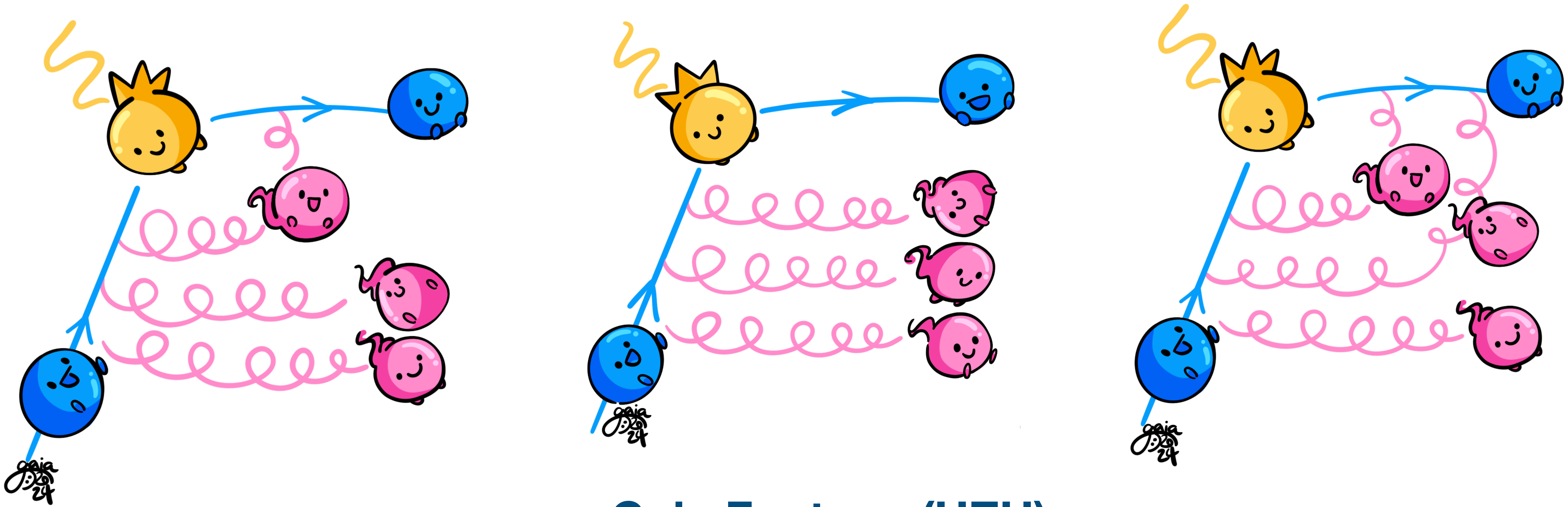
3 loops & 4 cuts:

towards N3LO RRR antenna functions



Gaia Fontana (UZH)

In collaboration with Thomas Gehrmann & Kay Schönwald

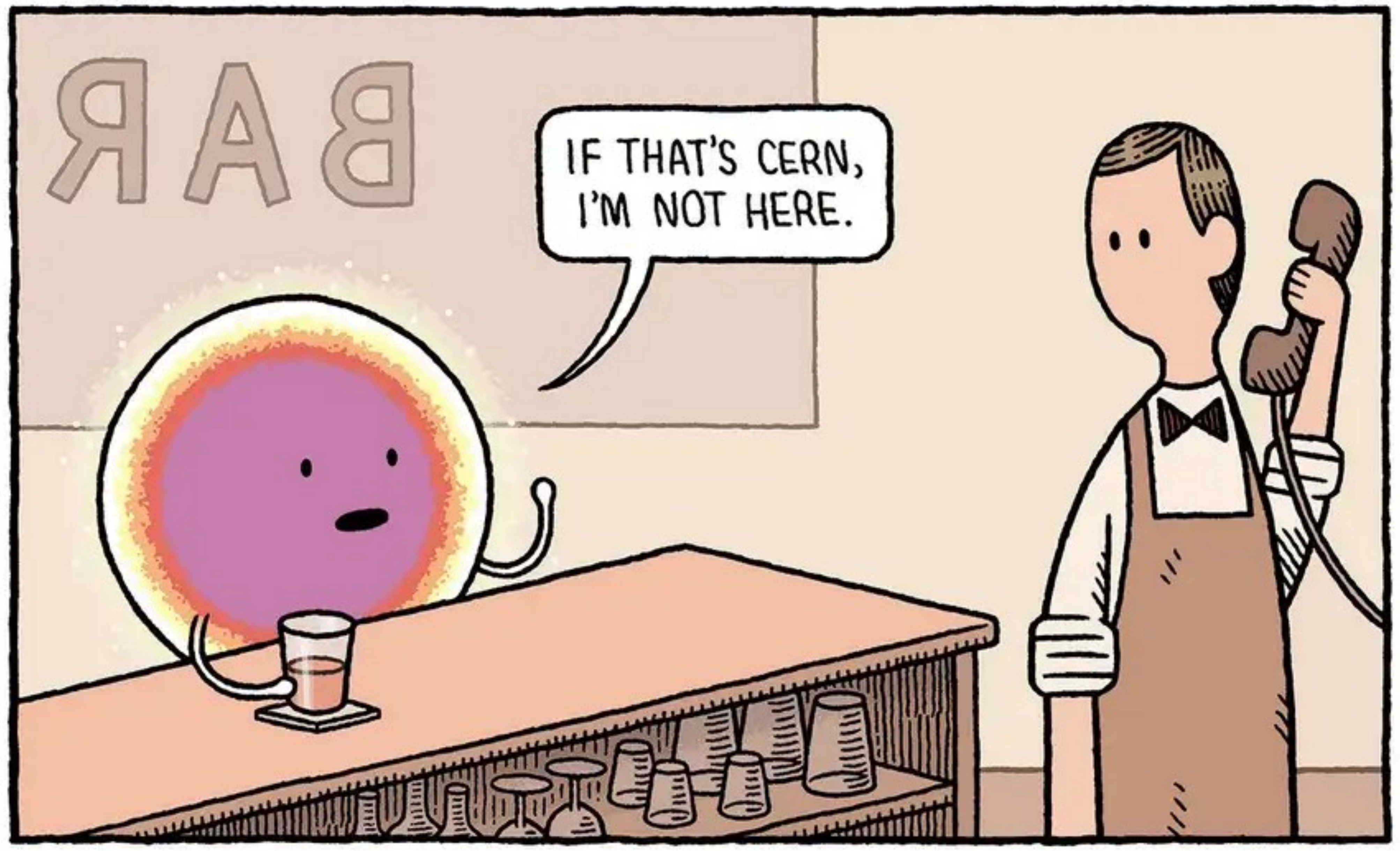
Universität Zürich UZH

Based on JHEP 03 (2024) 159 & upcoming works

Fall Bonn HEP meeting 2024
Bonn, 07/10/2024

Why precision?

- Precision physics as
  - test of the Standard model
  - gate to new physics



TOM GAULD for NEW SCIENTIST

- High-Lumi upgrade of LHC :
  - theory and experiments must have comparable uncertainties
  - needed: %-level accuracy:

perturbation theory **@ NNLO** and often **N3LO**

Hard Scattering

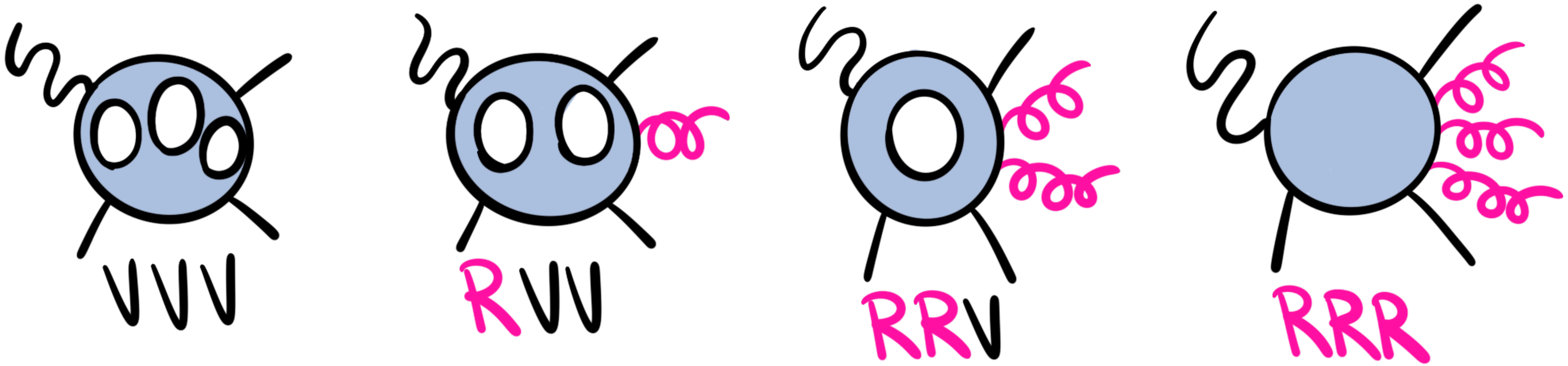
Looking @ QCD corrections:

$$\mathrm{d}\sigma = \mathrm{d}\sigma_{LO} + \alpha_S\, \mathrm{d}\sigma_{NLO} + \alpha_S^2\, \mathrm{d}\sigma_{NNLO} + \alpha_S^3\, \mathrm{d}\sigma_{N3LO} + \ldots$$

Perturbative series in the strong coupling

Beyond LO: contributions from diagrams with increasing loops and legs

@$\mathcal{O}(\alpha_s^3)$:



Real corrections!

looking @ m-jets cross section:

@ LO

$$d\sigma_{LO} = \int_{d\Phi_m} d\sigma_{Born}$$

@ NLO

$$d\sigma_{NLO} = \int_{d\Phi_{m+1}} d\sigma^{R}_{NLO} + \int_{d\Phi_m} d\sigma^{V}_{NLO}$$

@ NNLO

$$d\sigma_{NNLO} = \int_{d\Phi_{m+2}} d\sigma^{RR}_{NNLO} + \int_{d\Phi_{m+1}} d\sigma^{RV}_{NNLO} + \int_{d\Phi_m} d\sigma^{VV}_{NNLO}$$

