Non-perturbative QCD for flavour physics

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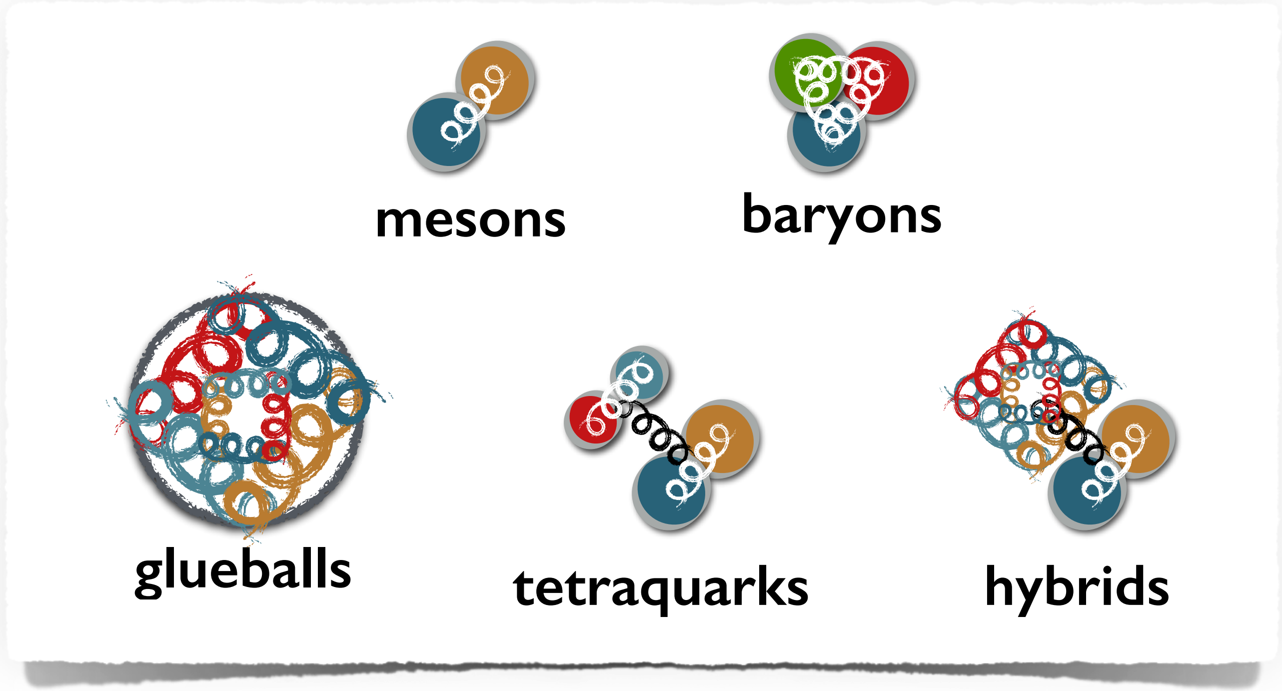
THE UNIVERSITY *of* EDINBURGH

Flavour physics

- flavour anomalies = opportunity for BSM
- QCD = crucial for confirming significance and interpreting

experiment = **SM** x *perturbative QCD* x **(non-perturbative QCD)**
+ **BSM** x *perturbative QCD* x **(non-perturbative QCD)**

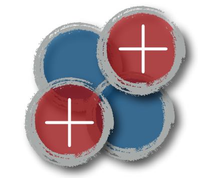
- QCD is complicated
- Difficult to extract non-perturbative predictions



Recipe for strong force predictions

1. Lagrangian defining QCD
2. Formal / numerical machinery (lattice field theory)
3. A few experimental inputs (e.g. M_π, M_K, M_Ω)

$$\mathcal{L}_{\mathrm{QCD}} = \sum_f \overline{\Psi}_f (i\not{D} - m_f)\Psi_f - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a$$

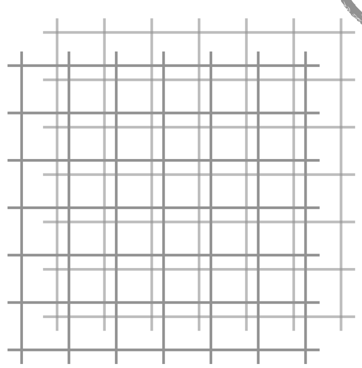


Wide range of precision pre-/post-dictions

$m_q^{\overline{\mathrm{MS}}}(\mu)$ $\alpha_{\overline{\mathrm{MS}}}(\mu)$

m_p

m_n



f_π

g_A

f_K

$(g-2)_\mu^{\mathrm{HVP}}$ f_D

f_B

FLAG 2021 Flavour Lattice Averaging Group

$\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1184(8)$ lattice average

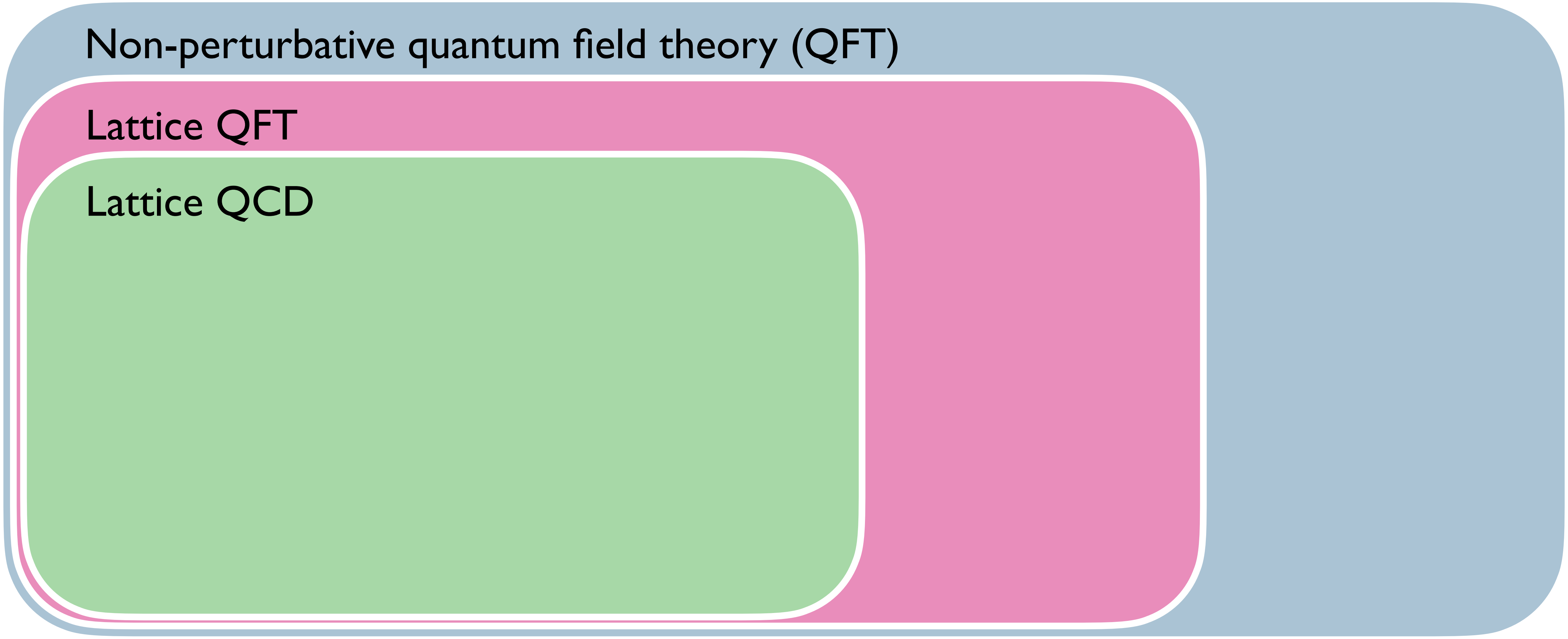
$\alpha^{(5)}_{\overline{\mathrm{MS}}}(M_Z) = 0.1176(11)$ PDG 20 (non-lattice)

Overwhelming evidence for QCD ✓ Tool for new-physics searches ✓

Lattice QCD

- a non-perturbative regularization of QCD
- a definition that is well-suited to numerical evaluation

render the quantum path-integral finite-dimensional → *evaluate using Monte Carlo importance sampling*



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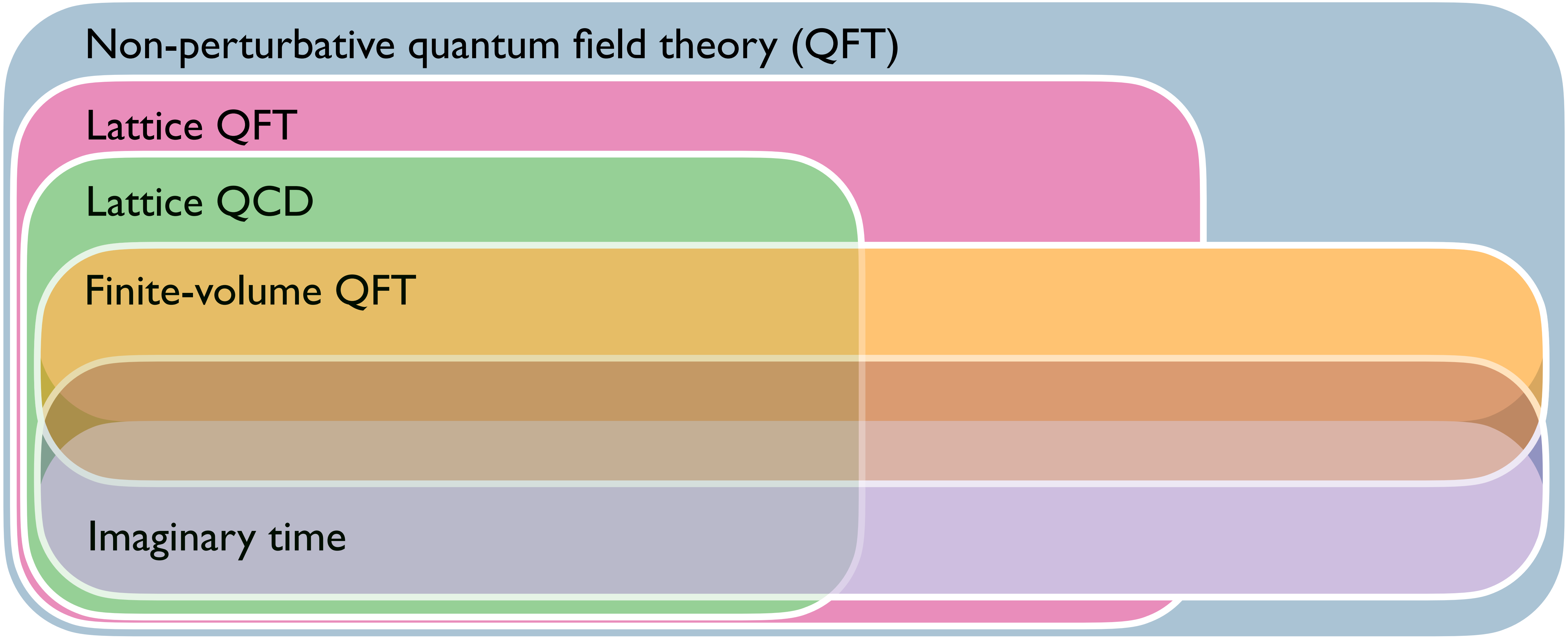

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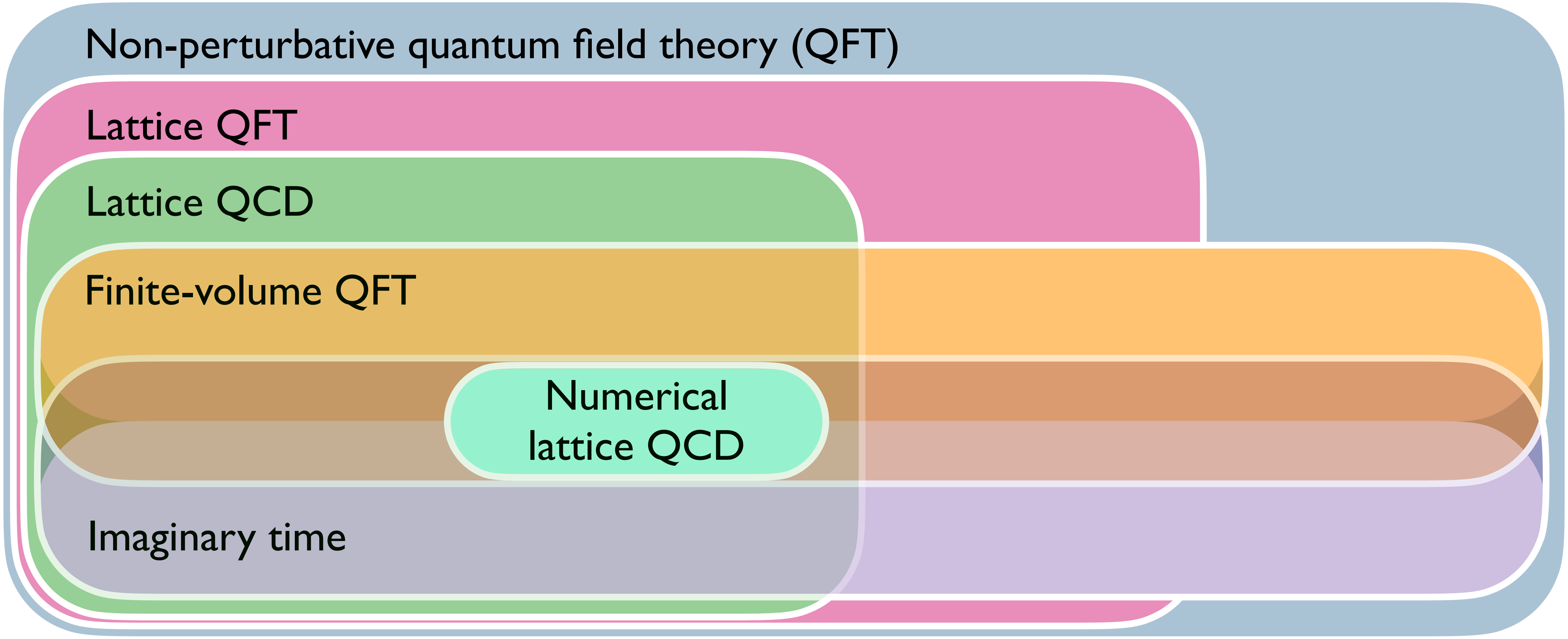


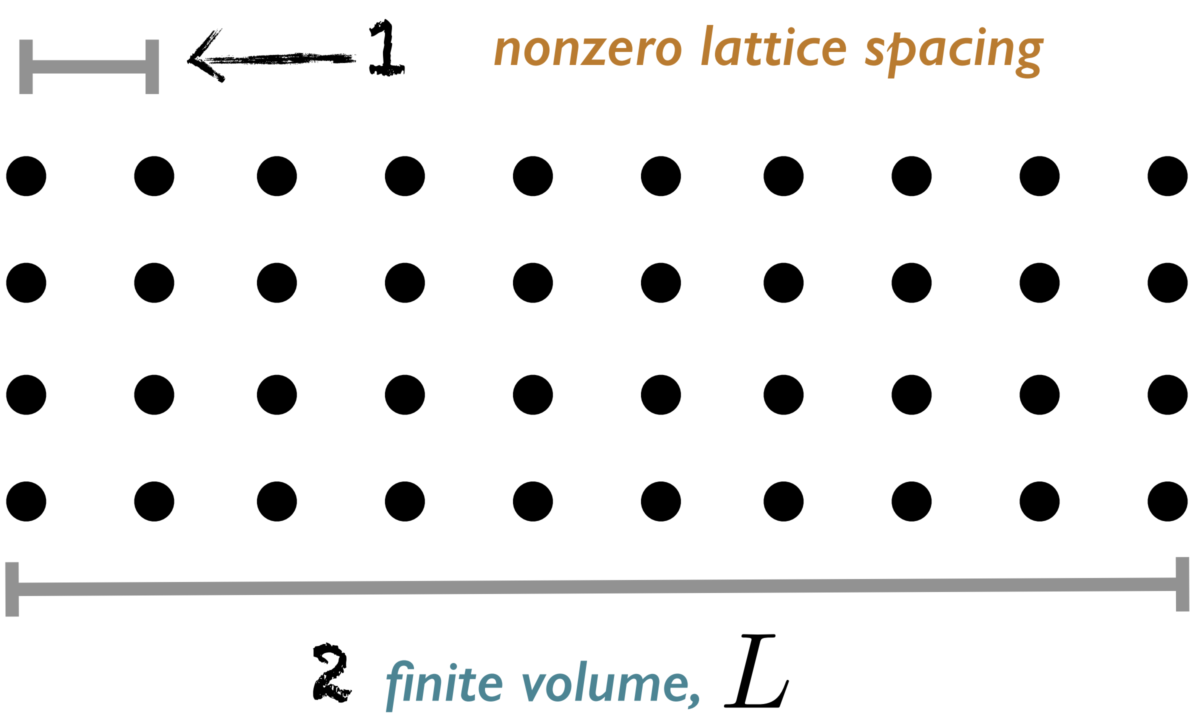
Challenges for lattice QCD

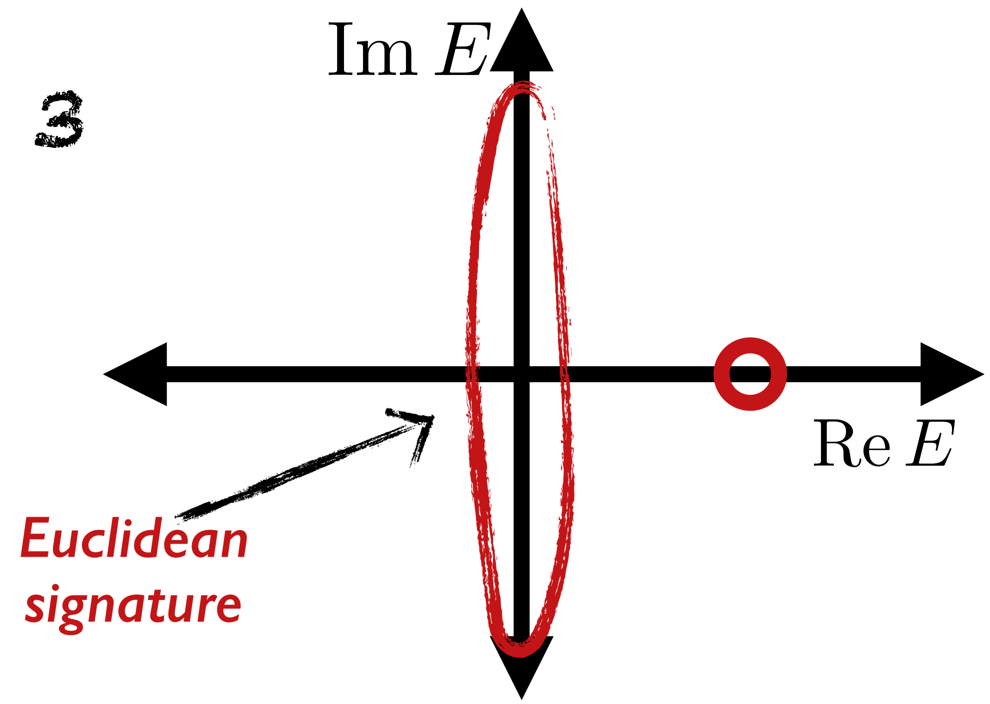
$$\text{observable} = \int \mathcal{D}\phi \; e^{iS} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

Challenges for lattice QCD

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c}\textbf{interpolator} \\ \textbf{for observable}\end{array} \right]$$

To proceed we have to make *three modifications*





Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

(but physical masses → increasingly common)

Some more details of LQCD

- Continuum theory is defined as…

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_f \overline{\Psi}_f \left(i\gamma^\mu D_\mu - m_f \right) \Psi_f$$

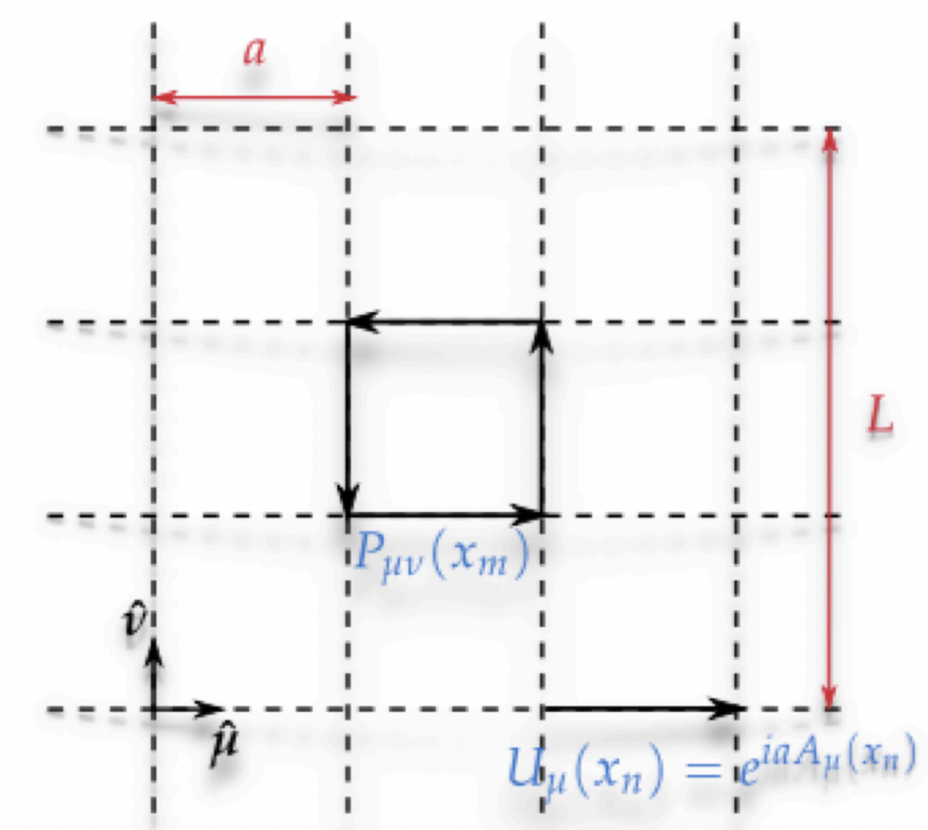
$$D_\mu = \partial_\mu - i g T^a A^a_\mu$$

- First step to putting on a lattice is to replace derivative with finite difference

$$\overline{\Psi}_f(x)\gamma^\mu \partial_\mu \Psi_f(x) \longrightarrow \overline{\Psi}_f(x)\gamma^\mu \frac{1}{a}\left[\Psi_f(x+a\hat{\mu}) - \Psi_f(x)\right]$$

or, for the covariant derivative…

$$\overline{\Psi}_f(x)\gamma^\mu D_\mu \Psi_f(x) \longrightarrow \overline{\Psi}_f(x)\gamma^\mu \frac{1}{a}\left[U_\mu(x)\Psi_f(x+a\hat{\mu}) - \Psi_f(x)\right]$$



- Some key messages

$U_\mu(x)$ is the link variable

exact gauge invariance for nonzero lattice spacing

field strength tensor from a "plaquette"

$$U_\mu(x) = e^{iaA_\mu(x)} \qquad P_{\mu\nu}(x) = e^{ia^2 F_{\mu\nu}(x) + O(a^3)}$$

Many kinds of quarks…

- Naive implementation of a quark on the lattice does not work

  put in one flavour… turns out you are simulating 16 flavours!

- Fixing this issue leads to many possibilities

  domain-wall quarks, staggered quarks, Wilson quarks, twisted-mass quarks

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- Quarks are integrated out analytically
- Resulting integral is very high dimensional ($\sim 100^4 \times 10 = 10^9$ dims)

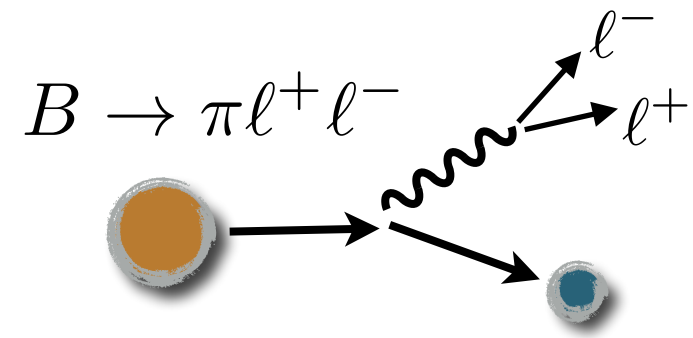
$$\langle \mathcal{O} \rangle = \int \mathcal{D}U \prod_f \det(D_f[U]) \, e^{-S_G[U]} \, \mathcal{O}(D^{-1}, U)$$

$$\sim \sum_{U \in E} \mathcal{O}(D^{-1}, U)$$

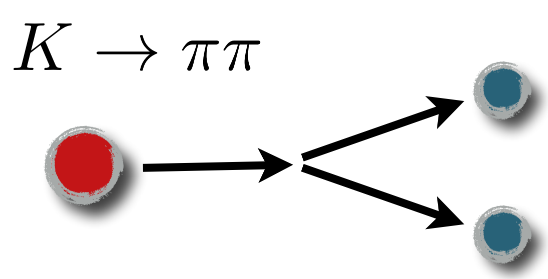
observable *Dirac operator*

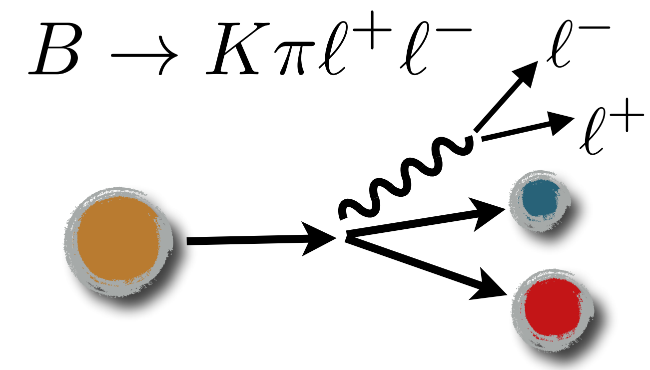
Matrix elements and LQCD

- Single-hadron initial and final states
  - Calculated directly in LQCD
  - New theory challenge = QED
  - See FLAG averages

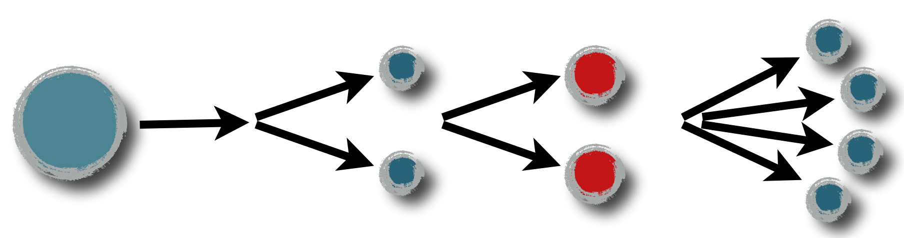


- Two-hadron final states
  - Significantly more challenging
  - Subtle finite volume issues





- Multi-hadron states for $\sqrt{s} > 4M_\pi$
  - All or nothing (must constrain all channels for a prediction)



$D \to \pi\pi, K\overline{K}$

Processes with QCD-stable hadrons

- Three categories:
  - Decay constants
  - Form factors
  - Mixing parameters

Decay constants	Form factors	Mixing parameters
$\langle 0 \vert \mathcal{J} \vert \mathbf{1} \rangle$	$\langle \mathbf{1} \vert \mathcal{J} \vert \mathbf{1}' \rangle$	$\langle \overline{\mathbf{1}} \vert \mathcal{H}^{\Delta F=2} \vert \mathbf{1} \rangle$
f_π, f_K, f_B	$f_+^{K^0\pi^-}(q^2),\ f_{B\to\pi}(q^2)$	$B_{B_d}^{(n)},\ B_{B_s}^{(n)}$

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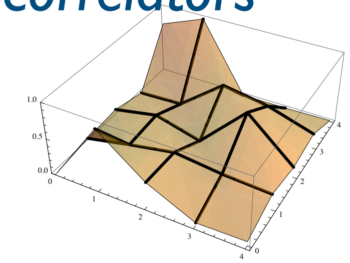
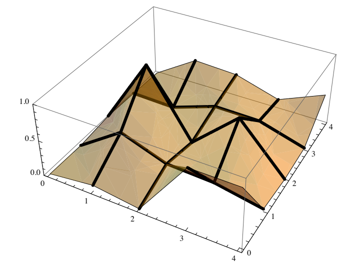
- Summary of the approach…
  - Importance sampling QCD gauge fields → *correlators*

$$\langle A_\mu^{\rm bare}(0)\ \pi_{\boldsymbol{p}}(-\tau)\rangle_{T,L,m_q,a} =$$  $+$  $+\cdots$

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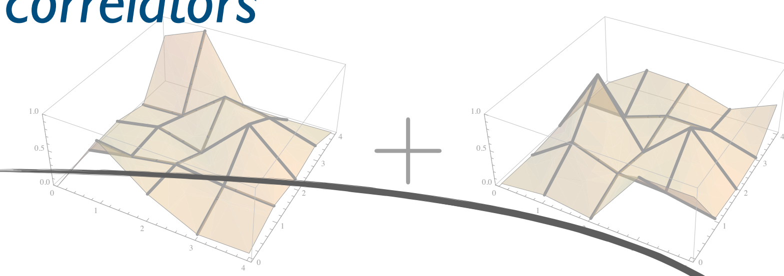

temporal length, volume, quark masses, lattice spacing

Processes with QCD-stable hadrons

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$$f_\pi,\, f_K,\, f_B \qquad f_+^{K^0\pi^-}(q^2),\;\; f_{B\to\pi}(q^2) \qquad B_{B_d}^{(n)},\;\; B_{B_s}^{(n)}$$

- Summary of the approach…
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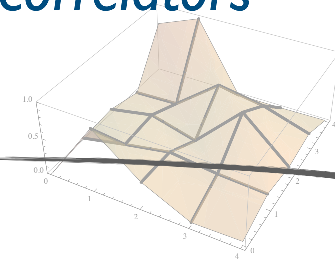
$$Z^{\rm renorm}\,\langle A_\mu^{\rm bare}(0)\;\pi_{\boldsymbol{p}}(-\tau)\rangle_{T,L,m_q,a}$$



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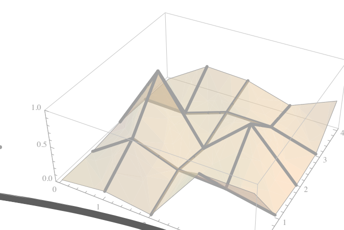

$+\cdots$

temporal length — volume — quark masses — lattice spacing

  - *Renormalization* of currents required (typically non-perturbative)

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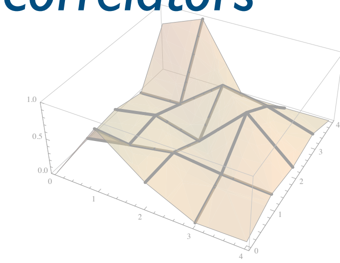
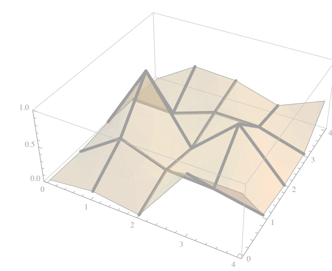
 +  + ⋯

temporal length volume quark masses lattice spacing

  - *Renormalization* of currents required (typically non-perturbative)
  - *Large time separation* filters excited states

$$Z^{\text{renorm}}\,\langle A_\mu^{\text{bare}}(0)\ \pi_{\boldsymbol{p}}(-\tau)\rangle_{T,L,m_q,a} = \langle A_\mu^{\text{renorm}}(0)\ e^{-\hat{H}\tau}\ \pi_{\boldsymbol{p}}(0)\rangle$$

$$\xrightarrow[\tau\gg\delta E_\pi]{} Z_\pi e^{-E_\pi\tau}\ ip_\mu f_\pi(T,L,m_q,a)$$

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temporal length volume quark masses lattice spacing

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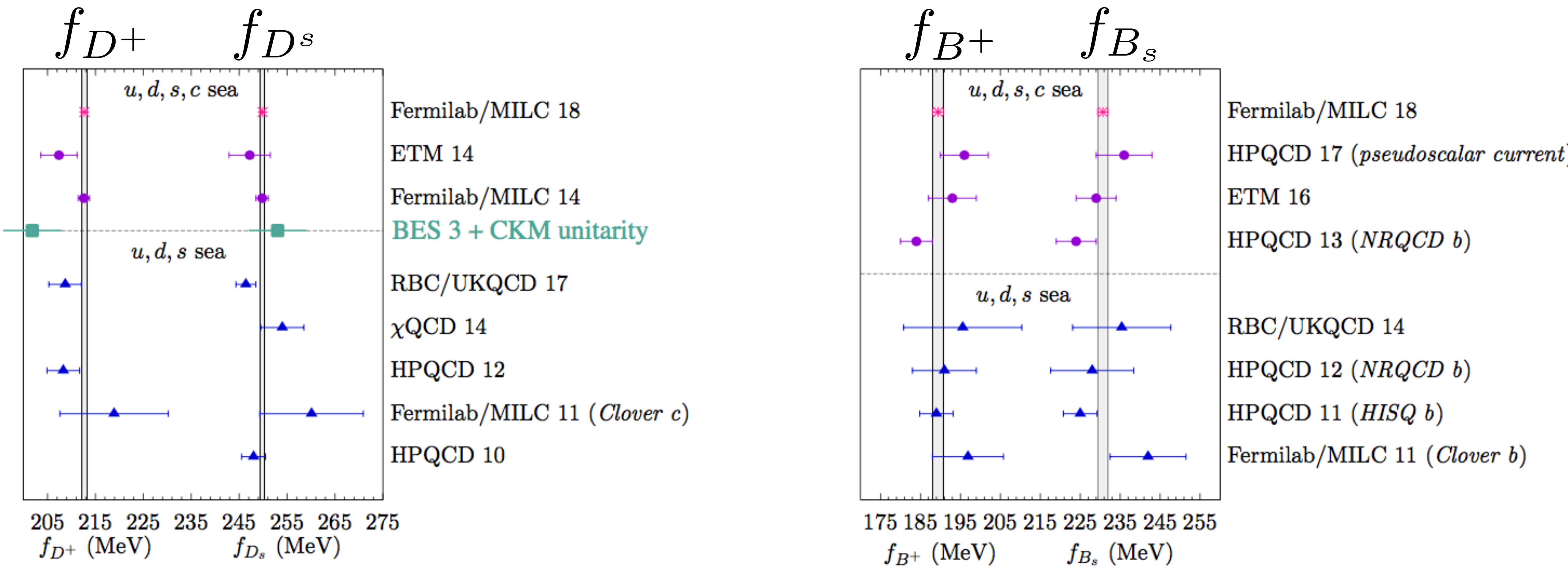
temporal length volume quark masses lattice spacing

  - *Renormalization* of currents required (typically non-perturbative)
  - *Large time separation* filters excited states
  - *Extrapolation/interpolation* to physical point

$$\lim_{T,L\to\infty}\ \lim_{a\to 0} f_\pi(T,L,m_q^{\rm phys},a) = f_\pi^{\rm phys}$$

Decay constants $\langle 0|\mathcal{J}|\mathbf{1}\rangle$

- Summary (from Bazavov et. al. [Fermilab/MILC] 2018)



- Current precision sufficient for BES III, BELLE II
- Fermilab/MILC includes QED uncertainty (not yet rigorous)
- MILC quoting higher precision than any other 2+1(+1) calculation

Need comparable precision from other calculations to cross-check

Decay constants: *latest FLAG update for charm*

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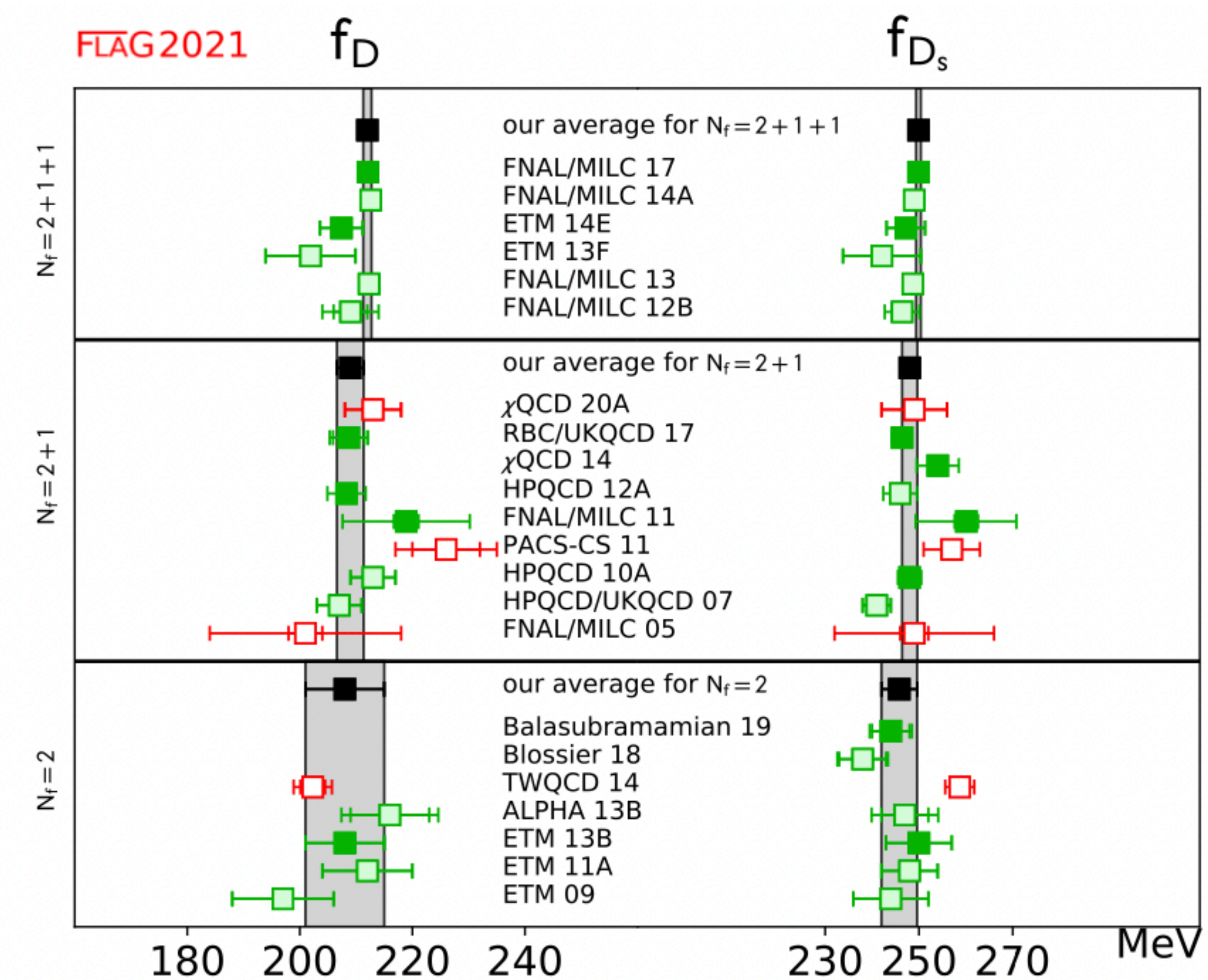

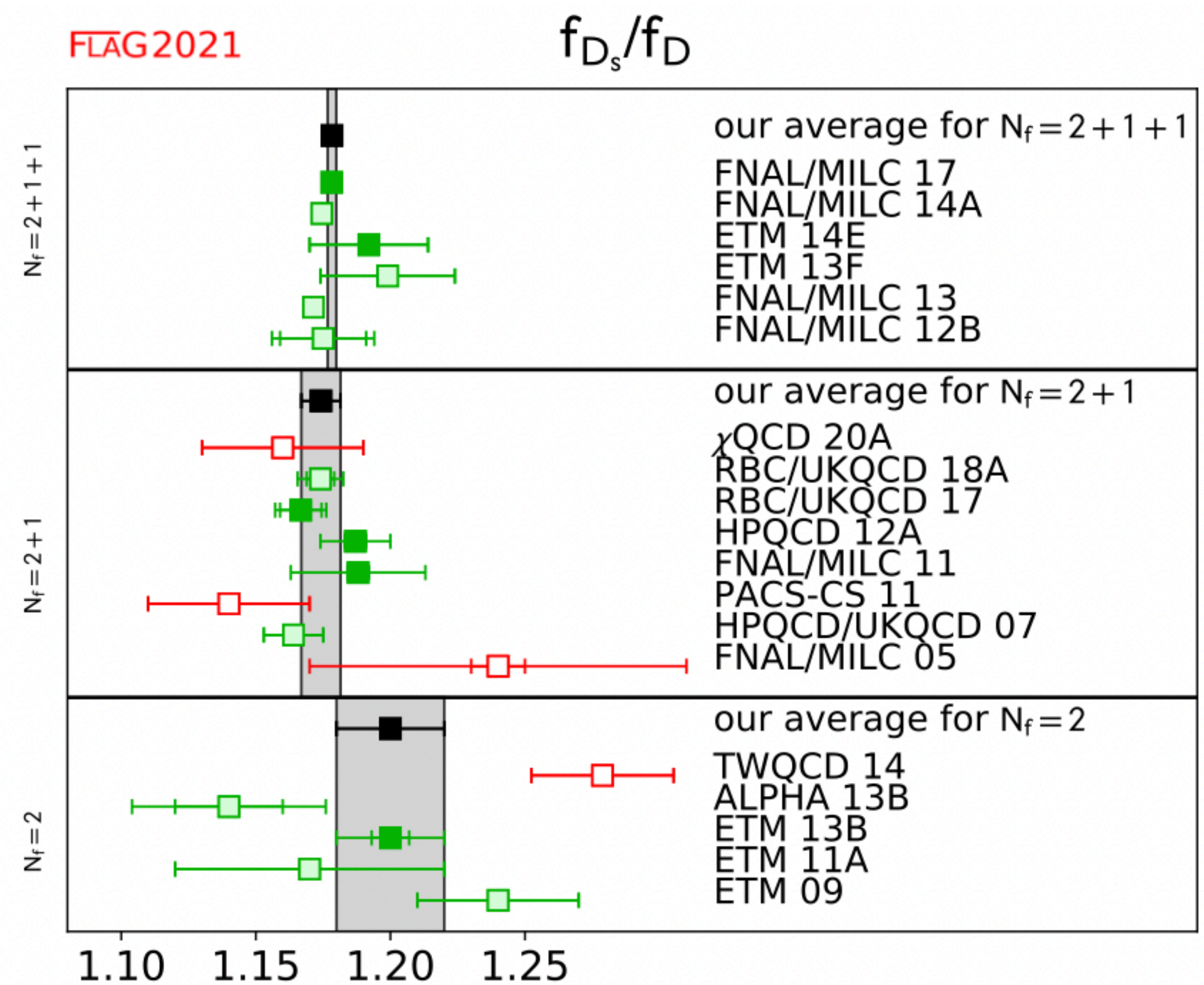

$$N_f = 2+1+1: \qquad f_D = 212.0(0.7)\ \text{MeV}$$

$$N_f = 2+1+1: \qquad f_{D_s} = 249.9(0.5)\ \text{MeV}$$

$$N_f = 2+1+1: \qquad \frac{f_{D_s}}{f_D} = 1.1783(0.0016)$$

Decay constants: *latest FLAG update for bottom*

FLAG 2021
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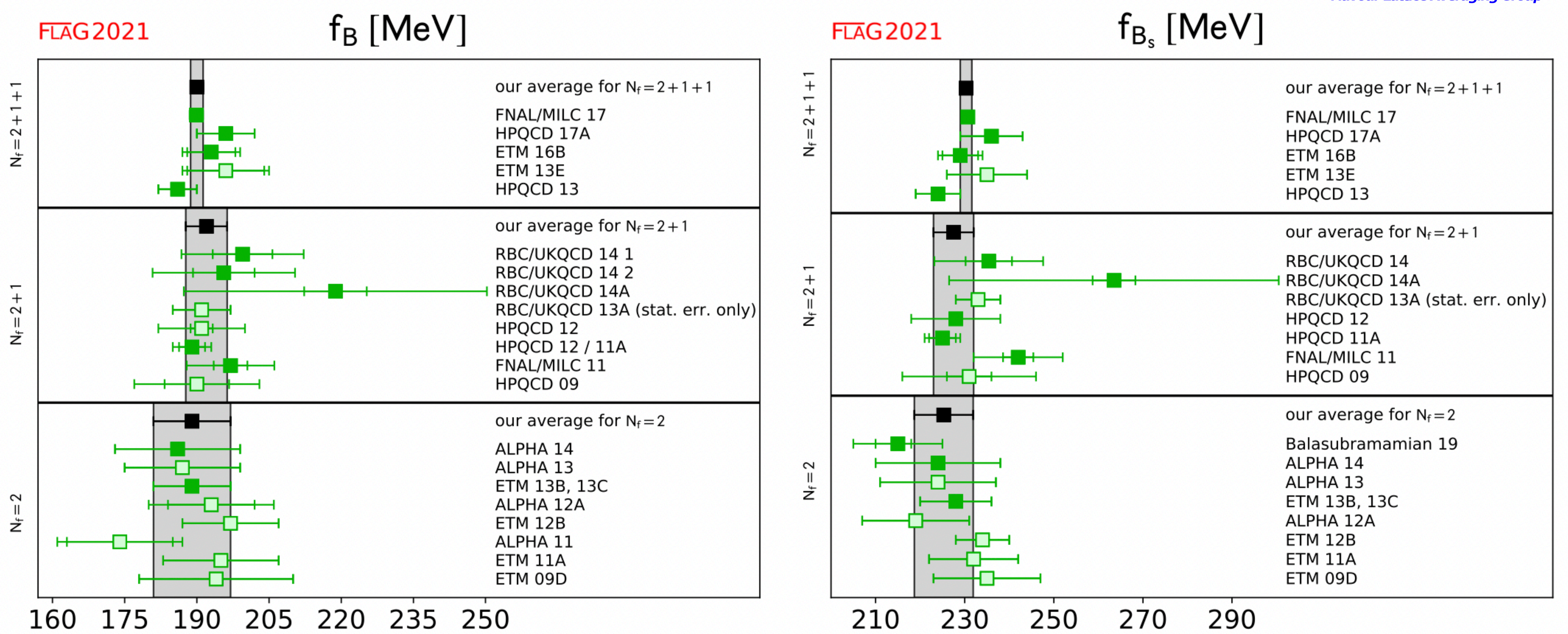


$$N_f = 2+1: \qquad f_B = 192.0(4.3)\ \text{MeV}$$

$$N_f = 2+1: \qquad f_{B_s} = 228.4(3.7)\ \text{MeV}$$

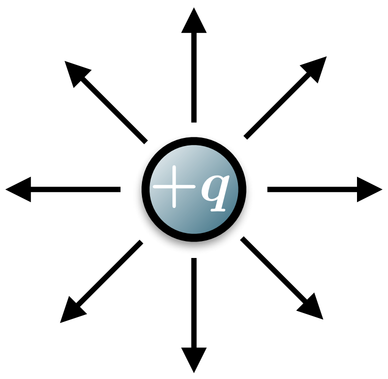
$$N_f = 2+1: \qquad \frac{f_{B_s}}{f_B} = 1.201(0.016)$$

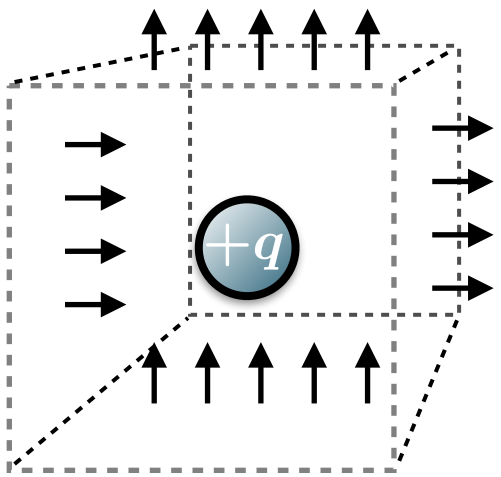
lattice QCD + **QED**

$$\alpha_{\text{QED}} \sim \frac{m_u - m_d}{\Lambda_{\text{QCD}}} \sim 1\%$$

- Relevant for sub-percent uncertainties
- Meaning of decay constants
  - Pure QCD
    $$\Gamma(K^- \to \ell^- \overline{\nu}_\ell) = \frac{G_F^2 |V_{us}|^2 f_K^2}{8\pi} m_K m_\ell^2 \left(1 - \frac{m_\ell^2}{m_K^2}\right)^2$$
  - QCD + QED (GRS scheme)
    $$\Gamma(K^- \to \mu^- \overline{\nu}_\mu [\gamma]) = (1.0032 \pm 0.0011) \Gamma^{(0)}(K^- \to \mu^- \overline{\nu}_\mu)$$
    C. Sachrajda *(Durham flavour workshop)* • Di Carlo et al.

