1,1) - \frac{D-2}{p^2(p^2+4)} I(1,0)$

Master integrals

- Baikov representation [Baikov, '96,'97]: propagators as integration variables

$$I \propto \int_{\mathcal{C}} d^{N_V} z [\det G(k_1, ..., k_l, p_1, ..., p_e)]^{\frac{D-l-e-1}{2}} \prod_{s=1}^{N_V} z_s^{-\nu_s}$$

- "Cut" all the propagators in a topology  $\longrightarrow \delta$

$$\mathrm{MaxCut} I \propto \int_{\mathcal{C}_{\mathrm{MaxCut}}} d^{N_V} z [\det G(k_1, ..., k_l, p_1, ..., p_e)]^{\frac{D-l-e-1}{2}} \prod_{s=1}^{N_V} z_s^{-\nu_s}$$

- Powerful way to understand the analytic structure $d\vec{I}_r = B\vec{I}_r + \cancel{\vec{N}_r}$

How do we see the geometry?

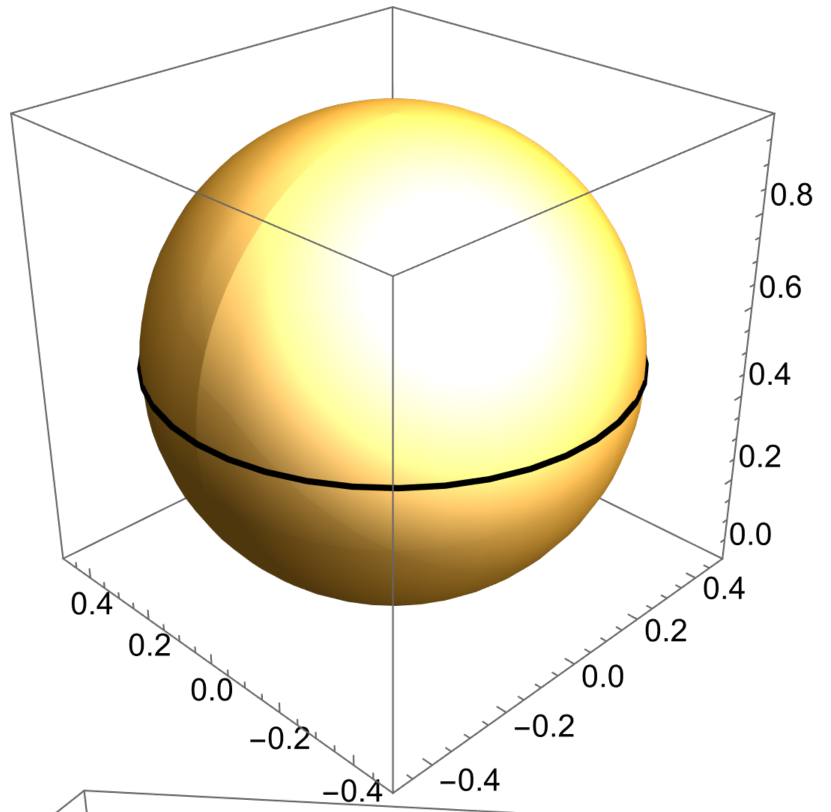
- Let's suppose we manage to do $N_V - 1$ integrations

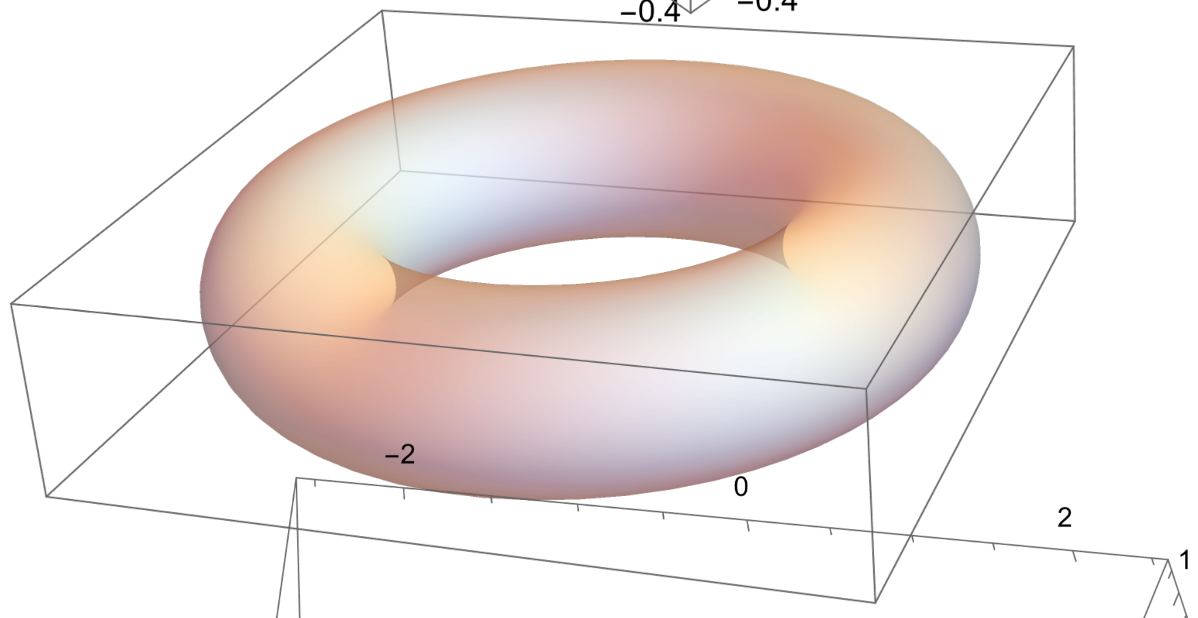
$$\text{MaxCut} I \propto \int_C \frac{dz}{\sqrt{P(z)}}$$