@ N3LO

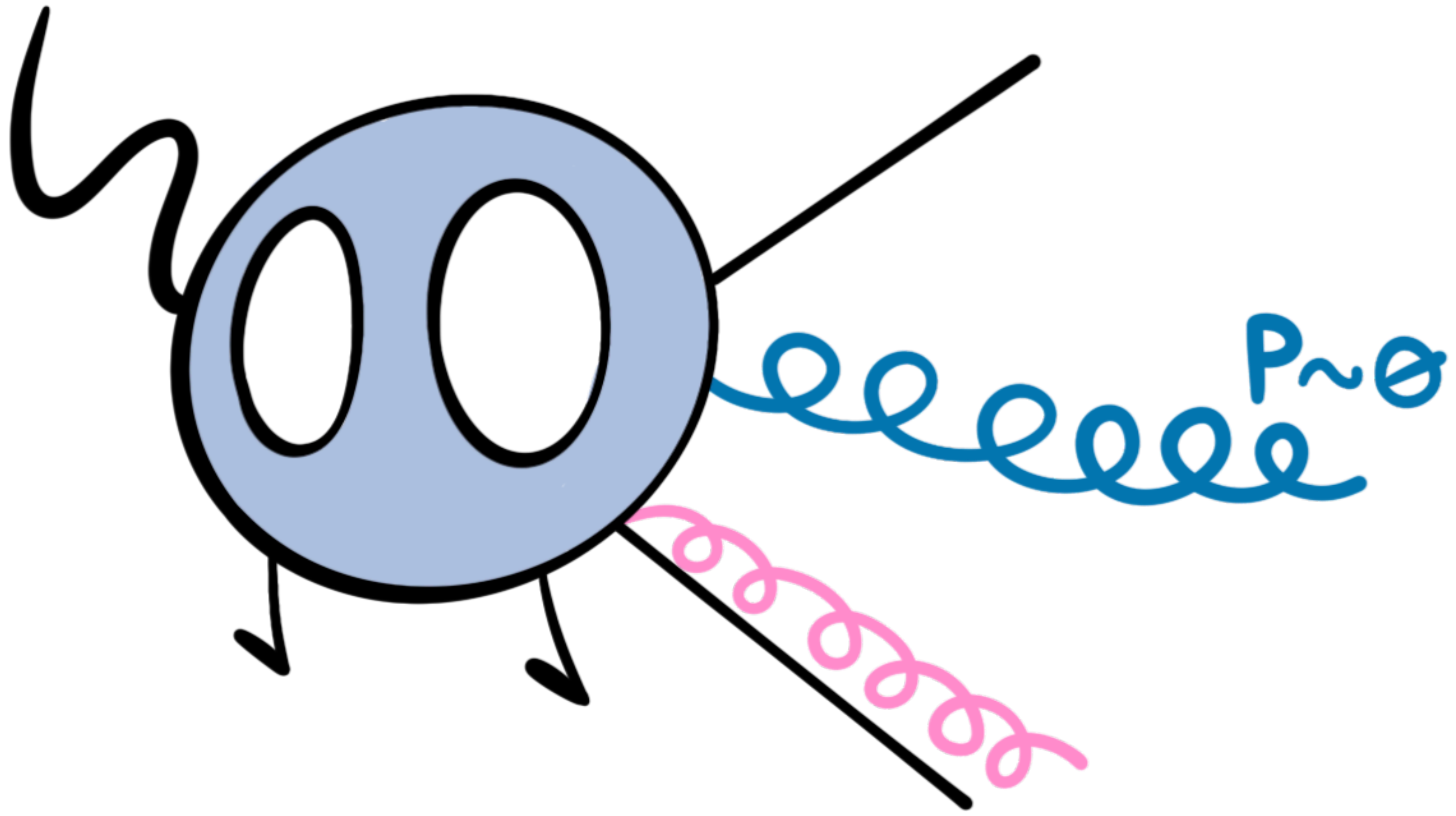
$$d\sigma_{N3LO} = \int_{d\Phi_{m+3}} d\sigma^{RRR}_{N3LO} + \int_{d\Phi_{m+2}} d\sigma^{RRV}_{N3LO} + \int_{d\Phi_{m+1}} d\sigma^{RVV}_{N3LO} + \int_{d\Phi_m} d\sigma^{VVV}_{N3LO}$$

KLN thm Kinoshita (1962); Lee, Nauenberg (1964)

finiteness when summing over all unresolved configurations

- Separate pieces are IR-divergent:
  - **Explicit** poles in ϵ after **loop** integration
  - **Implicit** divergencies from **real** radiation

- **soft** or **collinear** partons



How do we deal with these divergencies?

$$\mathrm{d}\sigma_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\sigma^{R}_{NLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\sigma^{V}_{NLO}$$

⛔ Hard to solve analytically ⛔

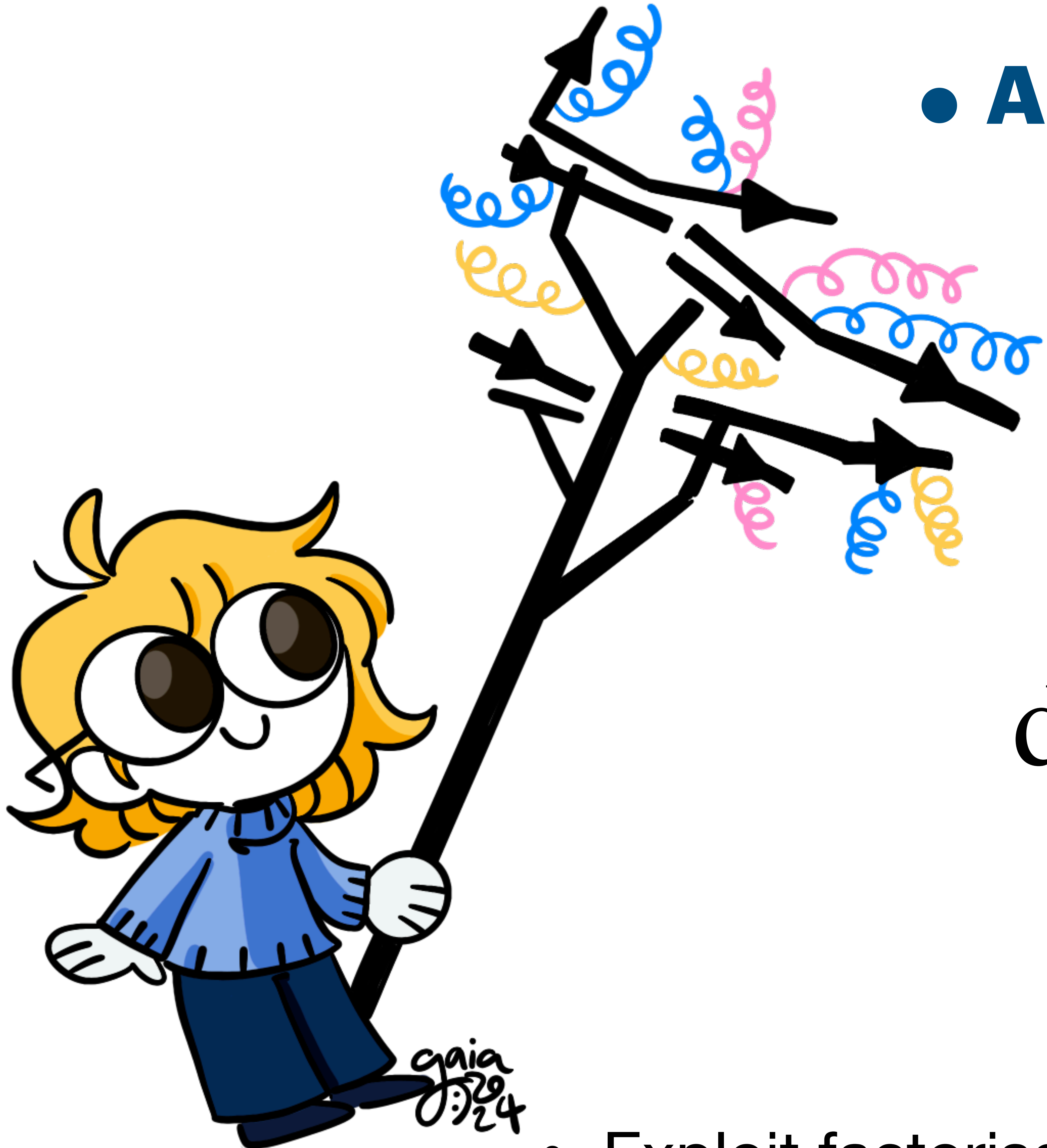
idea: Subtraction Schemes

$$\mathrm{d}\sigma_{NLO} = \int_{\mathrm{d}\Phi_{m+1}} \underbrace{\left(\mathrm{d}\sigma^{R}_{NLO} - \mathrm{d}\sigma^{S}_{NLO}\right)}_{\bullet\ \text{finite}} + \int_{\mathrm{d}\Phi_{m}} \underbrace{\left[\mathrm{d}\sigma^{V}_{NLO} + \int_{1} \mathrm{d}\sigma^{S}_{NLO}\right]}_{\bullet\ \text{finite}}$$

- Add and subtract the same quantity $\mathrm{d}\sigma^{S}$
    - Mimics singular behaviour in IR-limits of $\mathrm{d}\sigma^{V}, \mathrm{d}\sigma^{R}$
    - Makes the integrals individually finite
    - Simple enough to be analytically integrated over $\mathrm{d}\Phi_{1}$