- QED in a box
  - Periodicity incompatible with Gauss law
  - QED = long range
  - Require modification (vanishes as L → ∞)

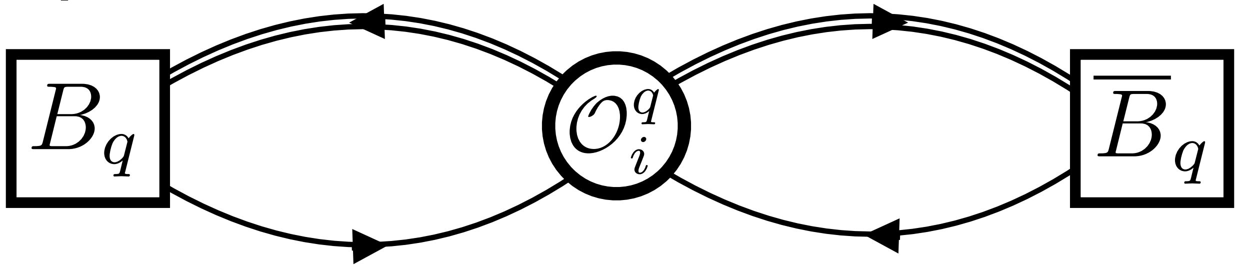




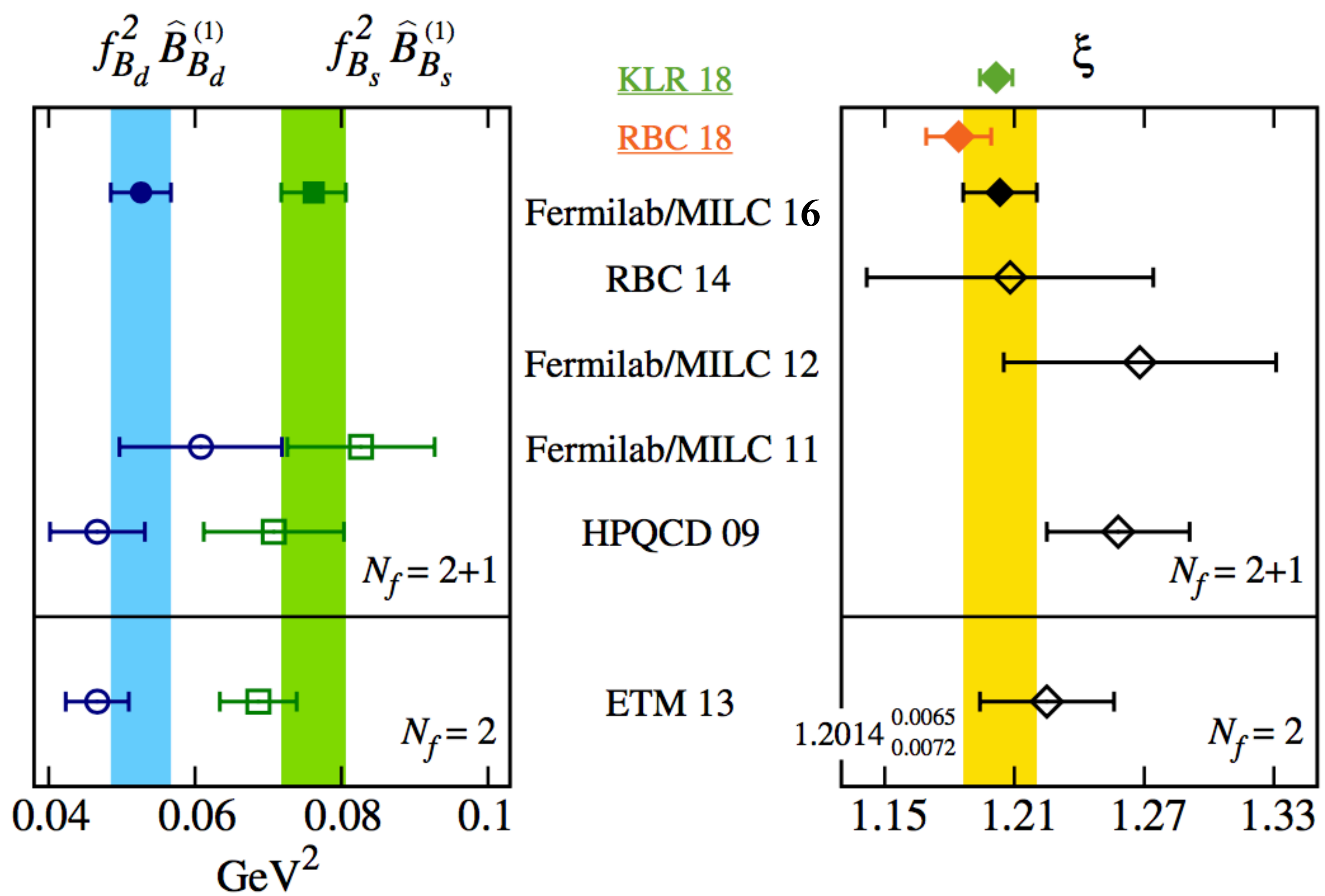
- Different soft scales for different particles
  - Well-understood for pions and kaons
  - *B* and *D* = different soft scale → requires theory developments

Neutral meson mixing $\langle \overline{\mathbf{1}} | \mathcal{H}^{\Delta F=2} | \mathbf{1} \rangle$

- B-mixing dominated by local matrix element



- Summary (from Bazavov et al. [Fermilab/MILC] 2016)



$$\xi = \frac{f_{B_s}\sqrt{\hat{B}_{B_s}}}{f_{B_d}\sqrt{\hat{B}_{B_d}}}$$

KLR 18 =
King, Lenz, Rauh *(2018)*
(QCD sum rules)

plot from Kronfeld
(Durham workshop 2019)

- Lattice precision (~3-4%) is well behind even older experiments (~0.06 - 0.2%)
- Challenging to find optimal 'discretization' (lattice definition of quarks)

Neutral meson mixing $\langle \overline{\mathbf{1}} | \mathcal{H}^{\Delta F=2} | \mathbf{1} \rangle$

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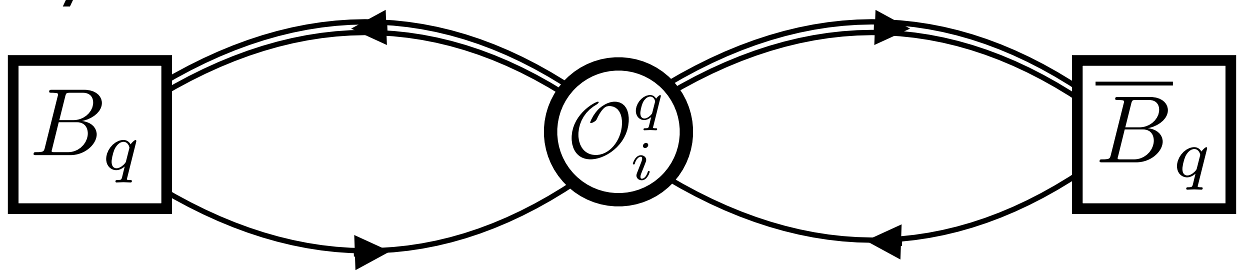
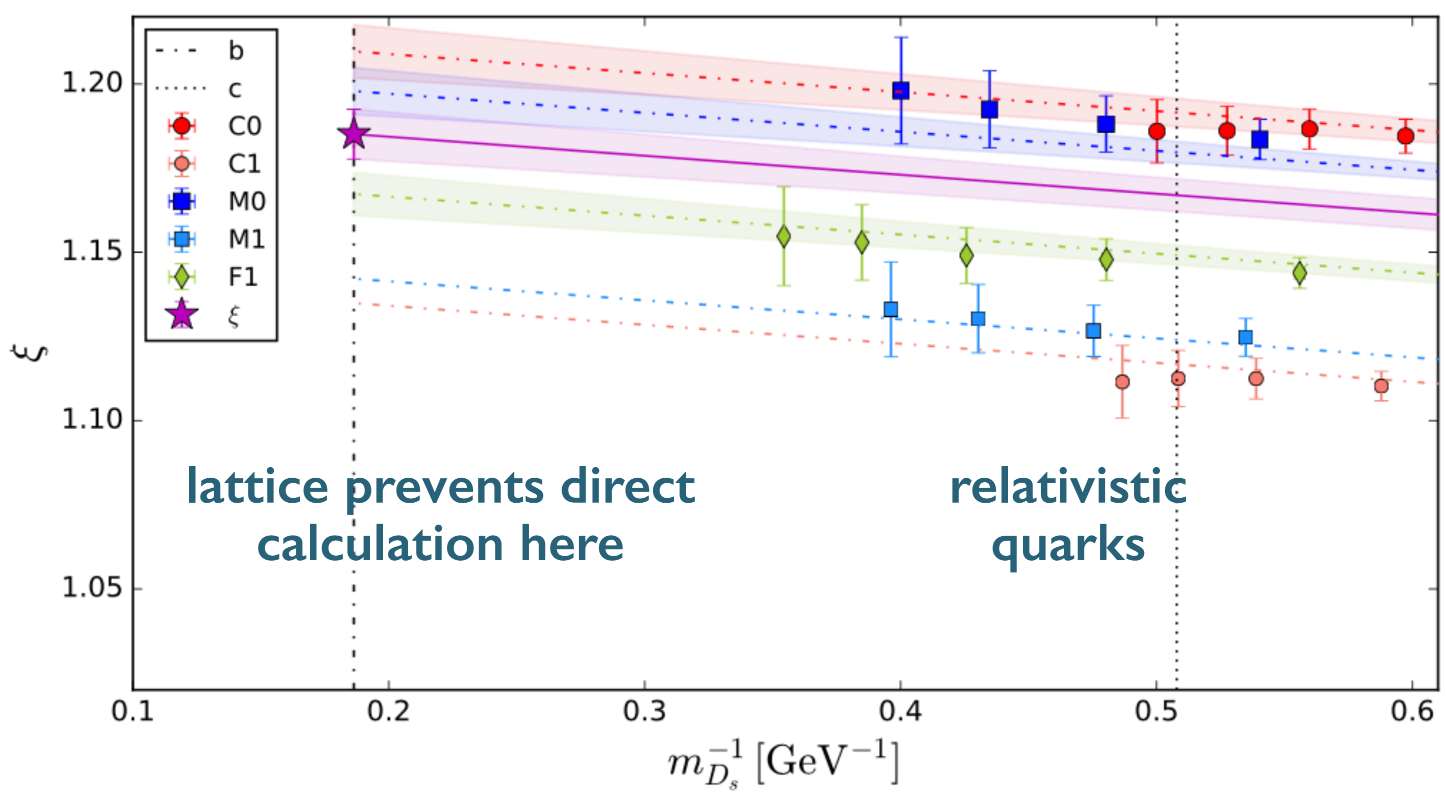


- RBC/UKQCD 2018

$$\xi = \frac{f_{B_s}\sqrt{\hat{B}_{B_s}}}{f_{B_d}\sqrt{\hat{B}_{B_d}}}$$

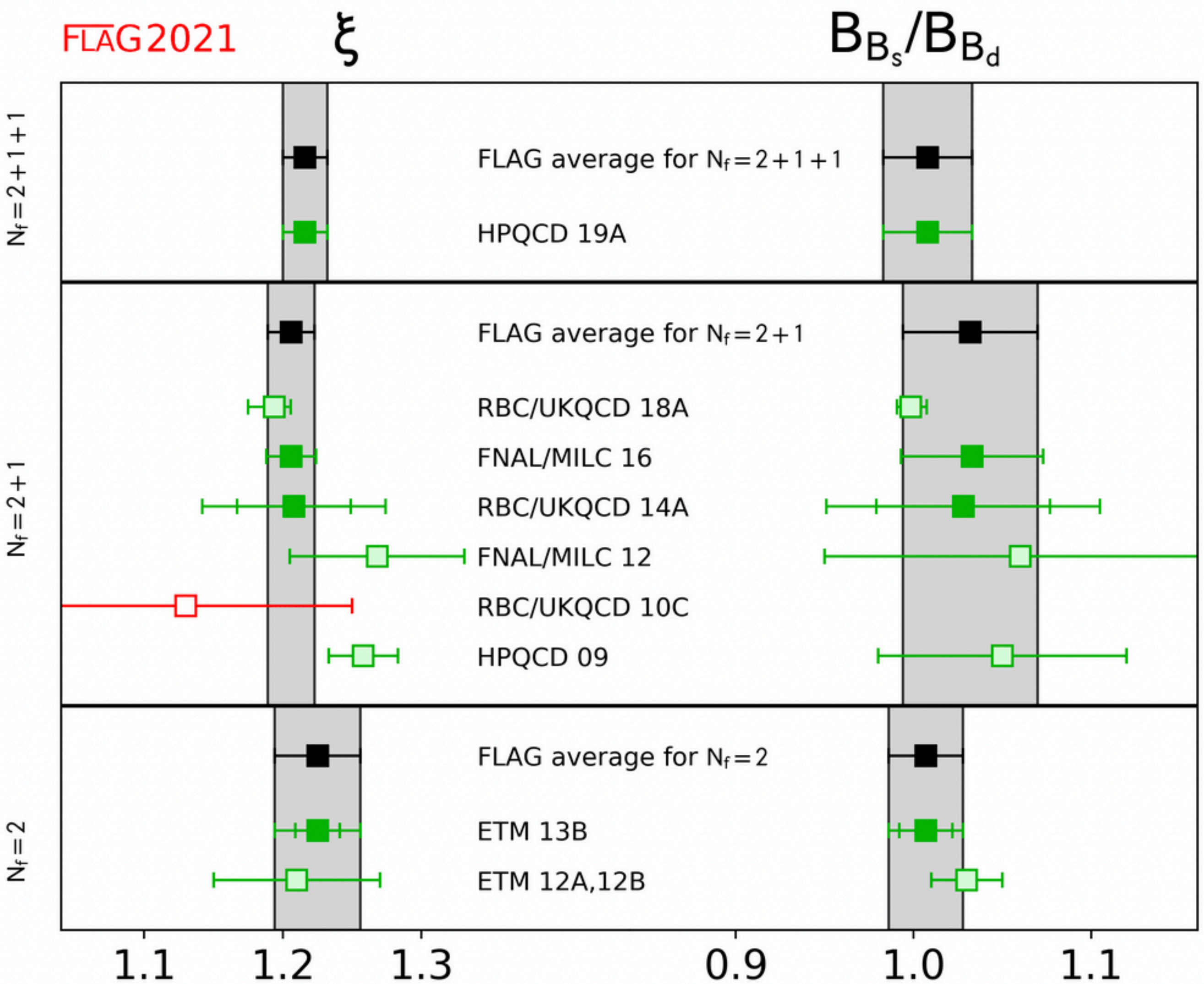
$$\xi(a, m_\pi, m_H)$$

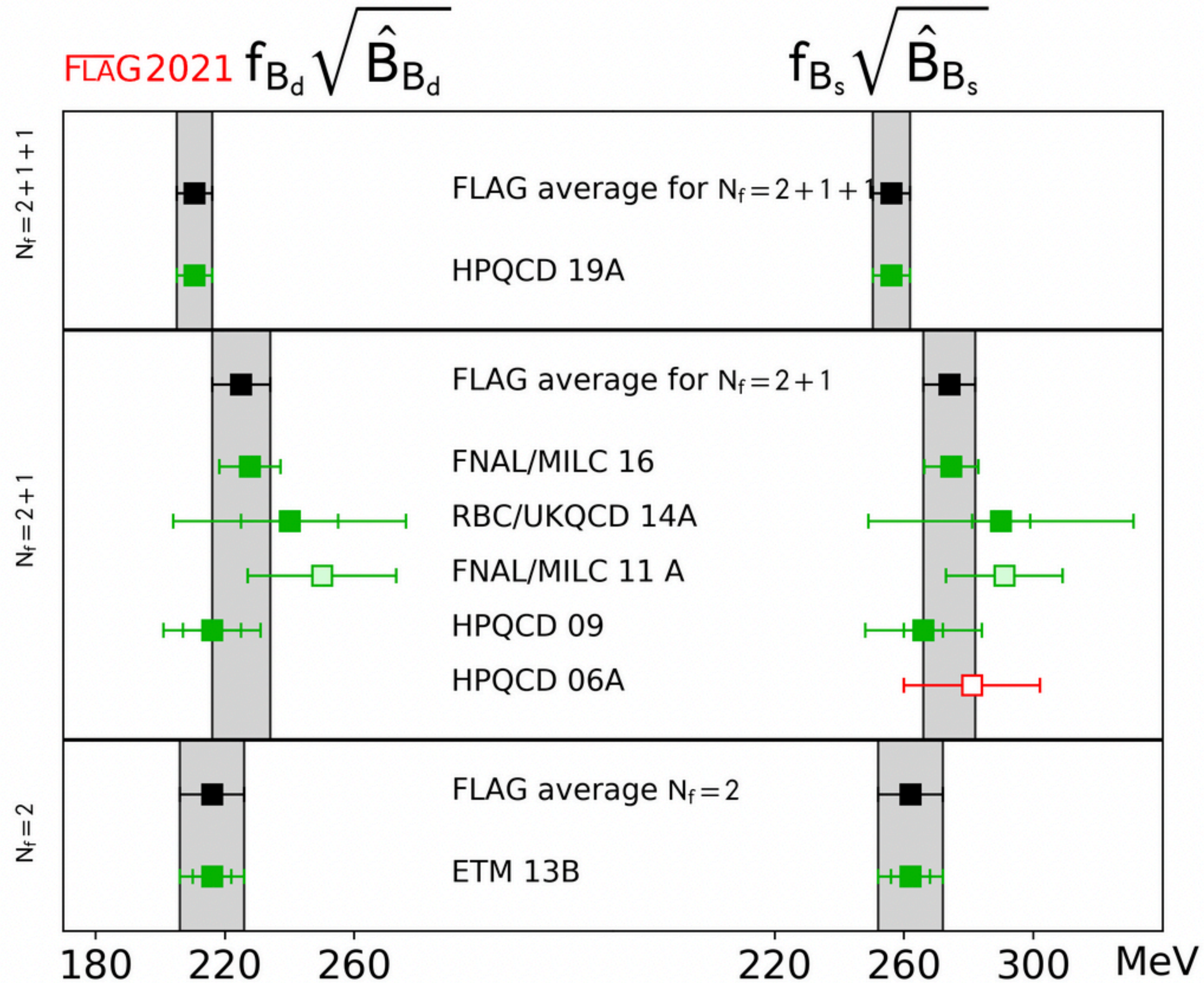


Uses a relativistic action for the *b* quark

Extrapolates to the heavy mass

B-mixing — FLAG plots





B-mixing — FLAG values

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$N_f = 2+1:$

$$f_{B_d}\sqrt{\hat{B}_{B_d}} = 225(9)\ \text{MeV} \qquad f_{B_s}\sqrt{\hat{B}_{B_s}} = 274(8)\ \text{MeV} \qquad \text{Refs. [19, 23, 47]},$$

$$\hat{B}_{B_d} = 1.30(10) \qquad \hat{B}_{B_s} = 1.35(6) \qquad \text{Refs. [19, 23, 47]},$$

$$\xi = 1.206(17) \qquad B_{B_s}/B_{B_d} = 1.032(38) \qquad \text{Refs. [19, 47]}.$$

$N_f = 2+1+1:$

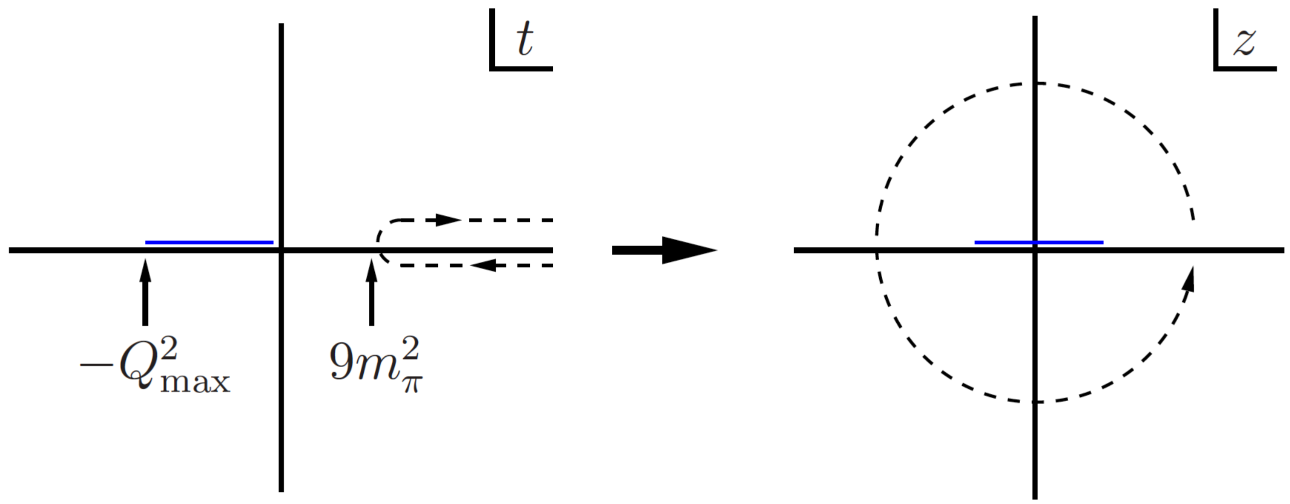
$$f_{B_d}\sqrt{\hat{B}_{b_d}} = 210.6(5.5)\ \text{MeV} \qquad f_{B_s}\sqrt{\hat{B}_{B_s}} = 256.1(5.7)\ \text{MeV} \qquad \text{Ref. [48]},$$

$$\hat{B}_{B_d} = 1.222(61) \qquad \hat{B}_{B_s} = 1.232(53) \qquad \text{Ref. [48]},$$

$$\xi = 1.216(16) \qquad B_{B_s}/B_{B_d} = 1.008(25) \qquad \text{Ref. [48]}.$$

Form factors $\langle \mathbf{1}|\mathcal{J}|\mathbf{1}'\rangle$

- Significantly more information (functions vs numbers)
- Conformal mapping → z-expansion → wider kinematic range



Bhattacharya, Hill, Paz *(2011)*

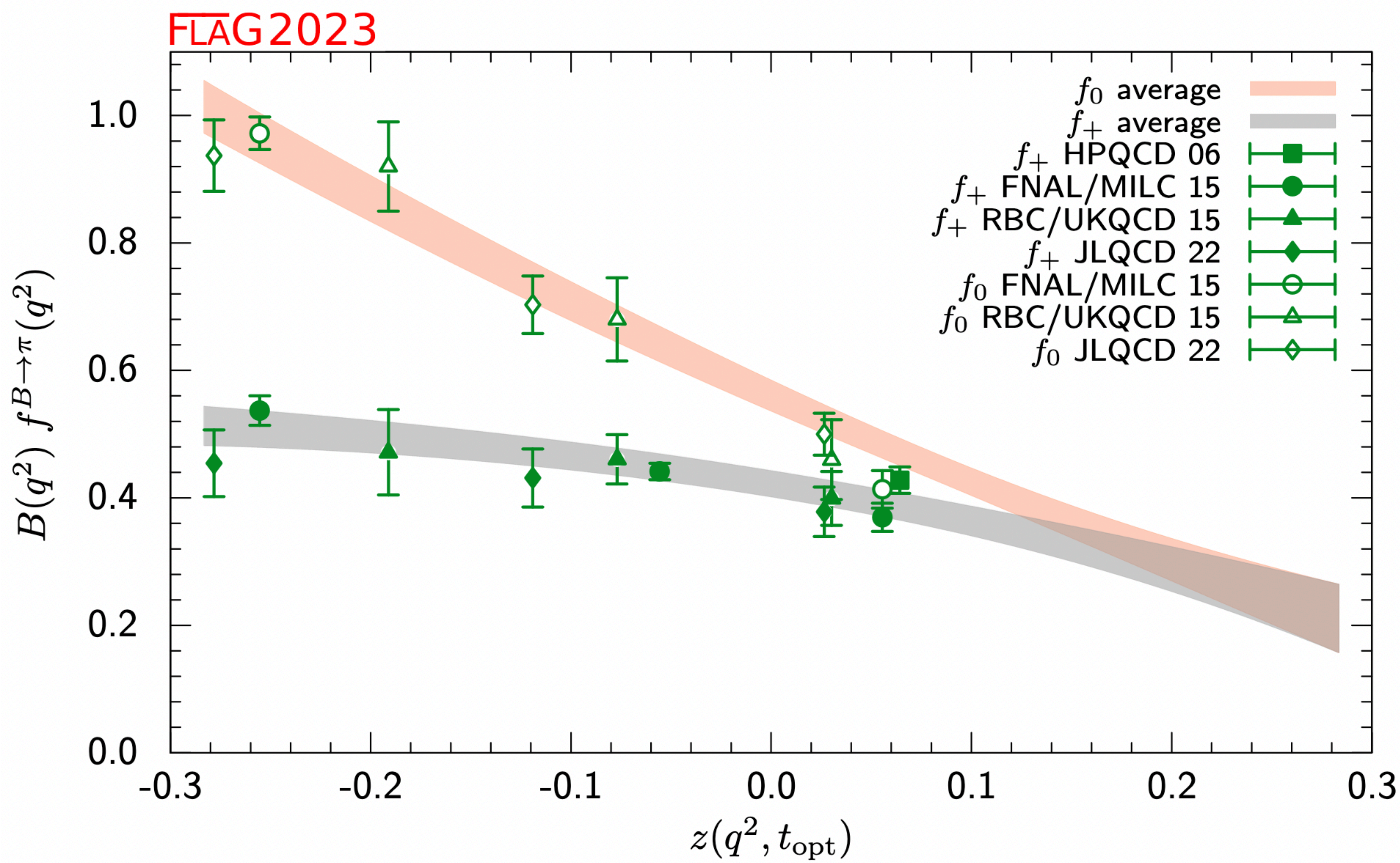
- Report z coefficients + correlations
- Joint fit to LQCD and experiment → CKM
- Better precision needed for BES III, LHCb and BELLE II

$\|V_{ud}\|$	$\|V_{us}\|$	$\|V_{ub}\|$
$\pi^+\to l^+\nu$	$K^+\to l^+\nu$	$B^+\to \tau^+\nu$
$\pi^+\to \pi^0 e^+\nu$	$K\to \pi l^+\nu$	$B\to \pi l^+\nu$
$\|V_{cd}\|$	$\|V_{cs}\|$	$\|V_{cb}\|$
$D^+\to l^+\nu$	$D_s^+\to l^+\nu$	$B_c^+\to \tau^+\nu$
$D\to \pi l^+\nu$	$D\to K l^+\nu$	$B\to \pi l^+\nu$
$\|V_{td}\|$	$\|V_{ts}\|$	$\|V_{tb}\|$
$B^0\to \pi^0 l^+ l^-$	$B^0\to K^0 l^+ l^-$	
$B^0\leftrightarrow \bar{B}^0$	$B_s^0\leftrightarrow \bar{B}_s^0$	

Kronfeld (Durham workshop) *(2019)*

Form factors $\langle \mathbf{1}|\mathcal{J}|\mathbf{1}' \rangle$

- Example: $f^{B\to\pi}(q^2)$

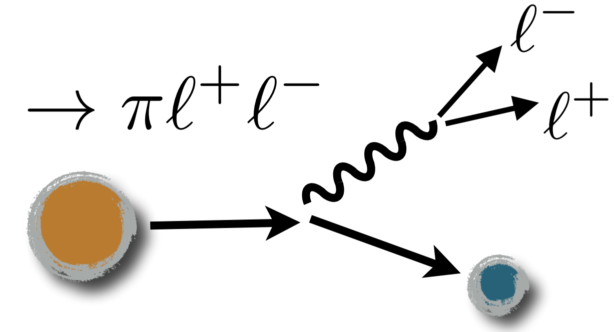


- See new FLAG report/website for details
- Please cite original work (each figure has a .bib)

Matrix elements and LQCD

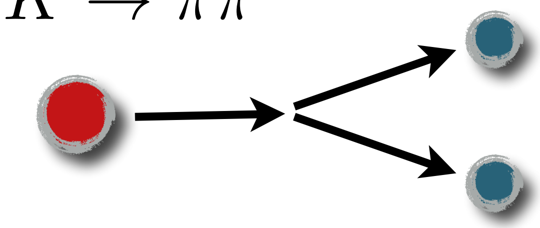
- ☑ Single-hadron initial and final states
  - ☐ Calculated directly in LQCD
  - ☐ New theory challenge = QED
  - ☐ See FLAG averages

$B \to \pi \ell^+ \ell^-$

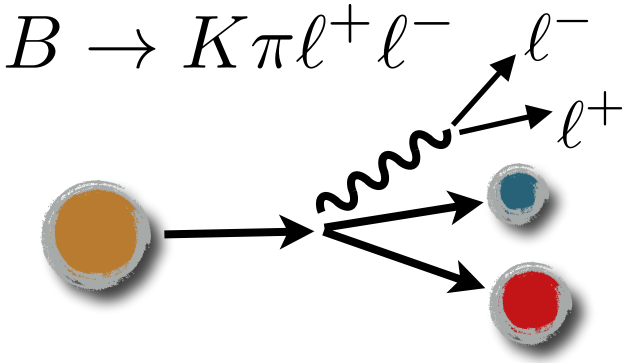


- ☐ Two-hadron final states
  - ☐ Significantly more challenging
  - ☐ Subtle finite volume issues

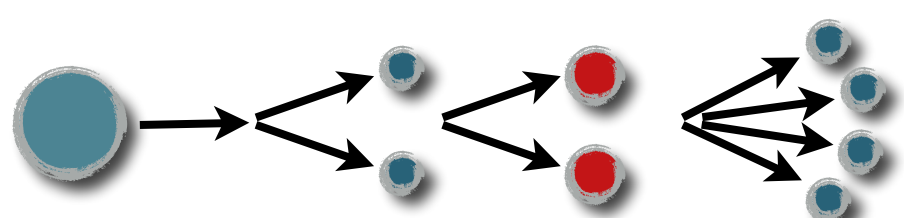
$K \to \pi\pi$



$B \to K \pi \ell^+ \ell^-$



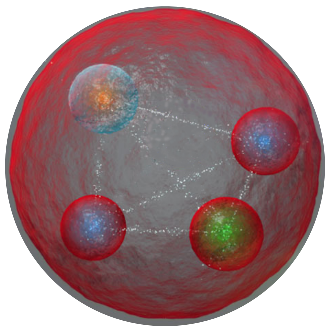
- ☐ Multi-hadron states for $\sqrt{s} > 4M_\pi$
  - ☐ All or nothing (must constrain all channels for a prediction)



$D \to \pi\pi, K\overline{K}$

Multi-hadron observables

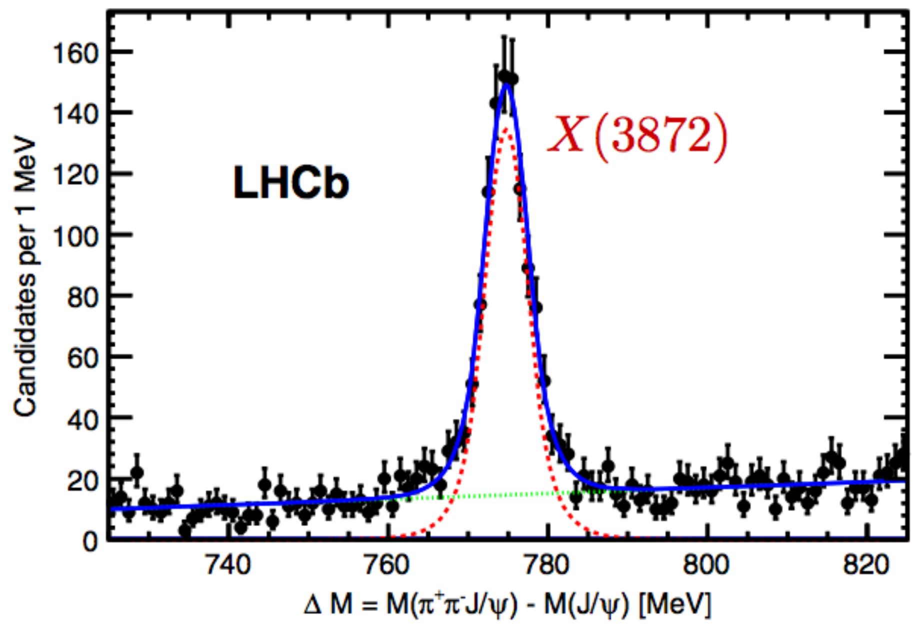
- Exotics, XYZs, tetra- and penta-quarks, *H* dibaryon



e.g. X(3872)

$\sim |D^0 \overline{D}^{*0} + \overline{D}^0 D^{*0}\rangle$?