- deg $P(z) \leq 2, \sqrt{P(z)}$ is rationalizable $\longrightarrow$ integral over the Riemann sphere
- deg $P(z) \geq 3, \sqrt{P(z)}$ is not rationalizable $\longrightarrow$ integral over non trivial geometry!

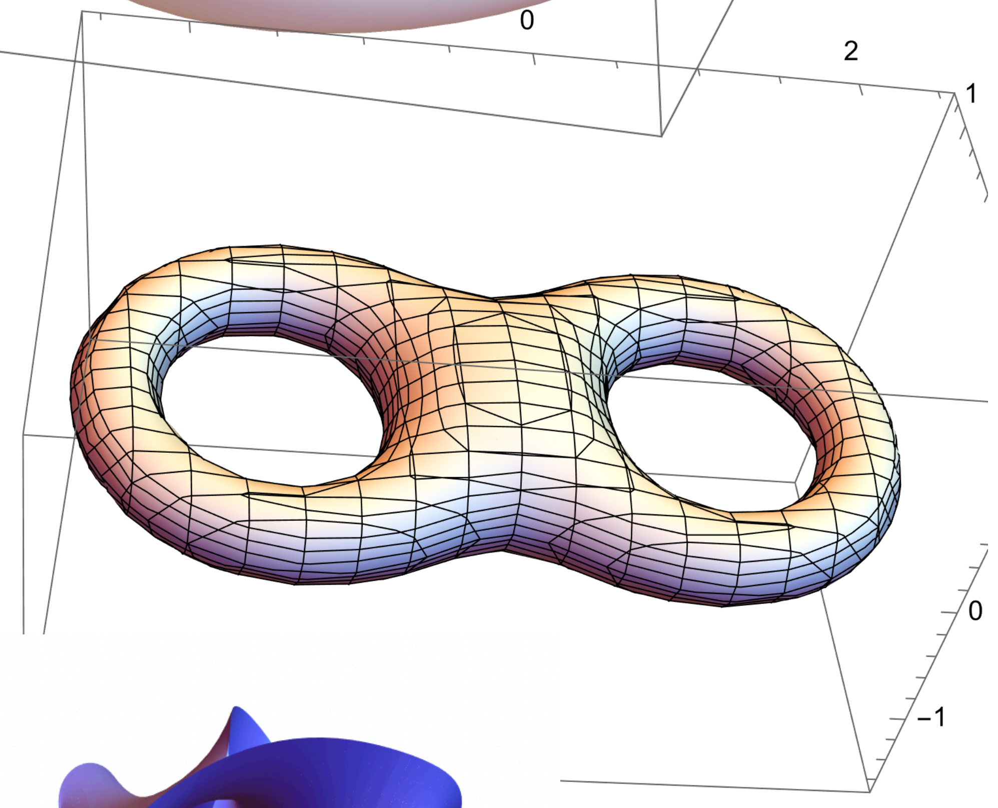
Zoos of geometry

- 1-loop Feynman integrals have the geometry of a Riemann sphere and by now are well understood: tools to compute the ϵ-form [Gituliar, Margery, '17][Meyer, '18].
  Solution space: $G(z_1, ..., z_k; y) = \int_0^y \frac{dt_1}{t_1 - z_1} \int_0^{t_1} \frac{dt_2}{t_2 - z_2} \cdots \int_0^{t_{k-1}} \frac{dt_k}{t_k - z_k}$