Antenna Subtraction scheme

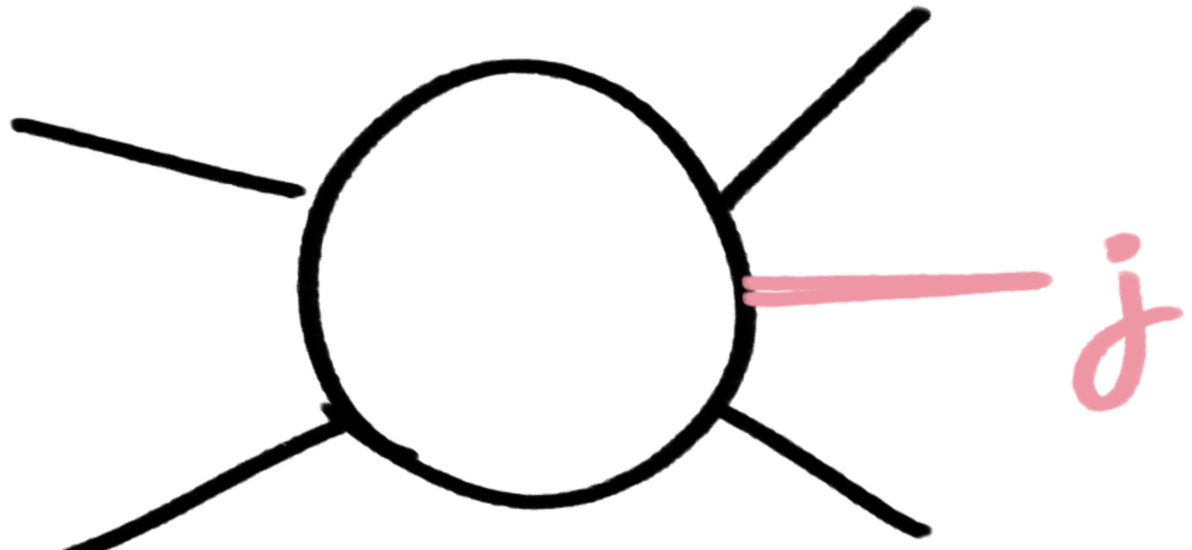
[Gehrmann-De Ridder, Gehrmann, Glover (2007)]



- **Antenna functions**
  - Built from simple matrix elements
  - Mimic the divergent behaviour in singular limits
  - Can be easily integrated over phase space

$$d\sigma^{S}_{NLO} \sim \boxed{X^{\ell}_{2+\,extra\,radiation}}\,\tilde{M}^{\ell}_{hard\,partons}\,J_m$$

- Exploit factorisation of matrix elements in IR limits



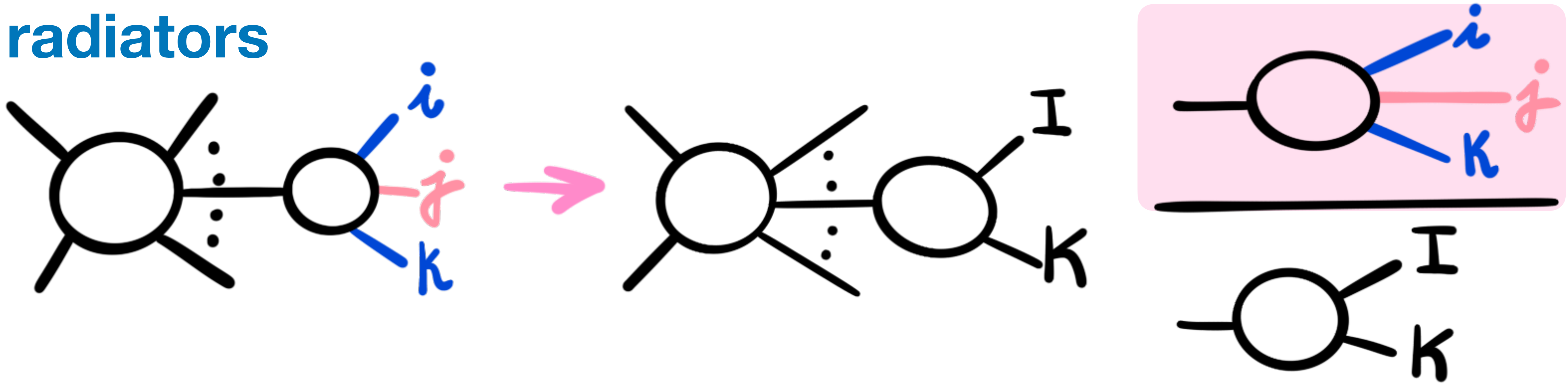
What can happen?

Matrix element with an extra radiation j

IR limit factorization

$j \parallel i, \quad j \parallel k, \quad j\,\text{soft}$

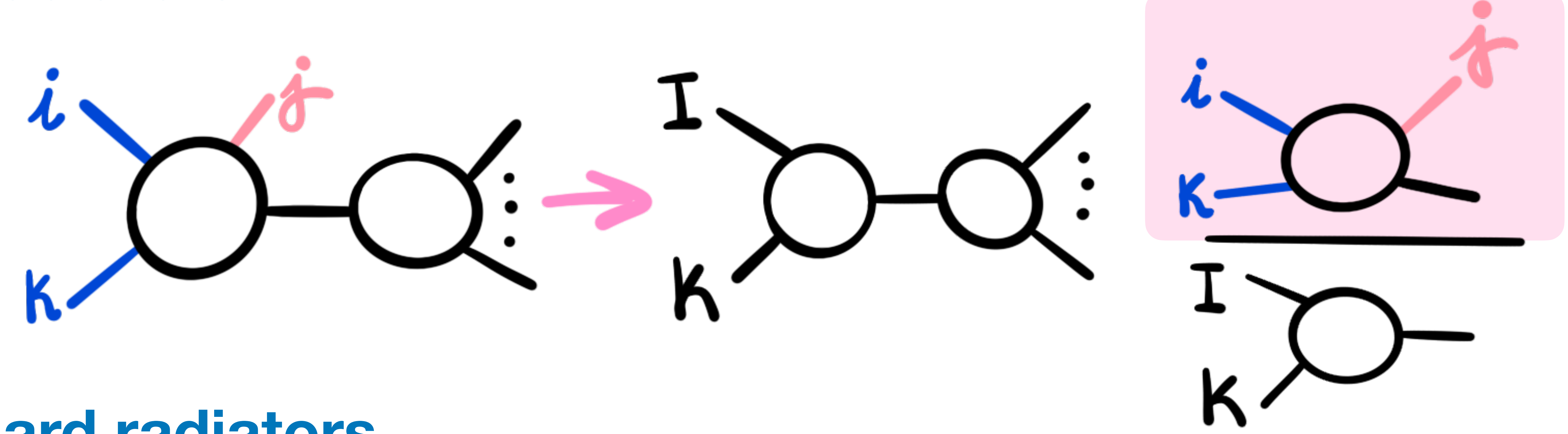
Final state hard radiators



Annihilation into hadrons

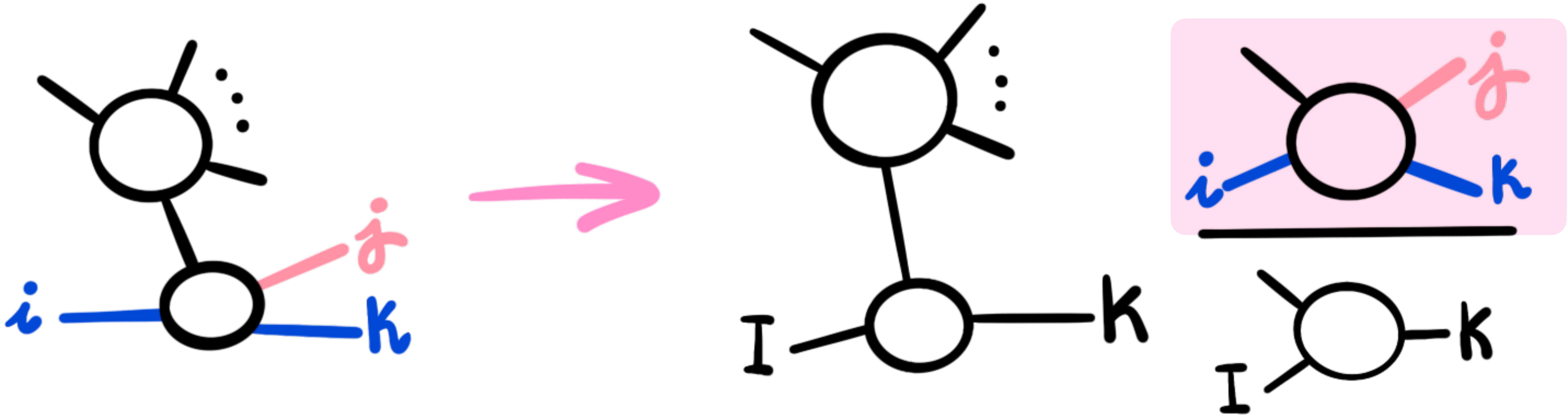
Solved! :)
[Chen,Jakubčík,Marcoli, Stagnitto ’23]