• LHCb (PRD92, 2015) •



- Eletroweak, CP violation, resonant enhancement

CP violation in charm $D \to \pi\pi, K\overline{K}$

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4}$$

• LHCb (PRL, 2019) •

$f_0(1710)$ could enhance ΔA_{CP}

• Soni (2017) •

Resonant B decays $B \to K^* \ell\ell \to K\pi\,\ell\ell$

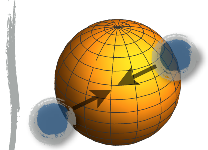
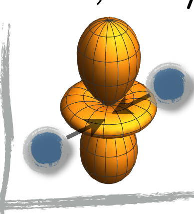
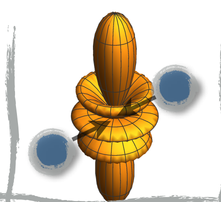
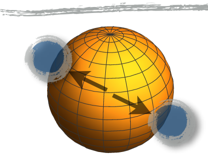
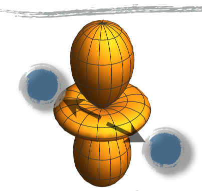
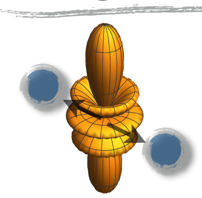
$|X\rangle, |\rho\rangle, |K^*\rangle, |f_0\rangle \quad \notin$ **QCD Fock space**

QCD Fock space

- ❑ At low-energies QCD = hadronic degrees of freedom $\pi \sim \bar{u}d, \; K \sim \bar{s}u, \; p \sim uud$
- ❑ Overlaps of multi-hadron ***asymptotic states*** → S matrix

$|\pi\pi, \text{in}\rangle$

$S(s) \equiv \langle \pi\pi, \text{out}|$

			
	$e^{2i\delta_0(s)}$	0	0
	0	$e^{2i\delta_1(s)}$	0
	0	0	$e^{2i\delta_2(s)}$

depends on $s = E_{\text{cm}}^2$
and angular variables

diagonal in angular momentum

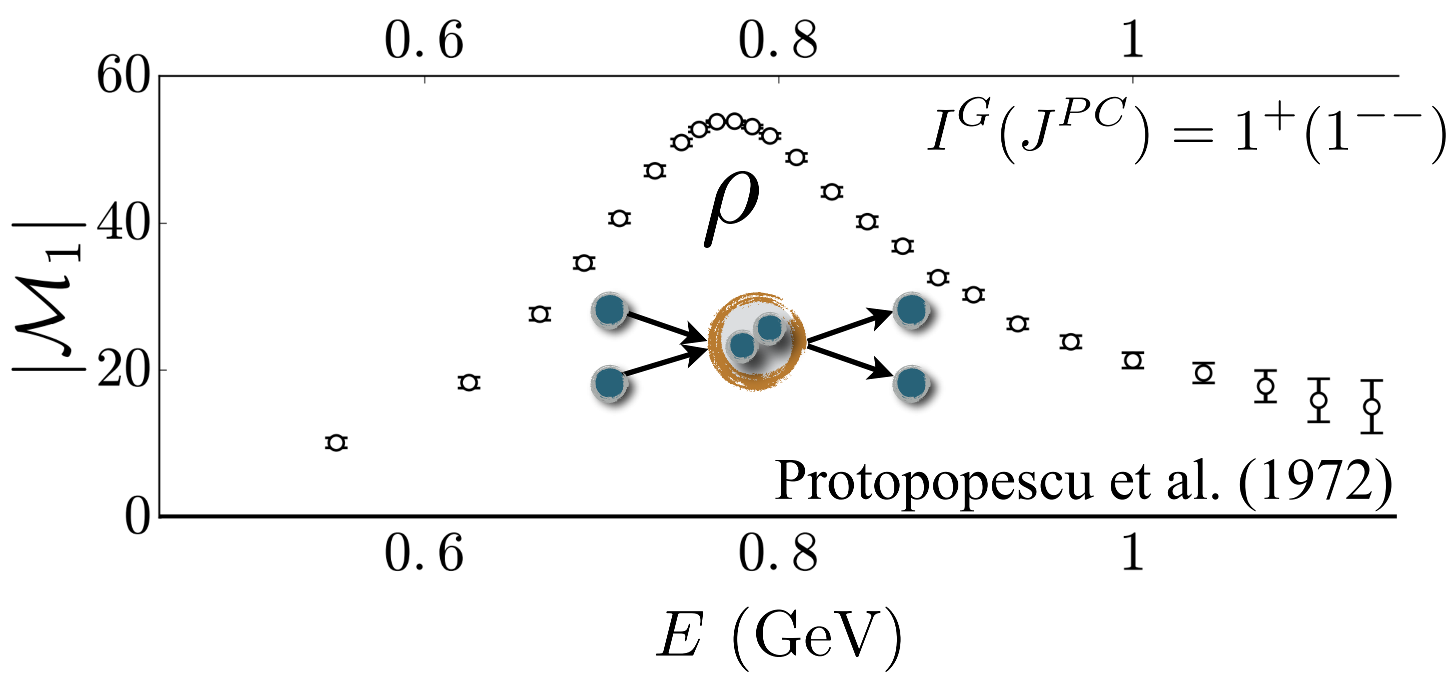
$$\mathcal{M}_\ell(s) \propto e^{2i\delta_\ell(s)} - 1$$

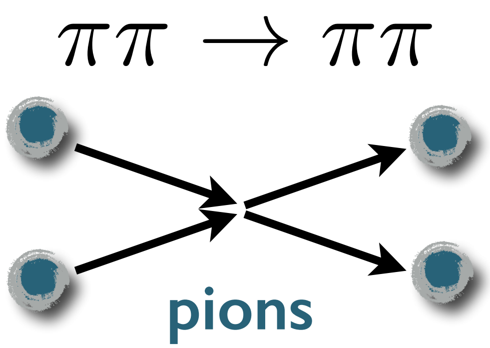
- ❑ ***An enormous space of information*** $|\pi\pi\pi\pi, \text{in}\rangle \quad |K\overline{K}, \text{in}\rangle \quad \cdots$

QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2 \delta_\ell(s)$

scattering rate





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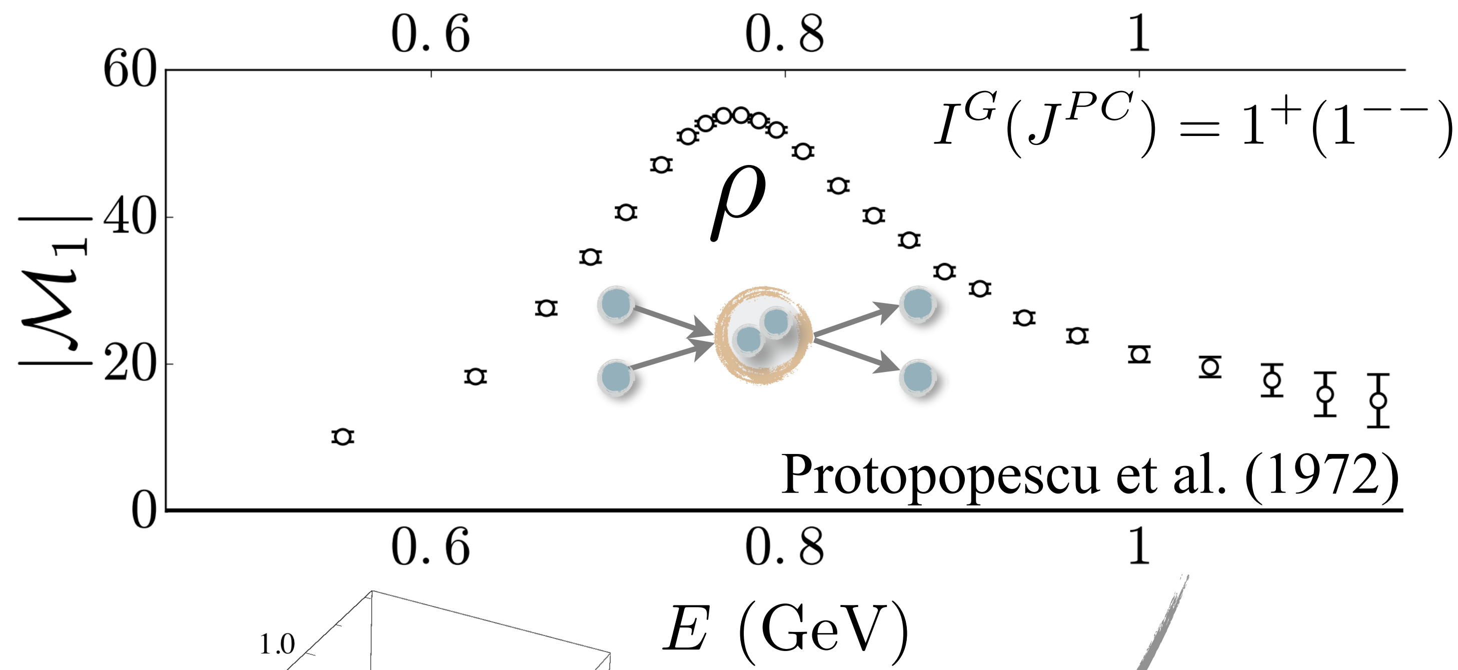
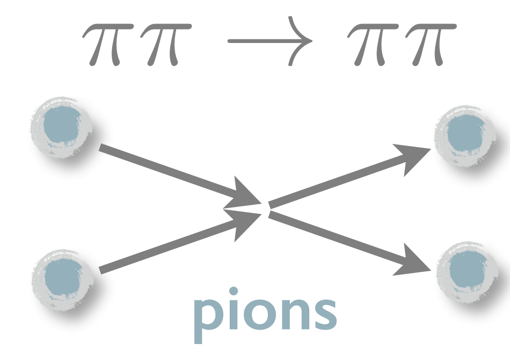
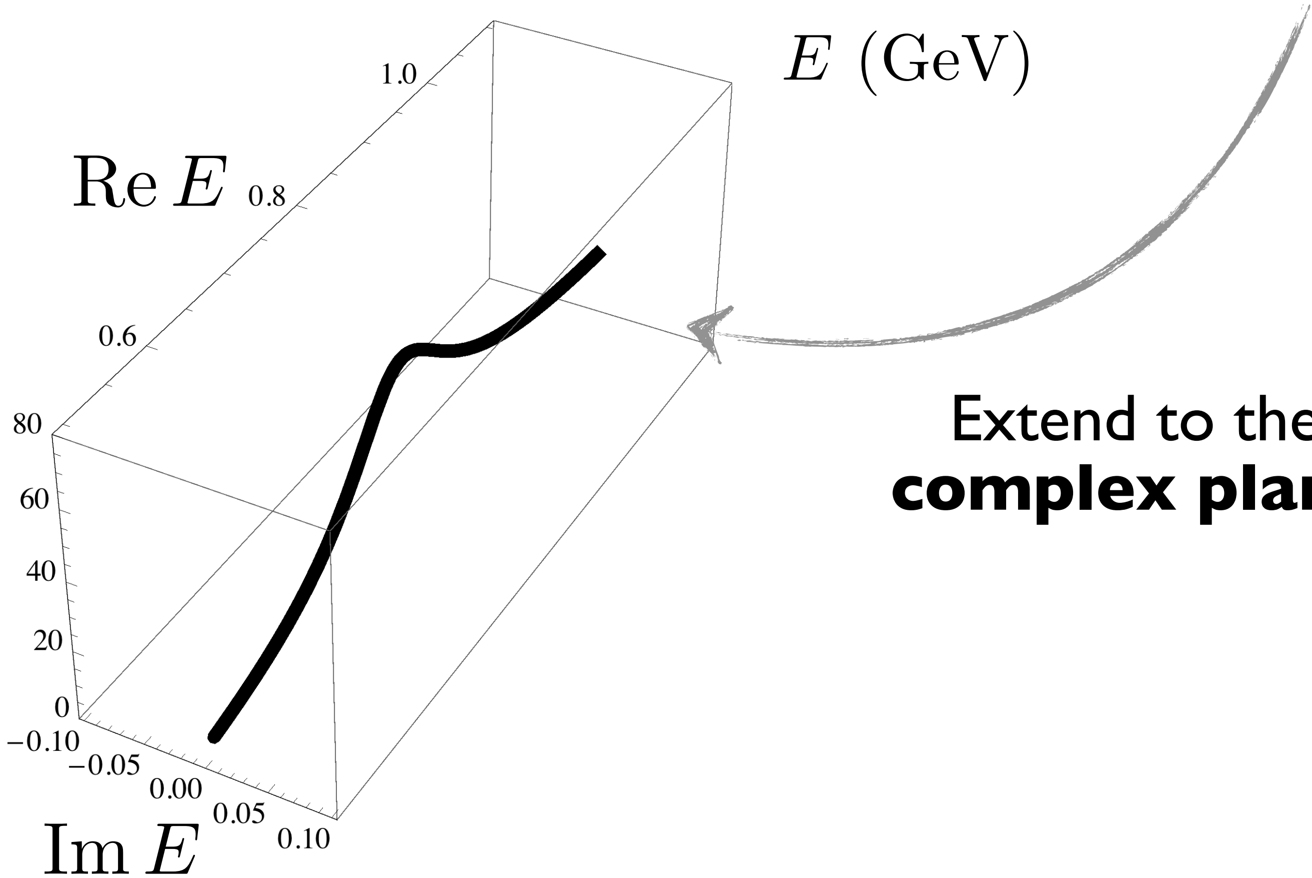




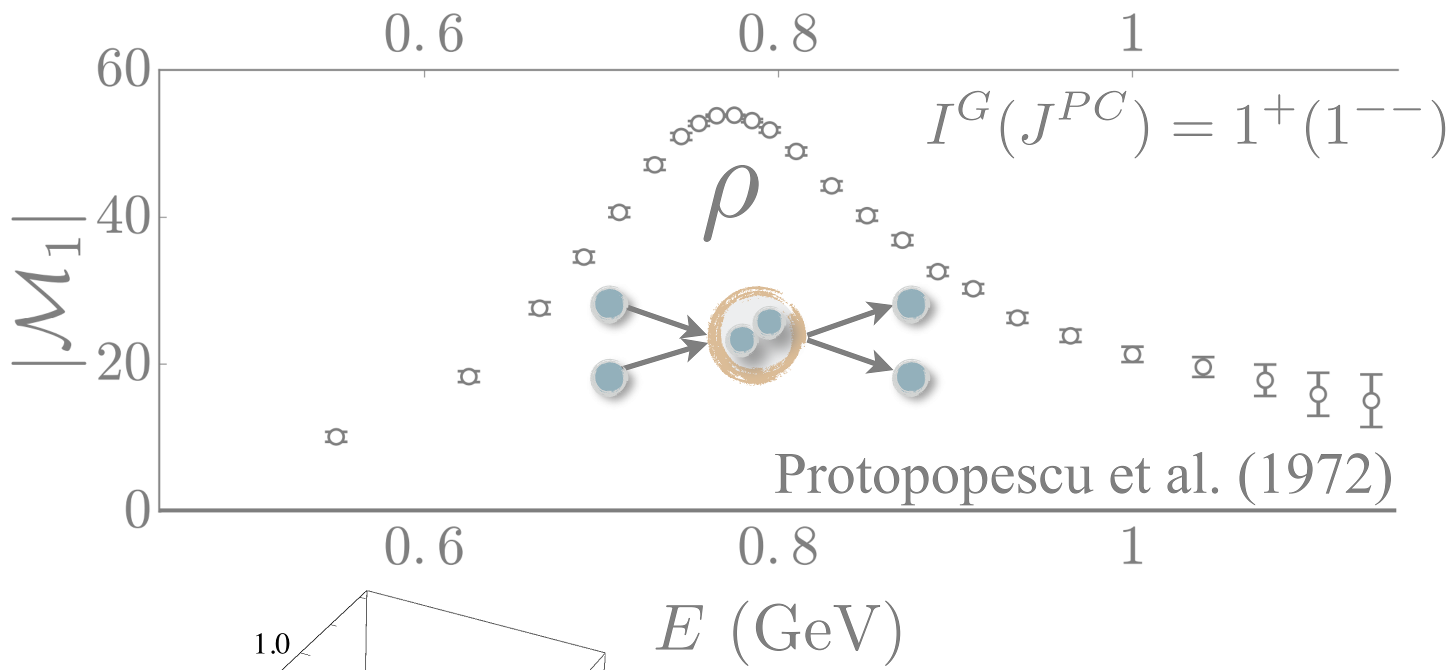


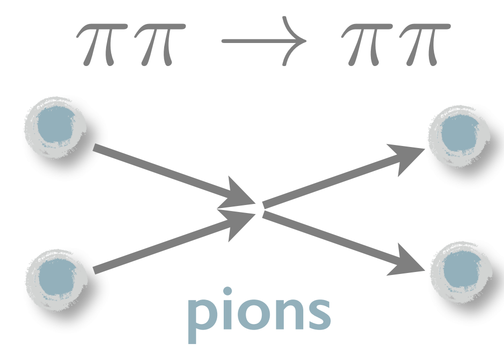
Extend to the **complex plane**

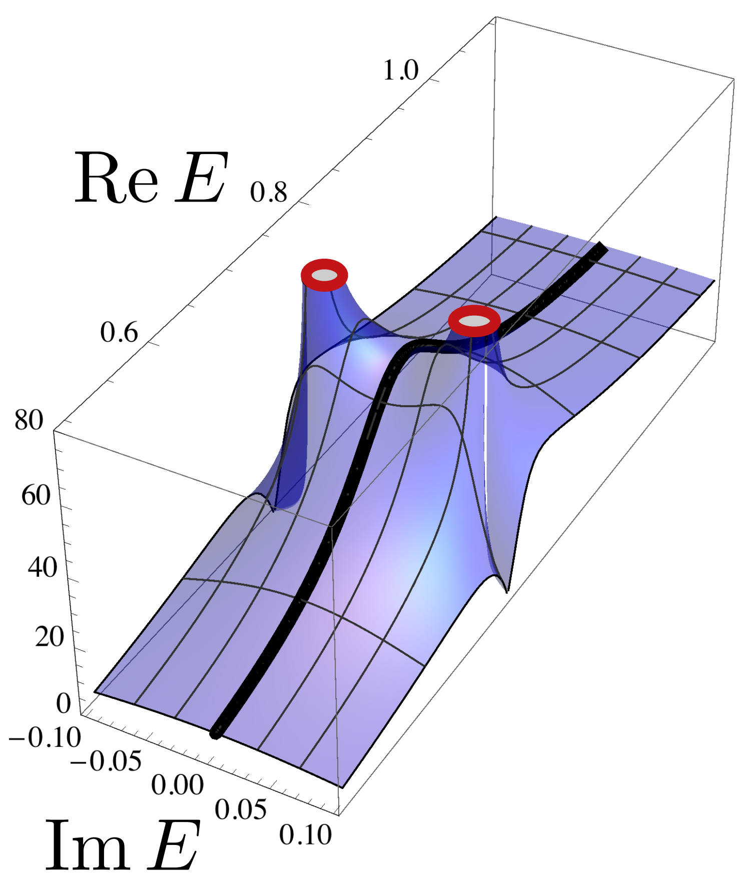
QCD resonances

- Roughly speaking, a bump in: $|\mathcal{M}_\ell(s)|^2 \propto |e^{2i\delta_\ell(s)} - 1|^2 \propto \sin^2\delta_\ell(s)$

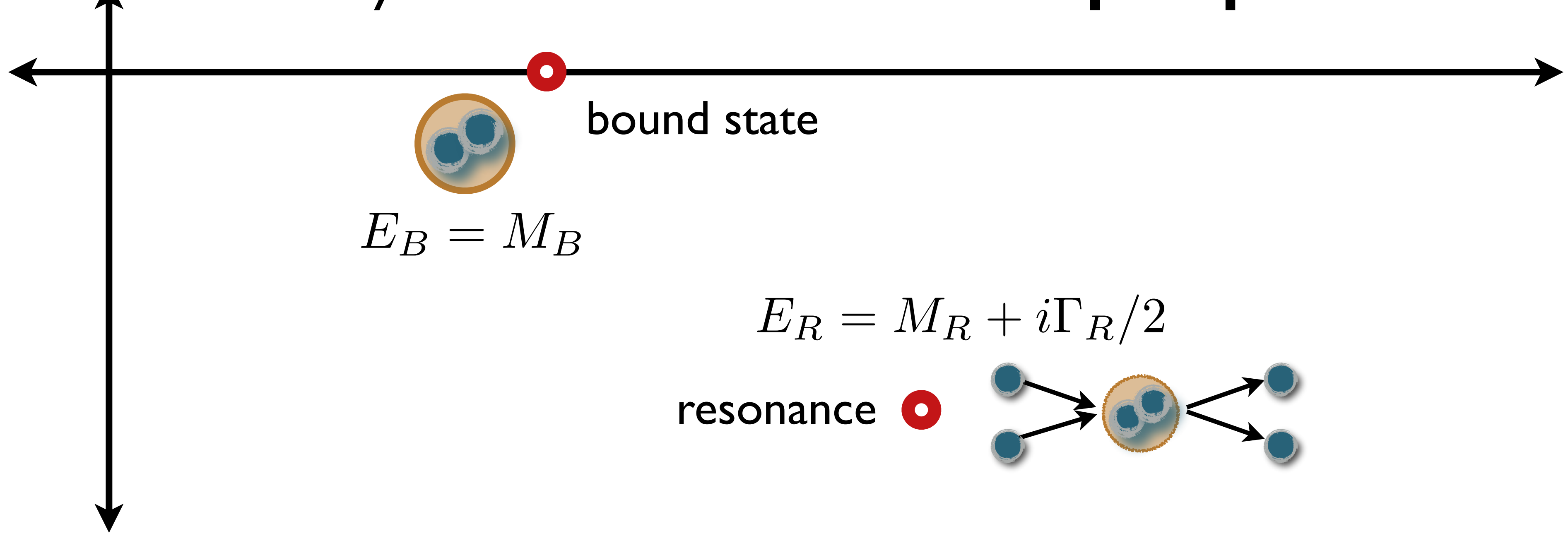
scattering rate





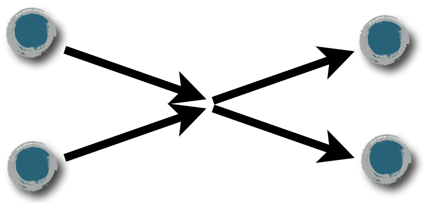


Analytic continuation reveals a **complex pole**



Analyticity

- Instead of $|\mathcal{M}(s)|^2$ → analytically continue the ***amplitude*** itself



  For two-particle energies $(2m)^2 < s < (4m)^2$, what is the analytic structure?

- The optical theorem tells us…

$$\rho(s)|\mathcal{M}_\ell(s)|^2 = \operatorname{Im}\mathcal{M}_\ell(s)$$

where $\rho(s) = \dfrac{\sqrt{1-4m^2/s}}{32\pi}$ is the two-particle phase space

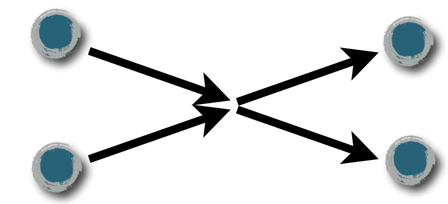
- Unique solution is…

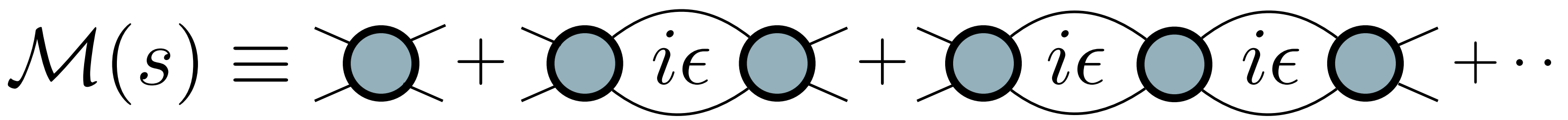
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)}$$

K matrix (short distance) — phase-space cut (long distance)

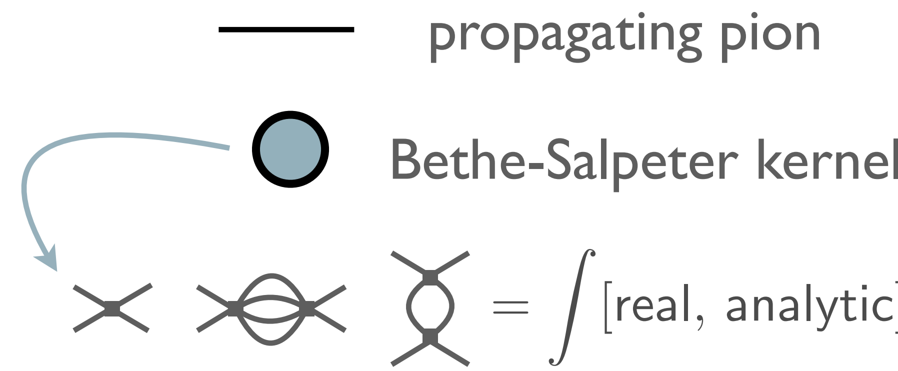
Key message: *The scattering amplitude has a square-root branch cut*

Analyticity (all orders diagrammatic)



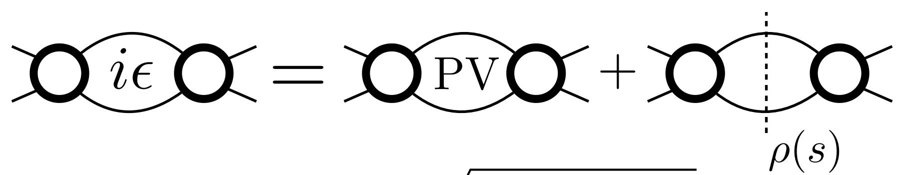


on-shell particles = singularities:
non-analytic for $(2m)^2 < s < (4m)^2$



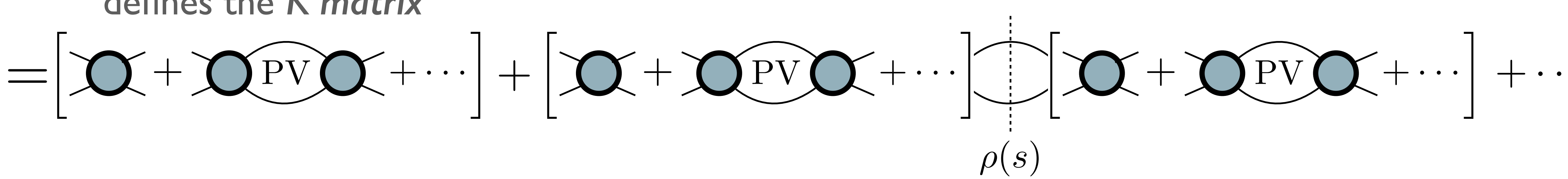
for $(2m)^2 < s < (4m)^2$

cutting rule



$\rho(s) \propto i\sqrt{s-(2m)^2}$

defines the *K* ***matrix***



$$= \mathcal{K}(s) + \mathcal{K}(s)\rho(s)\mathcal{K}(s) + \cdots = \frac{1}{\mathcal{K}(s)^{-1} - \rho(s)}$$

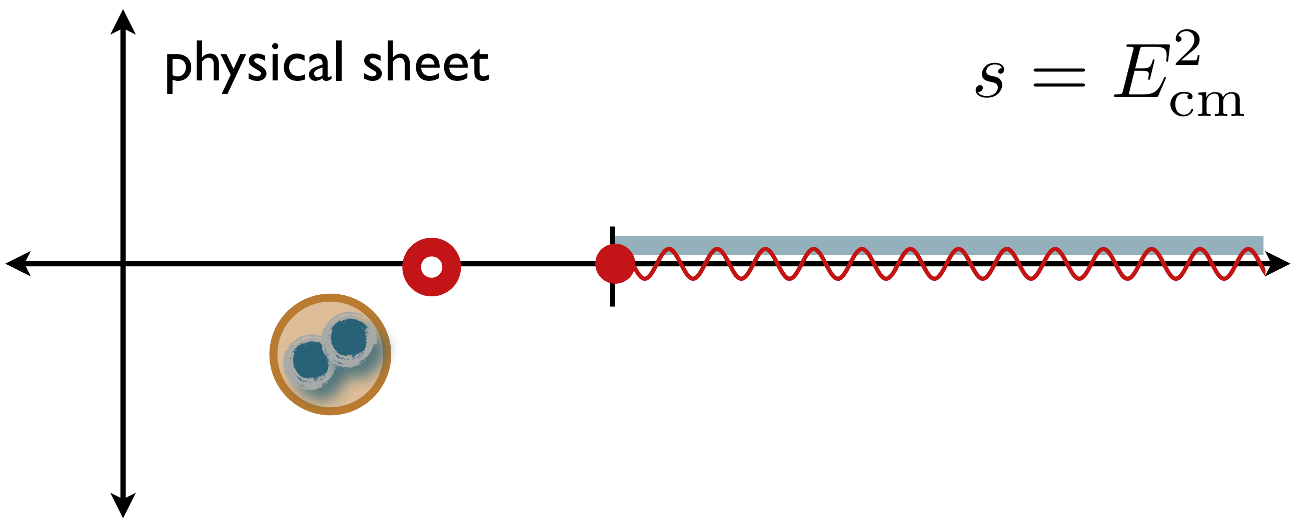
K matrix (short distance)

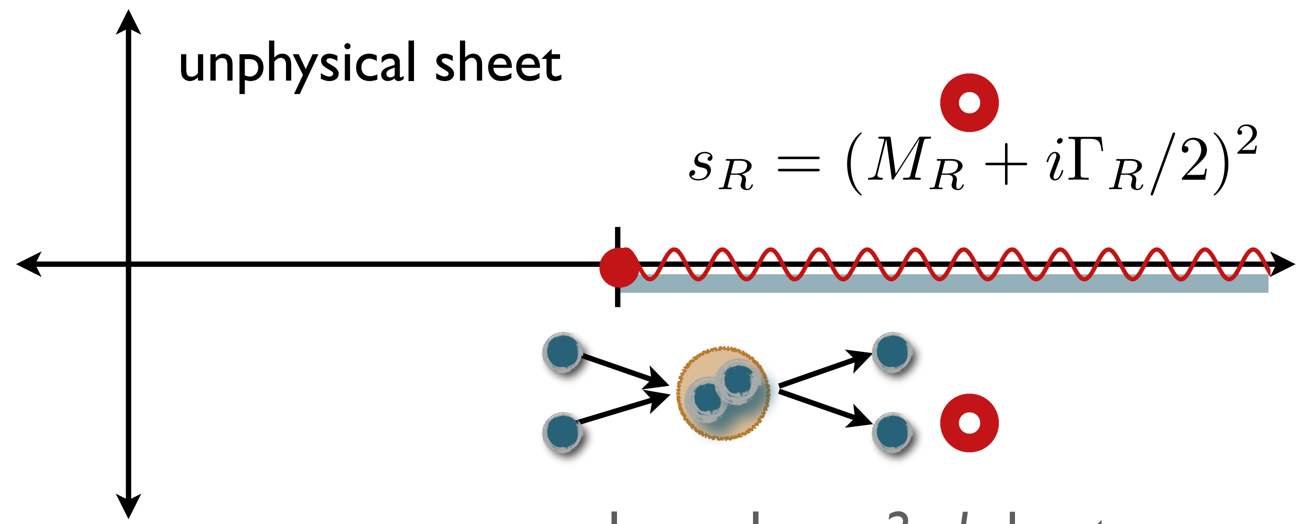
phase-space cut (long distance)

Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)} \propto \frac{1}{p\cot\delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto \sqrt{s-(2m)^2}$$

- Each channel generates a *square-root cut* → doubles the number of sheets





resonance poles only on *2nd* sheet

- Important lessons:

  Details of analyticity = important for quantitative understanding

  Possible to separate…
  (i) long-distance kinematic singularities
  (ii) short-distance/microscopic physics (depending on interaction details)

Non-perturbative QCD for flavour physics: Part II

Maxwell T. Hansen

March 19th, 2024

THE UNIVERSITY of EDINBURGH

Special request… bit more about different discretizations

- Naive approach leads to doubling
- Need a modification to remove the doublers… many options:

Staggered quarks (BMW/FNAL/MILC)
Reduce doublers from 16 to 4, then take a fourth root

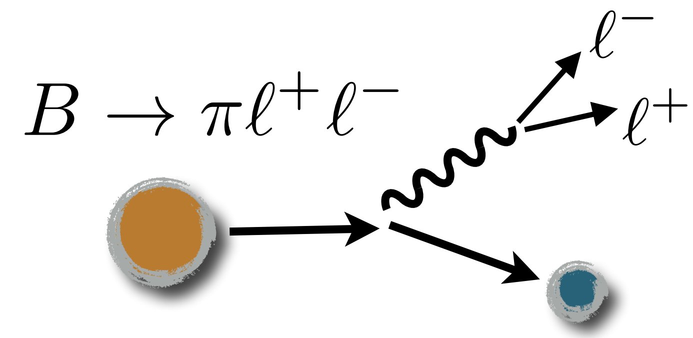
Domain-wall quarks (RBC/UKQCD)
Fifth dimension with theory living on a 4d membrane

Twisted-mass quarks (ETMC)
Break parity to automatically remove linear lattice effects

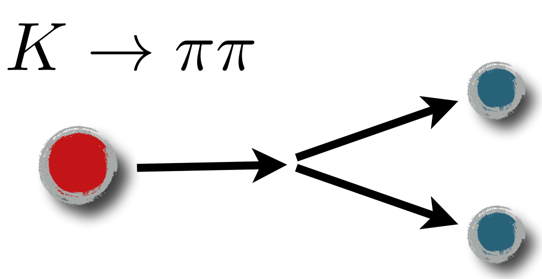
Wilson quarks (alpha, HadSpec)
Original approach

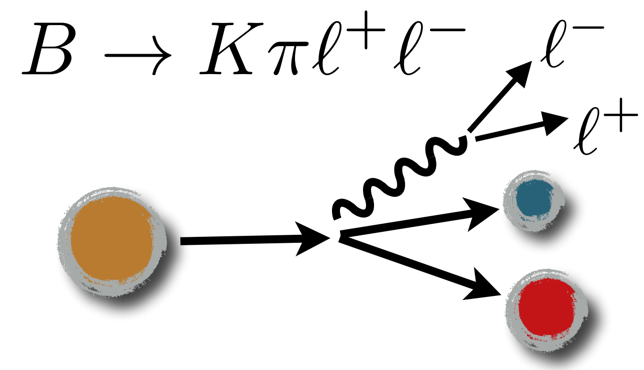
Matrix elements and LQCD

- ☑ Single-hadron initial and final states
  - ☐ Calculated directly in LQCD
  - ☐ New theory challenge = QED
  - ☐ See FLAG averages

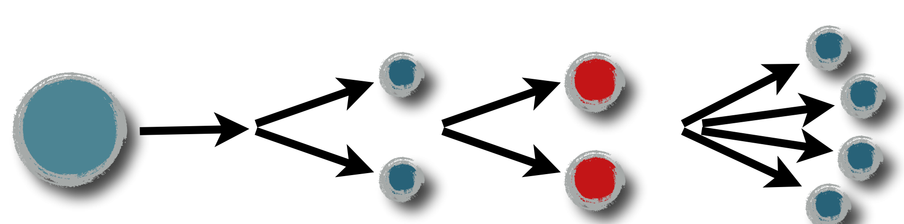


- ☑ Two-hadron final states
  - ☐ Significantly more challenging
  - ☐ Subtle finite volume issues





- ☐ Multi-hadron states for $\sqrt{s} > 4M_\pi$
  - ☐ All or nothing (must constrain all channels for a prediction)

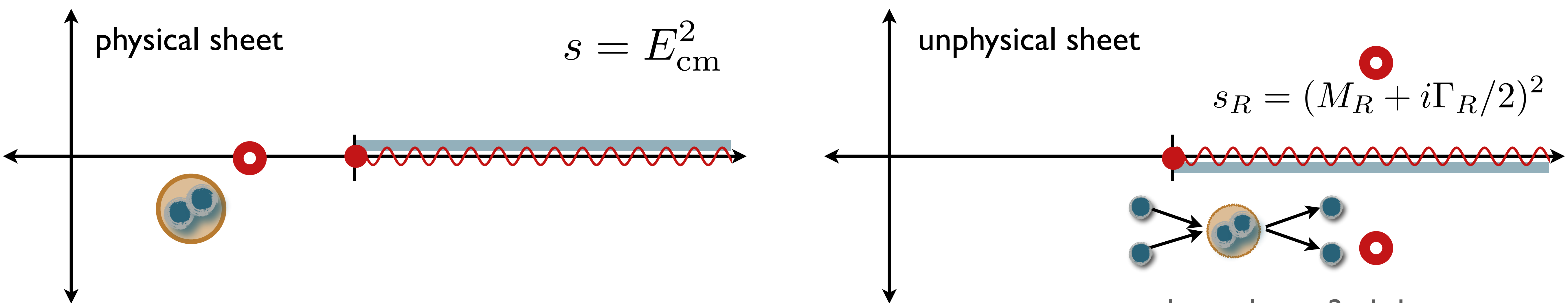


$D \to \pi\pi, K\overline{K}$

Cuts and sheets

$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} - i\rho(s)} \propto \frac{1}{p\cot\delta_\ell(s) - ip} \propto e^{2i\delta_\ell(s)} - 1 \qquad \rho(s) \propto \sqrt{s-(2m)^2}$$

- ❑ Each channel generates a *square-root cut* → doubles the number of sheets



resonance poles only on *2nd* sheet

- ❑ Important lessons:

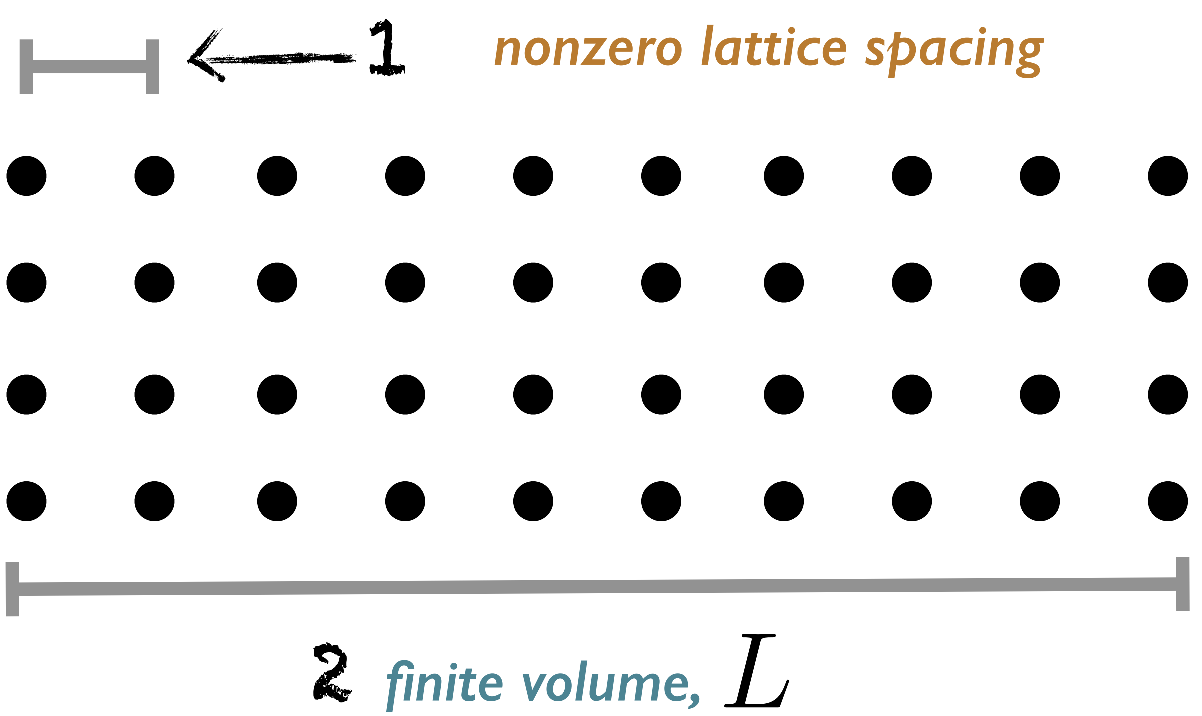
Details of analyticity = important for quantitative understanding

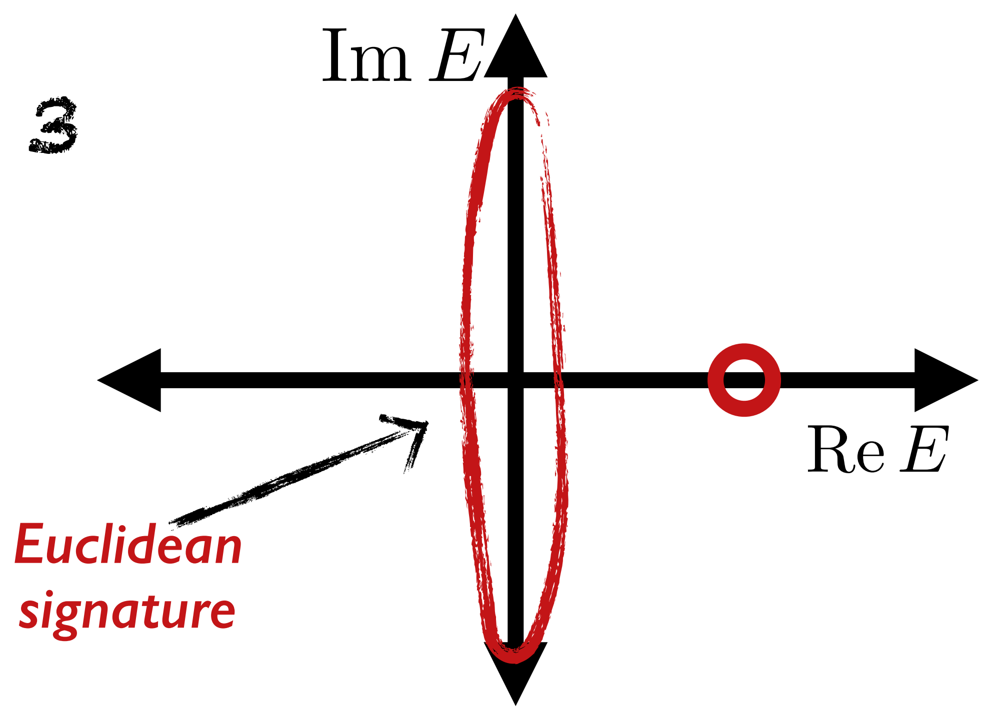
Possible to separate…
(i) long-distance kinematic singularities
(ii) short-distance/microscopic physics (depending on interaction details)

Challenges for lattice QCD

$$\text{observable?} = \int d^N\phi \, e^{-S} \left[\begin{array}{c} \textbf{interpolator} \\ \textbf{for observable} \end{array} \right]$$

To proceed we have to make *three modifications*



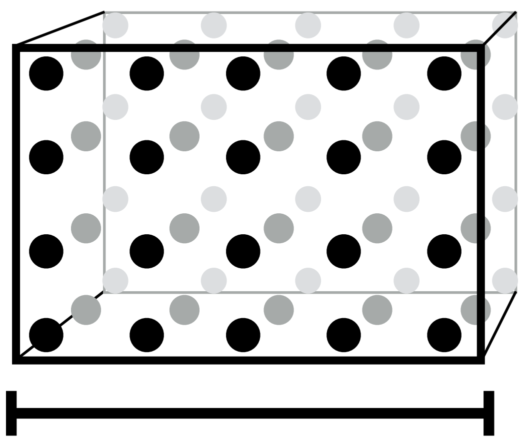


Also… $M_{\pi,\text{lattice}} > M_{\pi,\text{our universe}}$

(but physical masses → increasingly common)

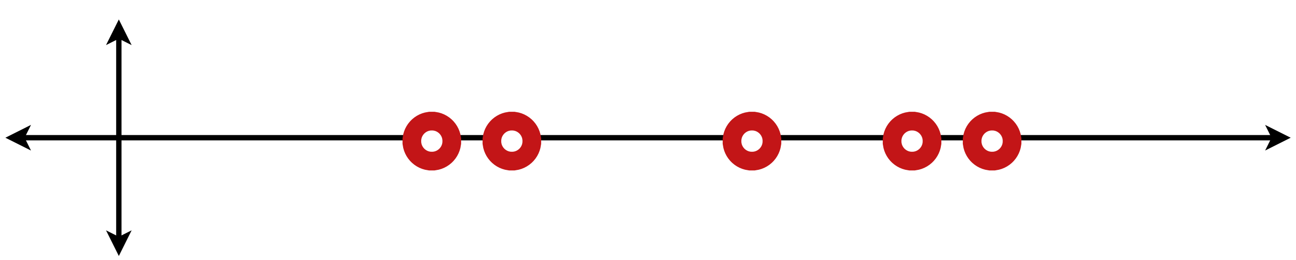
Difficulties for multi-hadron observables

- The ***Euclidean signature / imaginary time***…
  - ***Obscures*** real time evolution (that defines scattering)
  - ***Prevents*** normal LSZ (want $p_4^2 = -(p^2 + m^2)$, but we have only $p_4^2 > 0$)

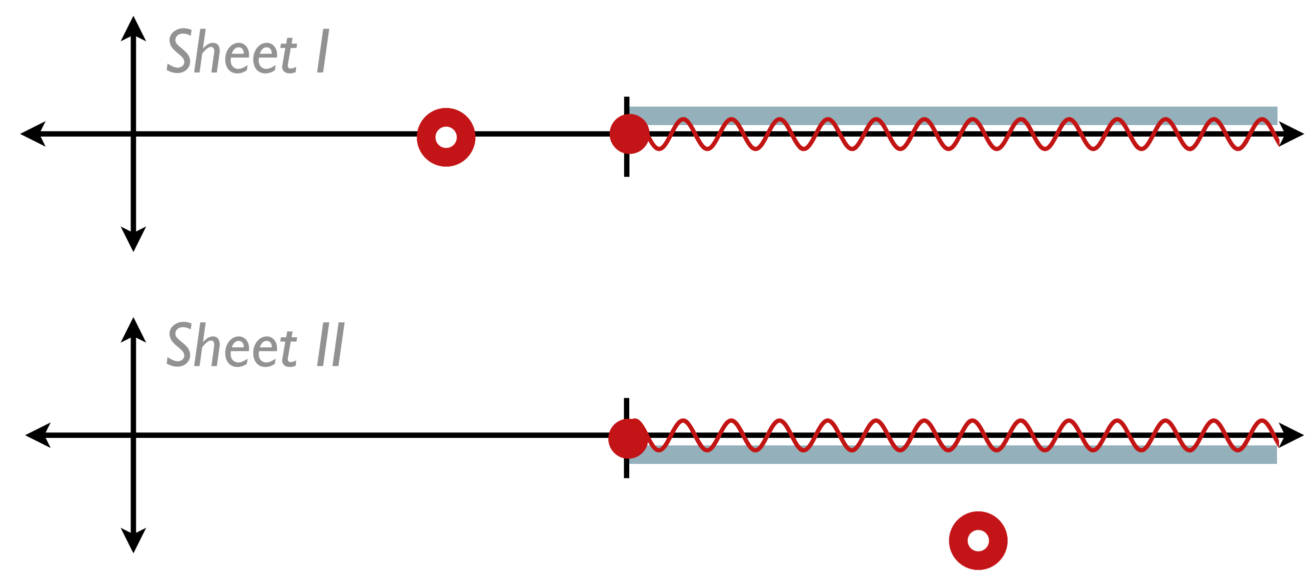


- The ***finite volume***…
  - ***Discretizes*** the spectrum
  - ***Eliminates*** the branch cuts and extra sheets
  - ***Hides*** the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure

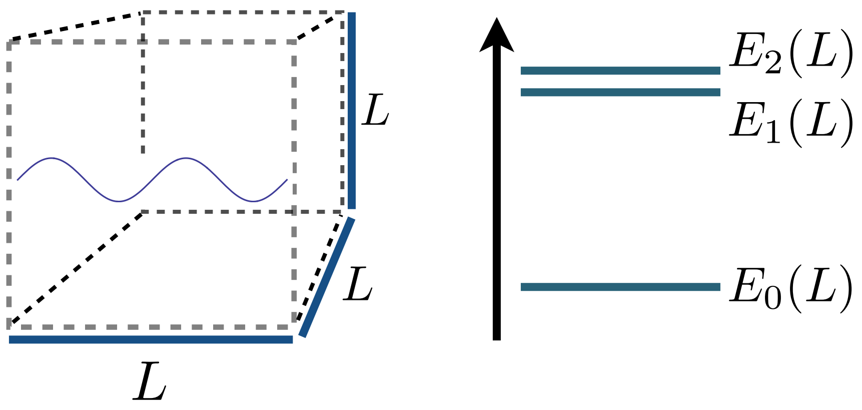


Importance of the finite volume

$|X\rangle, |\rho\rangle\,, |K^*\rangle\,, |f_0\rangle \quad \notin$ **QCD Fock**

$|\pi\pi, \text{out}\rangle, \quad |K\pi, \text{out}\rangle, \quad \cdots \in$ **QCD Fock space (continuum of states)**

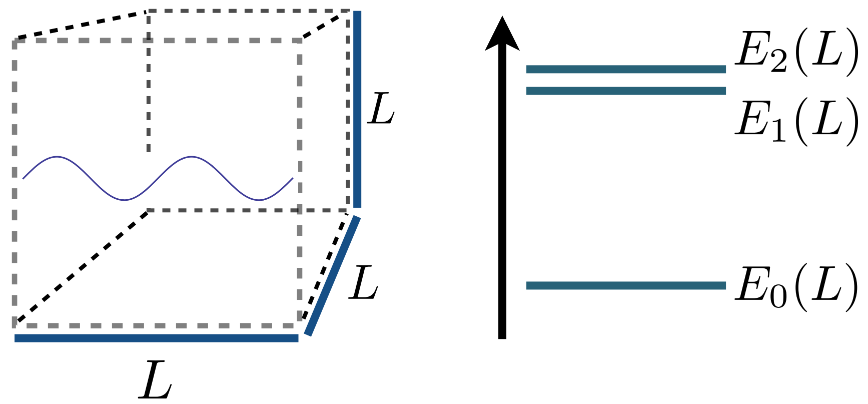
Relation is (highly) non-trivial



$\in$ **Discrete set of finite-volume states**

The finite-volume as a tool

- Finite-volume set-up

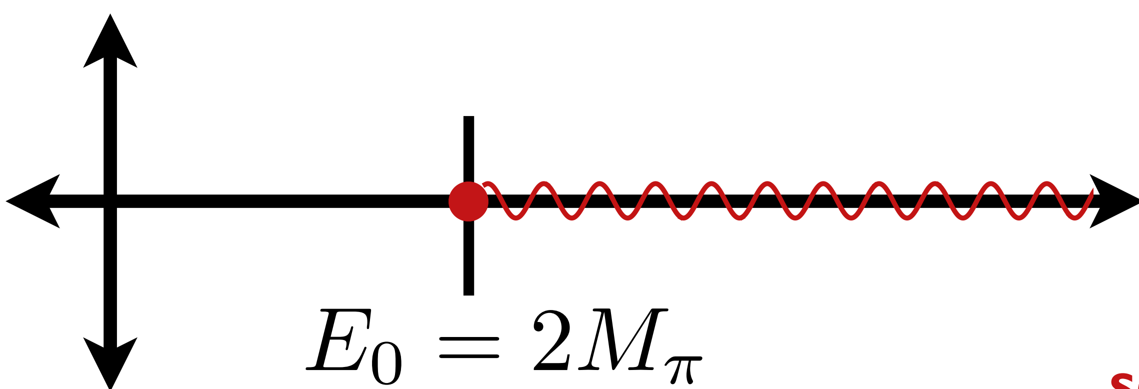


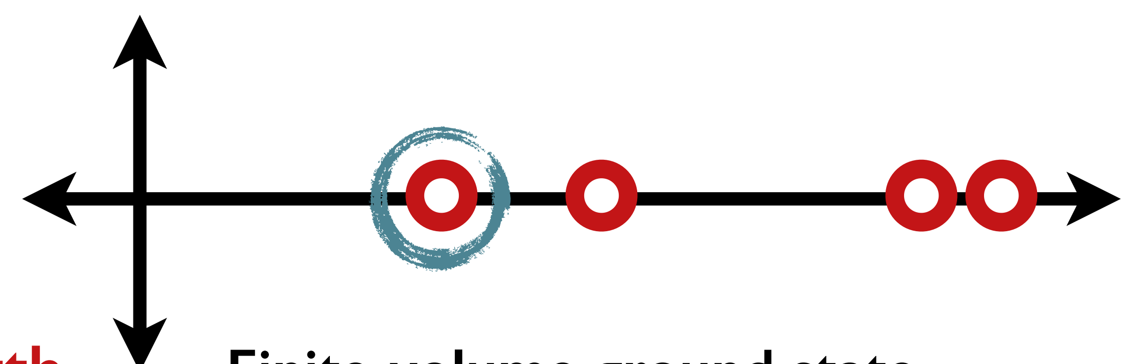
- cubic, spatial volume (extent L)
- periodic

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \qquad \vec{n} \in \mathbb{Z}^3$$

- L is large enough to neglect $e^{-M_\pi L}$
- T and lattice also negligible

- Scattering leaves an *imprint* on finite-volume quantities





scattering length

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

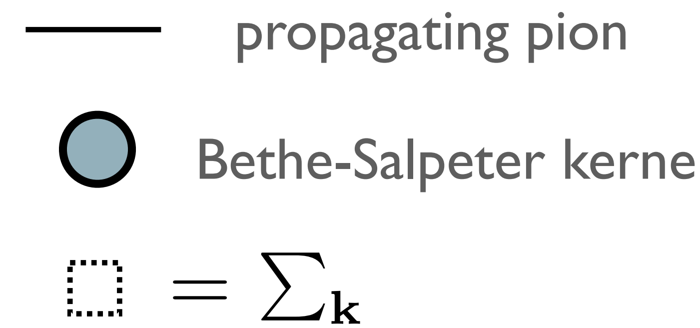
$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

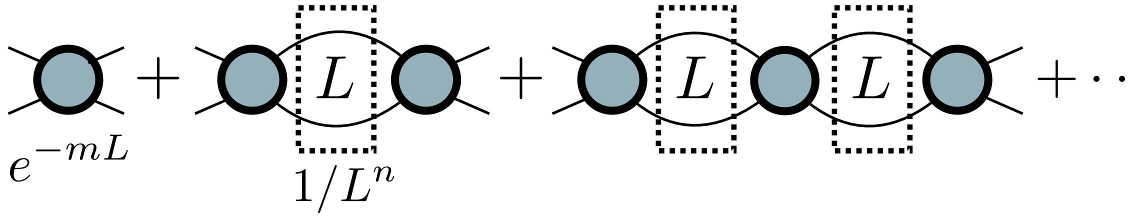
• Huang, Yang (1958) •

Derivation (all orders diagrammatic)

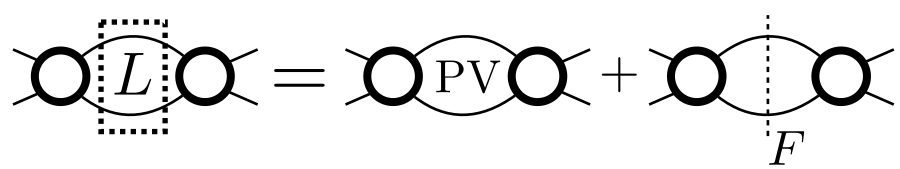
- Consider the finite-volume correlator:

$\mathcal{M}(s)$	$\mathcal{M}_L(P)$
probability amplitude	poles give f.v. spectrum



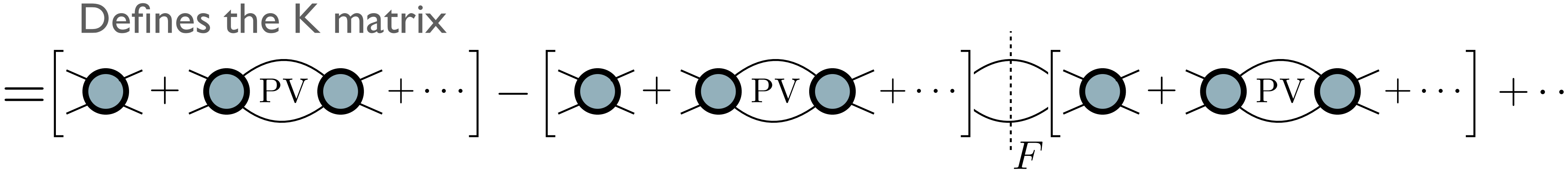
$\mathcal{M}_L(P) =$


For two-particle energies $(2m)^2 < s < (4m)^2$, what is the *L* dependence?