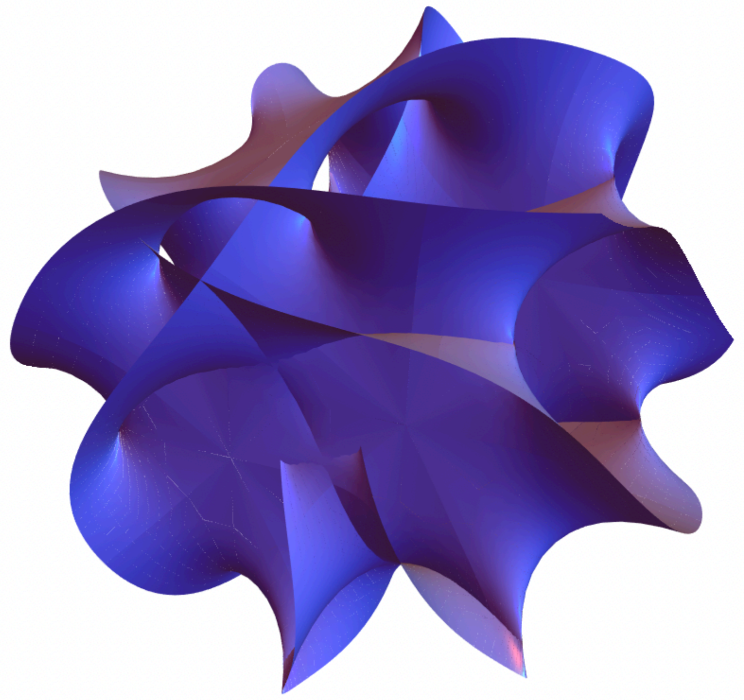


- At 2-loops we encounter the first non-trivial geometry:
  deg $P(z) = 3,4 \longrightarrow$ elliptic curve



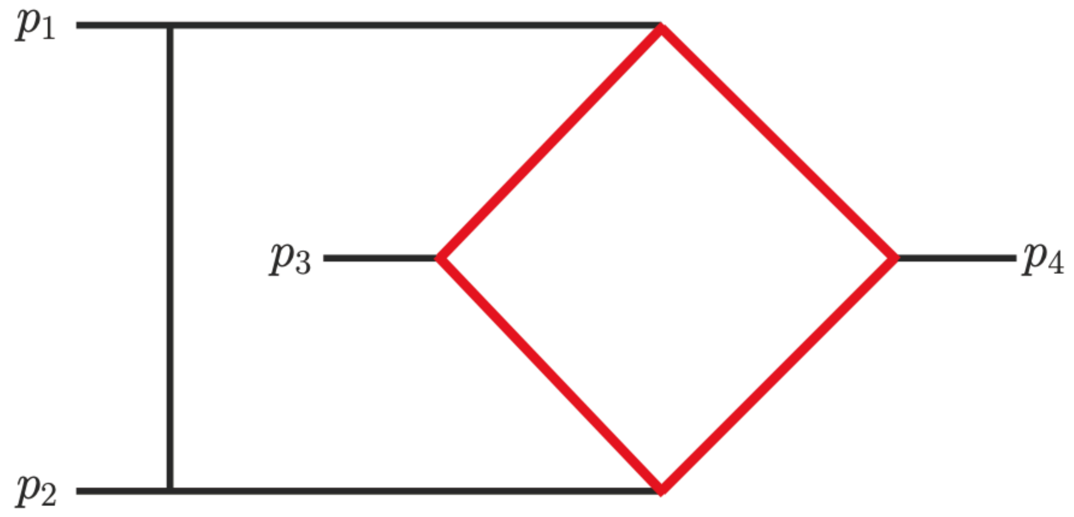
- deg $P(z) \geq 5 \longrightarrow$ hyperelliptic curve

- It can also happen that we can't do all the $N_V - 1$ integrations:
  $$\text{MaxCut}I \propto \int_C \frac{dx_1 \ldots dx_n}{\sqrt{P(x_1, \ldots, x_n)}} \longrightarrow \text{deg } P(z) = 2m \longrightarrow$$
  Calabi-Yau $(m-1)-$fold