Initial state hard radiators



Drell-Yan

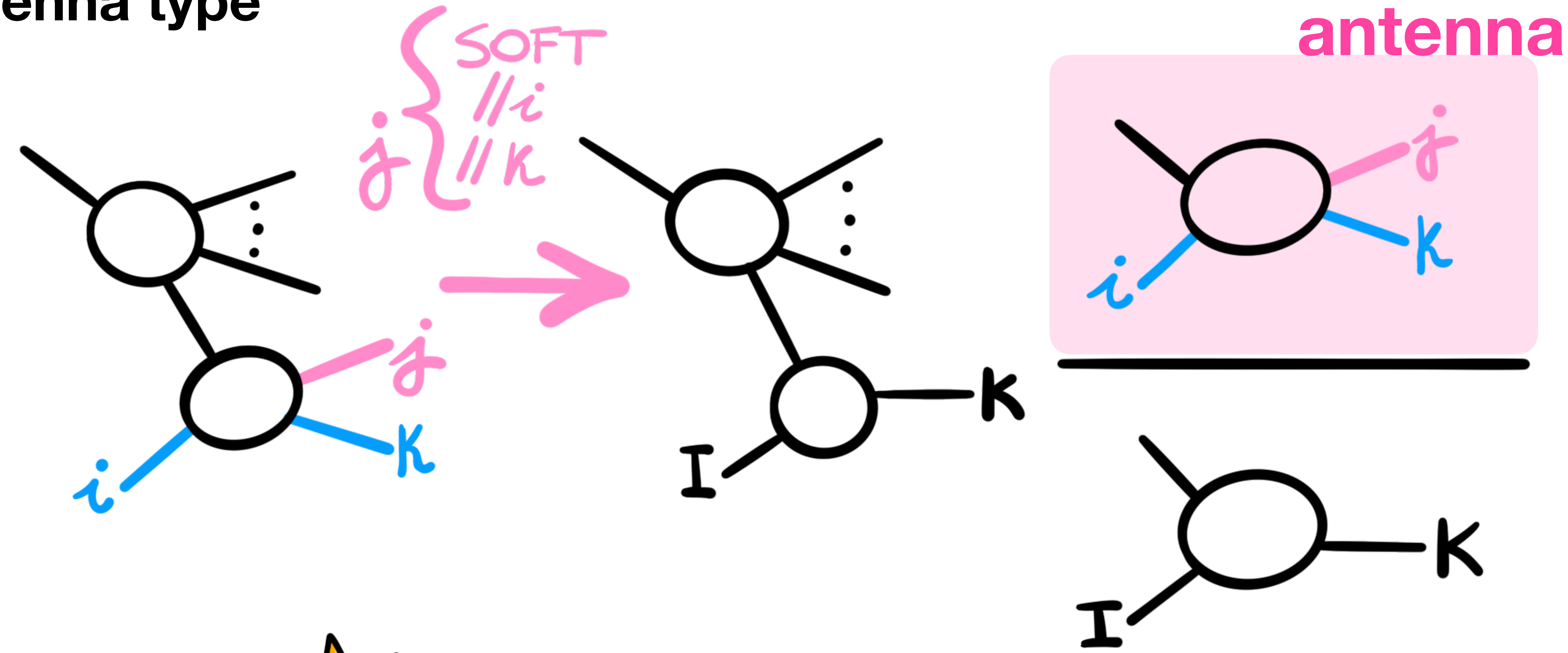
Initial-final state hard radiators



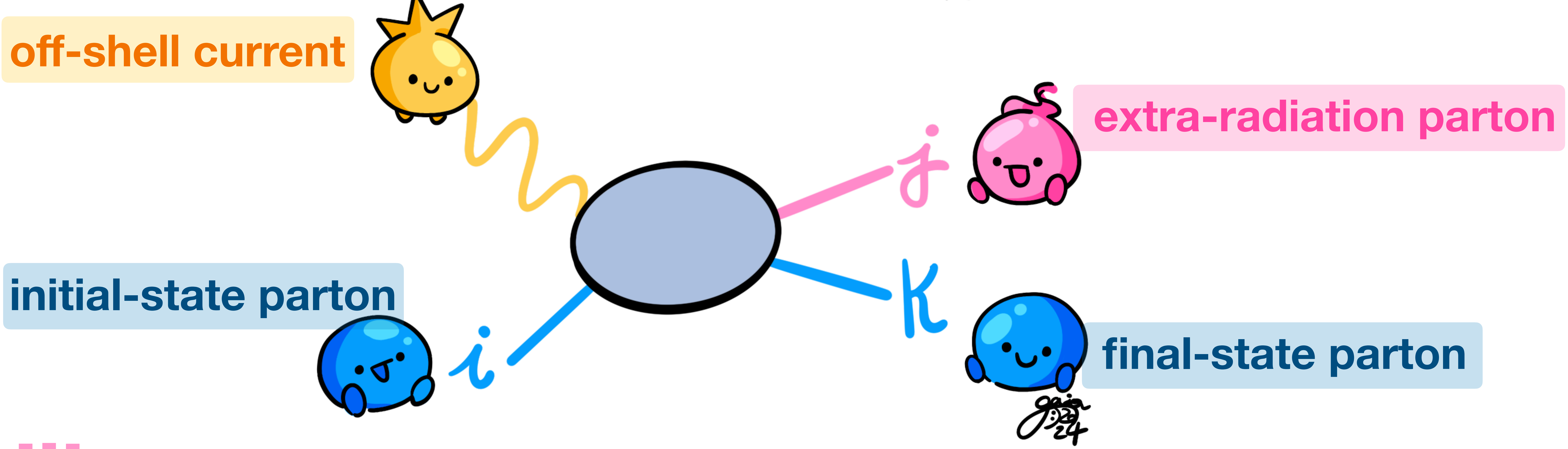
Deep Inelastic Scattering

- **Antenna functions**

Focus: initial-final antenna type



antenna function



off-shell current

extra-radiation parton

initial-state parton

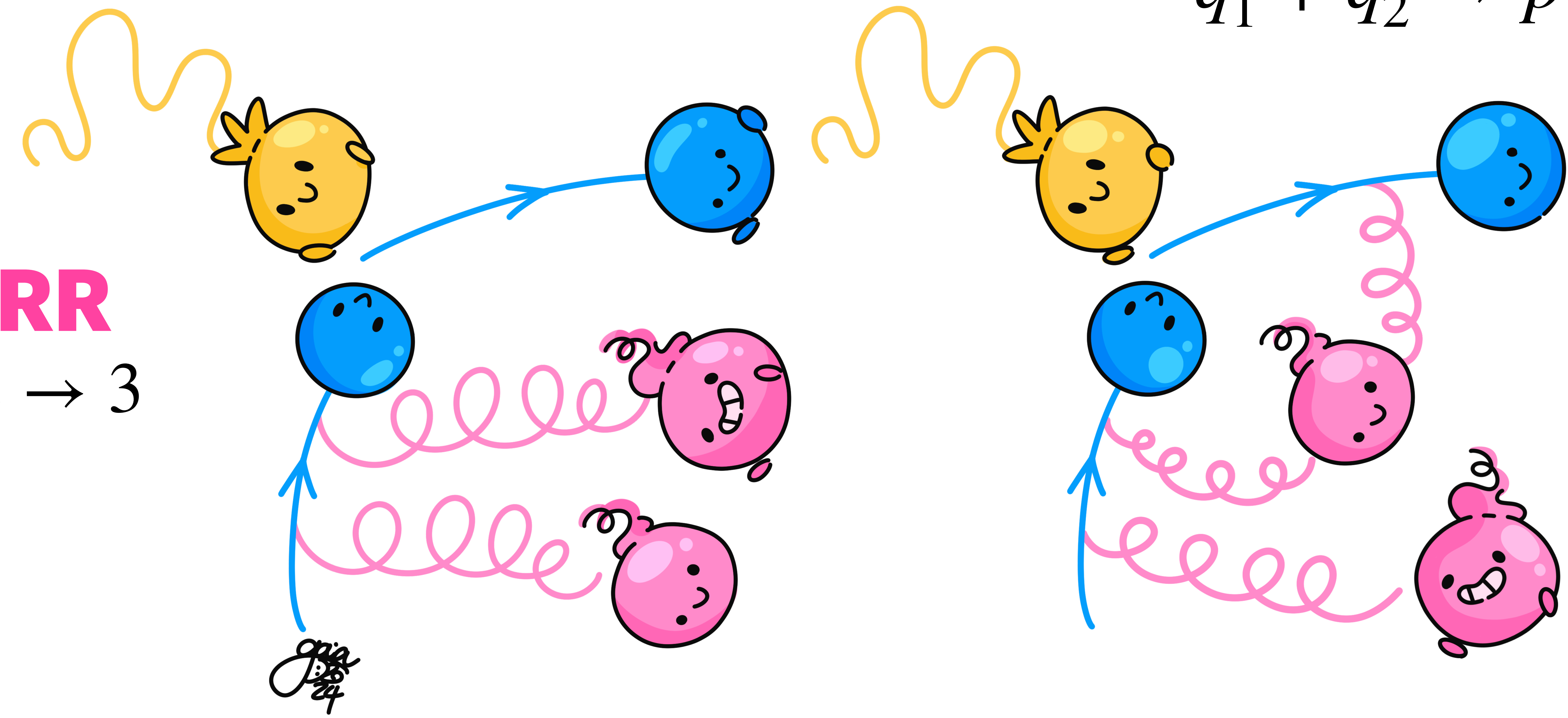
final-state parton

⇒DIS-like process

NNLO initial-final antennae

NNLO phase-space integrals for exclusive DIS

$$q_1 + q_2 \to p_1 + p_2 + (p_3)$$



RR
$2 \to 3$

RV
$2 \to 2$

kinematics

- $q_2^2 = -Q^2 < 0$
- $q_1^2 = 0$
- $p_i^2 = 0, \quad i = 1,2,3$

$$Q^2 = 1$$

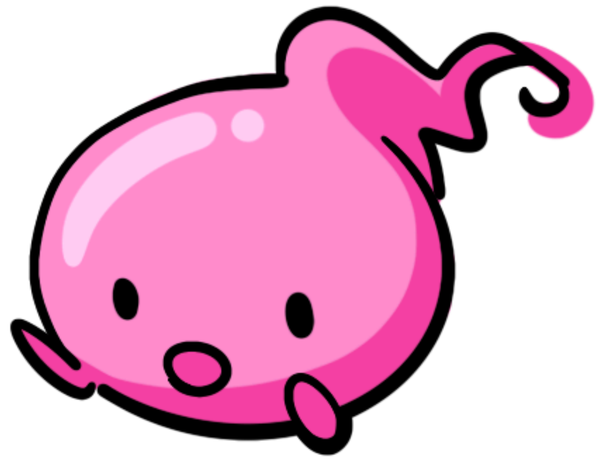
invariants

- $s = (q_1 + q_2)^2$
- $z = \dfrac{1}{2q_1q_2} \longrightarrow s = \dfrac{(1-z)}{z}$