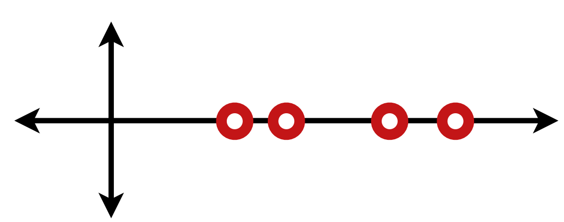


F = matrix of known geometric functions

Defines the K matrix



$$= \frac{1}{\mathcal{K}(s)^{-1} + F(P,L)}$$

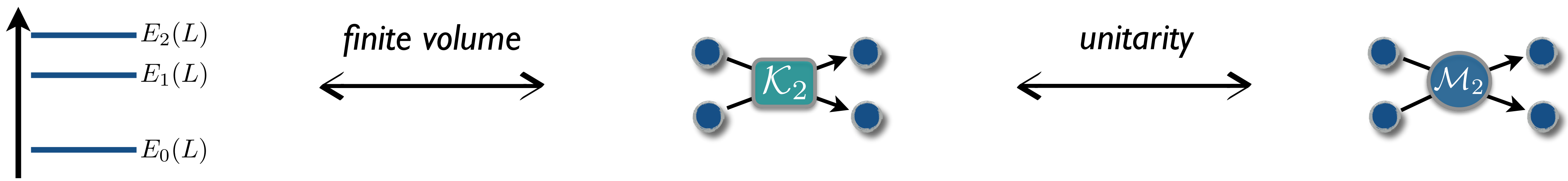


$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0$$

- Lüscher (1986) • Kim, Sachrajda, Sharpe (2005) • MTH, Sharpe (*coupled channels*, 2012) •

General relation

$$\det[\mathcal{K}^{-1}(s) + F(P,L)] = 0 \qquad F(P,L) \equiv \text{Matrix of known geometric functions}$$



Holds only for two-particle energies $s < (4m)^2$ Neglects e^{-mL}

Generalized to *non-degenerate masses, multiple channels, spinning particles*

Encodes angular momentum mixing

• Lüscher (1989) • *many others* •

Scattering processes and resonances from lattice QCD

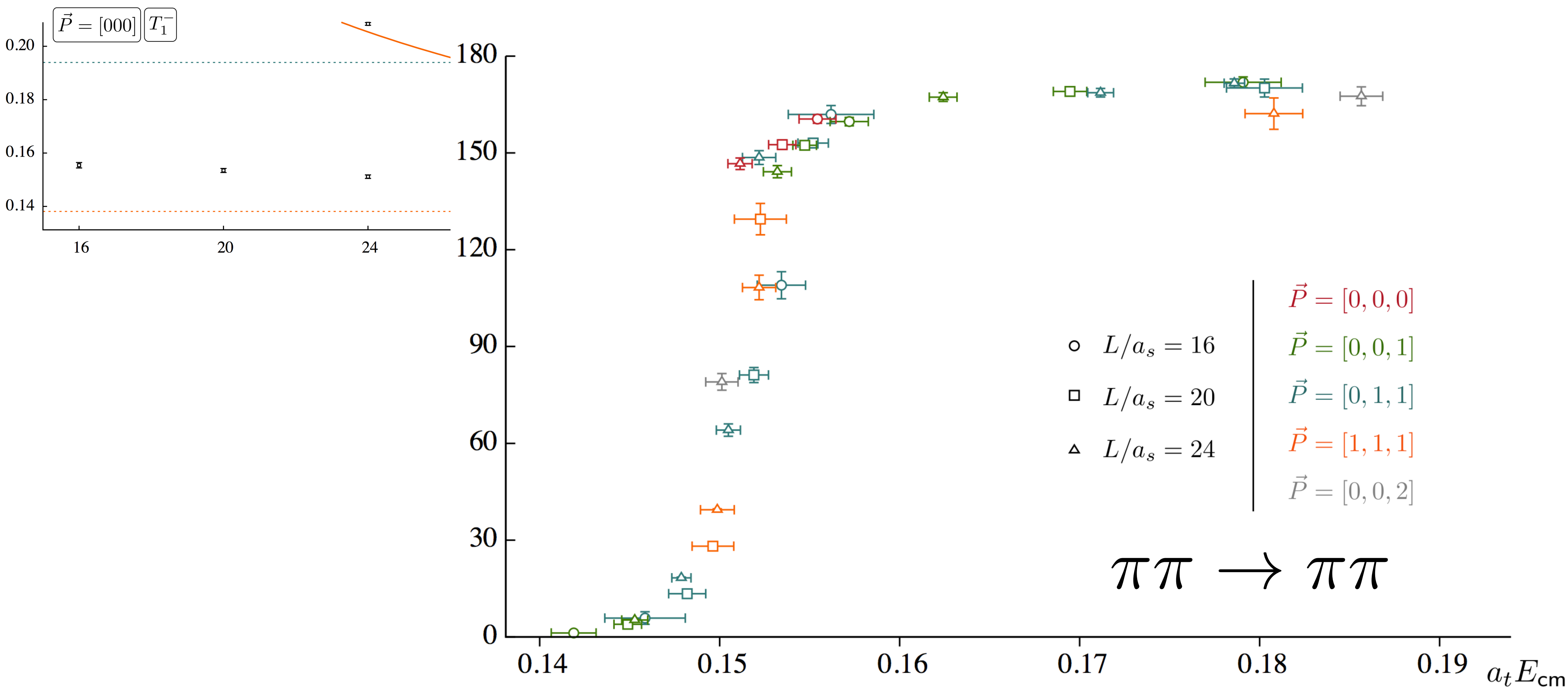
Raúl A. Briceño,[1,*] Jozef J. Dudek,[1,2,†] and Ross D. Young[3,‡]

REVIEWS OF MODERN PHYSICS

Using the result

- Single-channel case *(pions in a p-wave)*

$$\mathcal{K}(s_n)^{-1} = \rho \cot \delta(s_n) = -F(E_n, \vec{P}, L)$$



• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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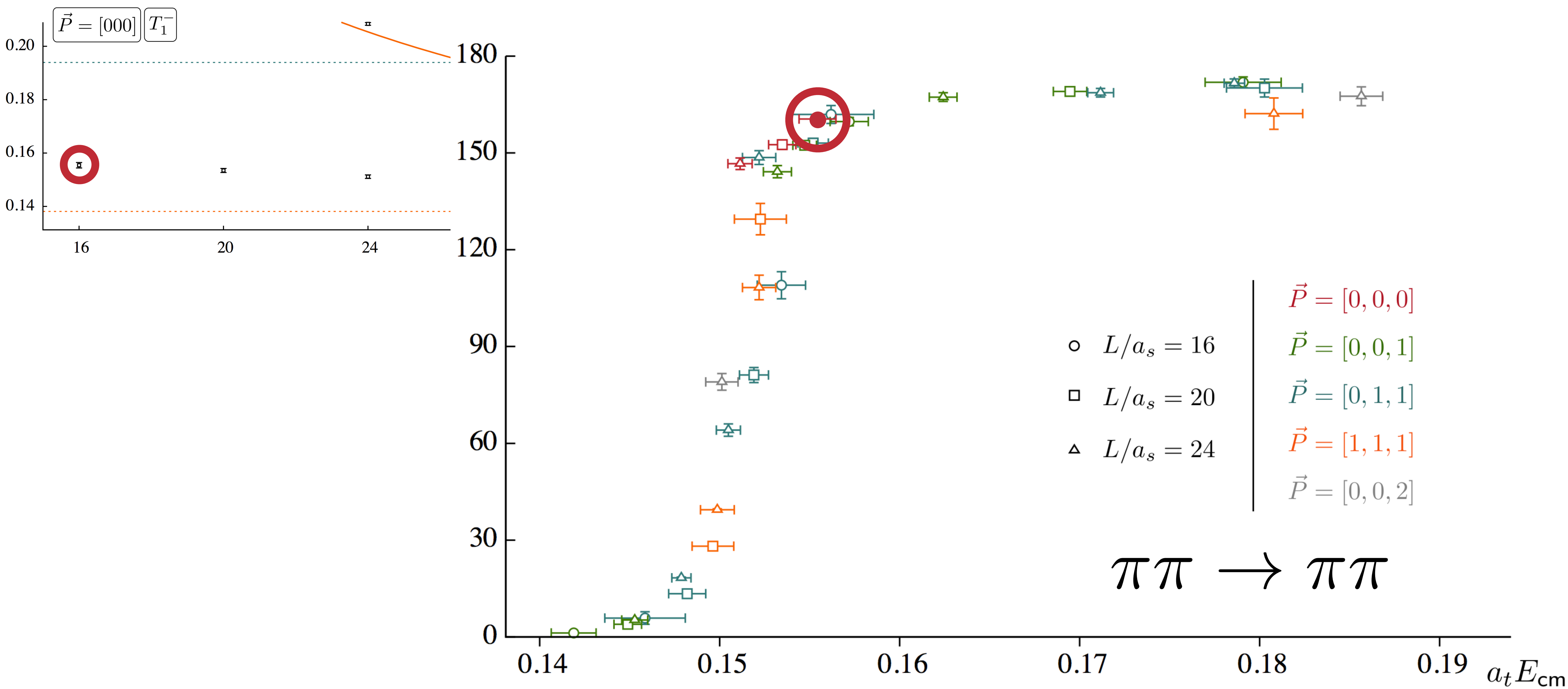

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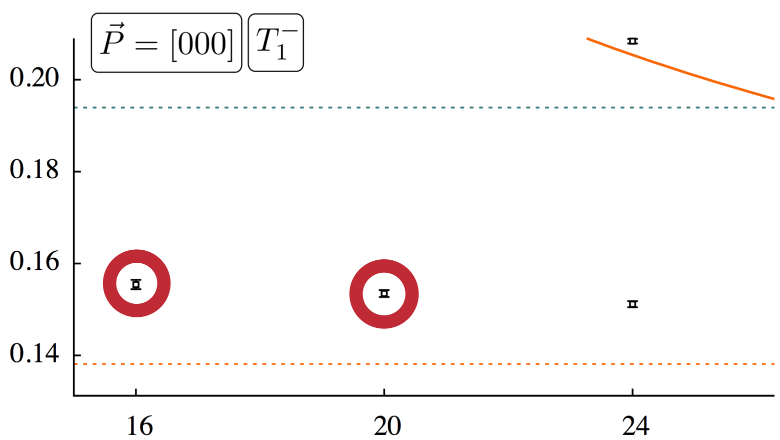

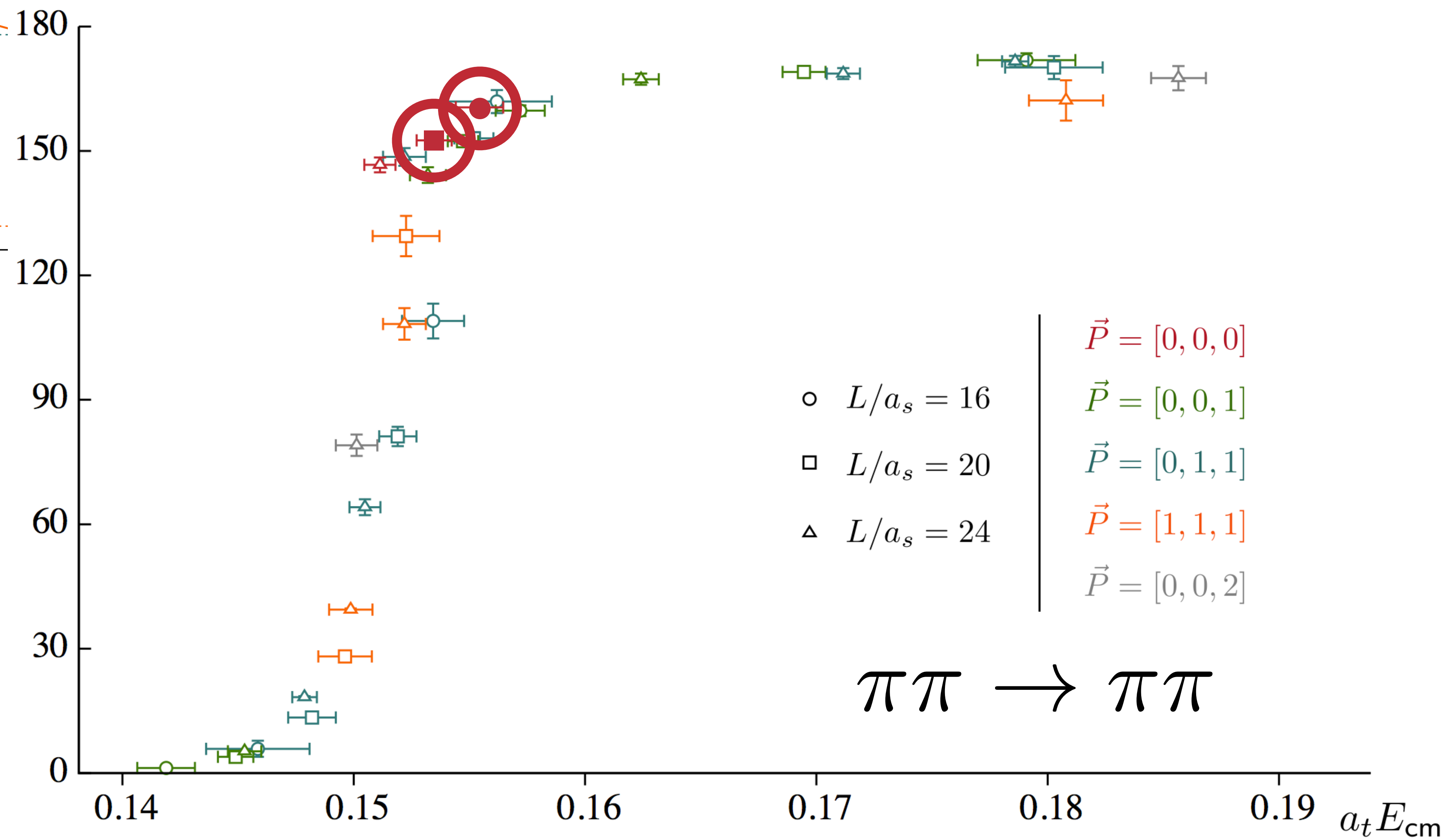

• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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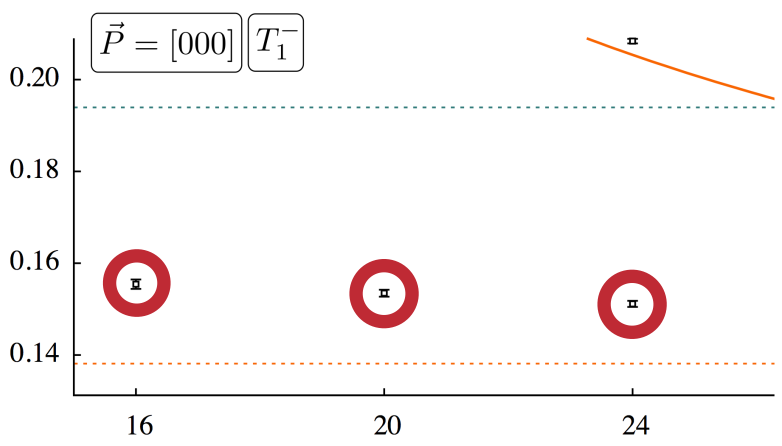

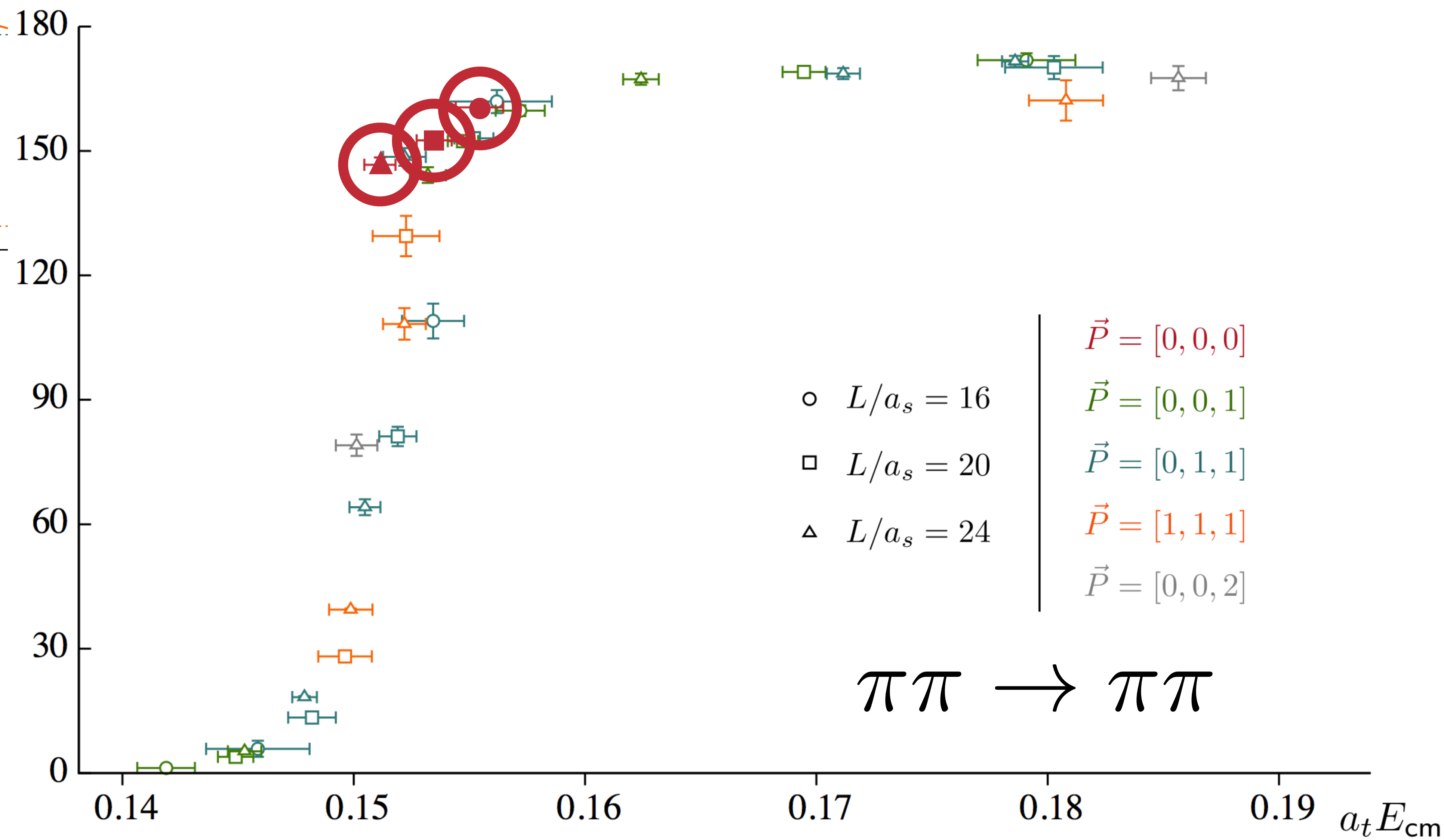

• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

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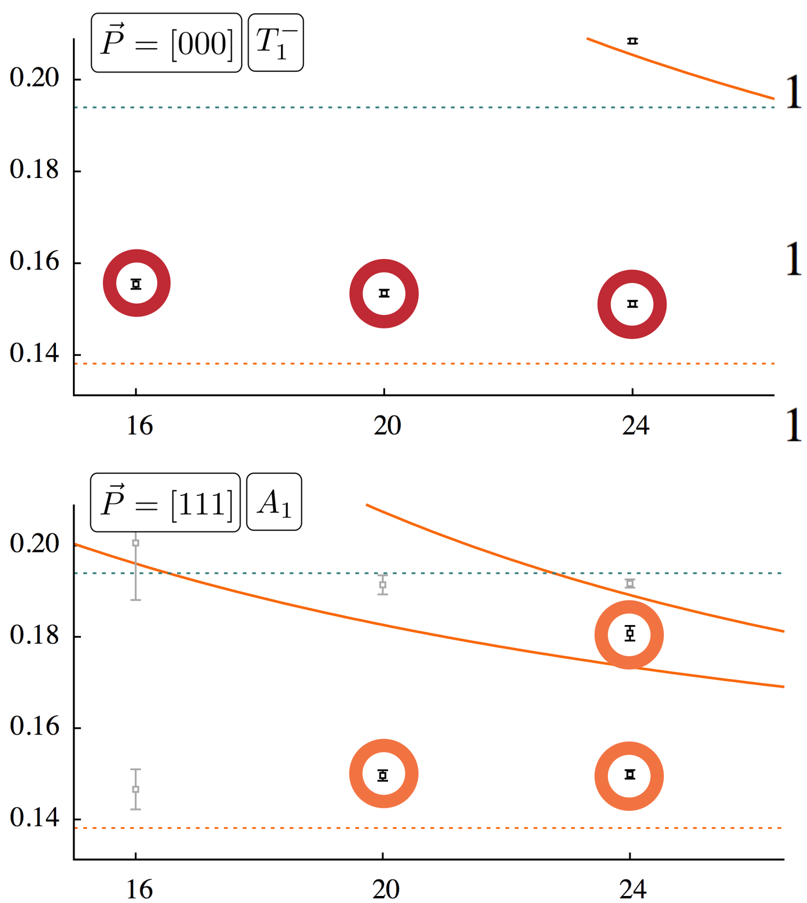
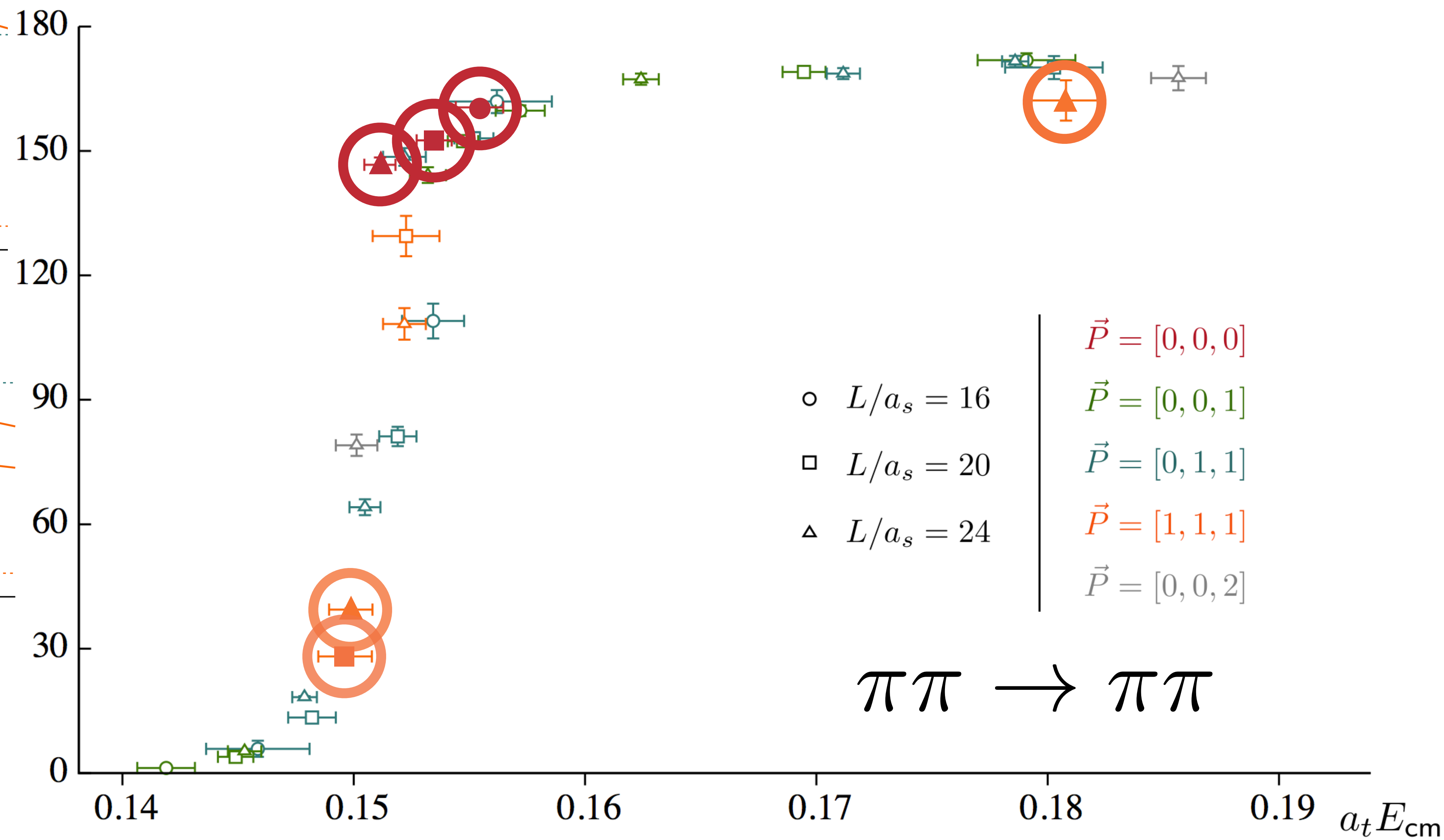
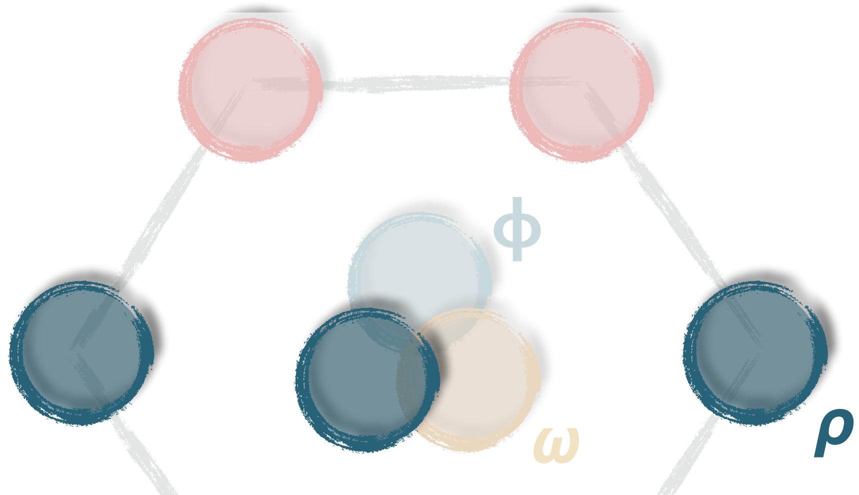


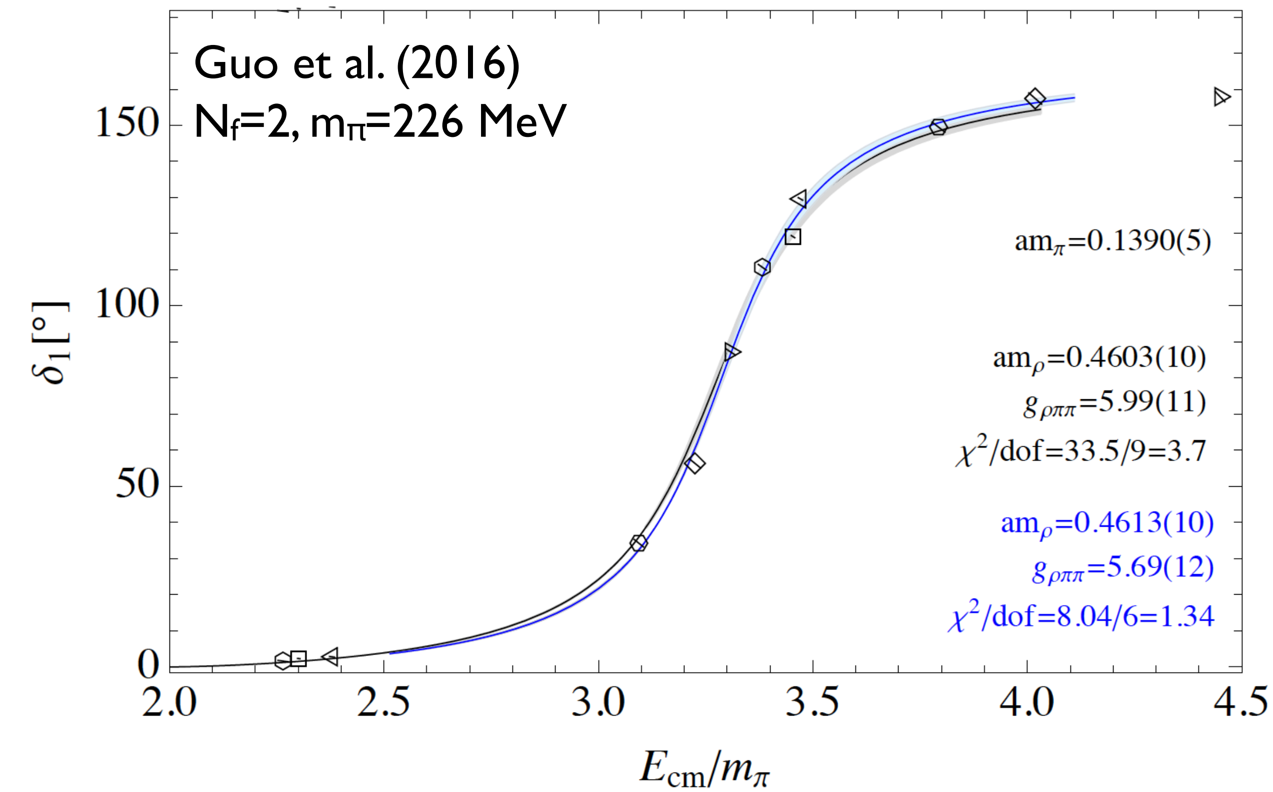


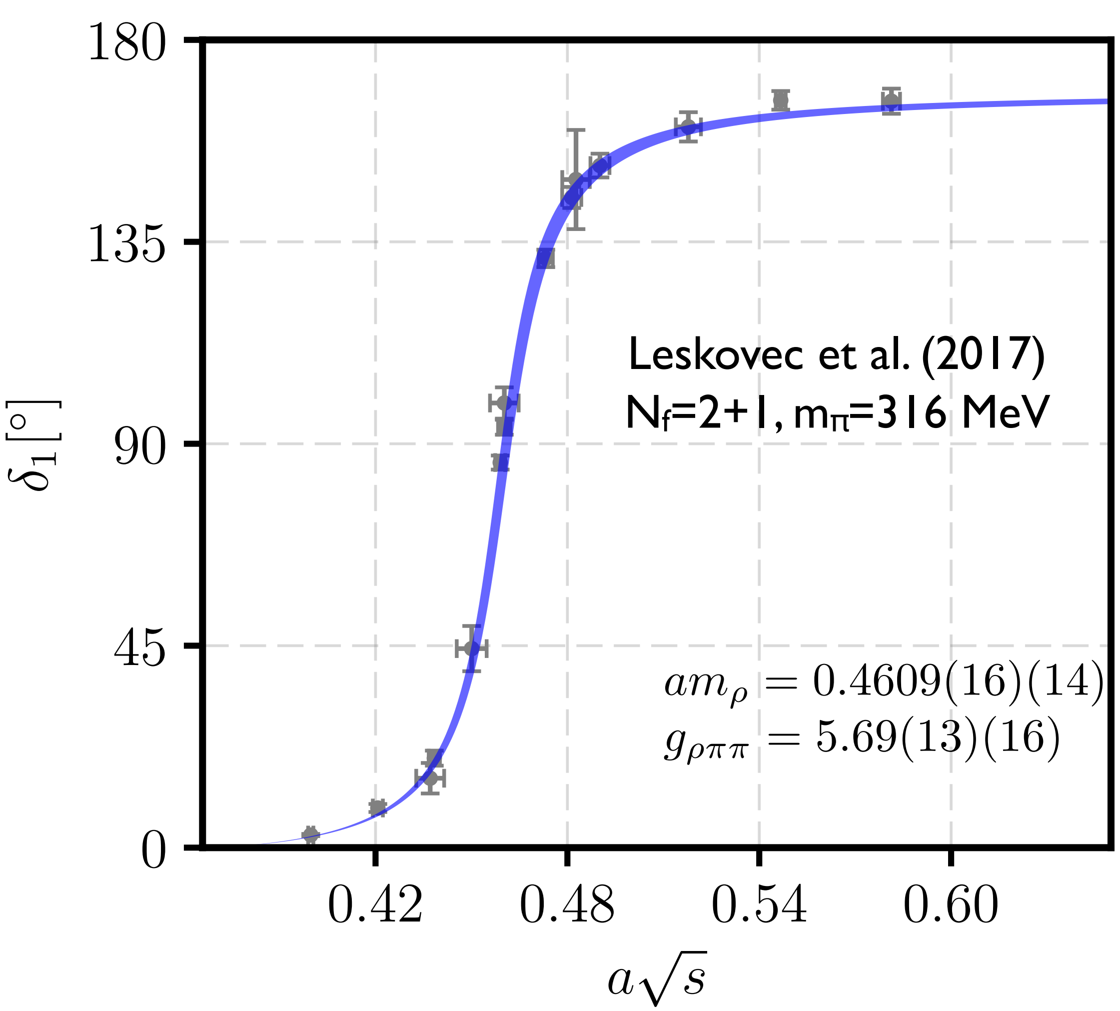
• Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505 •

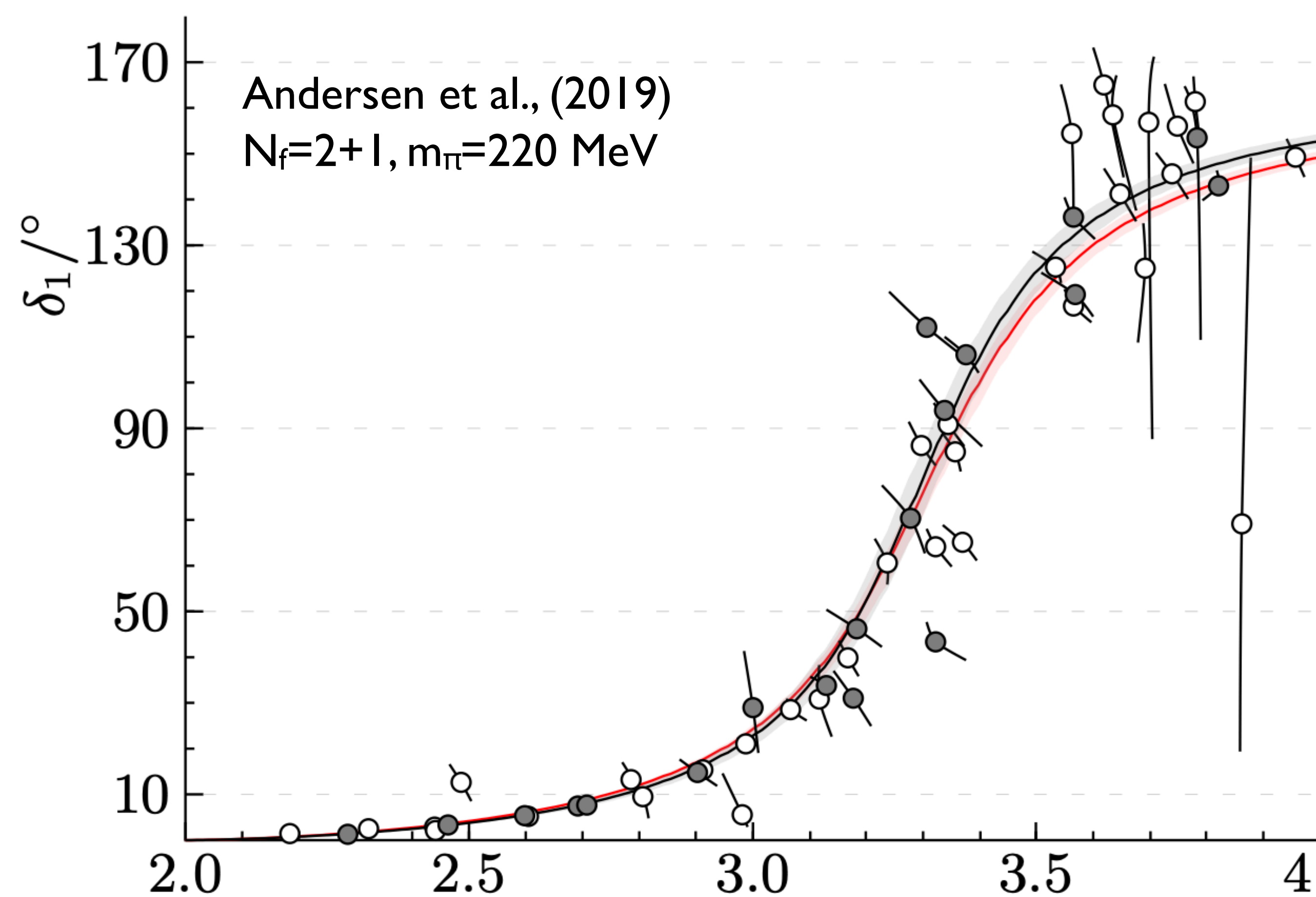
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$