Examples

2-loops



[Ahmed, Chaubey, Kaur, S.M. ’24]

$$I_{a_1,\cdots,a_9} = \left(\frac{e^{\epsilon\gamma_E}}{i\pi^{\frac{d}{2}}}\right)^2 \int \prod_{i=1}^{2} d^d k_i \frac{D_8^{a_8} D_9^{a_9}}{D_1^{a_1} D_2^{a_2} D_3^{a_3} D_4^{a_4} D_5^{a_5} D_6^{a_6} D_7^{a_7}}$$

$$\text{MaxCut} I_{111111100} = \frac{16}{\pi^4} \int_{\mathcal{C}} \frac{dP}{s\,(s+t+P)\,\sqrt{P}\,\sqrt{s+P}\,\sqrt{-4m_t^2 s + sP + P^2}} + O(\epsilon)$$

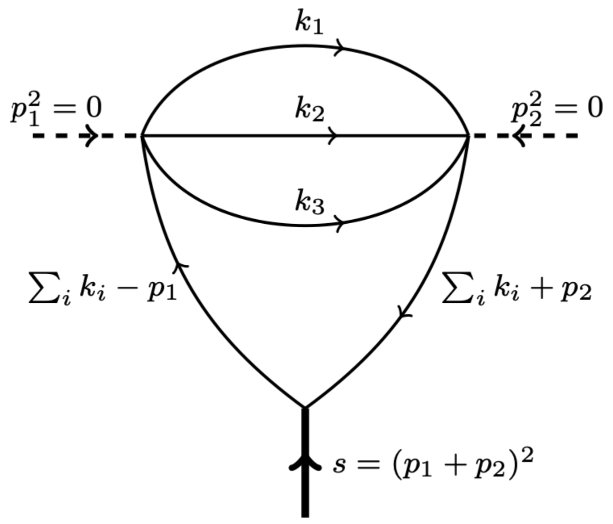
elliptic curve!

- A function space: Elliptic polylogarithms [Brown, Levin, ’11][Bloch, Vanhove, ’13][Adams, Bogner, Weinzierl, ’14’15][Broedel, Duhr, Dulat, Tancredi,’17]

$$\tilde{\Gamma}\left(\begin{smallmatrix} n_1 \ldots n_k \\ z_1 \ldots z_k \end{smallmatrix}; z; \tau\right) = \int_0^z dz' g^{(n_1)}(z' - z_1, \tau) \tilde{\Gamma}\left(\begin{smallmatrix} n_2 \ldots n_k \\ z_2 \ldots z_k \end{smallmatrix}; z; \tau\right)$$

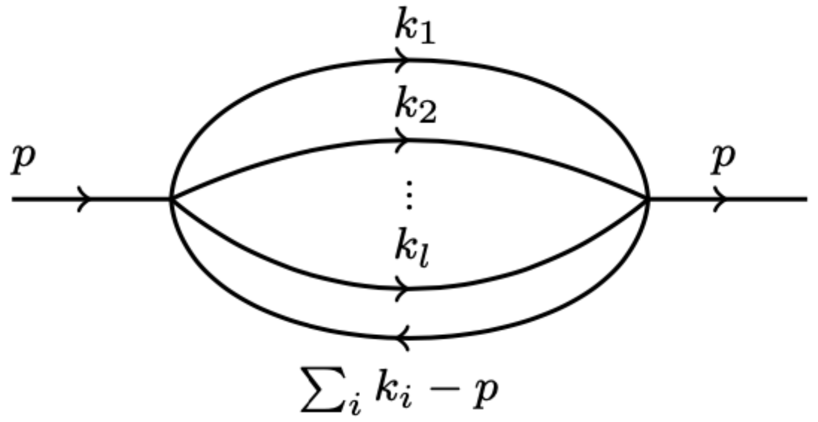
3-loops and higher

- 3-loops Ice-cone: 2 elliptic curves



[Görges, Nega, Tancredi, Wagner, '23]

- $n-$loop Banana integrals: Calabi-Yau $(n-1)-$fold



[Pögel, Wang, Weinzierl '22]

Where do we need the information on the geometry?

Solving the Feynman integral

[Pögel, Wang, Weinzierl '22][Görges, Nega, Tancredi, Wagner, '23][work in progress with Duhr, Nega, Tancredi, Wagner]

- A canonical basis for one variable Banana integrals:

$$M_1 = \frac{I_1}{\psi_0},$$

$$M_j = \frac{1}{Y_{j-1}}\left(\frac{\theta_q}{\epsilon} M_{j-1} - \sum_{k=1}^{j-1} F_{(j-1)k} M_k\right)$$

objects dependent on the geometry!

- This can be generalized for integrals that depend on more than 1 scale!

[work in progress with Duhr, Sohnle]

Conclusions

- The study of mathematical properties plays an important role for analytic computations.
- This is required to confront the current precision frontier.
- The algebraic properties are useful for finding an epsilon-form for the differential equations.