Workflow

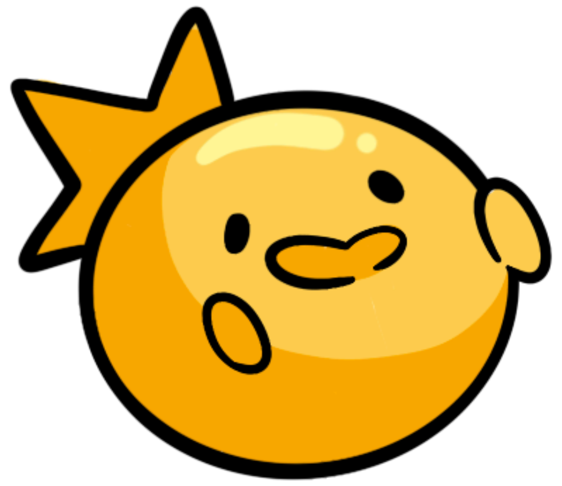
Phase space integral

Reverse unitarity



Reduction to master integrals

DE & canonical form



Boundaries

Reverse Unitarity

Anastasiou, Melnikov (2002)

phase space → (cut) loops

$$-2\pi i\delta^{+}(p_i^{+}) = \frac{1}{p_i^2+i0} - \frac{1}{p_i^2-i0} = \frac{1}{[p_i^2]_{cut}}$$

Notice!

$$\mathrm{d}\Phi_n = \prod_{i=1}^{n} \frac{\mathrm{d}^d p_i}{(2\pi)^d}\delta^{+}\left(p_i^2\right)$$

$$I_{RV} = \int \mathrm{d}\Phi_2\,(2\pi)^d\delta^{(d)}\left(q_1+q_2-\sum_{i=1}^{2}p_i\right)\int\frac{\mathrm{d}^d k}{(2\pi)^d}\prod_j\frac{1}{D_j^{\alpha_j}}$$

$$I_{RV} = \int\frac{\mathrm{d}^d k_1}{(2\pi)^d}\int\frac{\mathrm{d}^d k_2}{(2\pi)^d}\frac{1}{\not{D}_1}\frac{1}{\not{D}_2}\prod_j\frac{1}{D_j^{\alpha_j}}$$

$$I_{RR} = \int \mathrm{d}\Phi_3\,(2\pi)^d\delta^{(d)}\left(q_1+q_2-\sum_{i=1}^{3}p_i\right)\prod_j\frac{1}{D_j^{\alpha_j}}$$

$$I_{RR} = \int\frac{\mathrm{d}^d k_1}{(2\pi)^d}\int\frac{\mathrm{d}^d k_2}{(2\pi)^d}\frac{1}{\not{D}_1}\frac{1}{\not{D}_2}\frac{1}{\not{D}_3}\prod_j\frac{1}{D_j^{\alpha_j}}$$

Reduction to Master integrals

Reduction into a basis of linearly independent **master integrals**

$$\{G_j\} \subset \{I_j\}$$

$\{G_j\}$ = minimal linearly independent set

rational coefficients

$$I_j = \sum_k c_{jk} G_k$$

master integrals

Feynman integrals in dimensional regularization obey linear relations, e.g.

Integration By Parts identities + Lorentz Invariance ids, symmetry relations, …

[Chetyrkin, Tkachov (1981), Laporta (2000)]

$$\int \Big(\prod_{i=1}^{\ell} d^{\mathrm{D}} k_i \Big) \frac{\partial}{\partial k_i^\mu} \Big(\frac{v_j^\mu}{D_1^{a_1} \dots D_n^{a_n}} \Big) = 0,$$

$$v^\mu = \begin{cases} p_i^\mu & = \text{ external} \\ k_i^\mu & = \text{ loop} \end{cases}$$

reduction as solution of a large
and sparse system of identities

DE for Feynman integrals

- IBP → MI obeys first order differential equations $\partial_z \overrightarrow{G} = M \cdot \overrightarrow{G}$
- Magic transformation to new basis of MI $\{\vec{g}\}$: **canonical form** → solution is "straightforward"

[Henn (2013)]

$$\partial_z \vec{g} = \epsilon A \cdot \vec{g}$$

ϵ-dependence is factored out

Solution $\epsilon-$exp.

$$\vec{g}^{(k)} = \sum_{j=0}^{k} \int_{\gamma} \underbrace{\mathrm{d}\tilde{A} \cdot \ldots \cdot \mathrm{d}\tilde{A}}_{j} \cdot \vec{b}^{(k-j)}$$

Encodes the class of special functions needed for the solution

$$\mathrm{d}\tilde{A} = \sum_i a_i \, \mathrm{d}\log(\alpha_i)$$

- α_i **letters**
- independent letters: **alphabet**

Rational alphabet →results expressed as MPLs

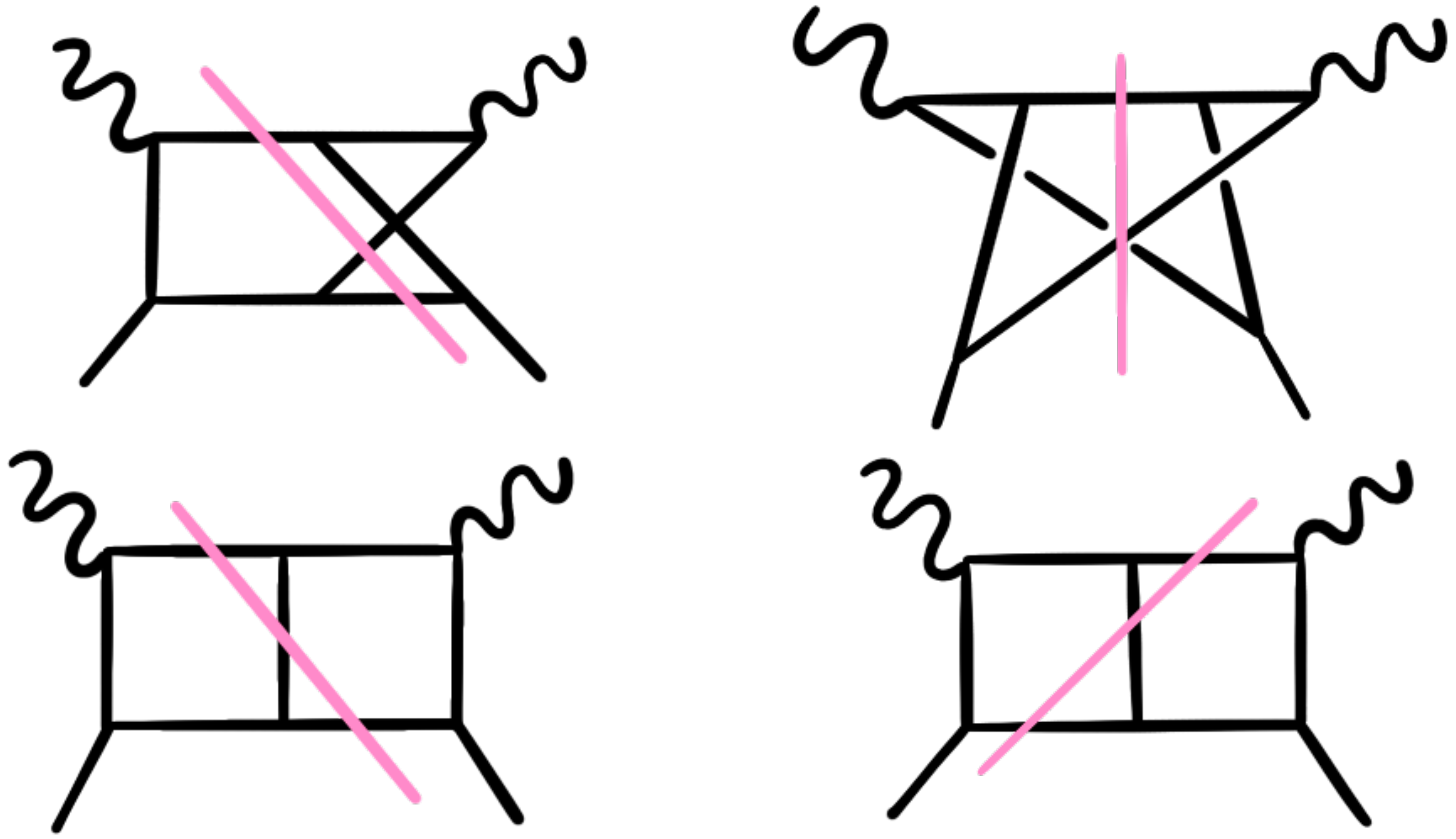
[Kummer (1840); Remiddi, Vermaseren (1999); Goncharov (2000)]

$$G(a_1, \ldots, a_n; t) \equiv \int_0^t \frac{\mathrm{d}x}{x - a_1} G(a_2, \ldots, a_n; x), \;\; \forall n \in \mathbb{N}, a_n \neq 0$$

NNLO calculation in a nutshell

RR

$$I_{RR} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{\not{D}_1} \frac{1}{\not{D}_2} \frac{1}{\not{D}_3} \prod_j \frac{1}{D_j^{\alpha_j}}$$