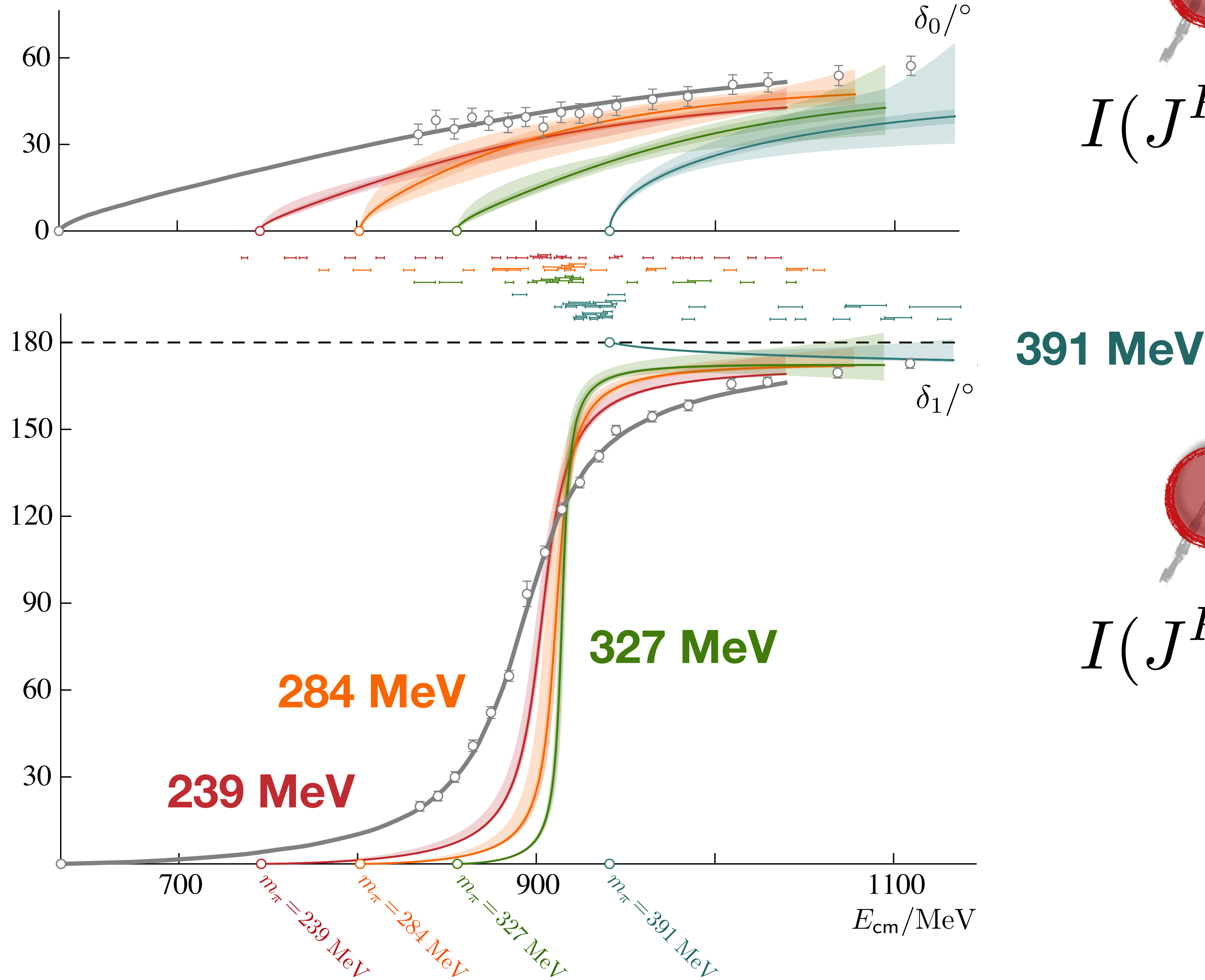


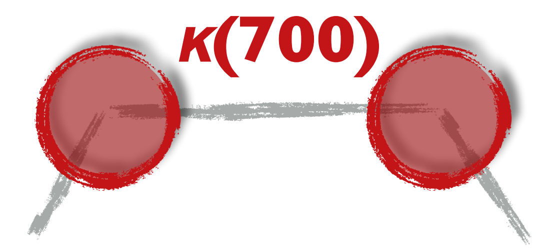




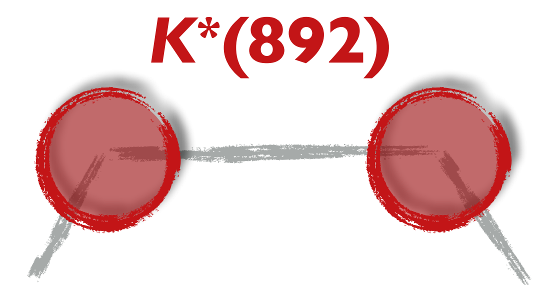


$\kappa, K^* \to K\pi$





$$I(J^P) = 1/2(0^+)$$

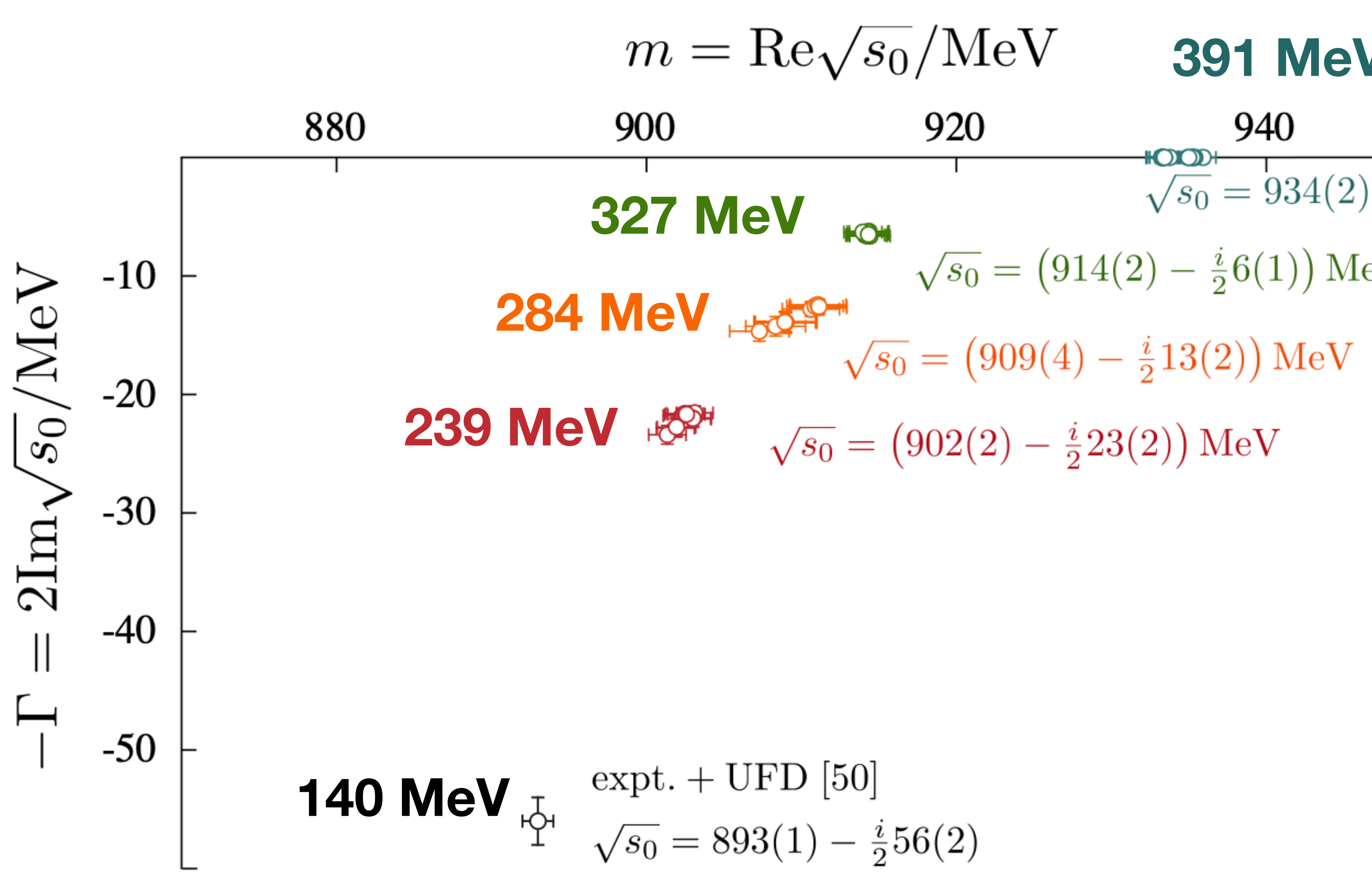


$$I(J^P) = 1/2(1^-)$$

• Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002* •

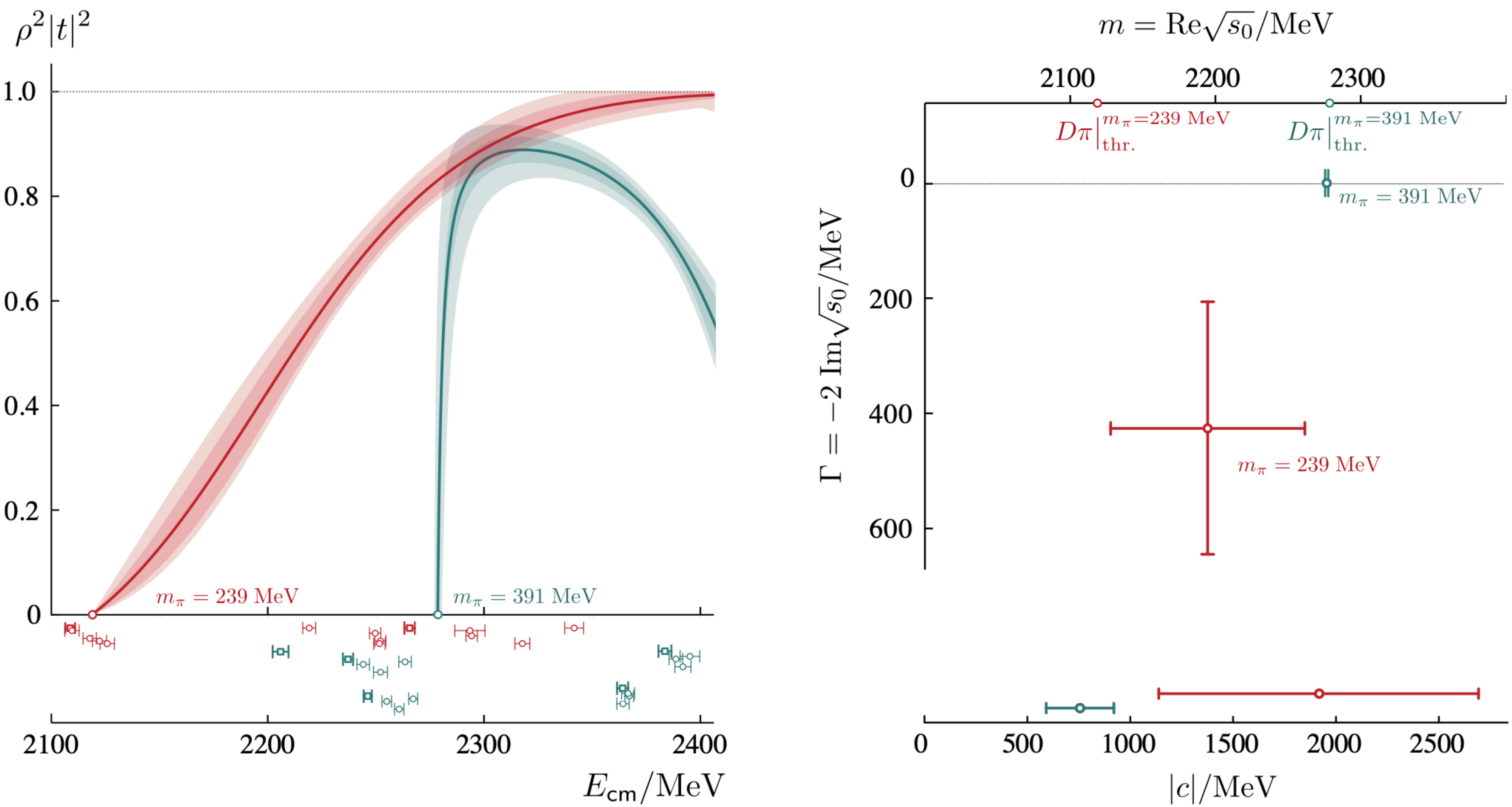
$\kappa, K^* \to K\pi$

$$I(J^P) = 1/2(1^-)$$



- Wilson et al. *Phys.Rev.Lett. 123 (2019) 4, 042002*

$D\pi \to D\pi$, $I = 1/2$

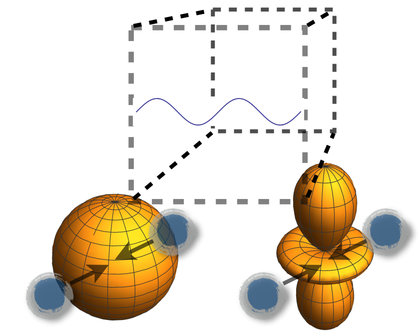


— *Isospin-1/2 Dπ scattering and the lightest D0* resonance from lattice QCD* —
Hadron Spectrum Collaboration — (2021) JHEP 07 (2021) 123

Coupled channels

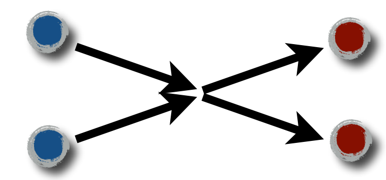
- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{K}_s^{-1} & 0 \\ 0 & \mathcal{K}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix}\right] = 0$



…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{K}_{a\to a} & \mathcal{K}_{a\to b} \\ \mathcal{K}_{b\to a} & \mathcal{K}_{b\to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix}\right] = 0$

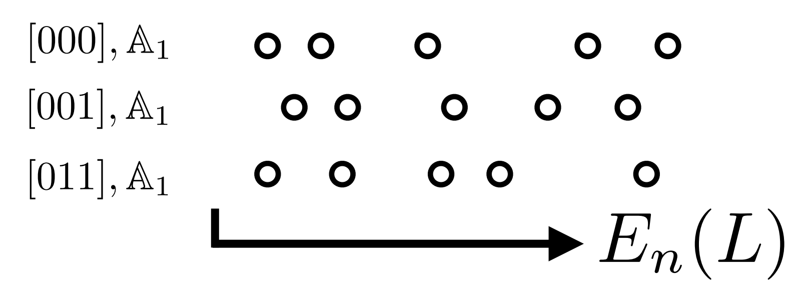


- Workflow…

Correlators with a large operator basis
$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

→ Reliably extract finite-volume energies
$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→ Vary L and P to recover a dense set of energies

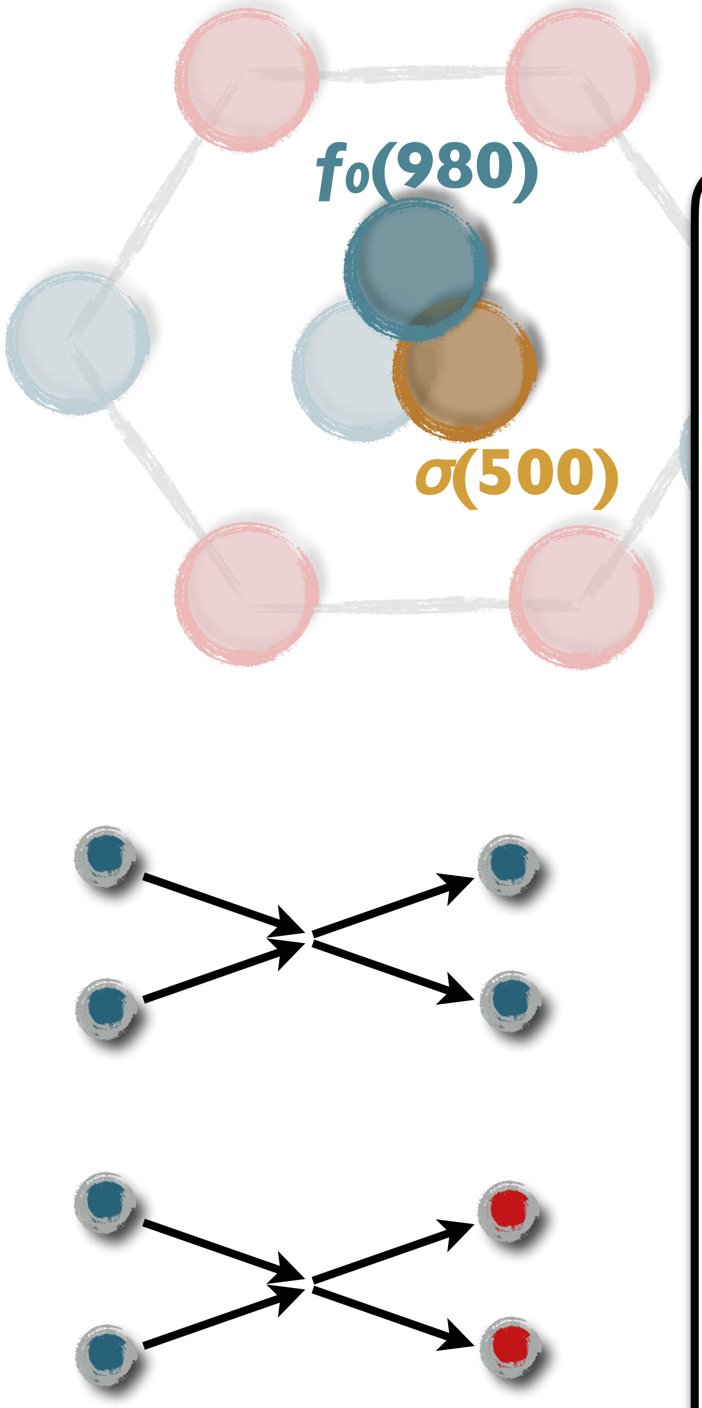


hadspec

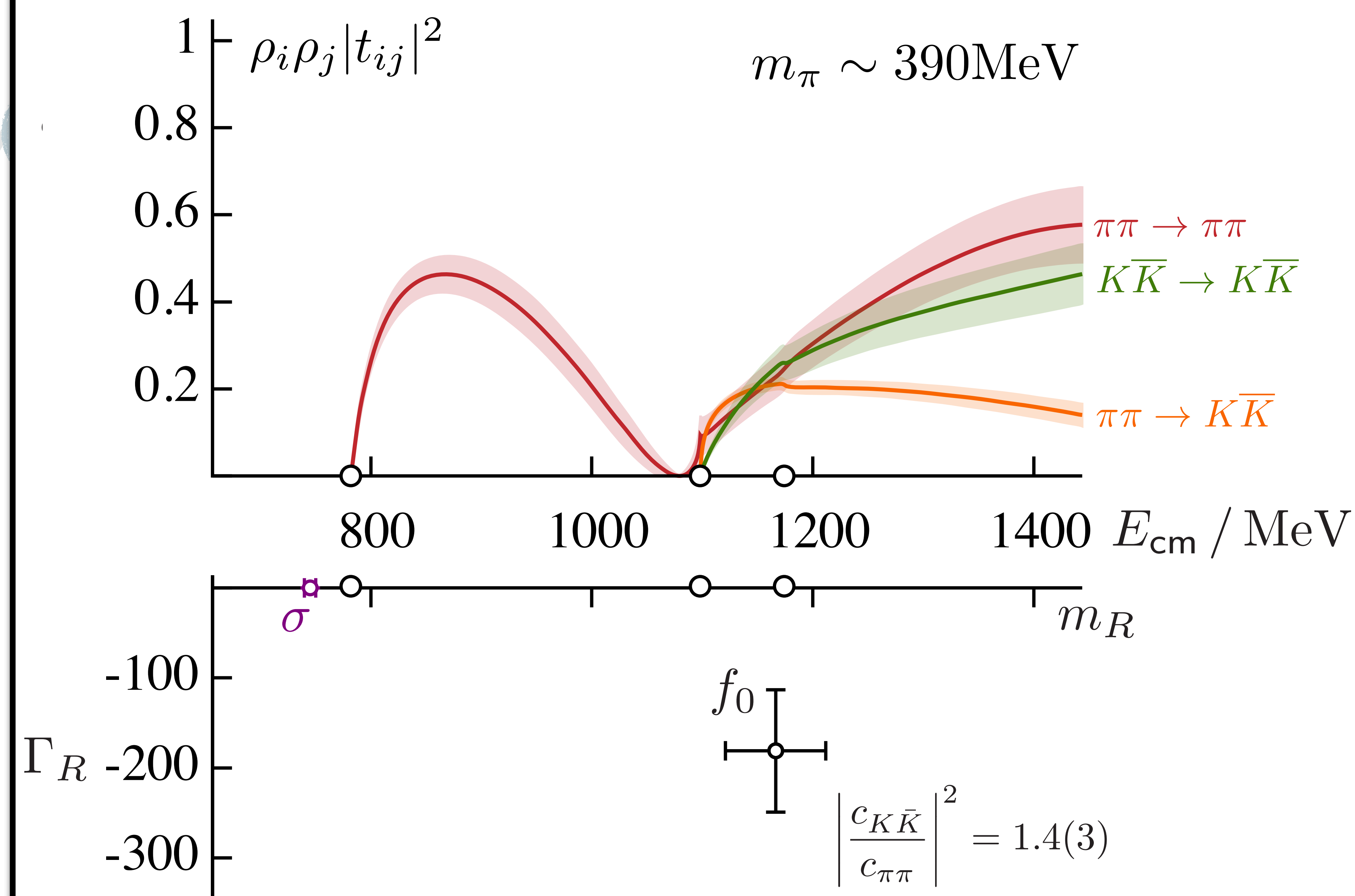
Identify a broad list of K-matrix parametrizations
polynomials and poles | EFT based | dispersion theory based

→ **Perform global fits to the finite-volume spectrum**

$$I^G(J^{PC}) = 0^+(0^{++})$$

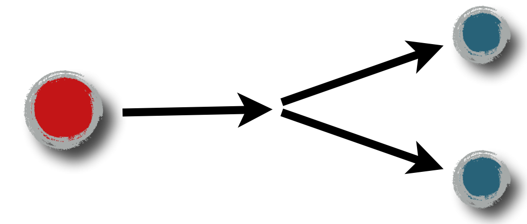


Coupled-channel scattering

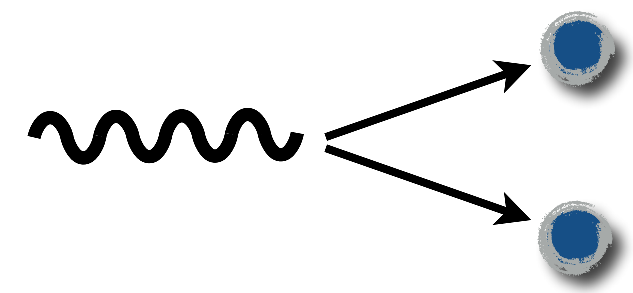


Briceño, Dudek, Edwards & Wilson (2017)

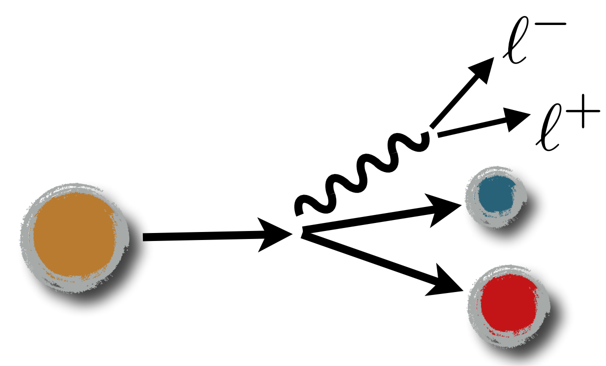
Formal progress: Transition amplitudes

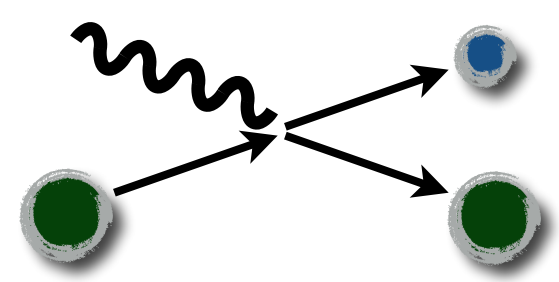
- Weak decay $\langle \pi\pi, \mathrm{out} | \mathcal{H} | K \rangle \equiv$ 

Lellouch, Lüscher *(2001)* • Kim, Sachrajda, Sharpe *(2005)* • Christ, Kim, Yamazaki *(2005)* • MTH, Sharpe *(2012)*

- Time-like form factors $\langle \pi\pi, \ \mathrm{out} | \mathcal{J}_\mu | 0 \rangle \equiv$ 

Meyer *(2011)*

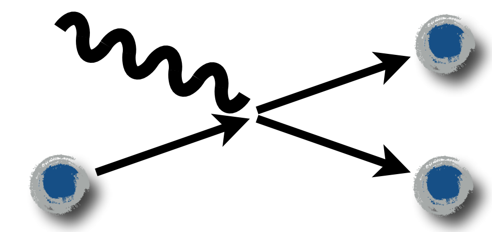
- Resonance form factors $\langle K\pi, \mathrm{out} | \mathcal{J}_{\alpha\beta} | B \rangle \equiv$ 

- Particles with spin $\langle N\pi, \mathrm{out} | \mathcal{J}_\mu | N \rangle \equiv$ 

Agadjanov *et al. (2014)* • Briceño, MTH, Walker-Loud *(2015)* • Briceño, MTH *(2016)*

Pion photo-production

$$\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$$



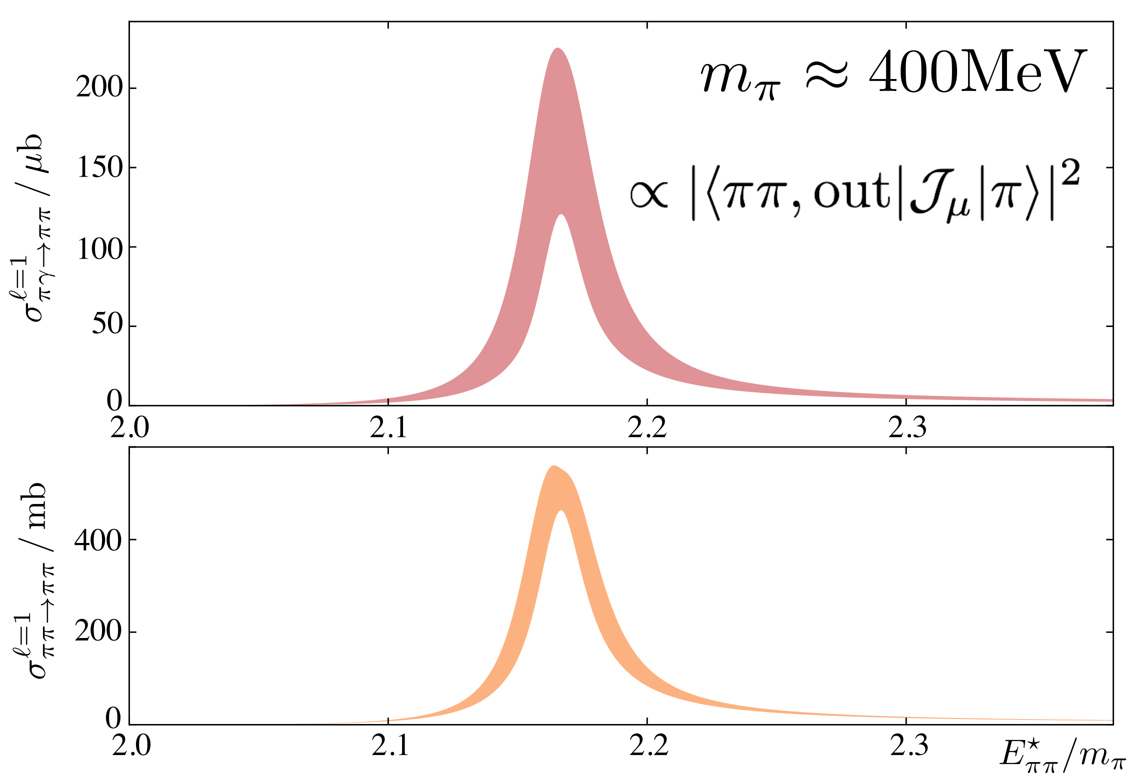
- Formal relation

get this from the lattice — experimental observable

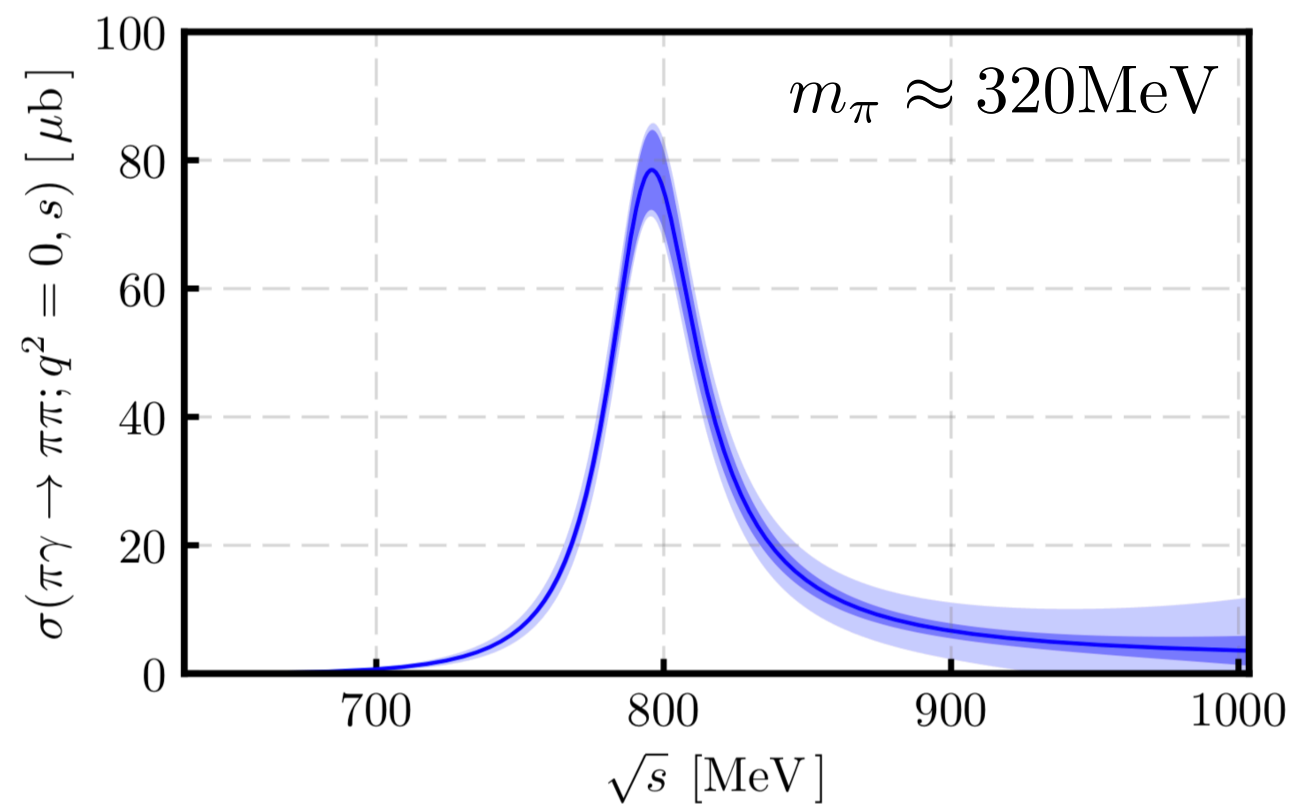
$$|\langle n, L|\mathcal{J}_\mu|\pi\rangle|^2 = \langle \pi|\mathcal{J}_\mu|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, L) \langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle$$

Briceño, MTH, Walker-Loud (2015)

- Numerical implementation



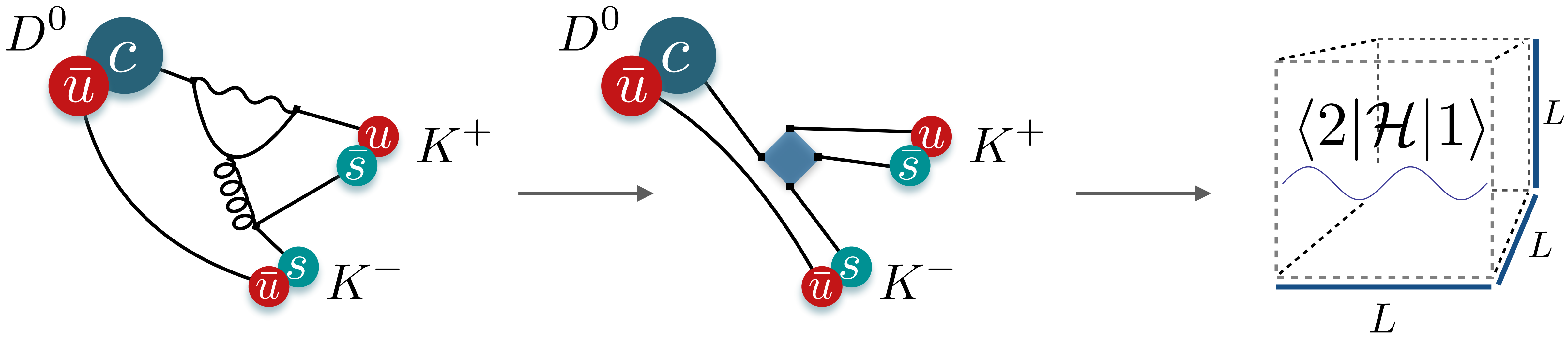
Briceño et. al., Phys. Rev. D93, 114508 (2016)



Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

Hadronic D decays

- Integrating out electroweak physics → basis of four-quark operators



- ***Complicated:*** non-perturbative ***renormalization***, many ***operators*** and ***contractions***

See the RBC/UKQCD calculation of K→ππ

multi-hadron final state

$$\langle n, L|\ \mathcal{H}^{\overline{\mathrm{MS}}}_{\mathrm{weak}}\ |D, L\rangle$$

renormalized weak Hamiltonian

incoming D meson

$e^{-M_\pi L}$ *volume effects*

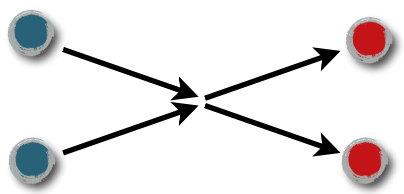
$\pi\pi, K\overline{K}, \pi\pi\pi\pi, \cdots$ ***have same quantum numbers + no asymptotic separation in the box***

How do we interpret $\langle n, L|$?

The finite-volume as a tool

- Coupled channels leave an *imprint* on finite-volume energies

infinite-volume
scattering amplitudes



$\pi\pi \to K\overline{K}$

(i) given these experimentally

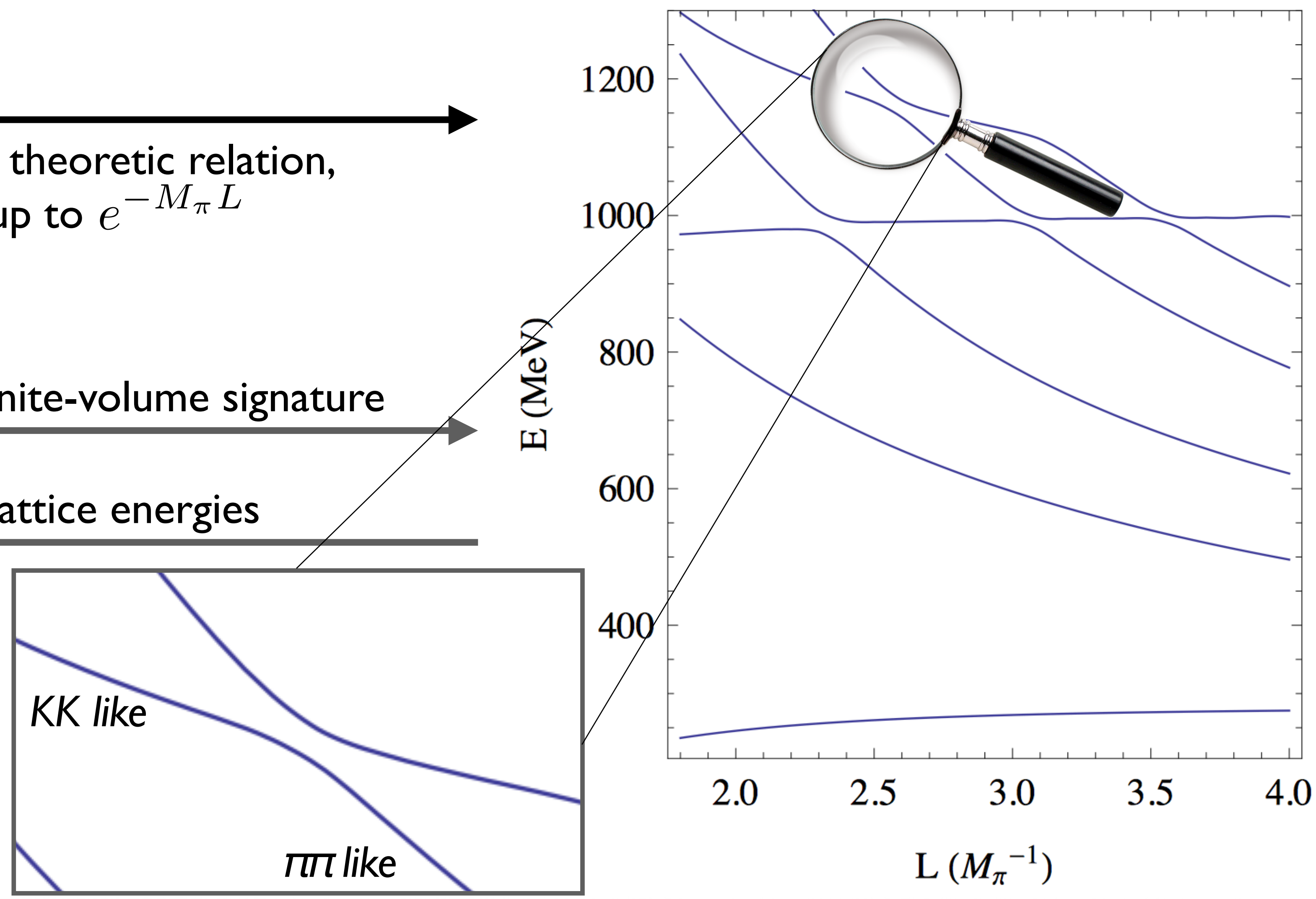
(ii) predict from first principles

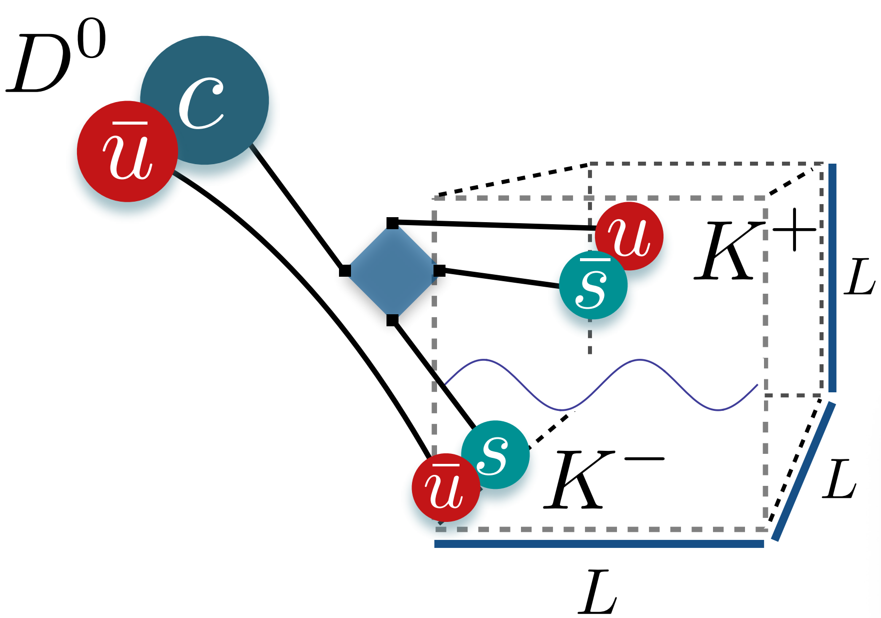
know field theoretic relation,
exact up to $e^{-M_\pi L}$

can predict finite-volume signature

given lattice energies

finite-volume energies



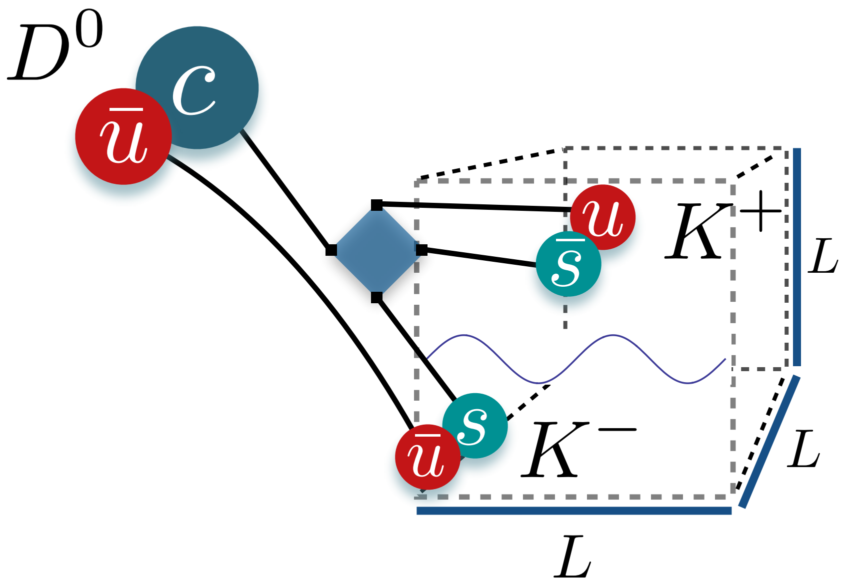


$$|n, L\rangle = c_\pi^{(n)}(L)\,|\pi\pi, \mathrm{out}\rangle + c_K^{(n)}(L)\,|K\overline{K}, \mathrm{out}\rangle$$

path to physical amplitude

• MTH, Sharpe, *Phys.Rev.* **D**86 (2012) 016007 •

How far in the future?



$$\langle n, L|\, \mathcal{H}^{\overline{\rm MS}}_{\rm weak}\, |D, L\rangle$$

$$|n, L\rangle = c_\pi^{(n)}(L)\,|\pi\pi, \text{out}\rangle + c_K^{(n)}(L)\,|K\overline{K}, \text{out}\rangle$$

- ❑ Pilot calculation underway at the University of Edinburgh
- ❑ Wilson-quark ensembles at the $SU(3)_F$ symmetric point
- ❑ See Fabian Joswig talks: Lattice2022 and MIT Colloquium

biggest challenge = still missing strategy for treating $\pi\pi\pi\pi$ etc, channels

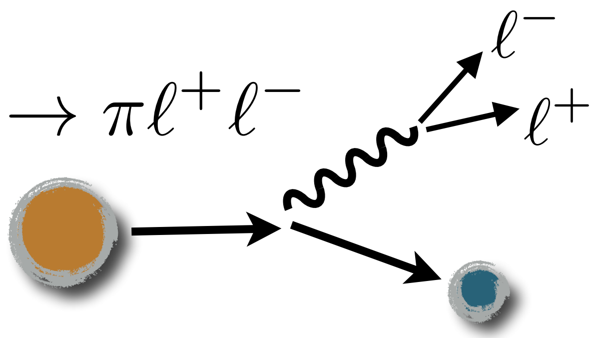
$$|n, L\rangle = c_\pi^{(n)}(L)\,|\pi\pi, \text{out}\rangle + c_K^{(n)}(L)\,|K\overline{K}, \text{out}\rangle + c_{4\pi}^{(n)}|\pi\pi\pi\pi, \text{out}\rangle + \cdots$$

Funded by UKRI Future Leaders Fellowship

UKRI UK Research and Innovation

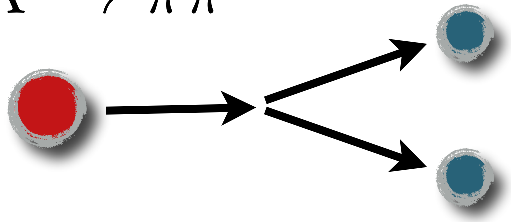
Matrix elements and LQCD

$B \to \pi \ell^+ \ell^-$

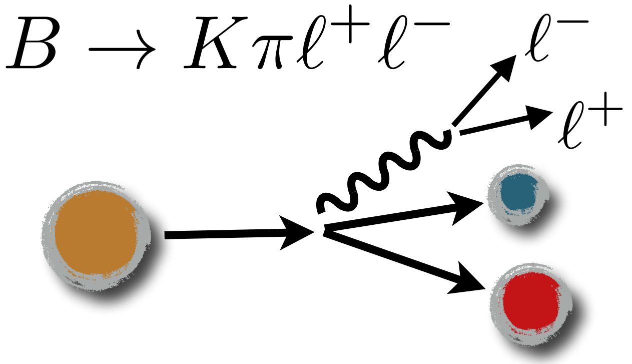


- ☑ Single-hadron initial and final states
  - ☐ Calculated directly in LQCD
  - ☐ New theory challenge = QED
  - ☐ See FLAG averages

$K \to \pi\pi$

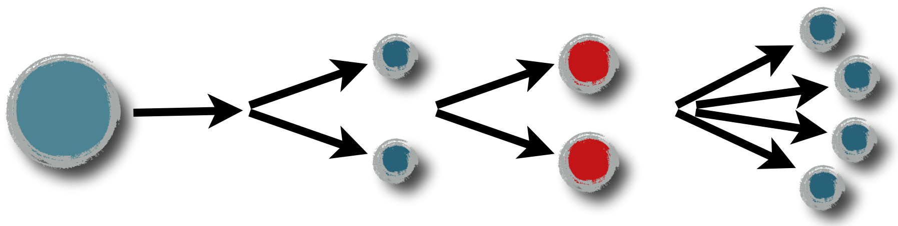


$B \to K \pi \ell^+ \ell^-$



- ☑ Two-hadron final states
  - ☐ Significantly more challenging
  - ☐ Subtle finite volume issues

- ☐ Multi-hadron states for $\sqrt{s} > 4M_\pi$
  - ☐ All or nothing (must constrain all channels for a prediction)

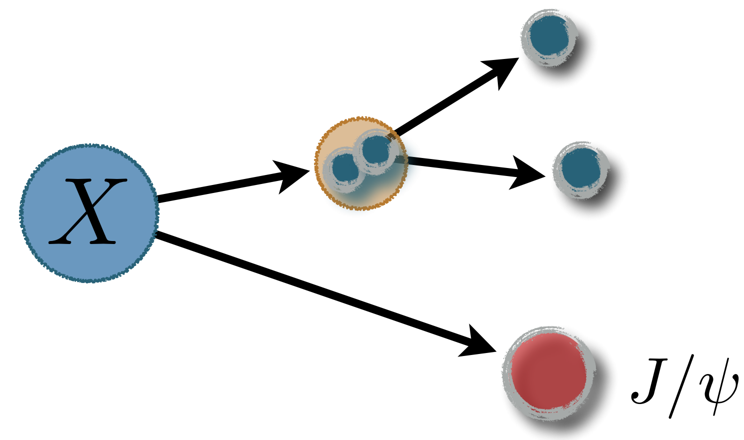


$D \to \pi\pi, K\overline{K}$

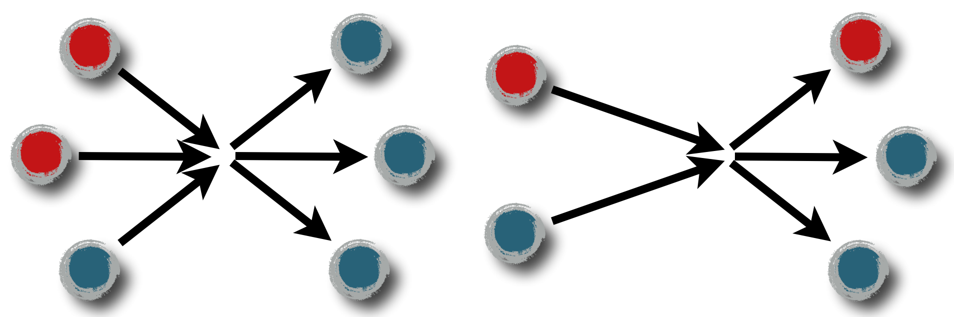
3-particle amplitudes

2-to-2 only samples J^P 0^+ 1^- $2^+ \cdots$

many interesting resonances have significant 3-body decays



Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



Applications…

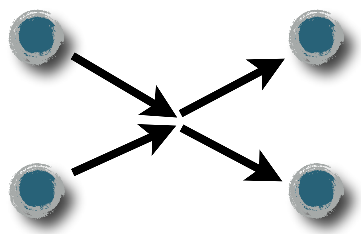
exotic resonance pole positions, couplings, quantum numbers

$\omega(782),\, a_1(1420) \to \pi\pi\pi$ $\quad X(3872) \to J/\psi\pi\pi$ $\quad X(3915)[Y(3940)] \to J/\psi\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

Complication: degrees of freedom

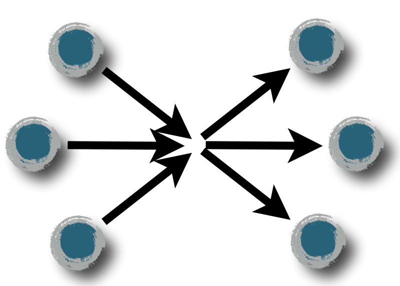


12 momentum components

-10 Poincaré generators

2 degrees of freedom

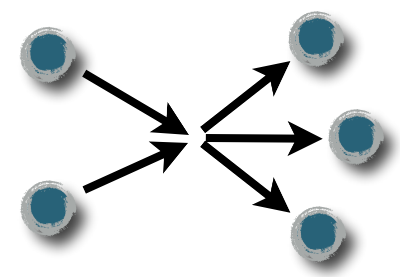
$$\vec{p}_1 + \vec{p}_2 \;\rightarrow\; \vec{p}_3 + \vec{p}_4 \;\longrightarrow\; \text{Mandelstam } s, t$$



18 momentum components

-10 Poincaré generators

8 degrees of freedom



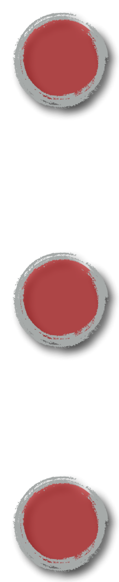
15 momentum components

-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering



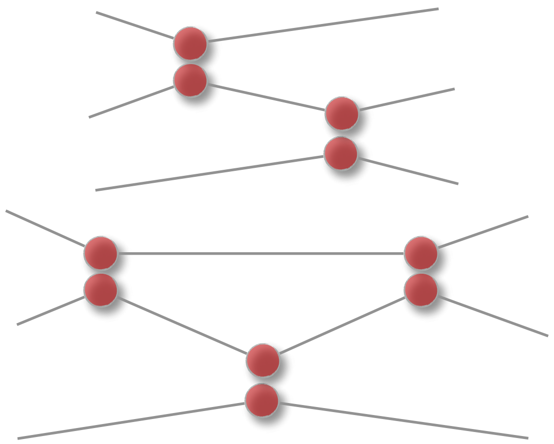
Complication: on-shell states

- Classical pairwise scattering

Complication: on-shell states

- Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to ***3*** binary collisions are possible



Dispersion Relations for Three-Particle Scattering Amplitudes. I*

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AND

ROBERT SUGAR
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AND

GEORGE TIKTOPOULOS
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

(Received 31 January 1966)

$$b=\frac{(m_1+m_3)(m_2+m_3)}{m_1m_2}$$

It follows that if

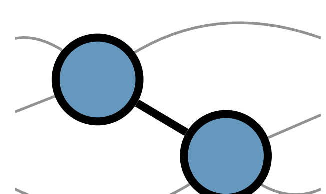
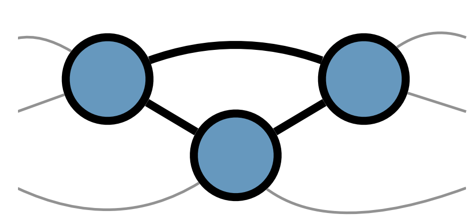
$$b^{n-2}(b-1)>1\,, \qquad \text{(IV.18)}$$

then $2n+1$ successive binary collisions are kinematically impossible.

$m_1 = m_2 = m_3 - \varepsilon$:
4 collisions possible
$\pi\pi K$

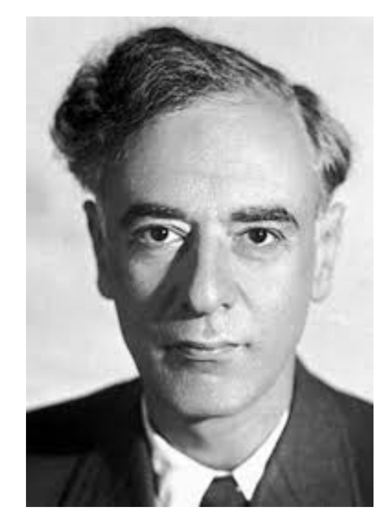
$b < 2$
5 collisions possible
$\pi K K$

- Correspond to Landau singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator $=$  $+$  $+\cdots$

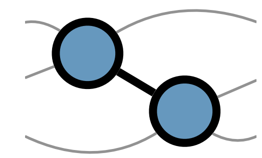
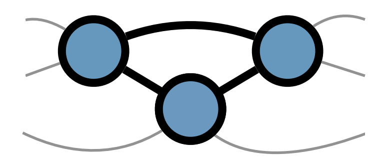
complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles



Two key observations

- Intermediate $K_{\mathrm{df},3}$ removes singularities

$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+ \cdots$

same degrees of freedom as M_3

smooth real function

relation to M_3 = known

- $K_{\mathrm{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\mathrm{df},3}(p_3, p_2, p_1; k_3, k_2, k_1) = \mathcal{K}^{\mathrm{iso},0}_{\mathrm{df},3} + \mathcal{K}^{\mathrm{iso},1}_{\mathrm{df},3}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

smooth real function

analogous to effective range expansion

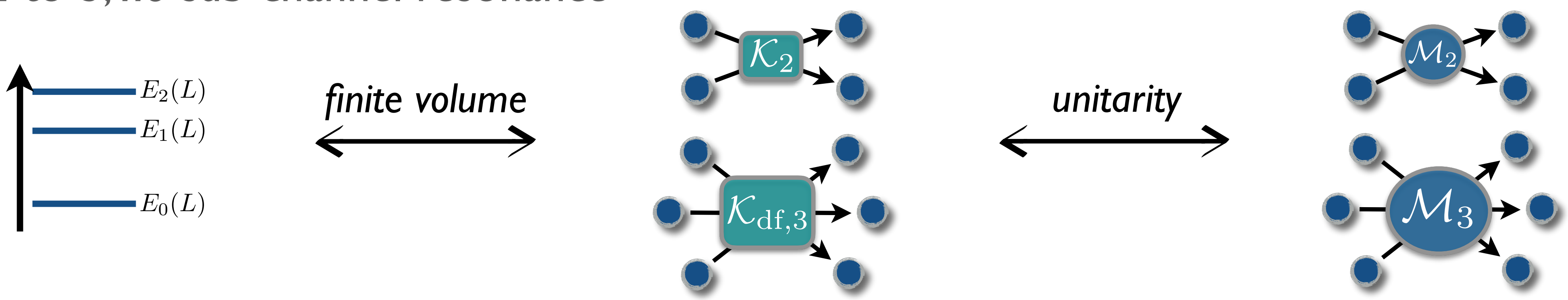
$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\,r\,p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status…

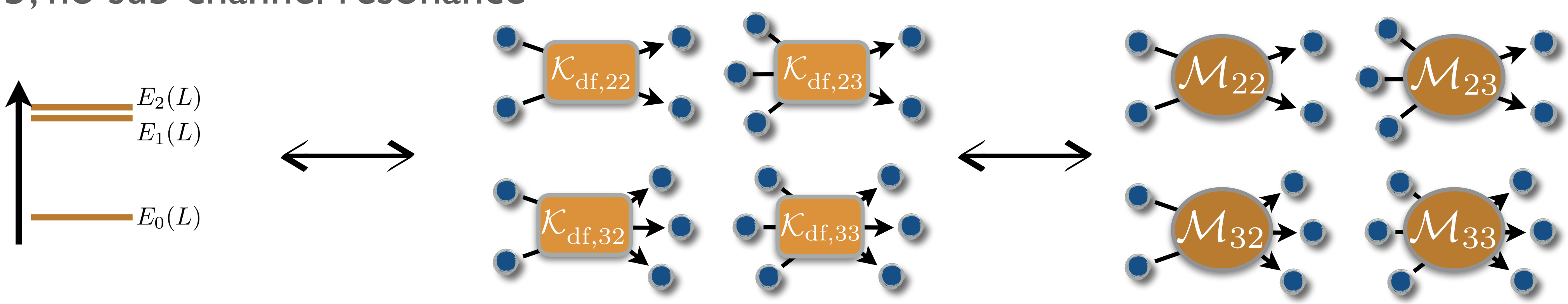
- General relation between *energies* and *two-and-three scalar scattering*

No **2-to-3,** ***no*** **sub-channel resonance**



- MTH, Sharpe (2014, 2015) •

2-to-3, no sub-channel resonance



- Briceño, MTH, Sharpe (2017) •

Including sub-channel resonances + ***different isospins*** + ***non-degenerate***

$$\pi\pi\pi \to \rho\pi \to \omega \to \rho\pi \to \pi\pi\pi$$

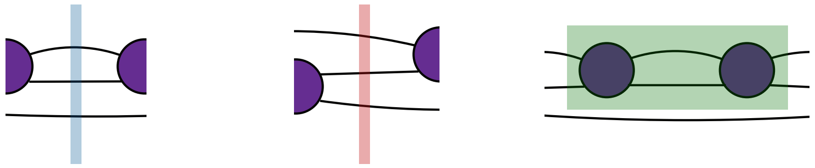
- Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

General relation

$$\det\left[\mathcal{K}_{\mathrm{df},3}^{-1}(s) + F_3(P, L|\mathcal{K}_2)\right] = 0$$

$F_3(P, L|\mathcal{K}_2) \equiv$ Matrix of functions depending on kinematics + two-particle dynamics

$$F_3 \equiv \frac{1}{3}F + F\mathcal{K}_2 \frac{1}{1-(F+G)\mathcal{K}_2} F$$



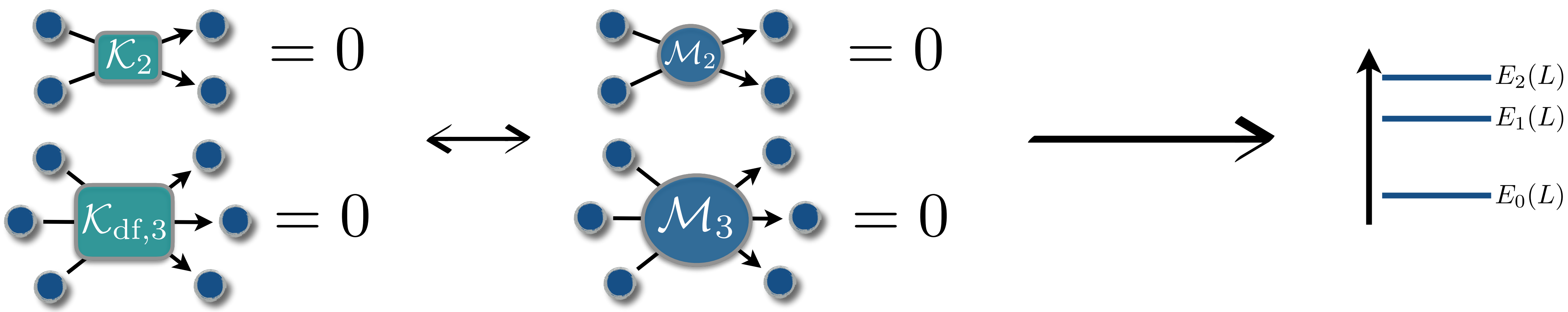
Holds only for three-particle energies

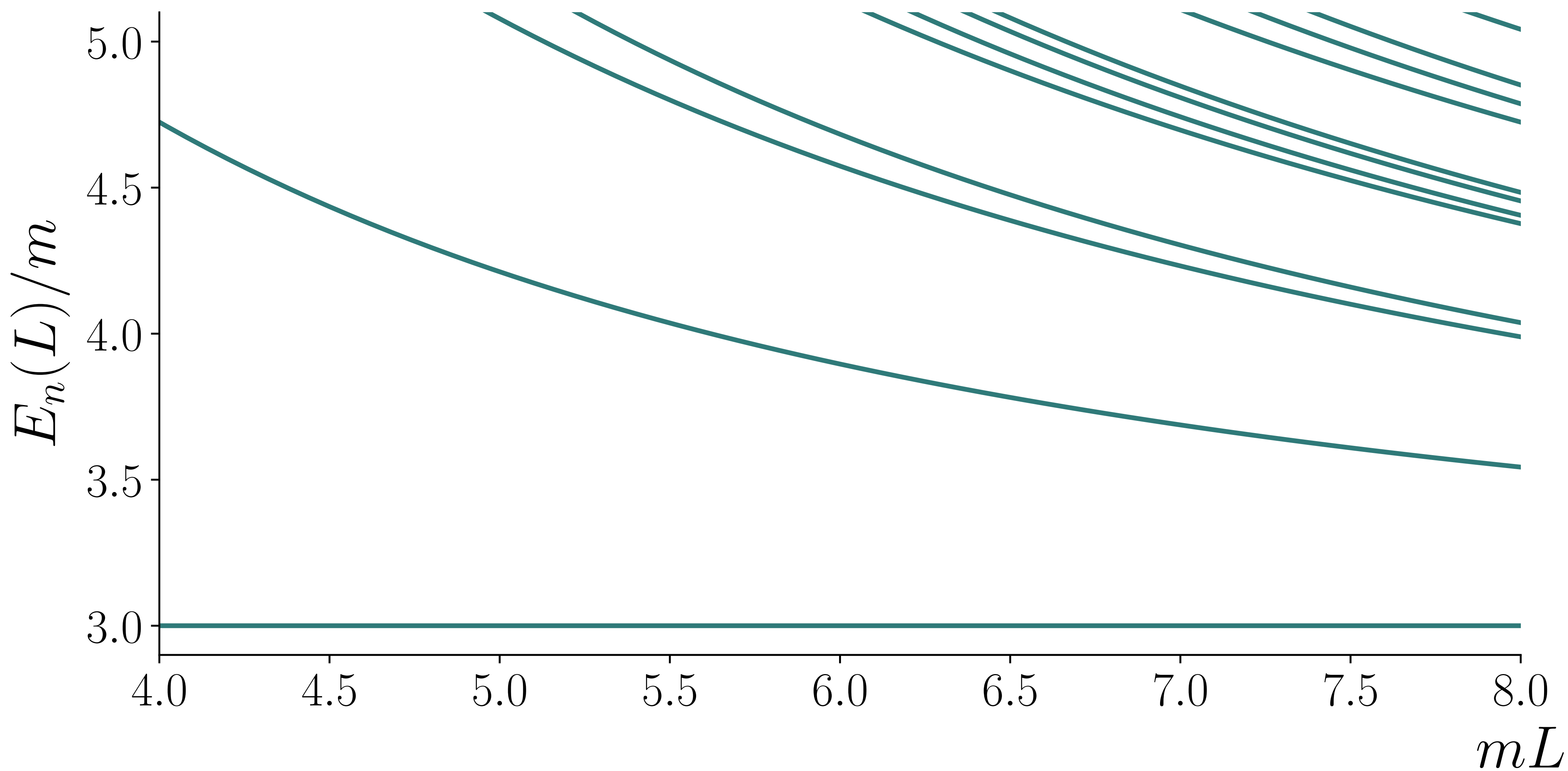
Neglects e^{-mL}

- MTH, Sharpe (2014-2016)
- *See also Döring, Mai, Hammer, Pang, Rusetsky*

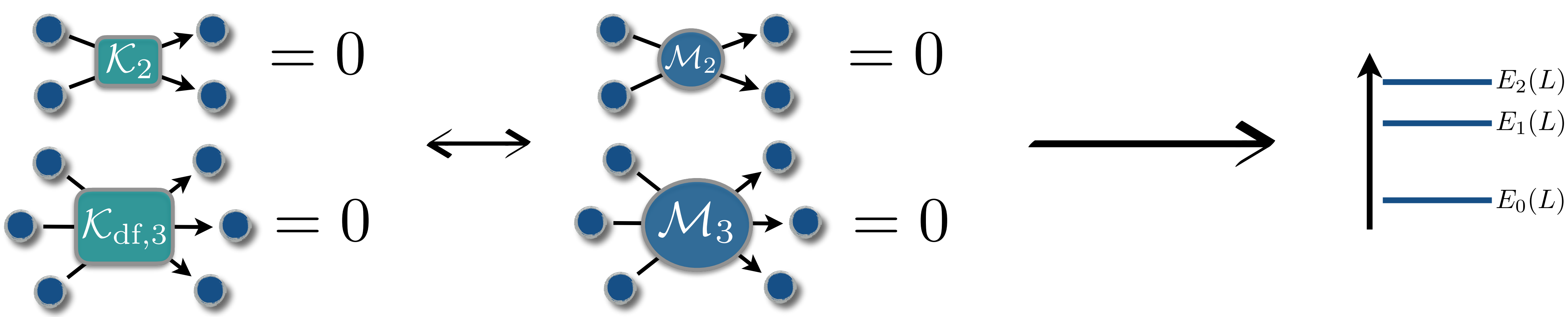
Review: *Lattice QCD and Three-particle Decays of Resonances*
MTH and Sharpe, *1901.00483*

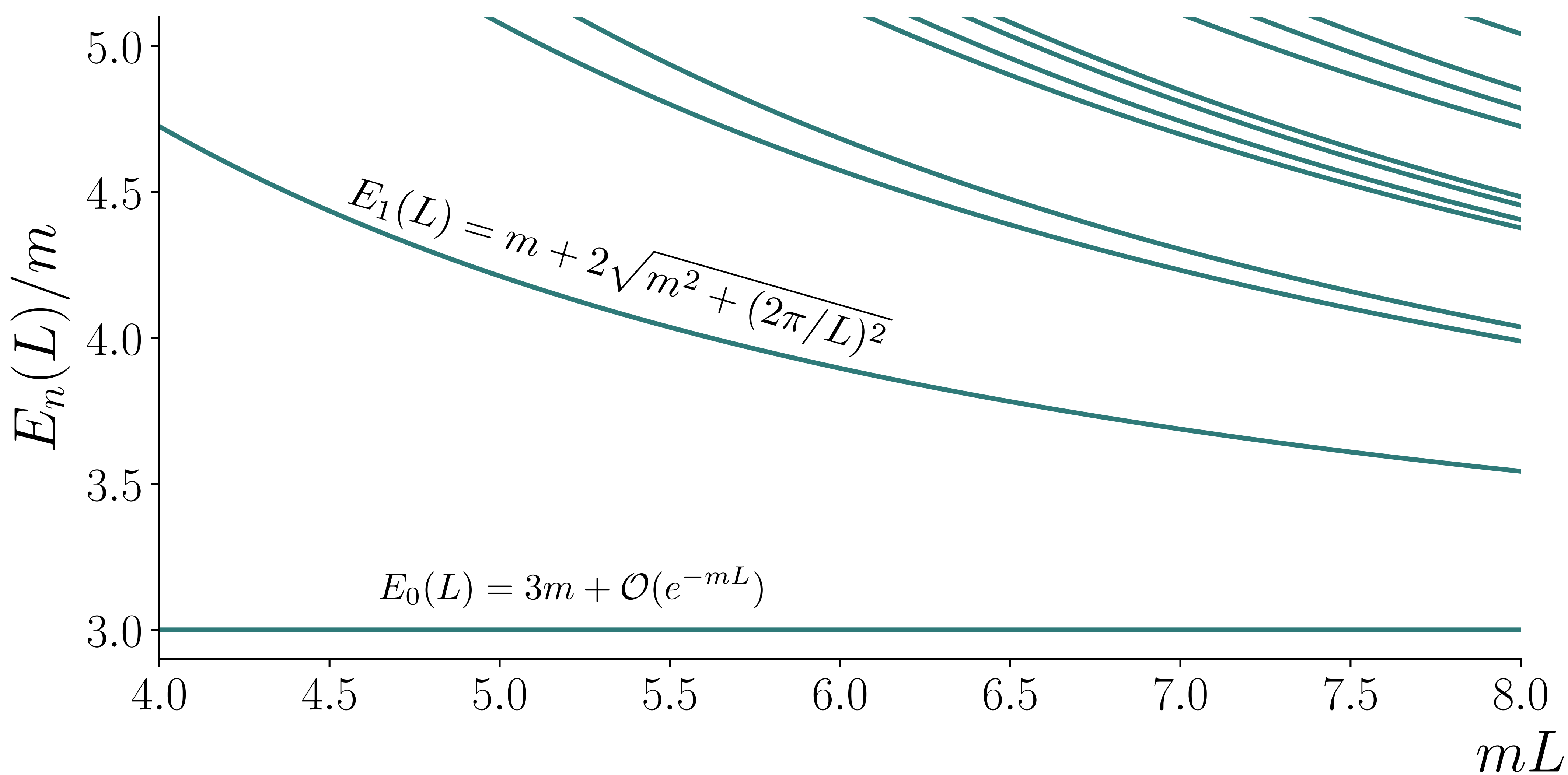
Non-interacting energies



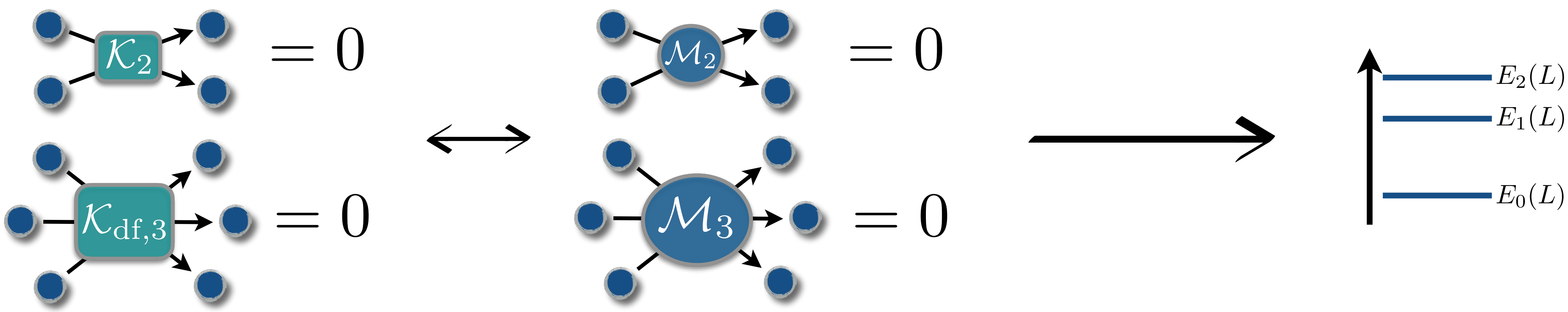


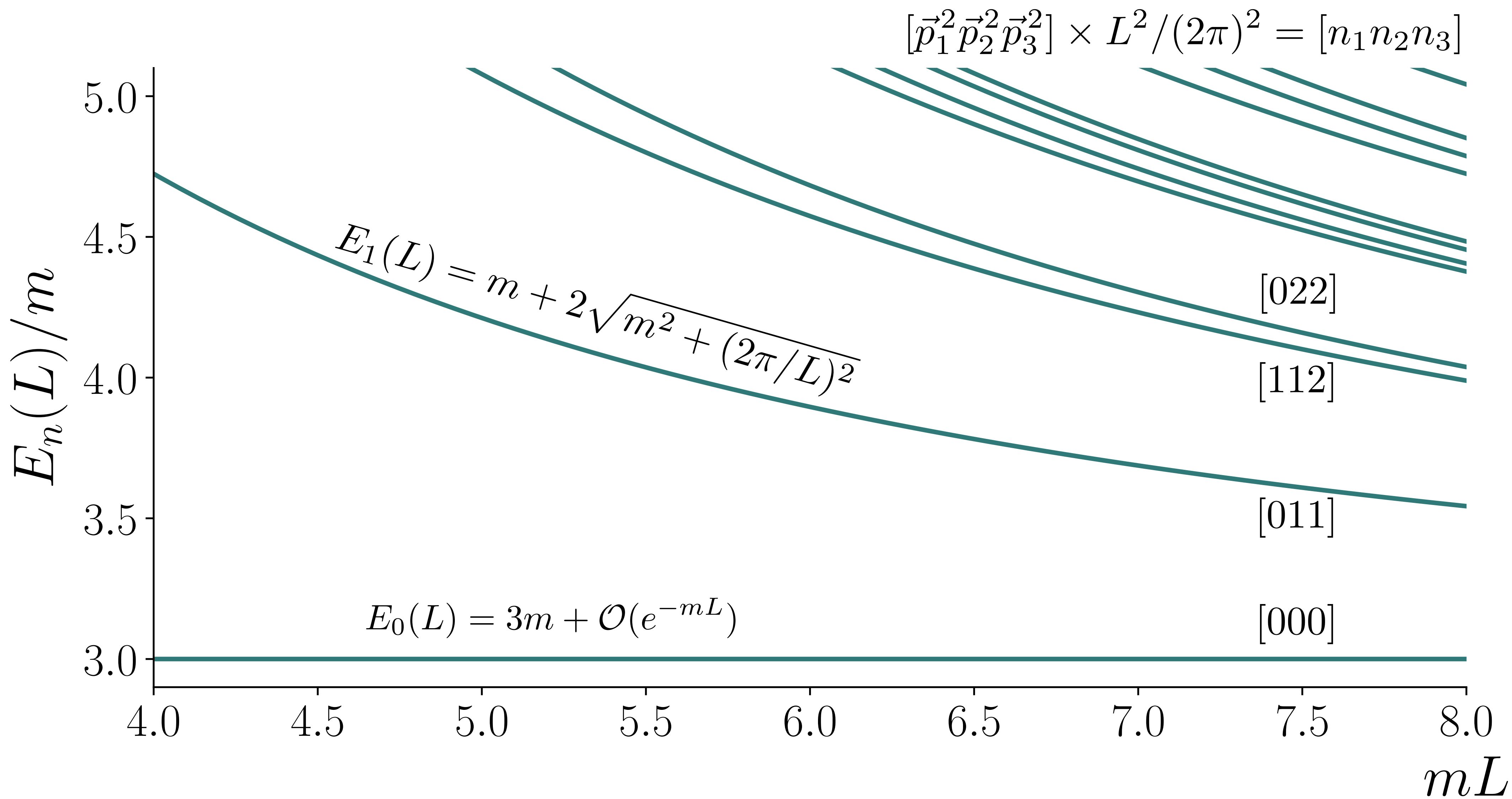
Non-interacting energies



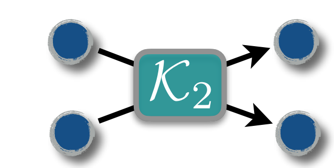


Non-interacting energies

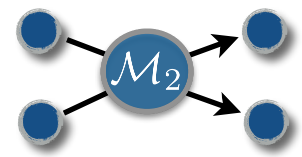




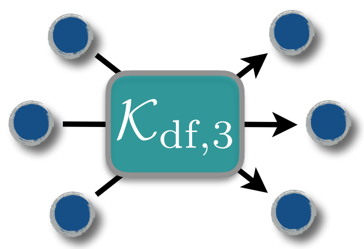
Two-particle interactions



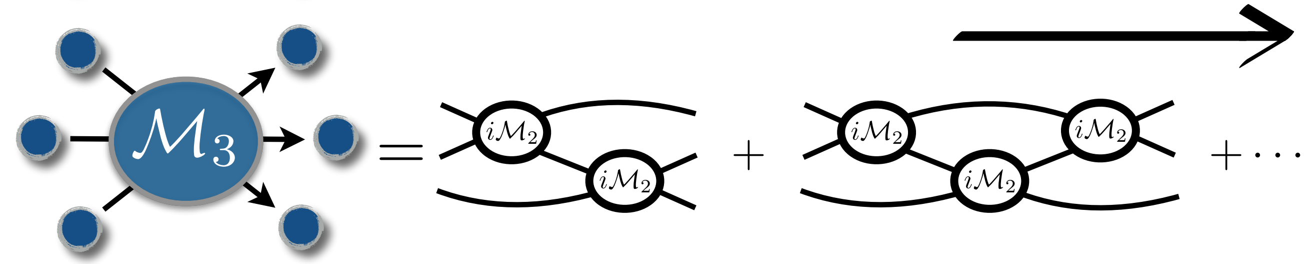
$= -16\pi\sqrt{s}\,a$

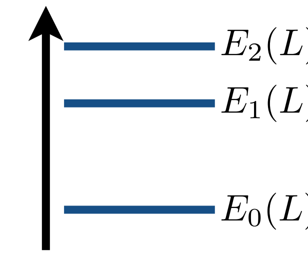


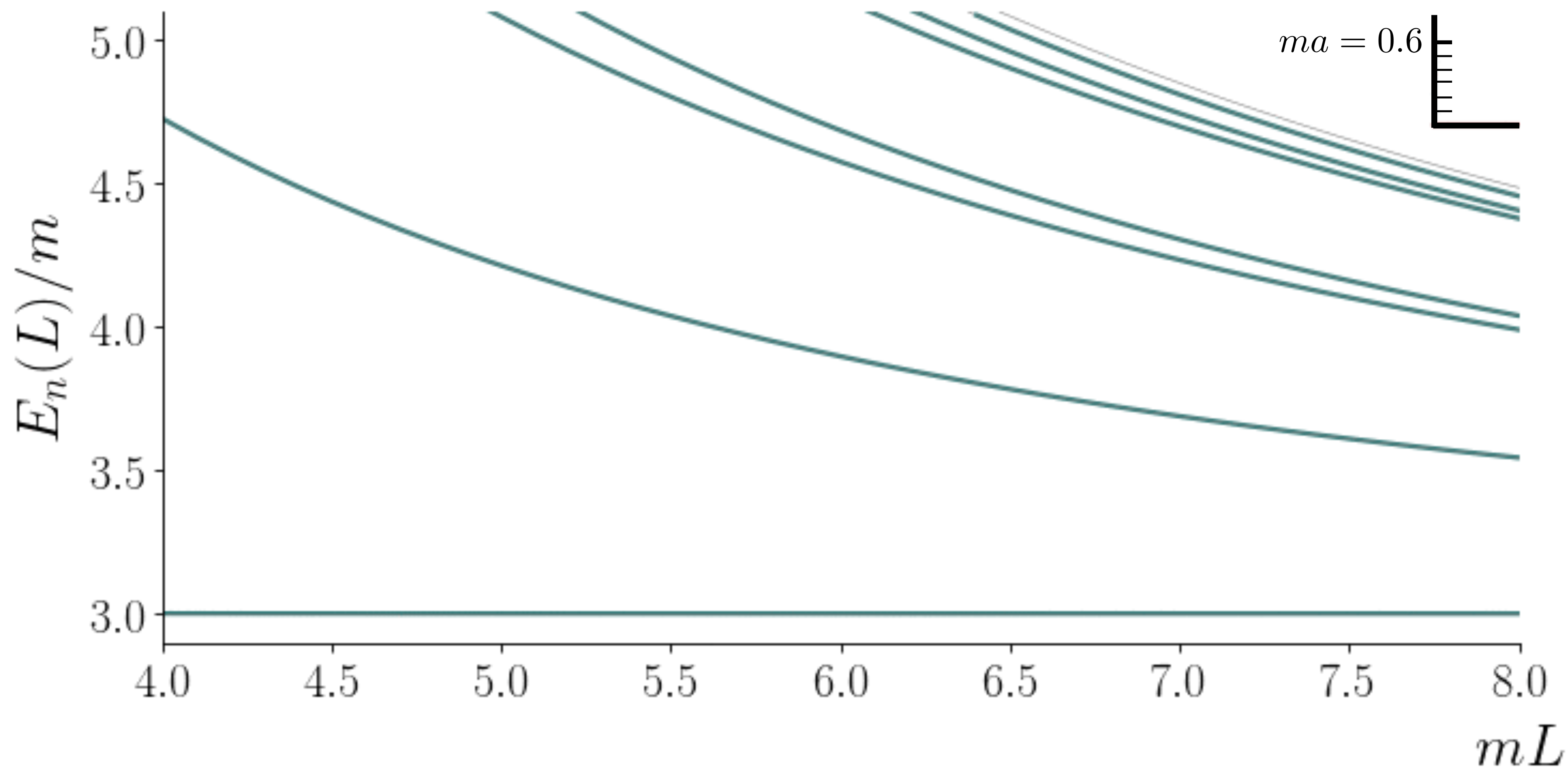
$= \dfrac{16\pi\sqrt{s}}{-1/a - ip}$



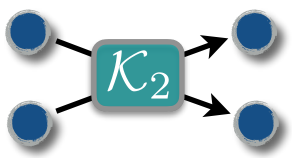
$= 0$







Two-particle interactions



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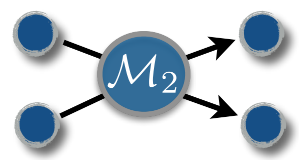

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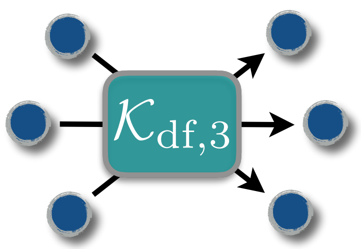
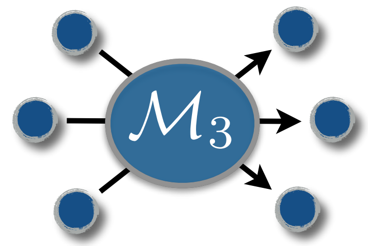
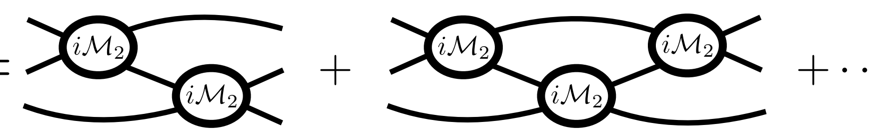
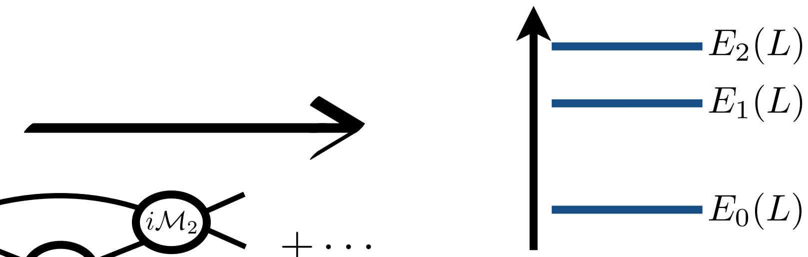