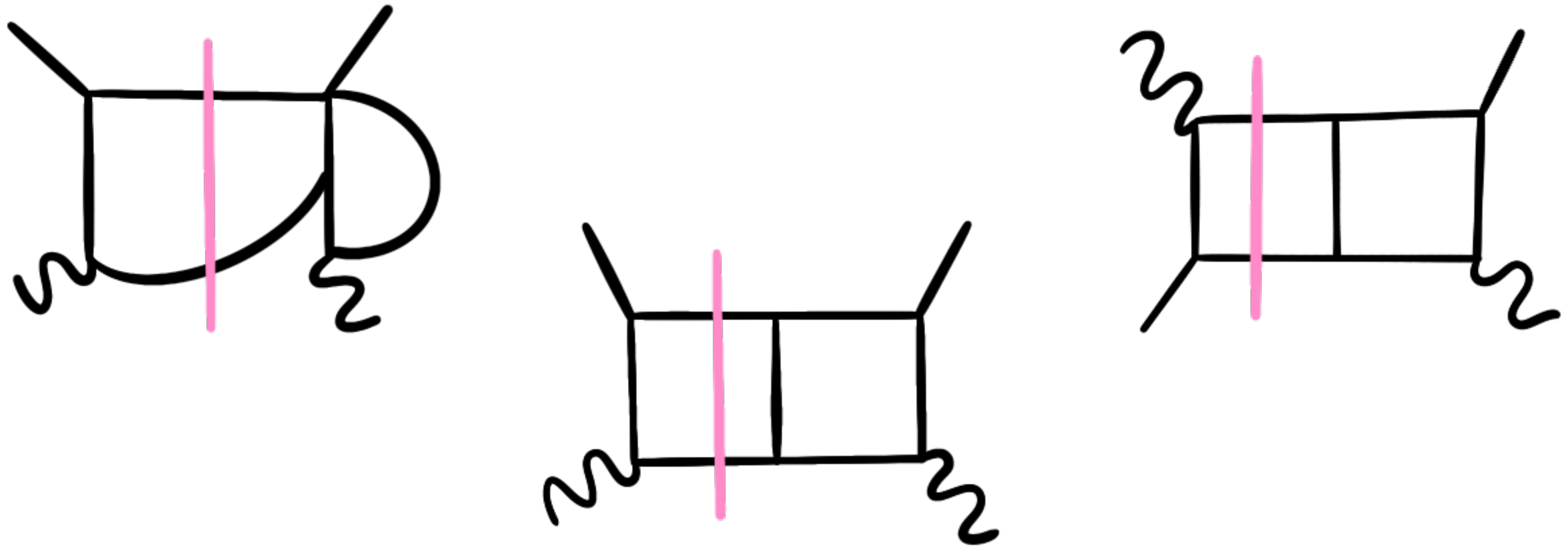


- 4 families, 9 MI
- 2 loops, 3 cut propagators
- DE in canonical form using FUCHSIA

[Gituliar & Magerya (2017)]

RV

$$I_{RV} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \frac{1}{\not{D}_1} \frac{1}{\not{D}_2} \prod_j \frac{1}{D_j^{\alpha_j}}$$



- 3 families, 6 MI
- 2 loops, 2 cut propagators
- DE in canonical form using FUCHSIA

Boundary Conditions

- Looking at $z \to 1$ **soft limit**, endpoint singularity
- Extract some relations from known behaviour of the integrals in the soft limit

$$I_i^{RR} \sim (1-z)^{n_i - 2\epsilon} \sum_j c_j(\epsilon)(1-z)^j, \quad n_i \in \mathbb{Z}$$

$$I_i^{RV} \sim (1-z)^{m_i - 2\epsilon} \sum_j d_j(\epsilon)(1-z)^j + (1-z)^{l_i - \epsilon} \sum_j e_j(\epsilon)(1-z)^j, \quad m_i, l_i \in \mathbb{Z}$$

- Extract the **leading behavior** of the MIs
- Rescaling the integrals w.r.t. their leading behavior → regularity
- Imposing that in this limit the terms $\log(1-z)$ and poles in $(1-z)$ vanish
- **Relations between boundaries** of different MIs

AAMFlow

[GF, Gehrmann, Schönwald (2024)]

Based on the `AMFlow` algorithm but **purely analytic**

[Liu, Ma (2022)]

Looking at $z \to 1$ **soft limit**, endpoint singularity

- Integral simplifies
- Allows to carry out this procedure analytically

$$I^{phys}(\epsilon, \vec{z}) \to I^{aux}(\epsilon, \vec{z}, \eta^2)$$

- Add aux mass η^2 to **chosen** propagators: **auxiliary family**
  - limits in kinematical variable and η^2 need to **commute**
- Derive DE with respect to the mass
  - $\partial_{\eta^2}\vec{I}^{aux} = A_\eta \cdot \vec{I}^{aux}$
  - Boundaries @ $\eta^2 \to \infty$ (large mass limit)
- Fix constants of integration in $\eta^2 \to \infty$ limit (easy!)
- "Flow" to $\eta^2 \to 0$ for physical solution:
  - **method of regions** to extract the physical solution