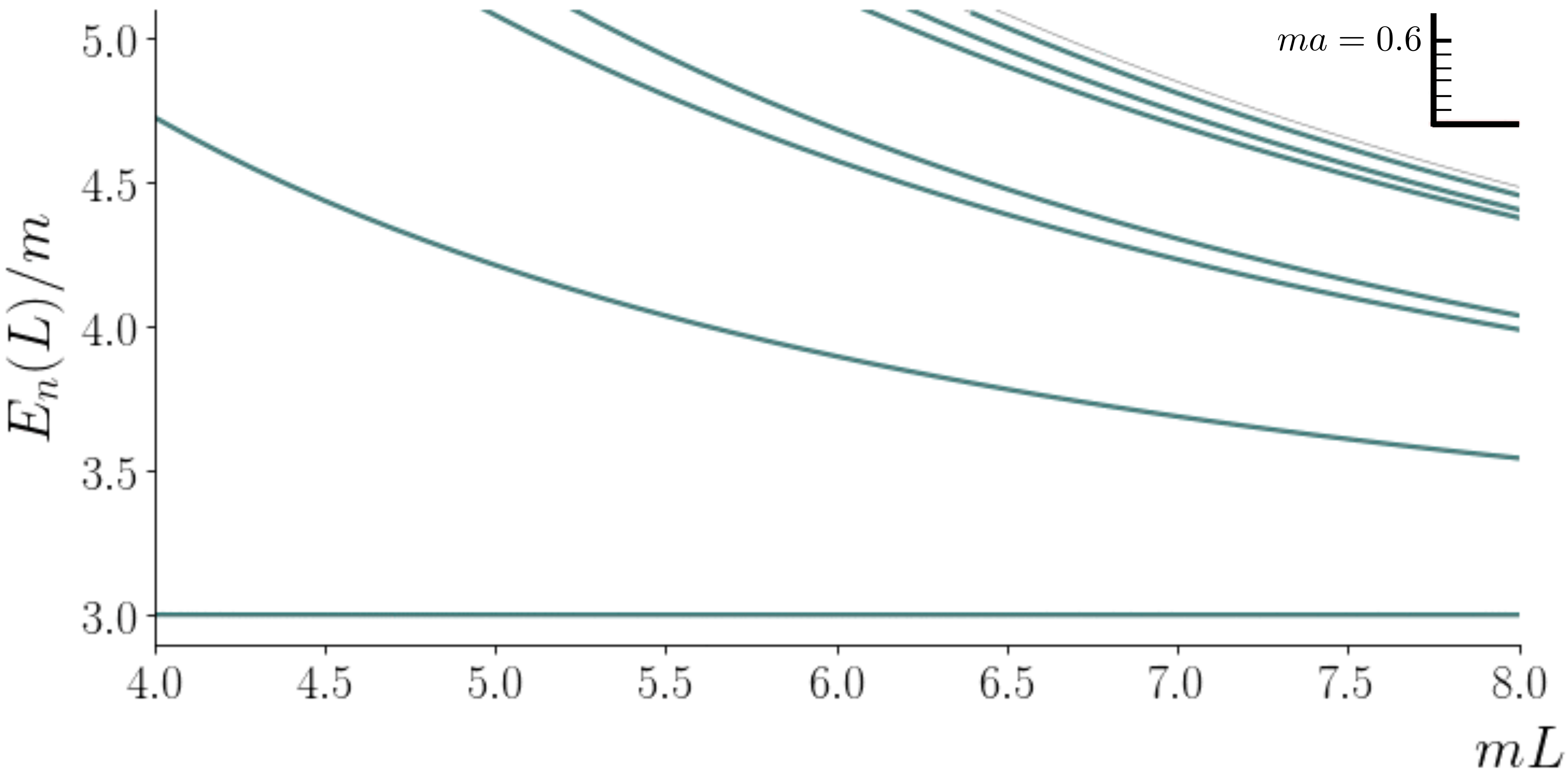
$= 0$



$=$







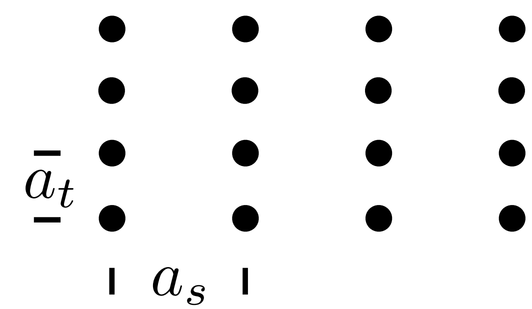
$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$ in lattice QCD

lattice details

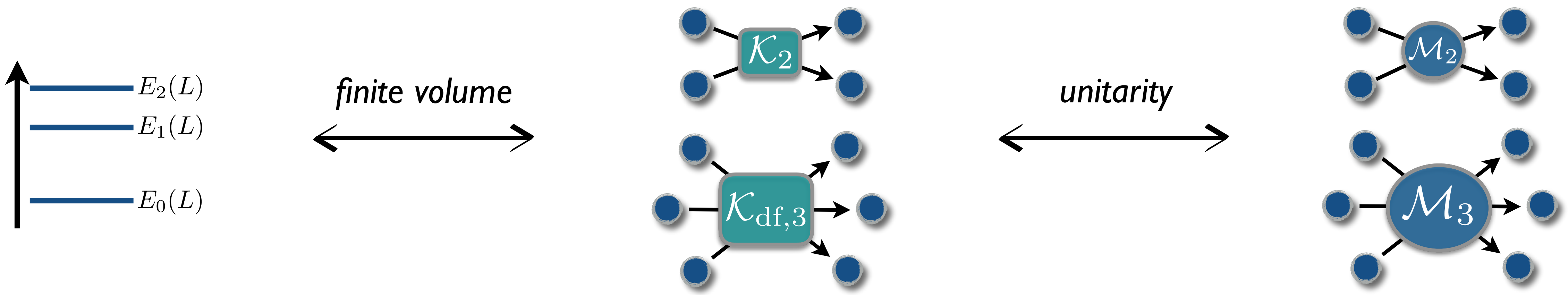
$N_f = 2+1 \qquad a_s/a_t = 3.444(6)$

$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$

$L_s/a_s = 20, 24$



❑ Workflow outline



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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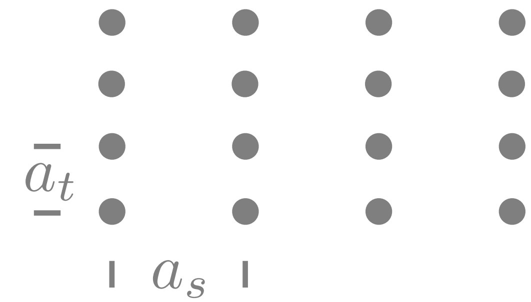
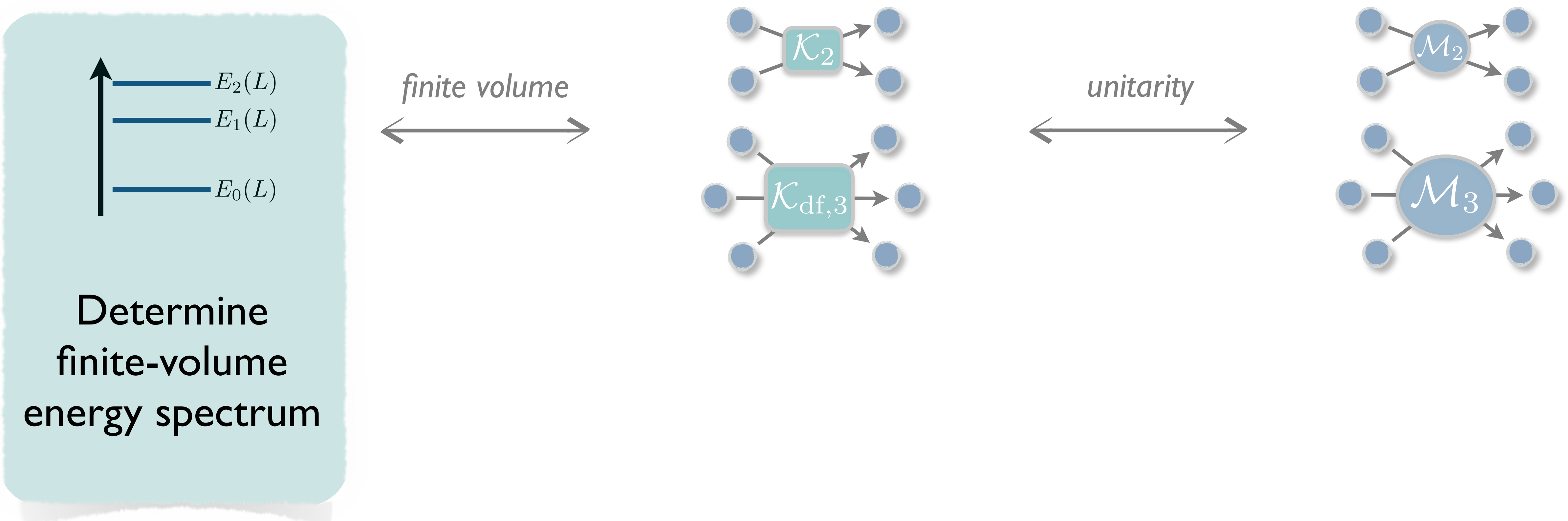
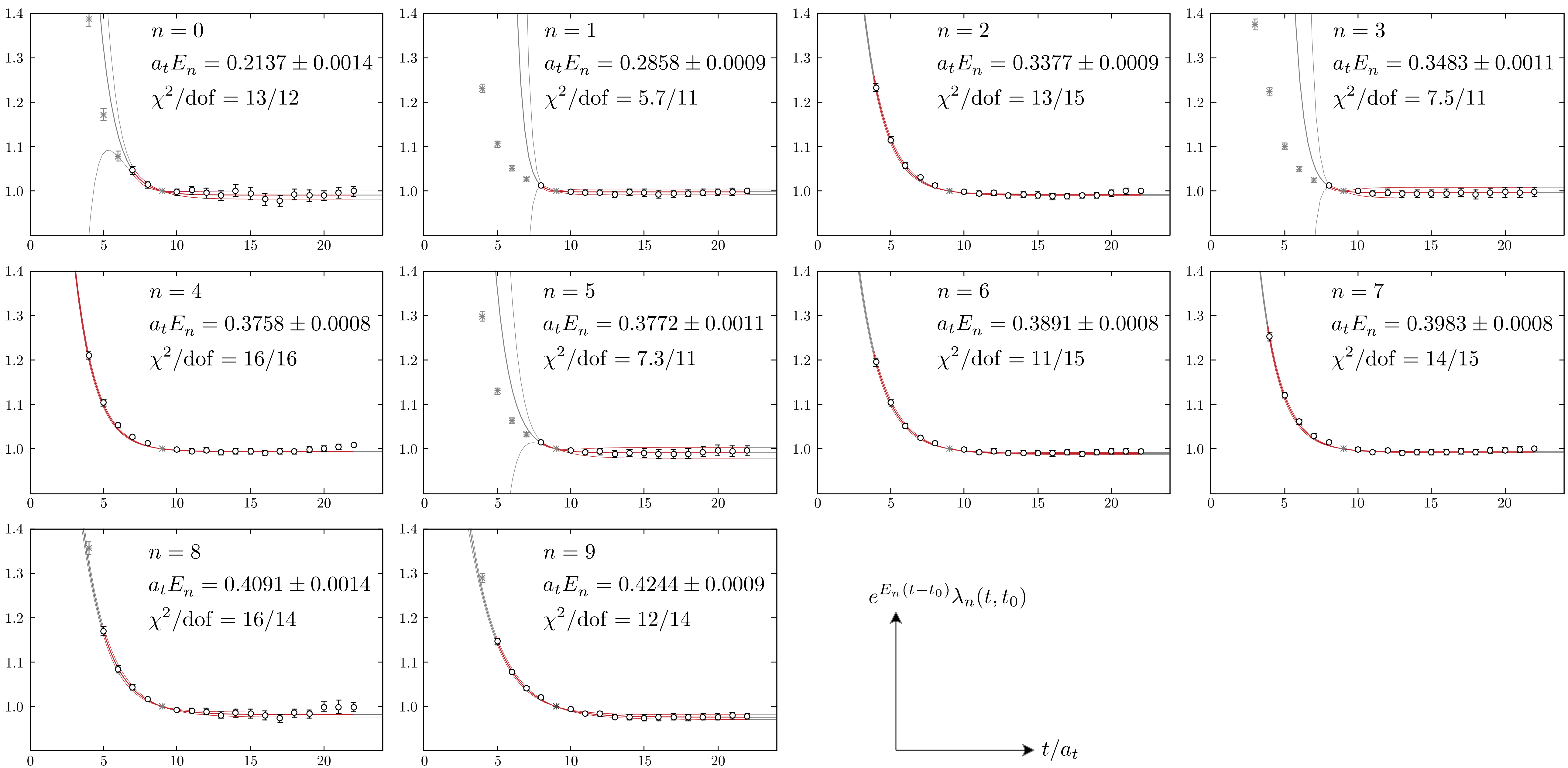


- [ ] Workflow outline



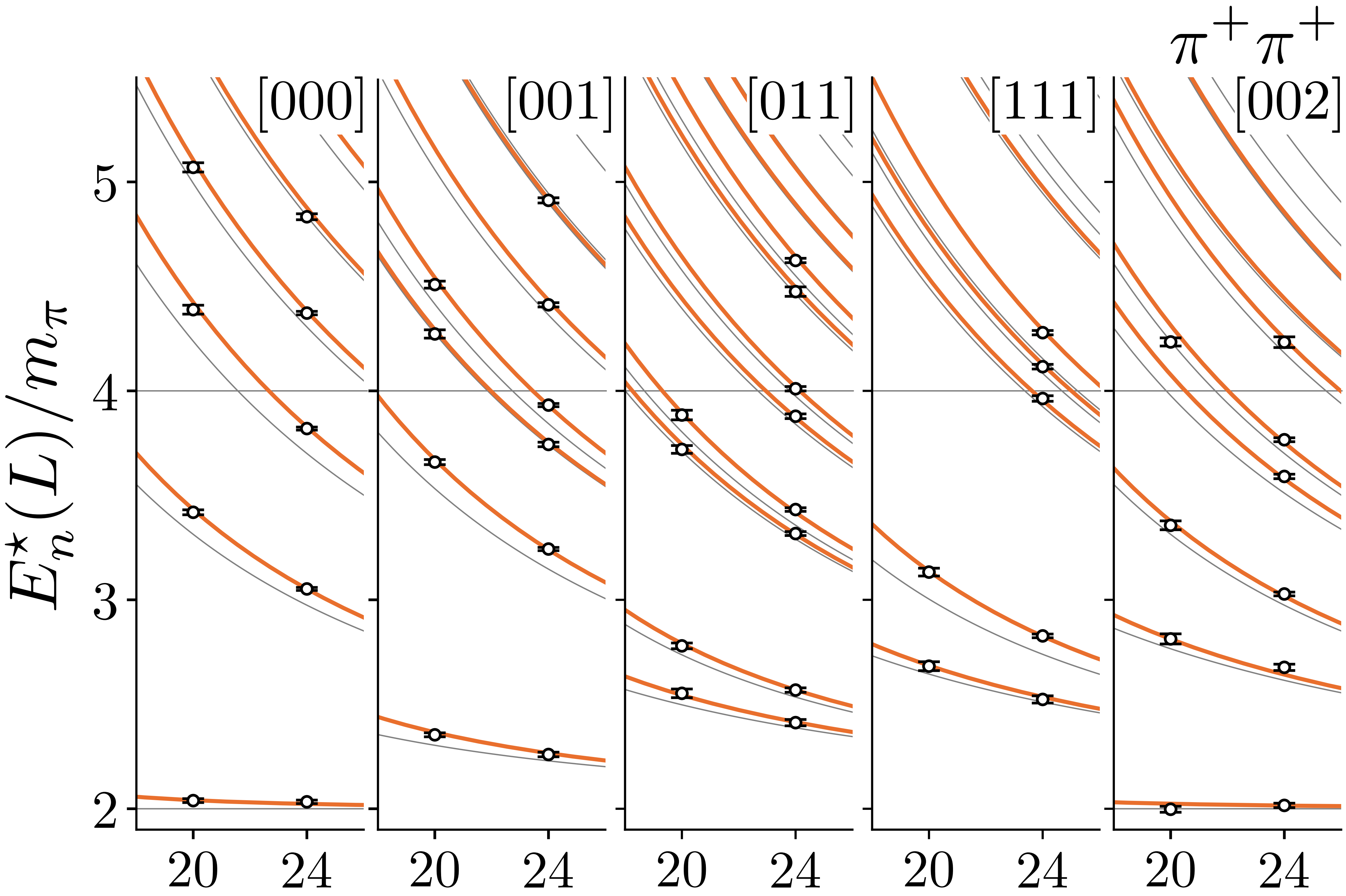
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$I = 3\,(\pi^+\pi^+\pi^+),\ \ \boldsymbol{P} = [000],\ \ \Lambda = A_1^-,\ \ L/a_s = 24$$



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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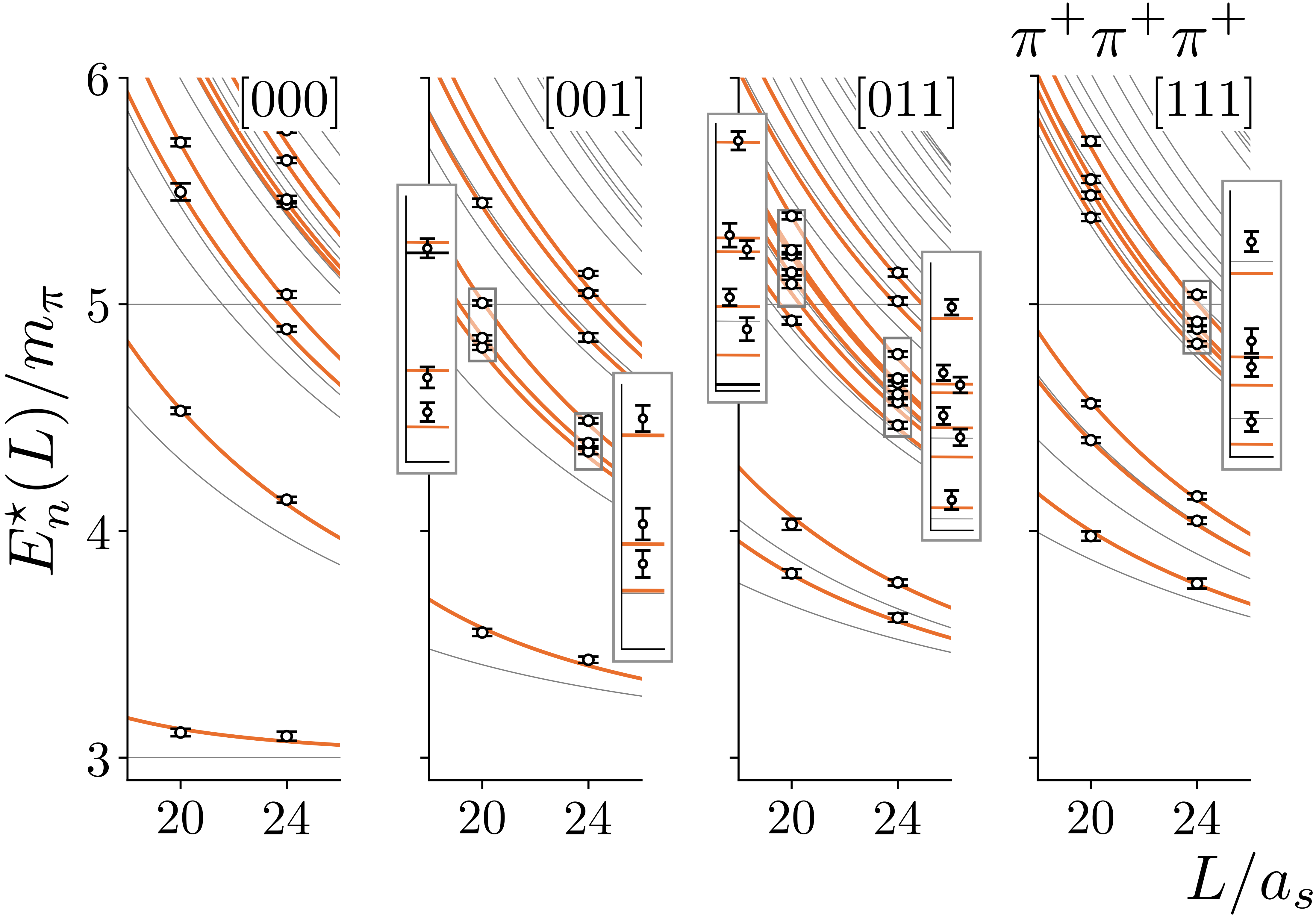

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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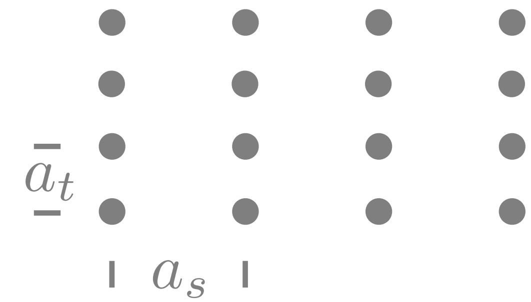
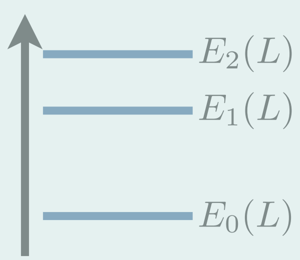


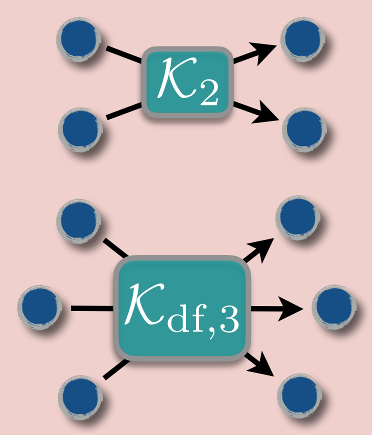
☐ Workflow outline



Determine finite-volume energy spectrum

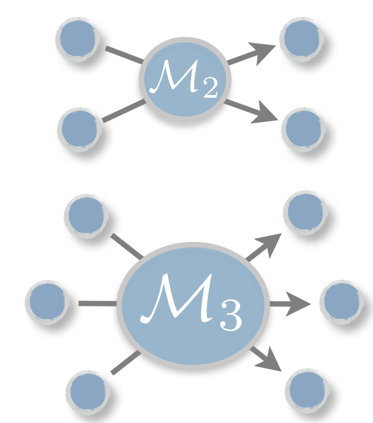
✓

finite volume



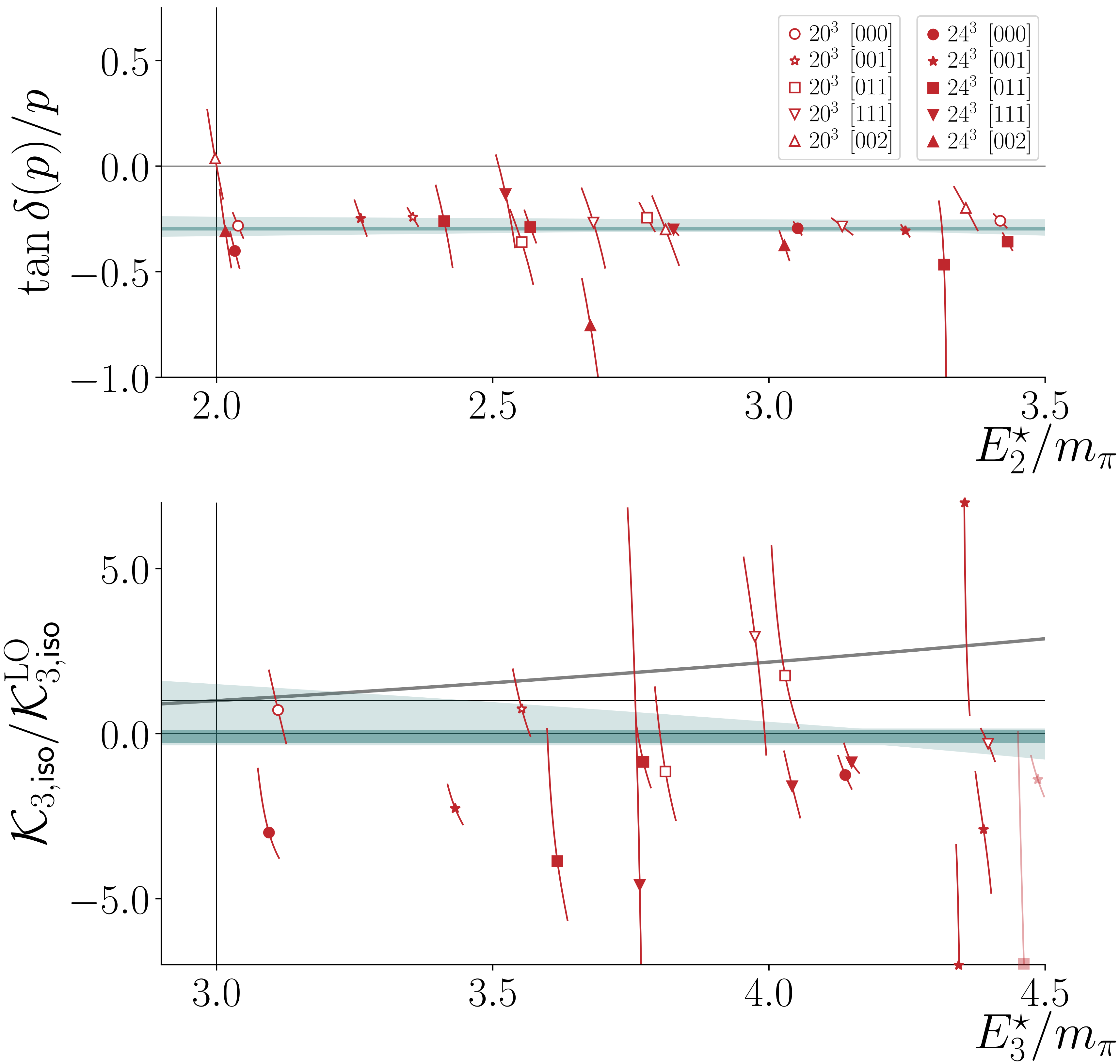
Fit to constrain two- and three-body K matrices

unitarity



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

K matrix fits



Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\mathrm{df},3}$ depending only on $E_{\mathrm{cm}} = E^\star$

Fit both two and three-body K to various polynomials

Cut on the CM energy in the fits

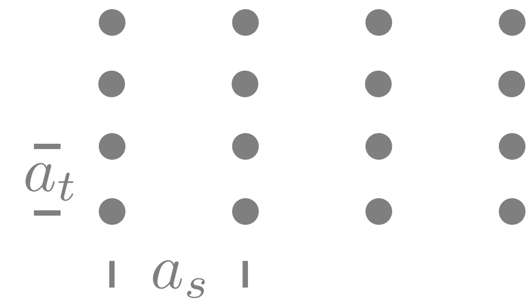
$K_{\mathrm{df},3}$ is scheme dependent (removed upon converting to $\mathcal{M}_3$)

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



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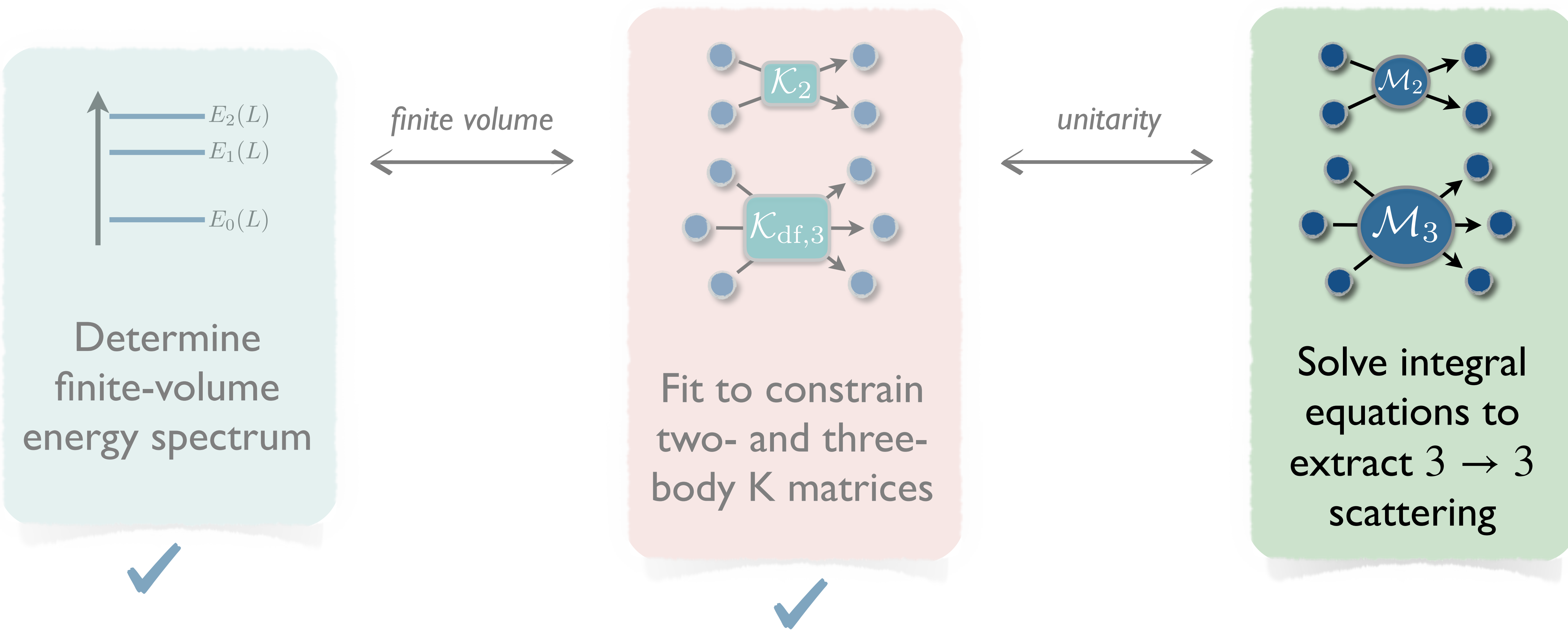

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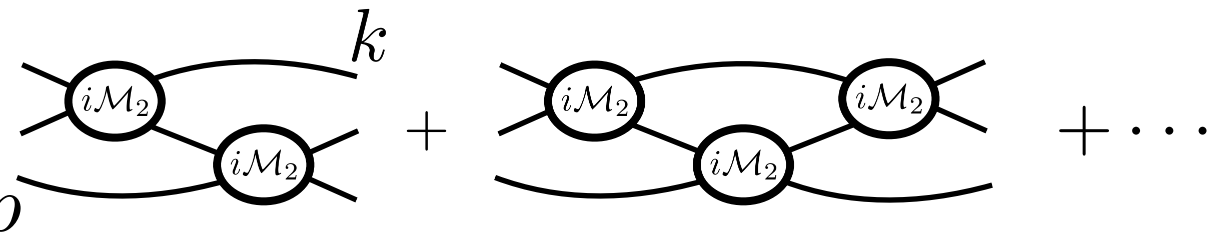
Integral equation

$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star, \boldsymbol{k})$$



Vanishes for $K_{\text{df},3} = 0$

$$\boldsymbol{D}(N,\epsilon) = -\boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}} \cdot \boldsymbol{G}(\epsilon) \cdot \boldsymbol{P} \cdot \boldsymbol{D}(N,\epsilon)$$

$$\mathcal{D}^{\mathsf{un}}(E_3^\star, \boldsymbol{p}, \boldsymbol{k}) = \lim_{\epsilon\to 0} \lim_{N\to\infty} \boldsymbol{D}_{pk}(N,\epsilon)$$

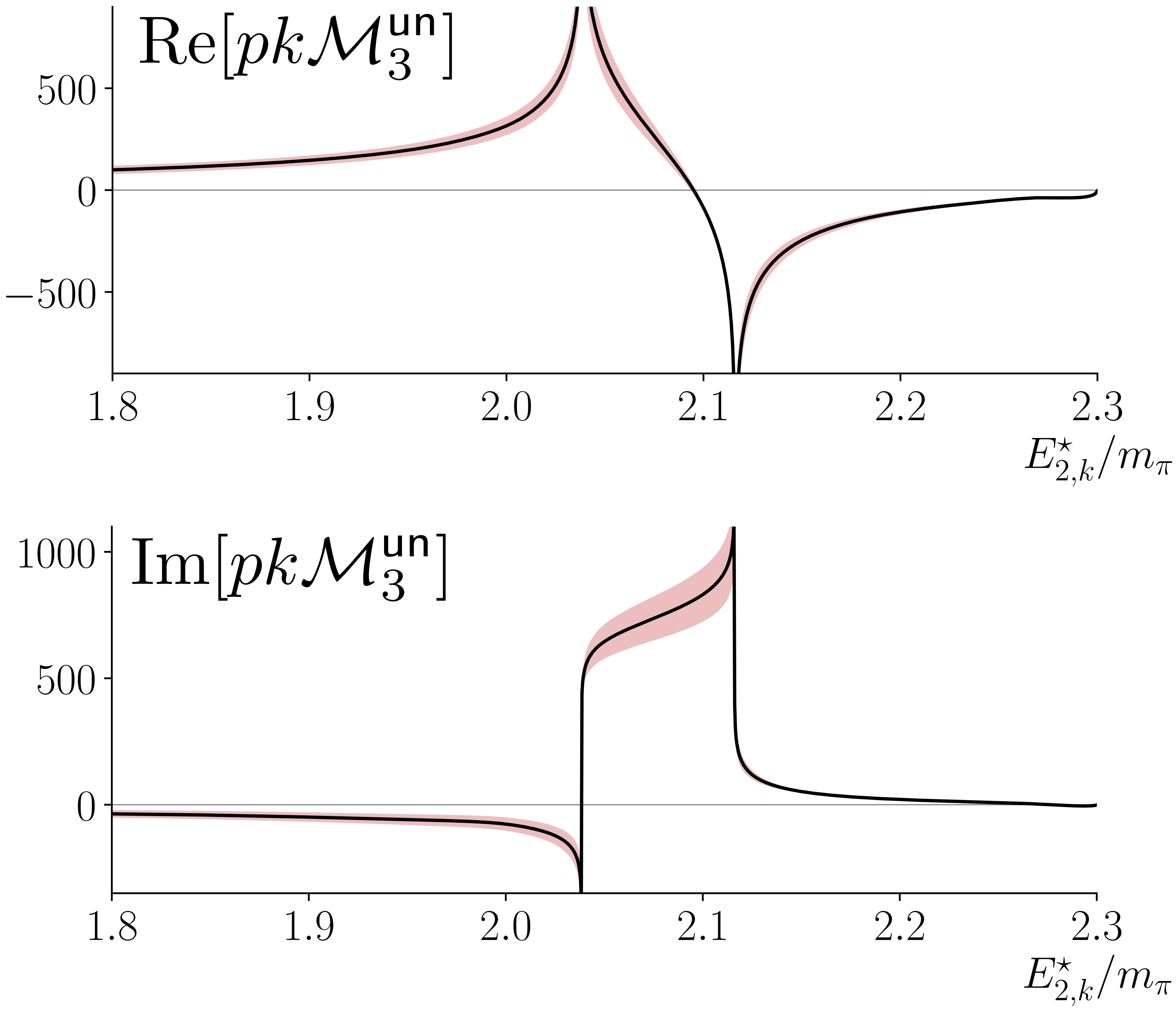
- See also…

Solving relativistic three-body integral equations in the presence of bound states

Andrew W. Jackura,[1, 2, *] Raúl A. Briceño,[1, 2, †] Sebastian M. Dawid,[3, 4, ‡] Md Habib E Islam,[2, §] and Connor McCarty[5, ¶]

arXiv: 2010.09820

Integral equation



Total angular momentum = 0

Two-particle sub-system angular momentum = 0

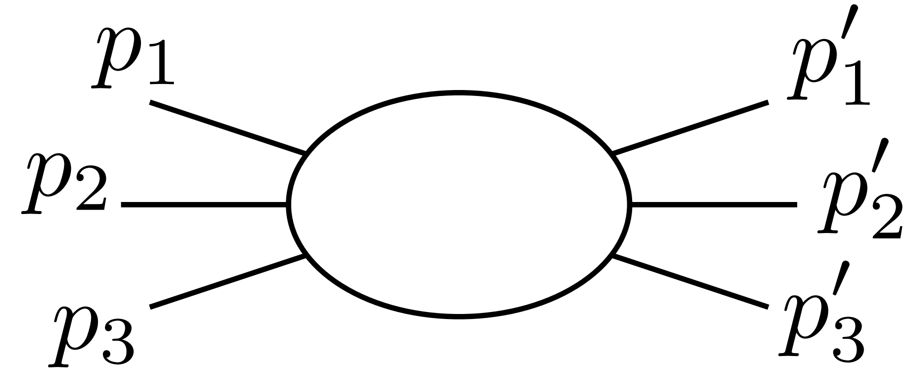
Plot at fixed $E_3^\star$ and p

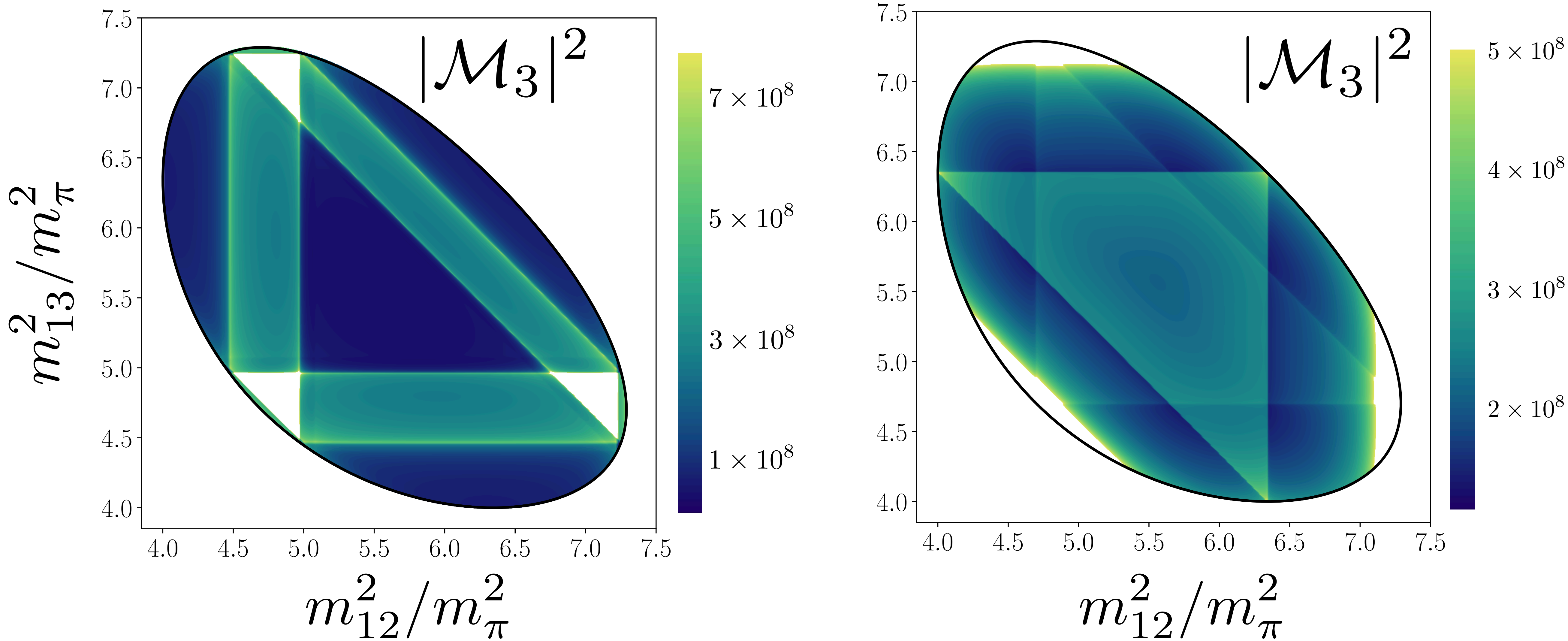
Both two- and three-body uncertainties estimated

Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j\in\{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p_i', p_j)$$





MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

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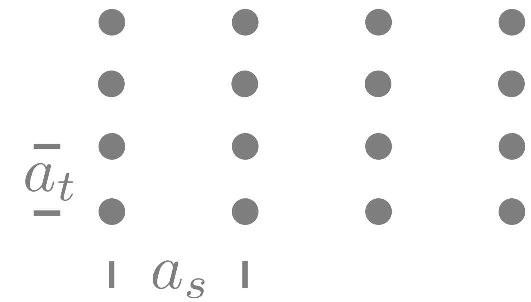
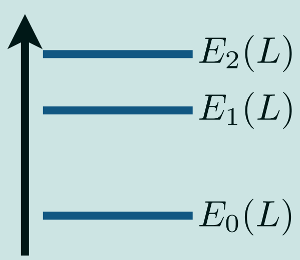
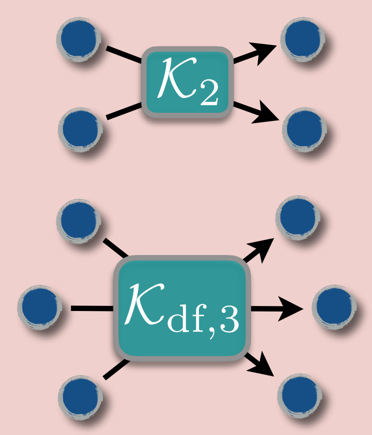


Workflow outline



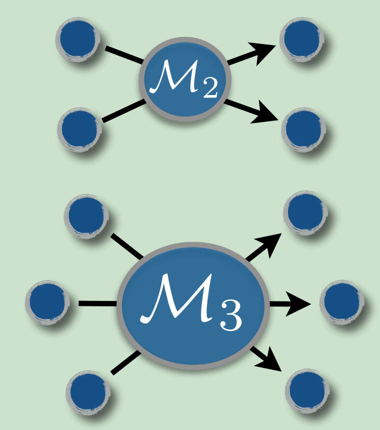
Determine finite-volume energy spectrum

finite volume ⟷



Fit to constrain two- and three-body K matrices

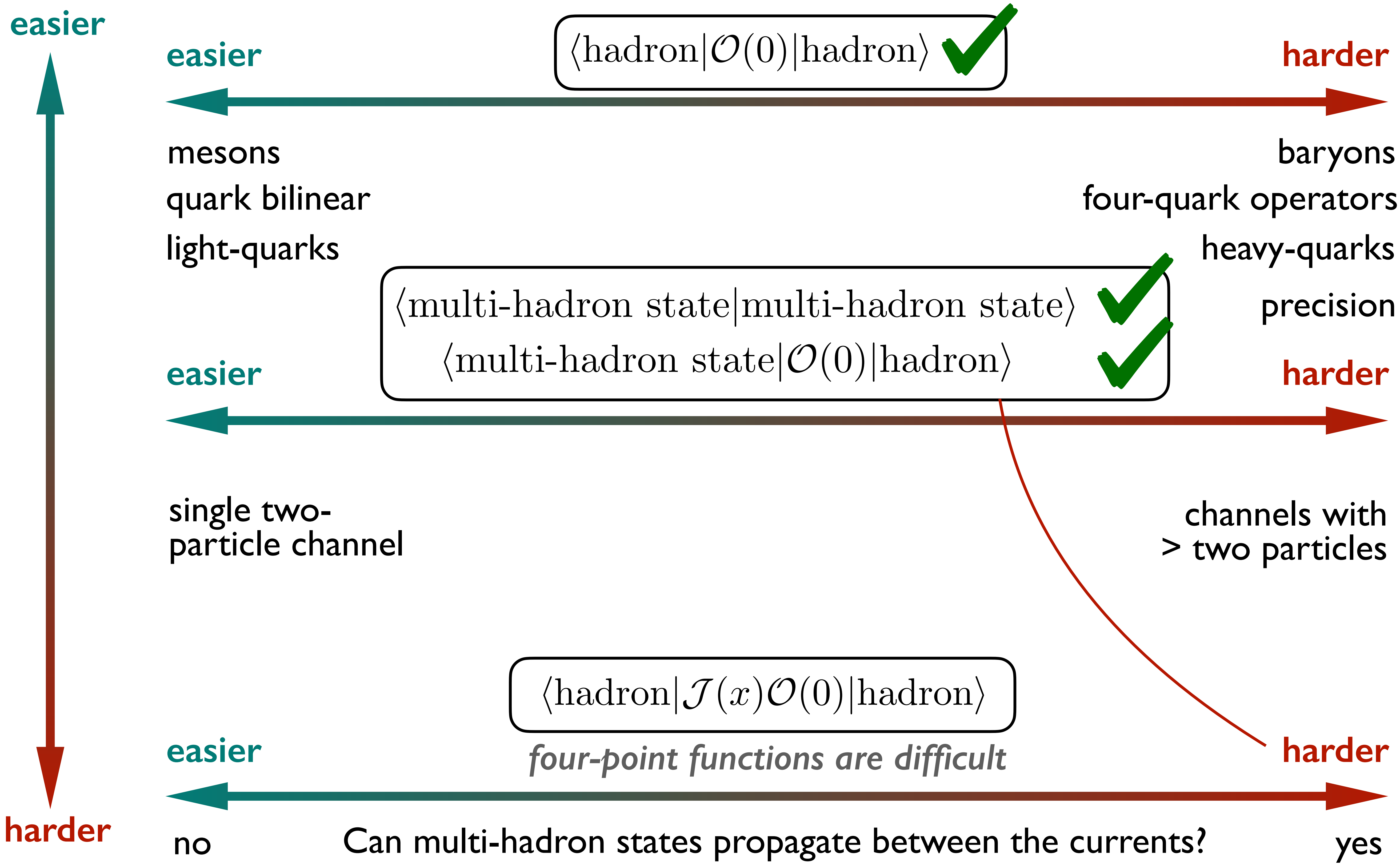
unitarity ⟷



Solve integral equations to extract $3 \to 3$ scattering

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

(Incomplete) landscape of lattice observables



Formal & numerical progress: Long-distance matrix elements

- Formal method understood… *assuming only two-hadron intermediate states*

$$\Sigma^+ \xrightarrow[H_W]{} N\pi \to p\gamma^* \qquad\qquad K^0 \xrightarrow[H_W]{} \pi\pi \xrightarrow[H_W]{} \overline{K}^0$$

  - Issue of growing exponentials *(Christ et al.)*

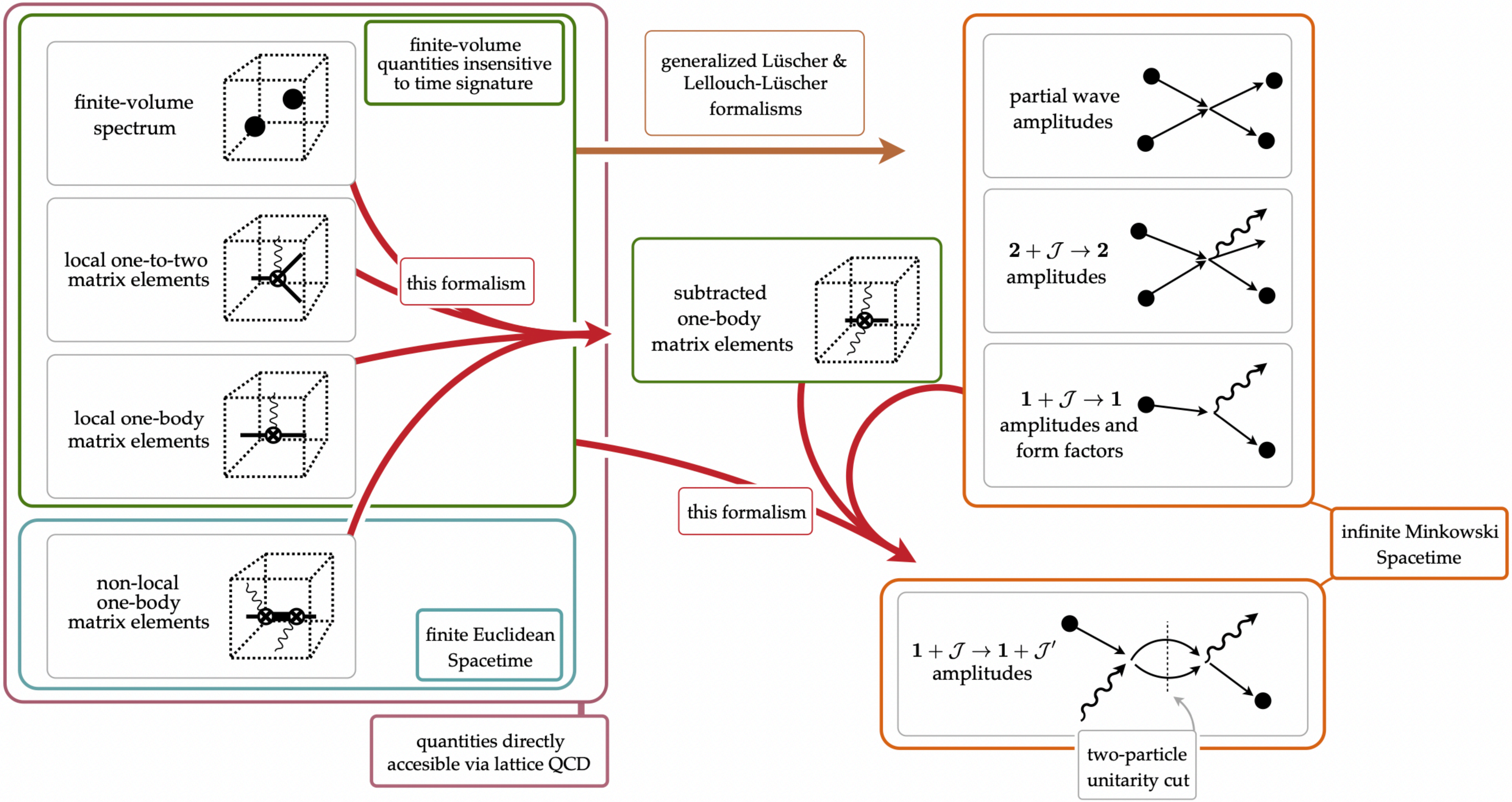
$$\langle \overline{K} | \mathcal{H}_W(0) \mathcal{H}_W(-|\tau|) | K \rangle_L = \sum_n c_n(L) e^{-(E_n(L)-M_K)|\tau|} \quad \underset{\int_{-T}^{0} d\tau}{\Longrightarrow} \quad \sum_n c_n \frac{1-e^{-(E_n-M_K)T}}{M_K - E_n}$$