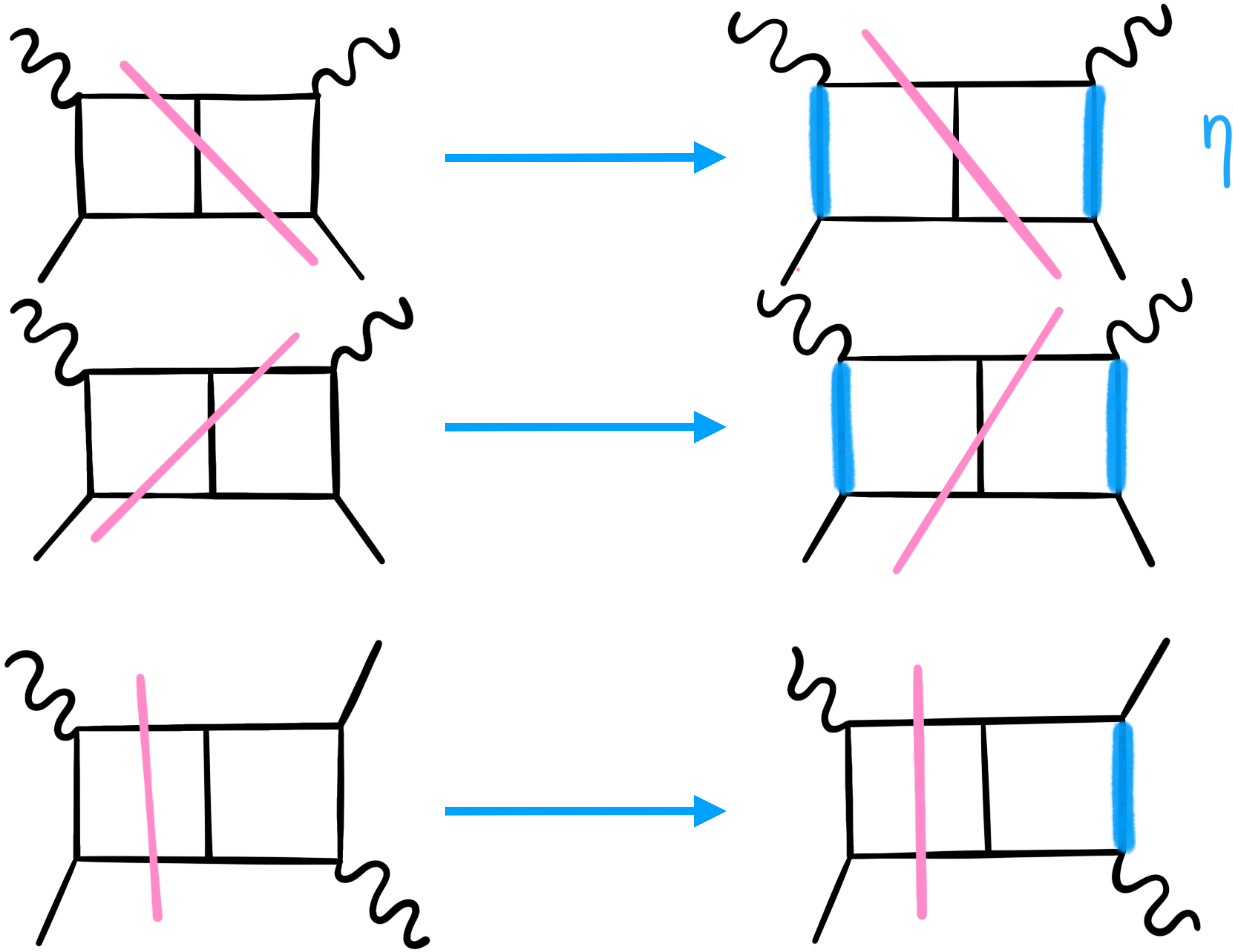
AAMFlow/2

Required families to be calculated for obtaining all the boundaries

auxiliary topologies

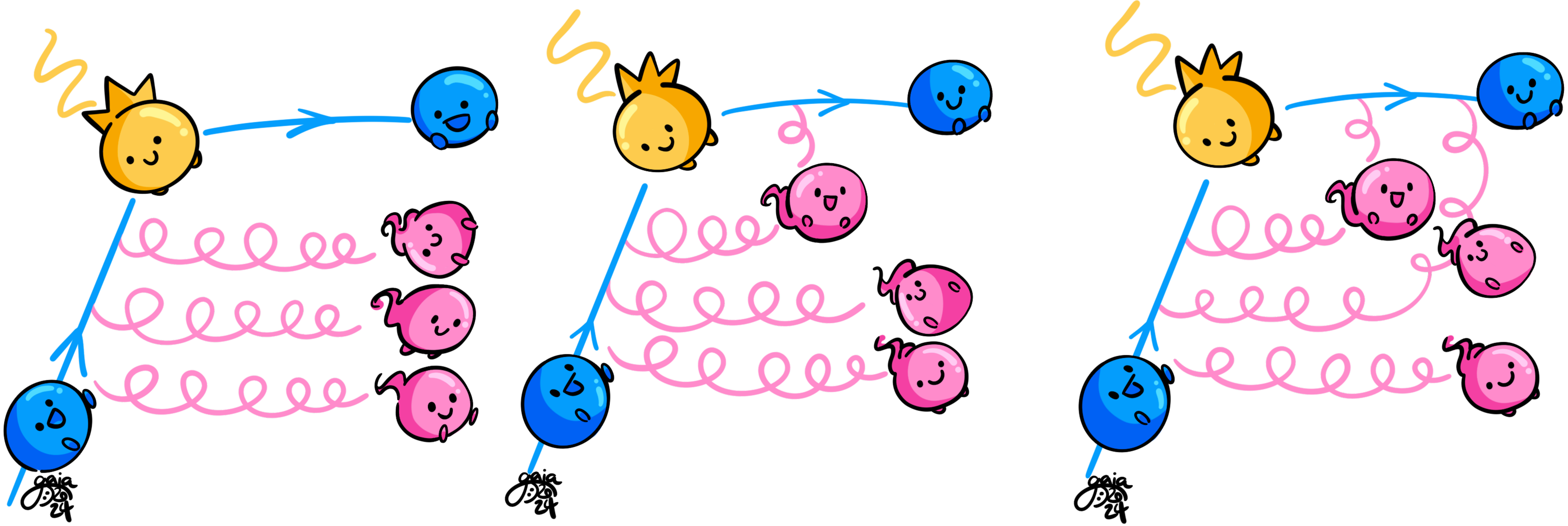
RR

RV



η^2 AUXILIARY MASS

Towards N3LO RRR antennae

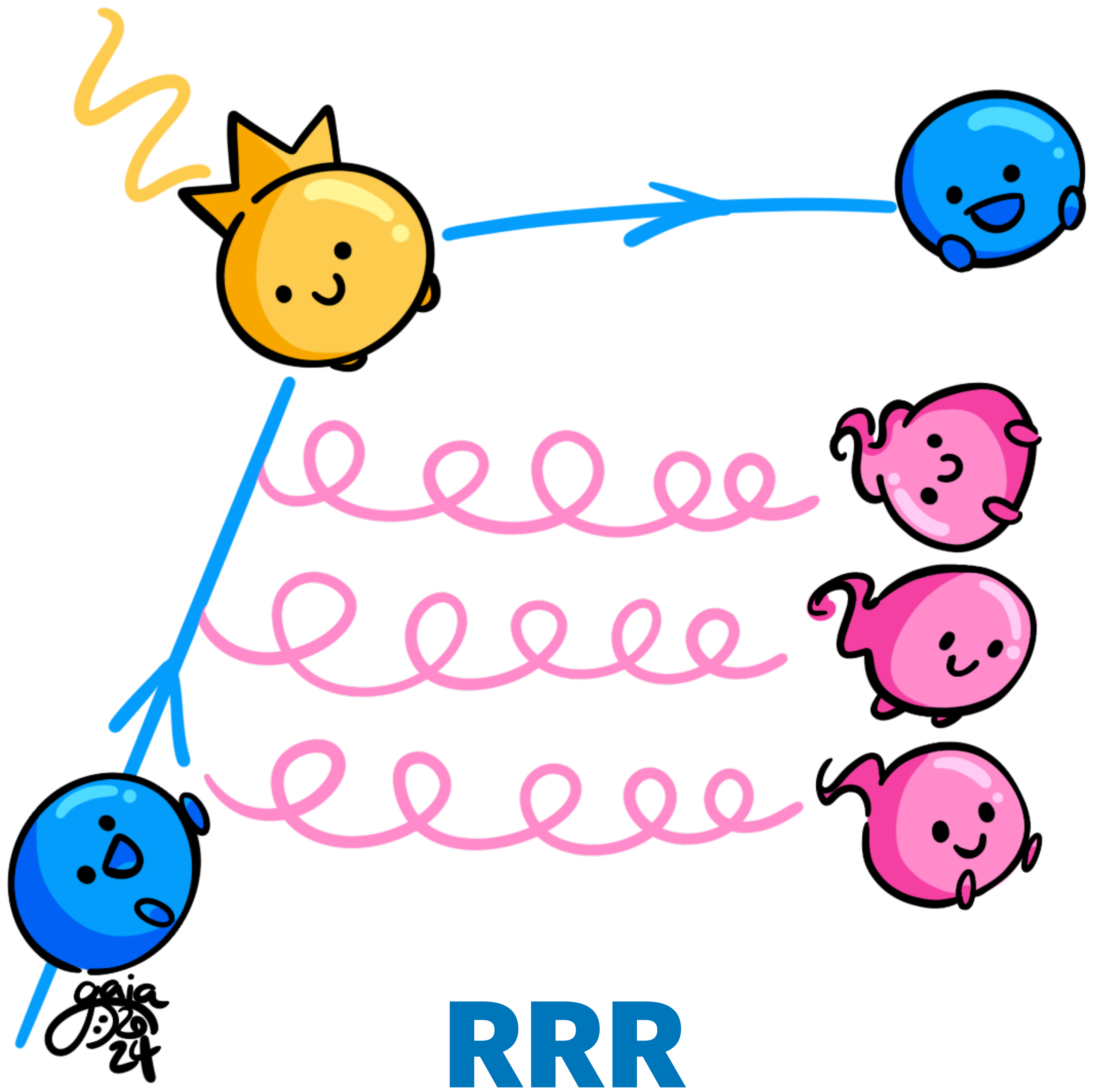


N3LO initial-final antennae

N3LO phase-space integrals for exclusive DIS

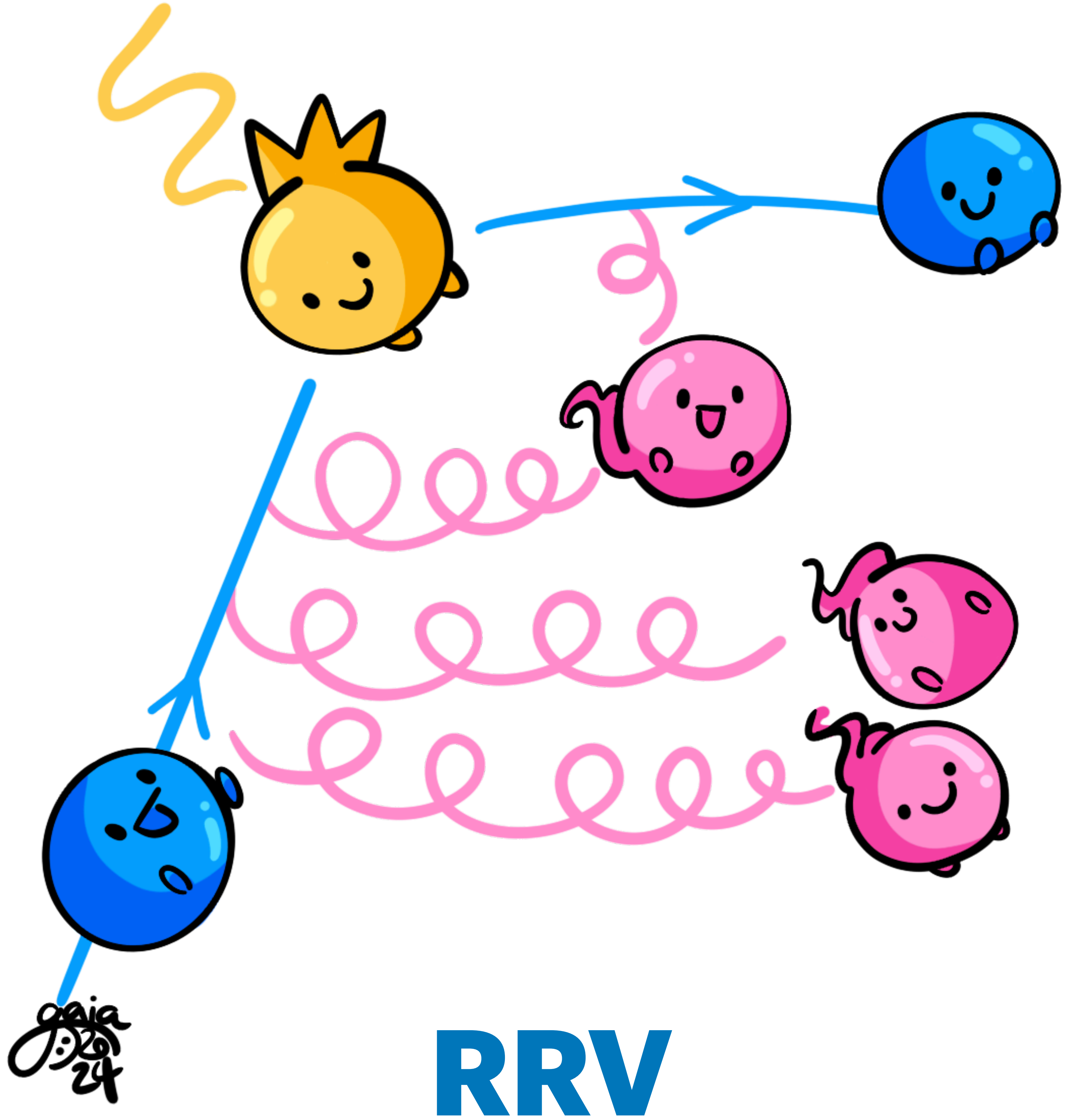
$$q_1 + q_2 \to p_1 + p_2 + (p_3) + (p_4)$$

$$q_2^2 = -Q^2 < 0,\ q_1^2 = 0,\ p_i^2 = 0, \quad i = 1{,}2{,}3{,}4$$



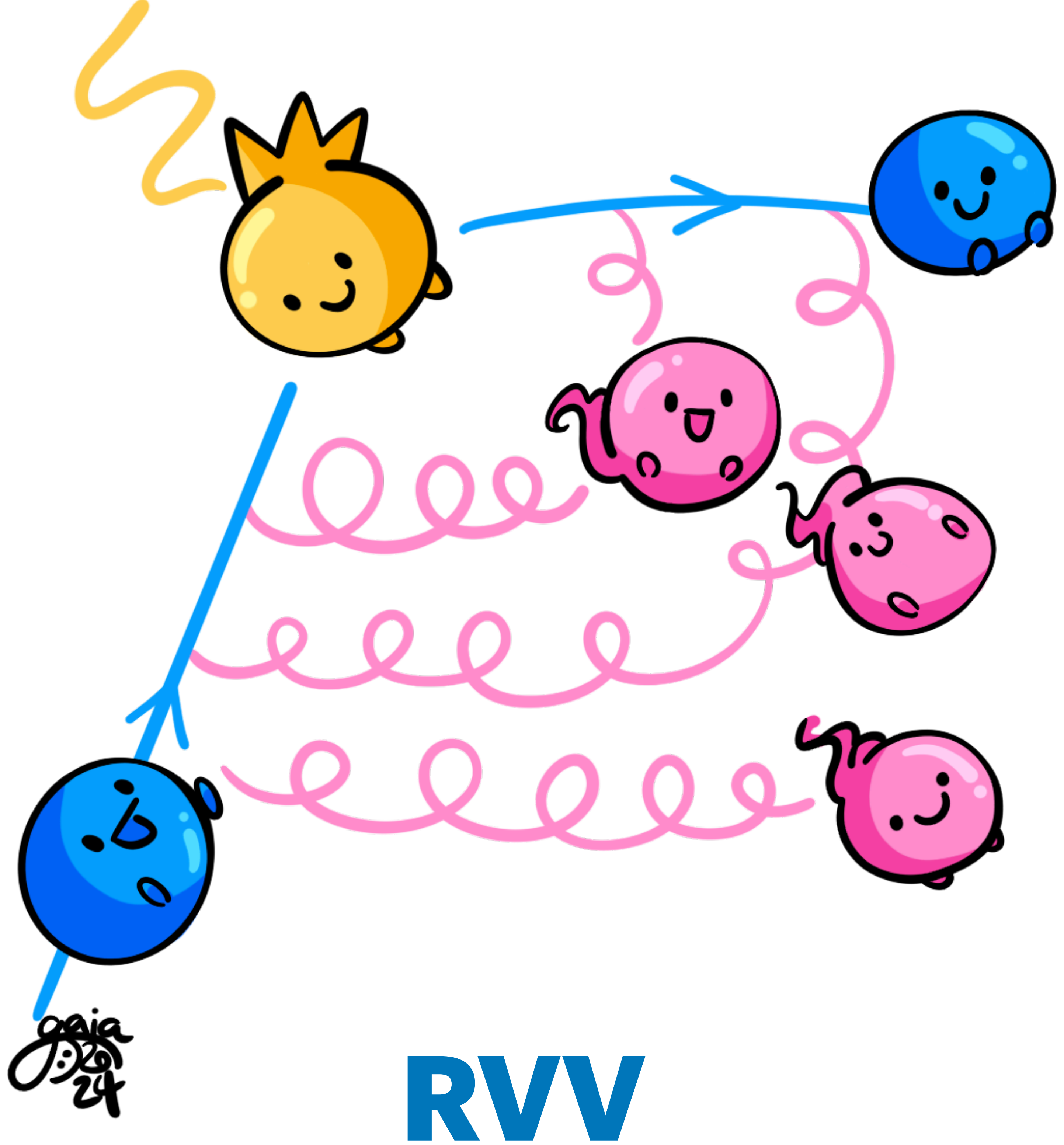
RRR

$2 \to 4$



RRV

$2 \to 3$



RVV

$2 \to 2$

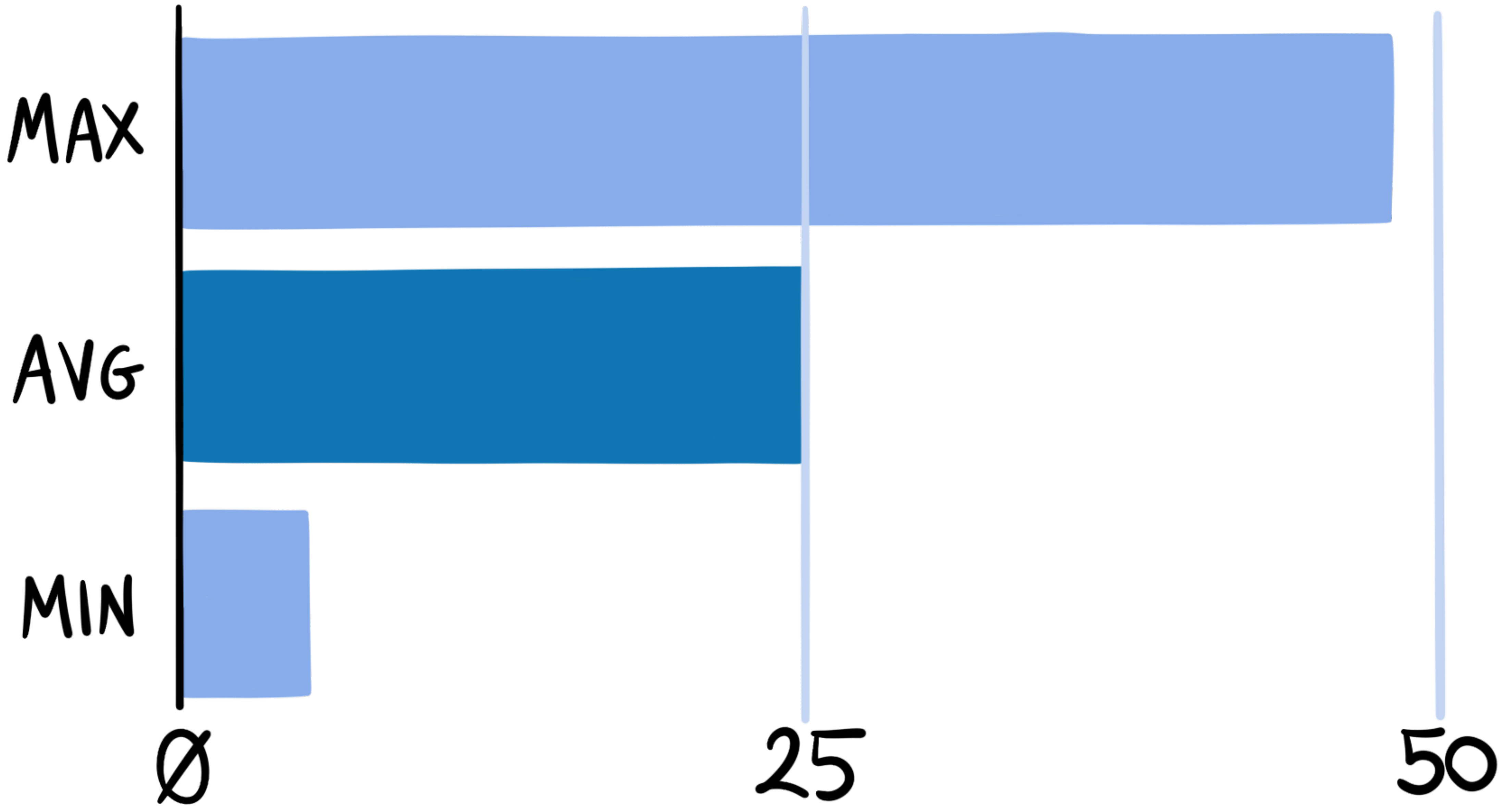
Getting to know the RRR families

Physical 4-cuts of the 3 loop inclusive DIS amplitude

$$I_{RRR} = \int \frac{\mathrm{d}^d k_1}{(2\pi)^d} \int \frac{\mathrm{d}^d k_2}{(2\pi)^d} \int \frac{\mathrm{d}^d k_3}{(2\pi)^d} \frac{1}{\not{D}_1} \frac{1}{\not{D}_2} \frac{1}{\not{D}_3} \frac{1}{\not{D}_4} \prod_j \frac{1}{D_j^{\alpha_j}}$$

- Few number of MI for each family →**4 cuts**
- **Total: 1620 MIs** (**No symmetries** between families included)

#MI PER FAMILY



- DE matrix M is a function of $M(z, \epsilon)$

⇒ **playground for automatic tools! eg** `LIBRA`

[Lee (2020)]

Getting to a Canonical Basis (1): Balancing acts

$M(z, \epsilon)$

Fuchsification: no higher order poles

$$\sum_k \frac{M_k(z,\epsilon)}{x - x_k}$$

Normalizing eigenvalues $\propto \epsilon$

$$\sum_k \frac{M_k^\star(z,\epsilon)}{x - x_k}$$

$\epsilon-$factorization

$$\epsilon \sum_k \frac{A(x)}{x - x_k}$$

- Build **balancing transformations** via graphical interface
    - Change pole order and eigenvalues of residue matrix around poles

- Strategy for multi loop calculations: exploit block triangular structure

Diagonal blocks → **Global fuchsification** → **Factoring out ϵ**

Getting to a Canonical Basis (2): good candidates

- Sectors with higher number of propagators (top sector (TS), Next-to-TS, NNTS)
- **Baikov representation**

$$I = \int \left(\prod_{i=1}^{\ell} \mathrm{d}^d k_i \right) \frac{1}{z_1^{\nu_1} \cdots z_n^{\nu_n}} = K \int \mathrm{d}z_1 \cdots \mathrm{d}z_n \, B(\mathbf{z})^{\gamma} \frac{1}{z_1^{\alpha_1} \cdots z_n^{\alpha_n}}$$

- Numerator Ansatz $N(\vec{z})$
- Check candidates with constant leading singularity with `DLogBasis` [Wasser (2020)]
- Keep only the linearly independent candidates for a new basis

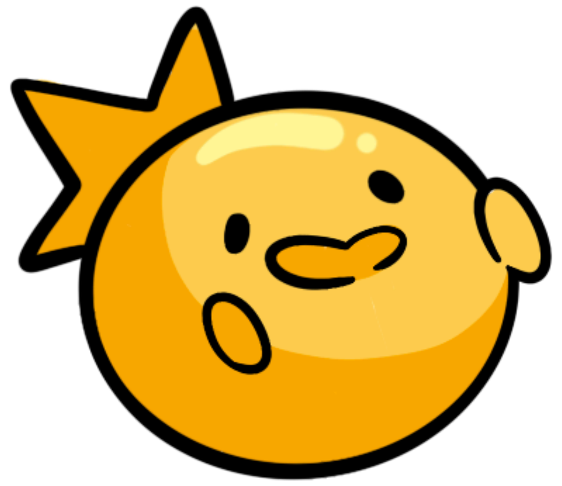
$$\int dz_1 \ldots dz_n \, B^{\gamma} \frac{N(\vec{z})}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}} \Bigg|_{\text{cut condition}}$$

`Libra`

Results

Canonical DE for all the families ✓

Letters: $$\left\{\frac{1}{z}, \frac{1}{1+z}, \frac{1}{1-z}, \frac{1}{1+2z}, \frac{1}{1-2z}\right\}$$



Boundary conditions ➡ SOON

- Numerical evaluation with `AMFlow` **@ 200 digits** (~80% done…) **& PSLQ**
- Constraints from symmetry relations between the families
- Calculation of the amplitude → which boundaries are actually needed

Outlook:

- Extend calculation to RVV and RRV layers
- Ultimate goal: obtaining the full set of integrated initial-final antennae

Thank you for your attention!