  - Issue of power-like finite-volume effects (after discarding exponential)

$$F_L = \sum_n \frac{c_n}{M_K - E_n}$$

Christ, Feng, Martinelli, Sachrajda *(2015)* • Christ *et al. (2016)*
• Briceño, Davoudi, MTH, Schindler, Baroni *(2019)* • Erben, Gülpers, MTH, Hodgson, Portelli *(2022)*

Formal & numerical progress: Long-distance matrix elements



Christ, Feng, Martinelli, Sachrajda *(2015)* • Christ *et al. (2016)* • Briceño, Davoudi, MTH, Schindler, Baroni *(2019)* • Erben, Gülpers, MTH, Hodgson, Portelli *(2022)*

Not discussed here

- Three-hadron transitions ($K \to \pi\pi\pi, \quad \gamma^\star \to \pi\pi\pi$)

  finite-volume methods exist

- Left-hand branch cuts

  finite-volume methods break on left-hand cuts (e.g. T_{cc}^+)

- Spectral densities from regulated inverse Laplace transform

- Resonance form factors ($\pi\pi \to \pi\pi\gamma$)

Conclusions

- LQCD is in the era of 'rigorous resonance spectroscopy'
- The finite-volume = ***a useful tool***
- Challenges and progress

  formal analysis was technical → ***ground work is now set***

  scattering demands high precision excited states → ***advanced algorithms make this possible***

  3-body amplitude is highly singular → ***intermediate K matrix is not***

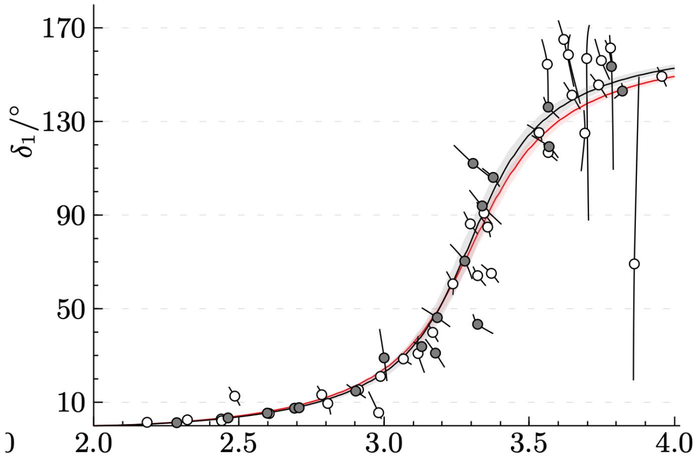
- Next steps…

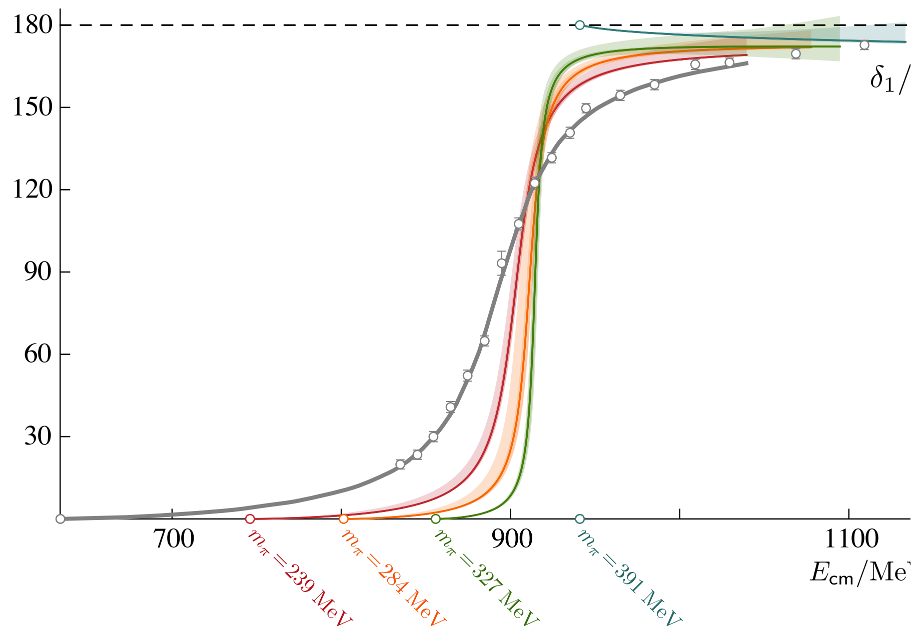
  complete 3-particle formalism → *extend to N-particle formalism*

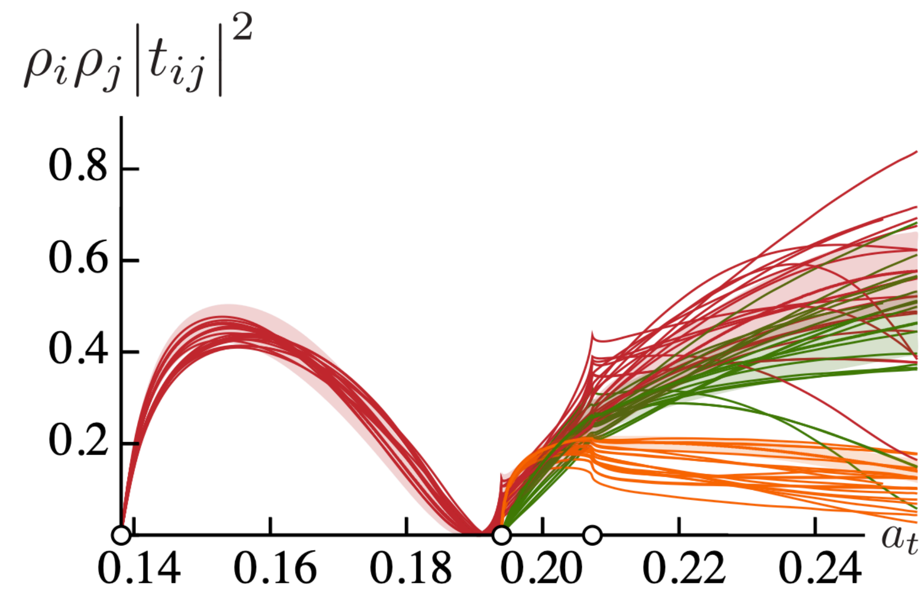
  extend studies involving an external current

  push more channels into the precision regime

Big Picture

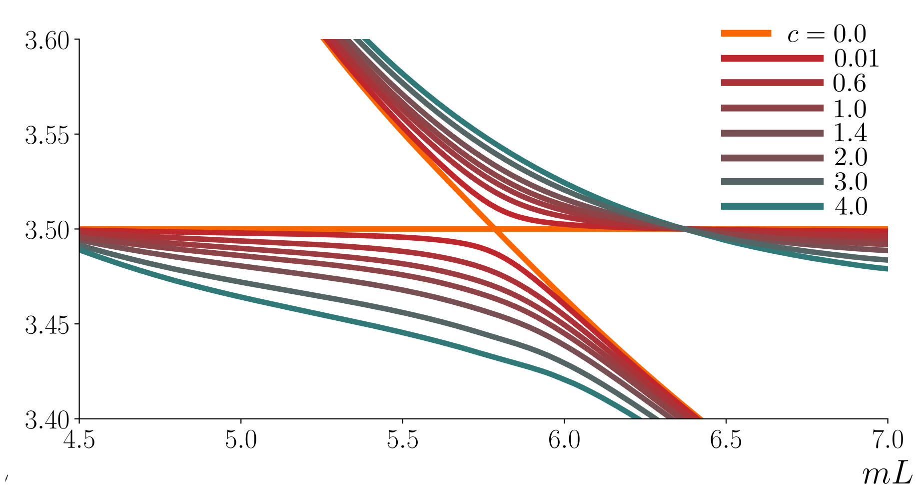


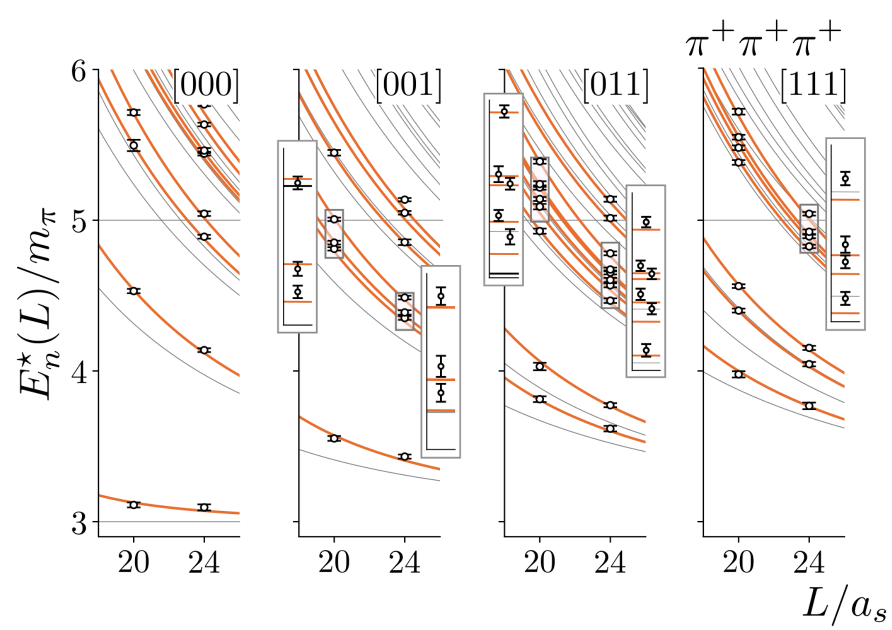


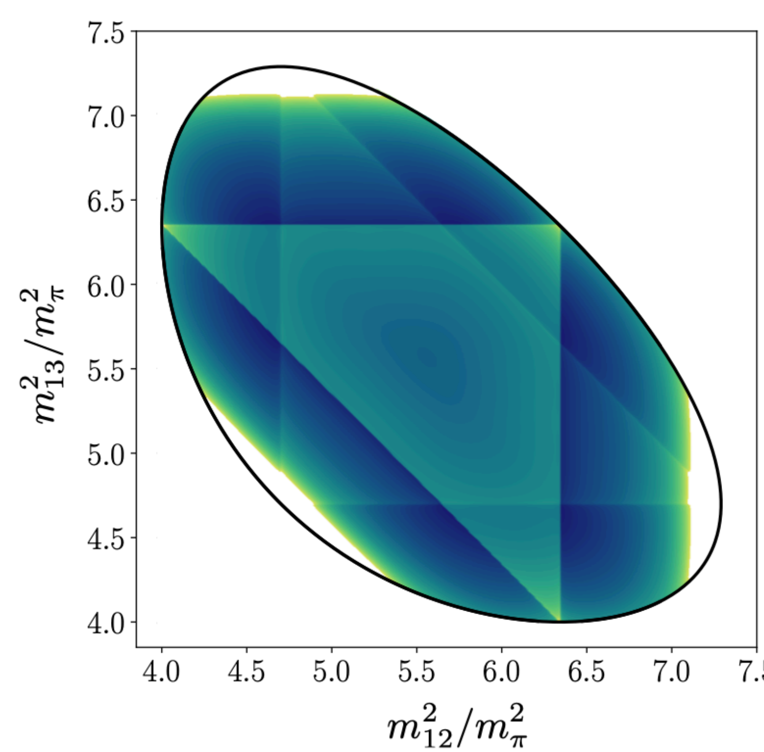


Stay tuned, and…

Thanks for listening!



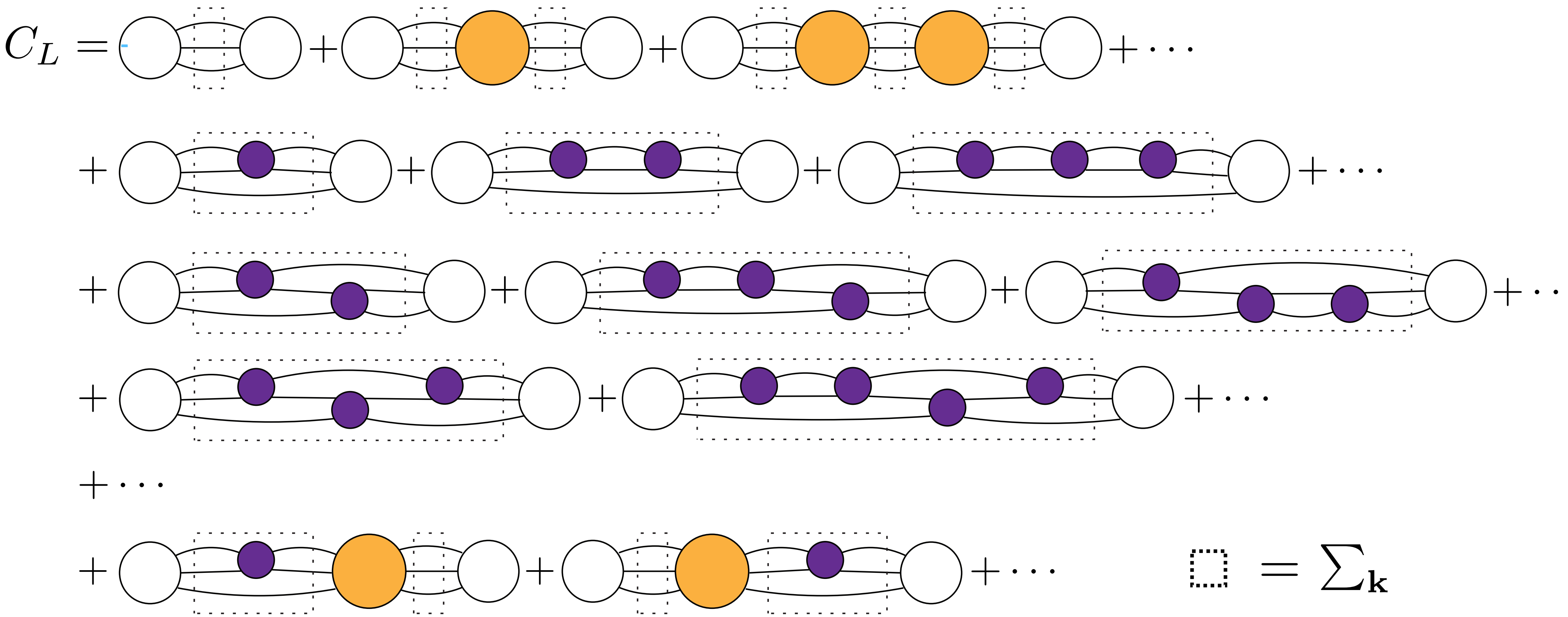


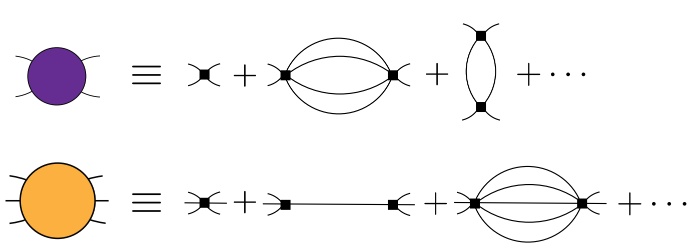


Back-up slides

3-particle derivation

- Study 3-body correlator in an ***all-orders skeleton expansion***

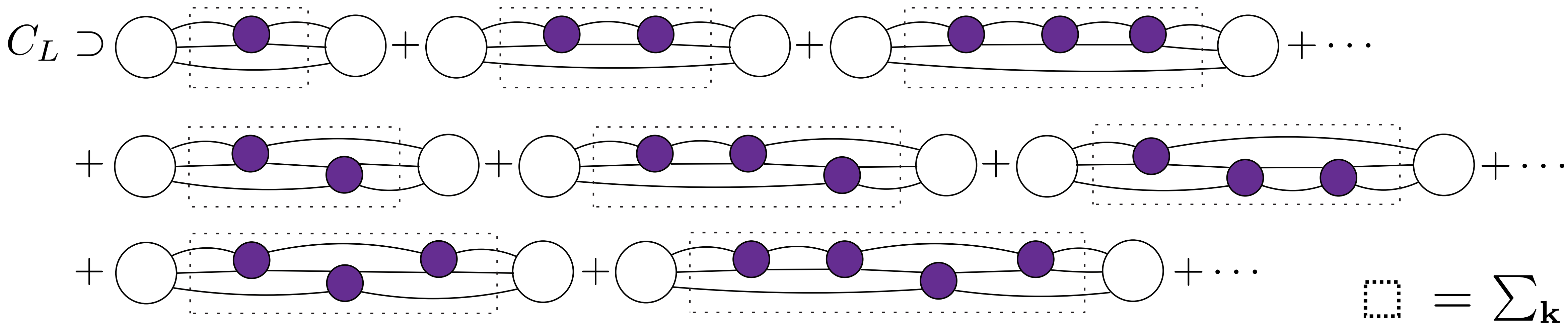




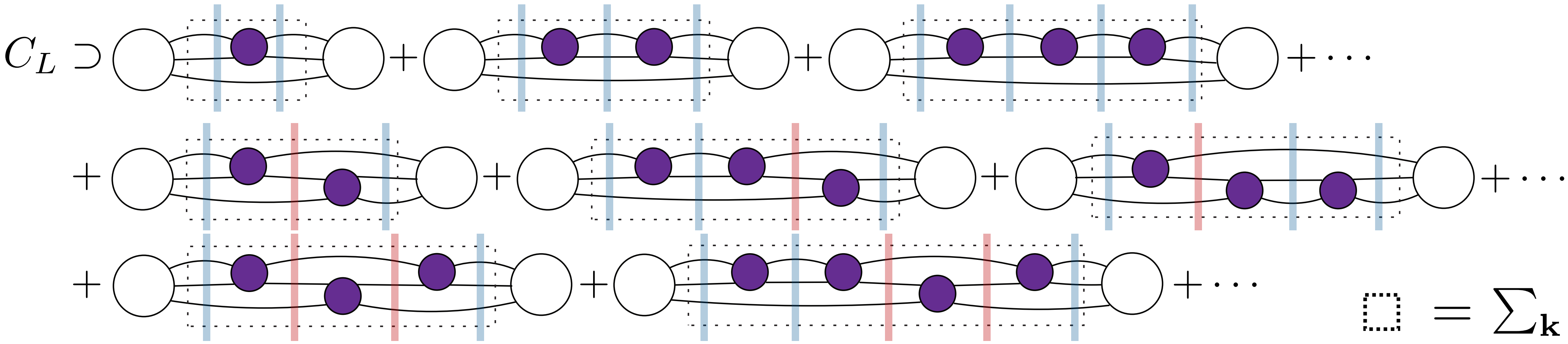
kernels have suppressed *L* dependence

lines = fully dressed hadrons

Two types of cuts



Two types of cuts



$$\mathbf{A}'_3\,\mathbf{F}\,\mathbf{K}_2\,\mathbf{F}\,\mathbf{A}_3 + \mathbf{A}'_3\,\mathbf{F}\,[\mathbf{K}_2\,\mathbf{F}]^2\,\mathbf{A}_3 + \mathbf{A}'_3\,\mathbf{F}\,[\mathbf{K}_2\,\mathbf{F}]^3\,\mathbf{A}_3 + \cdots = \mathbf{A}'_3\,\mathbf{F}\,\frac{1}{1-\mathbf{K}_2\mathbf{F}}\,\mathbf{K}_2\mathbf{F}\,\mathbf{A}_3$$

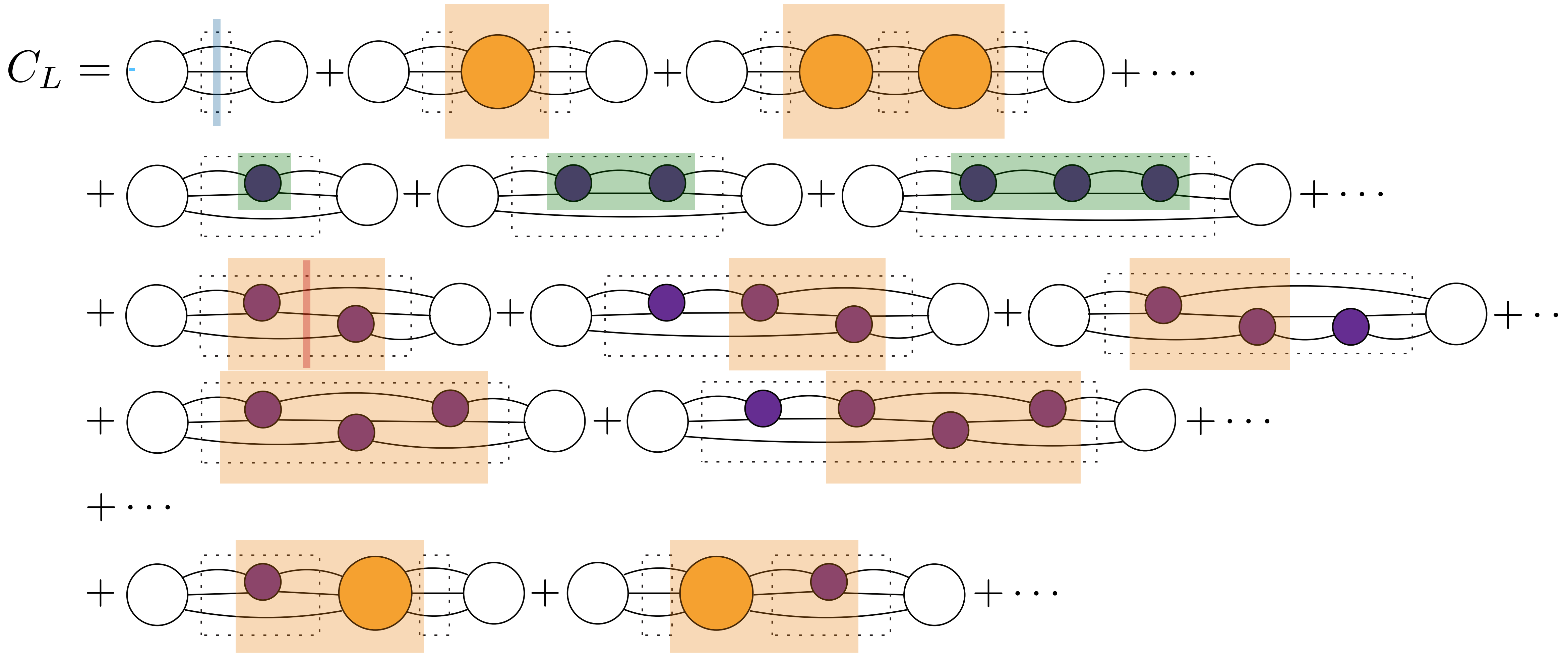
$$\mathbf{A}'_3\,\mathbf{F}\,\mathbf{K}_2\,\mathbf{G}\,\mathbf{K}_2\,\mathbf{F}\,\mathbf{A}_3 + \cdots$$

have not yet considered entire diagram contributions

missing contributions from ***off-shellness***

missing smooth terms (short-distance parts)

Short-distance parts & summation



$$C_L - C_\infty = \mathbf{A}_3' \, \mathbf{F}_{33} \, \mathbf{A}_3 + \mathbf{A}_3' \, \mathbf{F}_{33} \, \mathbf{K}_{\mathrm{df},3} \, \mathbf{F}_{33} \, \mathbf{A}_3 + \cdots$$

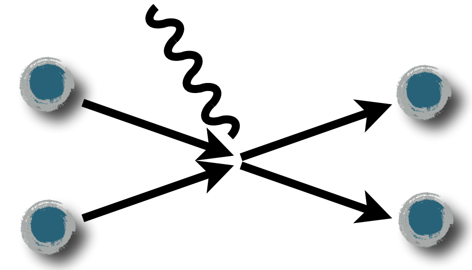
$$= \mathbf{A}_3' \frac{1}{\mathbf{F}_{33}^{-1} + \mathbf{K}_{\mathrm{df},3}} \mathbf{A}_3$$

$$\mathbf{F}_{33} \equiv \frac{1}{3}\mathbf{F} + \mathbf{F}\,\mathbf{K}_2 \frac{1}{1 - (\mathbf{F} + \mathbf{G})\,\mathbf{K}_2} \mathbf{F}$$

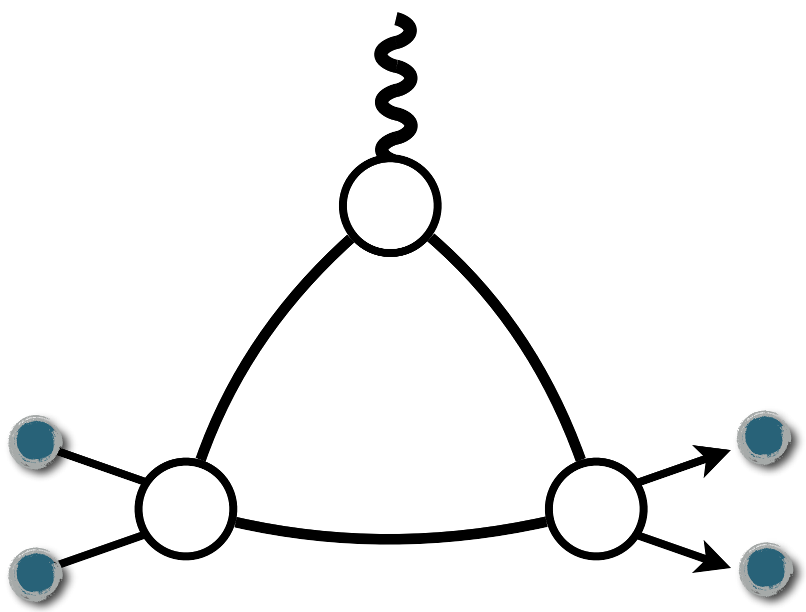
no term left behind

$2 + \mathscr{J} \to 2$

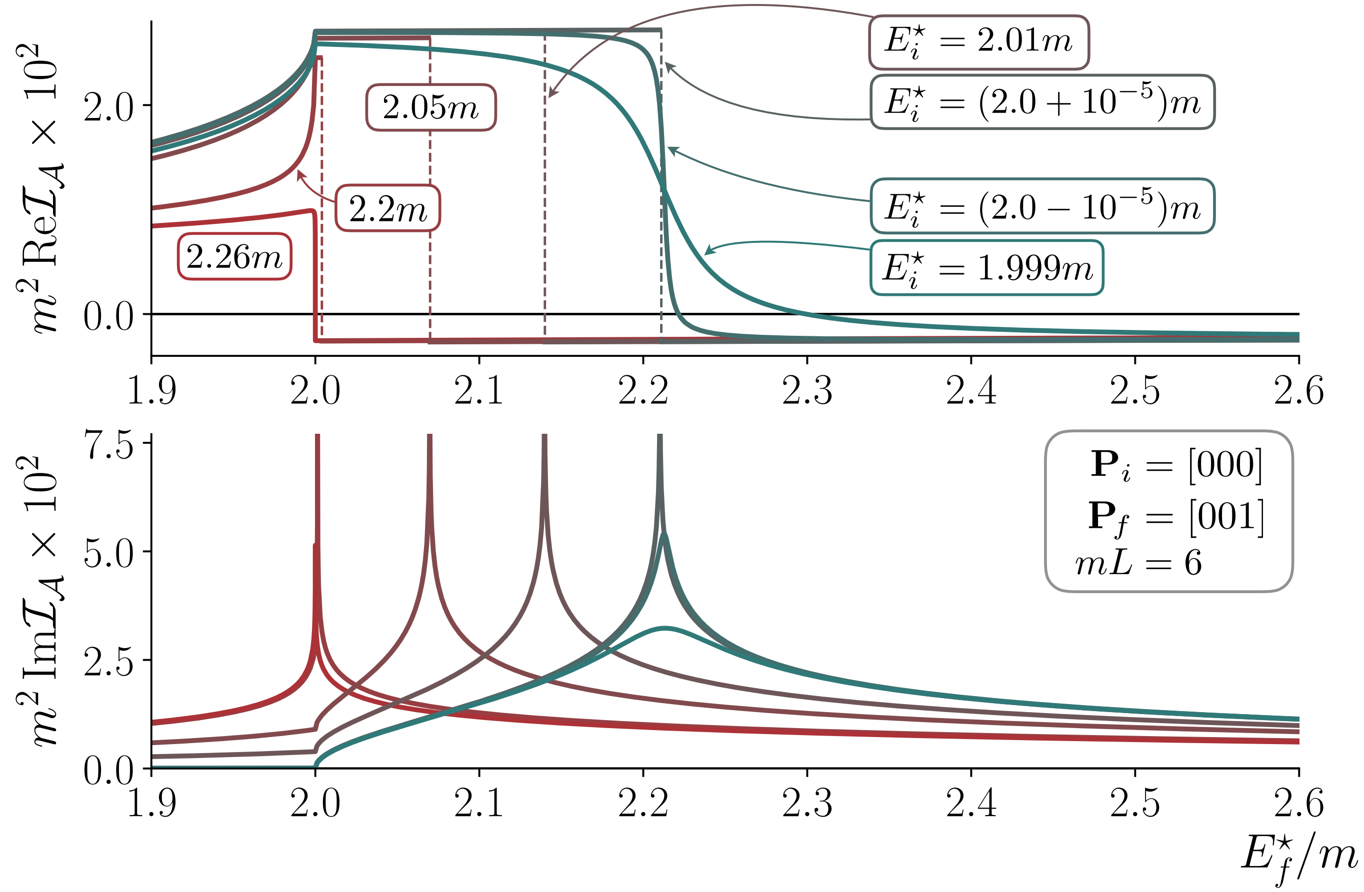
- Formalism for multi-hadron form factors

$$\langle \pi\pi, \mathrm{out} | \mathcal{J}_\mu | \pi\pi, \mathrm{in} \rangle \equiv$$ 

- Continuation to the pole → ***resonance form factors***
- Must carefully treat ***triangle*** singularities

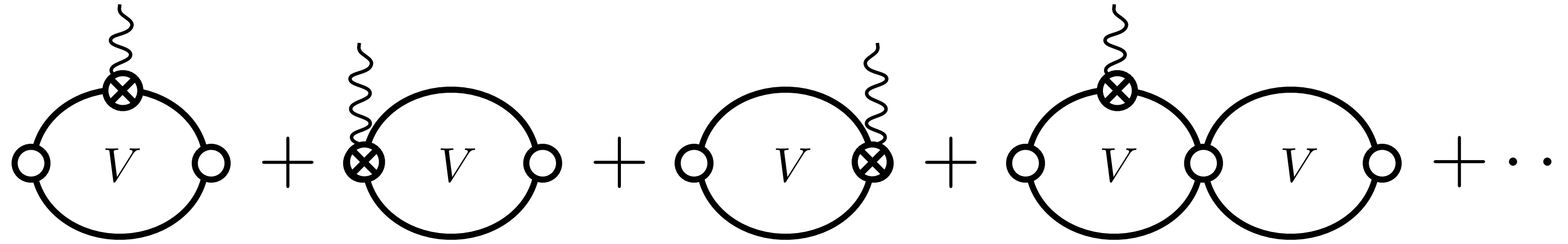


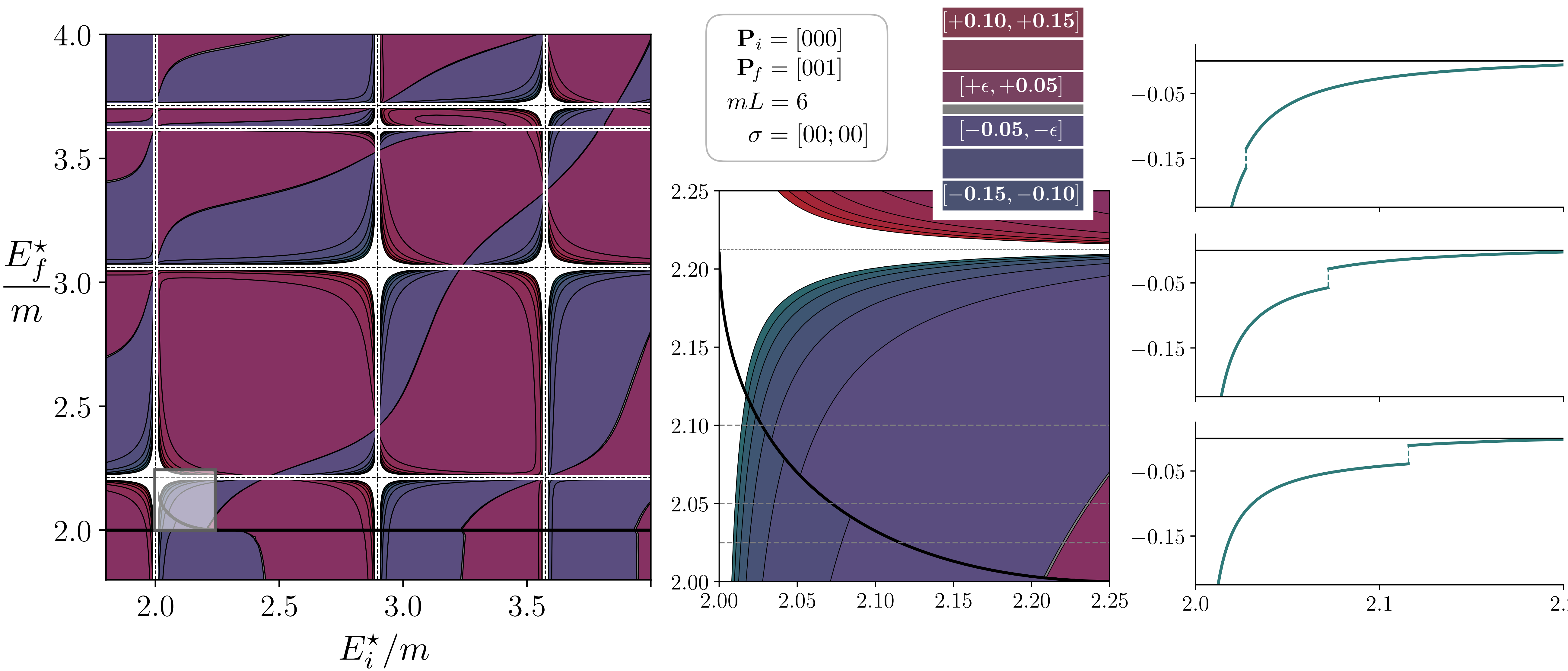
$\sim$



Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

$2 + \mathcal{J} \to 2$

$C_L^{2\to 2}(P_f, P_i) =$ 



Briceño, MTH (2017) • Baroni, Briceño, MTH, Ortega-Gama (2018) • Briceño, MTH, Jackura (2020)

Two strategies…

- ☑ Finite-volume as a tool
  - LQCD → Energies and matrix elements

$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

  - Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to ***experimental observables***
  - Applicable only in limited energy range for two- and three-hadron states

- ☐ Spectral function method
  - Formally applies for any number of particles / any energy range
  - An answer to the question… "Can't you just analytically continue?"
  - Still important challenges and limitations to consider

Two strategies…

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Correlation functions → observables

- Lattice QCD gives finite-volume Euclidean correlators

$$\langle 0 | \, \mathcal{O}_1(0) \, e^{-\hat{H}\tau} \, \mathcal{O}_2(0) \, | 0 \rangle_L$$ have

- Complete physical information is contained in…

$$\langle 0 | \, \mathcal{O}_1(0) \, f(\hat{H}) \, \mathcal{O}_2(0) \, | 0 \rangle_\infty$$ want

Correlation functions → observables

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$$\langle 0|\, \mathcal{O}_1(0)\, e^{-\hat{H}\tau}\, \mathcal{O}_2(0)\, |0\rangle_L$$ have

- Complete physical information is contained in…

$$\langle 0|\, \mathcal{O}_1(0)\, f(\hat{H})\, \mathcal{O}_2(0)\, |0\rangle_\infty$$ want

- Detailed choice of *f(E)* and operators determines the observable

R-ratio

$$\langle 0|\, j_\mu(0)\, \delta(\hat{H} - \omega)\, j_\mu(0)\, |0\rangle_\infty$$

Meyer • Bailas, Hashimoto, Ishikawa (2020)
Alexandrou et al. (2022)

D-meson total lifetime

$$\langle D|\, \mathcal{H}_W(0)\, \delta(M_D - \hat{H})\, \mathcal{H}_W(0)\, |D\rangle_\infty$$

MTH, Meyer, Robaina (2017)

ππ → ππ amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, \pi(0)\, |\pi\rangle_\infty$$

j → ππ amplitude

$$\langle \pi|\, \pi(0)\, \frac{1}{E - \hat{H} + i\epsilon}\, j_\mu(0)\, |0\rangle_\infty$$

Bulava, MTH (2019)

Linear reconstruction

$$\langle \mathcal{O}(0) e^{-\hat{H}\tau} \mathcal{O}(0) \rangle = \int d\omega \, e^{-\omega\tau} \, \langle \mathcal{O}(0) \delta(\omega - \hat{H}) \mathcal{O}(0) \rangle$$

have want

$$G(\tau) = \int d\omega \, e^{-\omega\tau} \, \rho(\omega)$$

have want

- ***Linear, model-independent reconstruction*** (e.g. Backus-Gilbert-like, Chebyshev)

$$\sum_\tau \mathcal{K}(\bar{\omega}, \tau) G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega}, \tau) \int d\omega \, e^{-\omega\tau} \, \rho(\omega)$$

Linear reconstruction

$$\langle \mathcal{O}(0) e^{-\hat{H}\tau} \mathcal{O}(0) \rangle = \int d\omega\, e^{-\omega\tau} \, \langle \mathcal{O}(0) \delta(\omega - \hat{H}) \mathcal{O}(0) \rangle$$

have — want

$$G(\tau) = \int d\omega\, e^{-\omega\tau} \, \rho(\omega)$$

have — want

- **_Linear, model-independent reconstruction_** (e.g. Backus-Gilbert-like, Chebyshev)

$$\sum_\tau \mathcal{K}(\bar{\omega}, \tau) G(\tau) = \sum_\tau \mathcal{K}(\bar{\omega}, \tau) \int d\omega\, e^{-\omega\tau} \, \rho(\omega) = \int d\omega \left[\sum_\tau \mathcal{K}(\bar{\omega}, \tau)\, e^{-\omega\tau} \right] \rho(\omega)$$

$$= \int d\omega\, \widehat{\delta}_\Delta(\bar{\omega}, \omega)\, \rho(\omega)$$

δ is exactly known

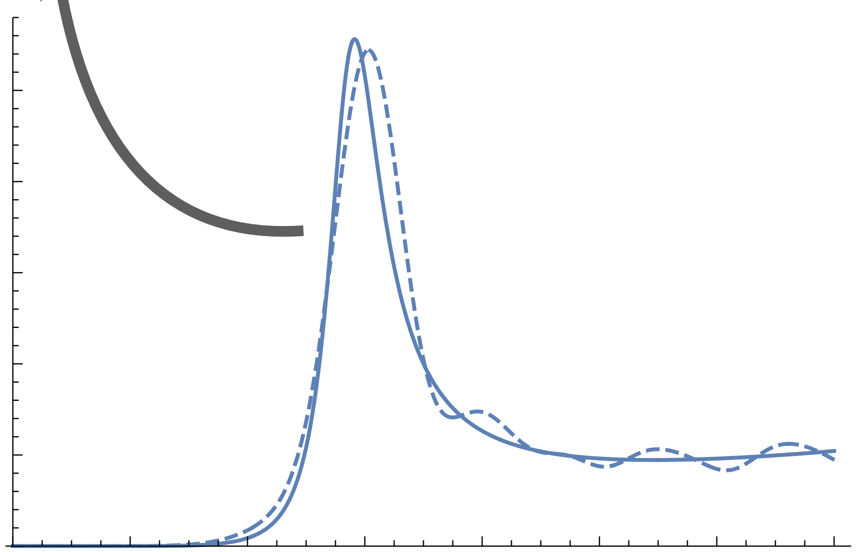
- Non-linear (***not discussed here…***)
  - Maximum Entropy Method (MEM)
  - Direct fits
  - Neural networks

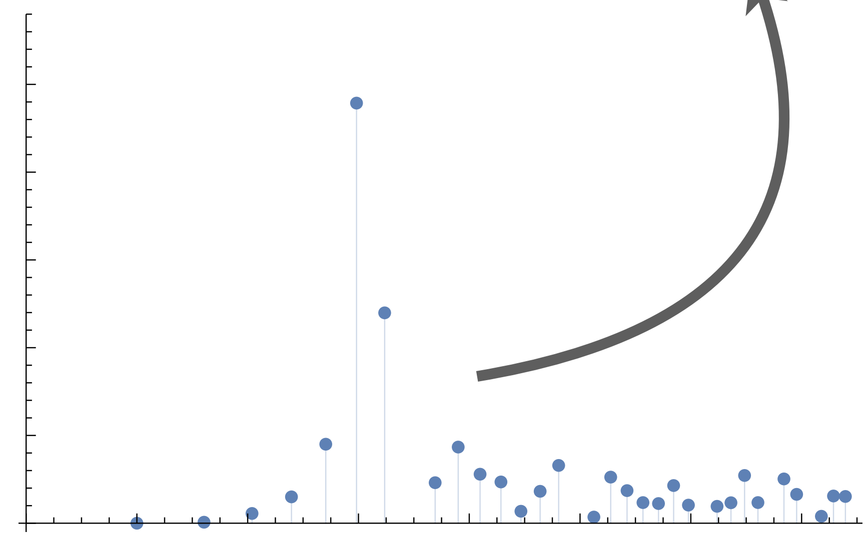
See multiple ECT and CERN workshops, work by Aarts, Allton, Amato, Brandt, Burnier, Del Debbio, Francis, Giudice, Hands, Harris, Hashimoto, Jäger, Karpie, Liu, Meyer, Monahan, Orginos, Robaina, Rothkopf, Ryan, …*

Role of the finite volume

$$\widehat{\rho}_L(\bar{\omega}) \equiv \int_0^{\infty} d\omega\, \widehat{\delta}_\Delta(\bar{\omega}, \omega)\, \rho_L(\omega)$$

$$G_L(\tau) = \int d\omega\, e^{-\omega\tau}\, \rho_L(\omega)$$





- Any reconstructed spectral function that ≠ forest of deltas…

  contains implicit smearing (or else $L \to \infty$)

We require…

$$1/L \ll \Delta \ll \mu_{\text{physical}}$$

smearing function covers many delta peaks

smearing does not overly distort observable

MTH, Meyer, Robaina (2017)

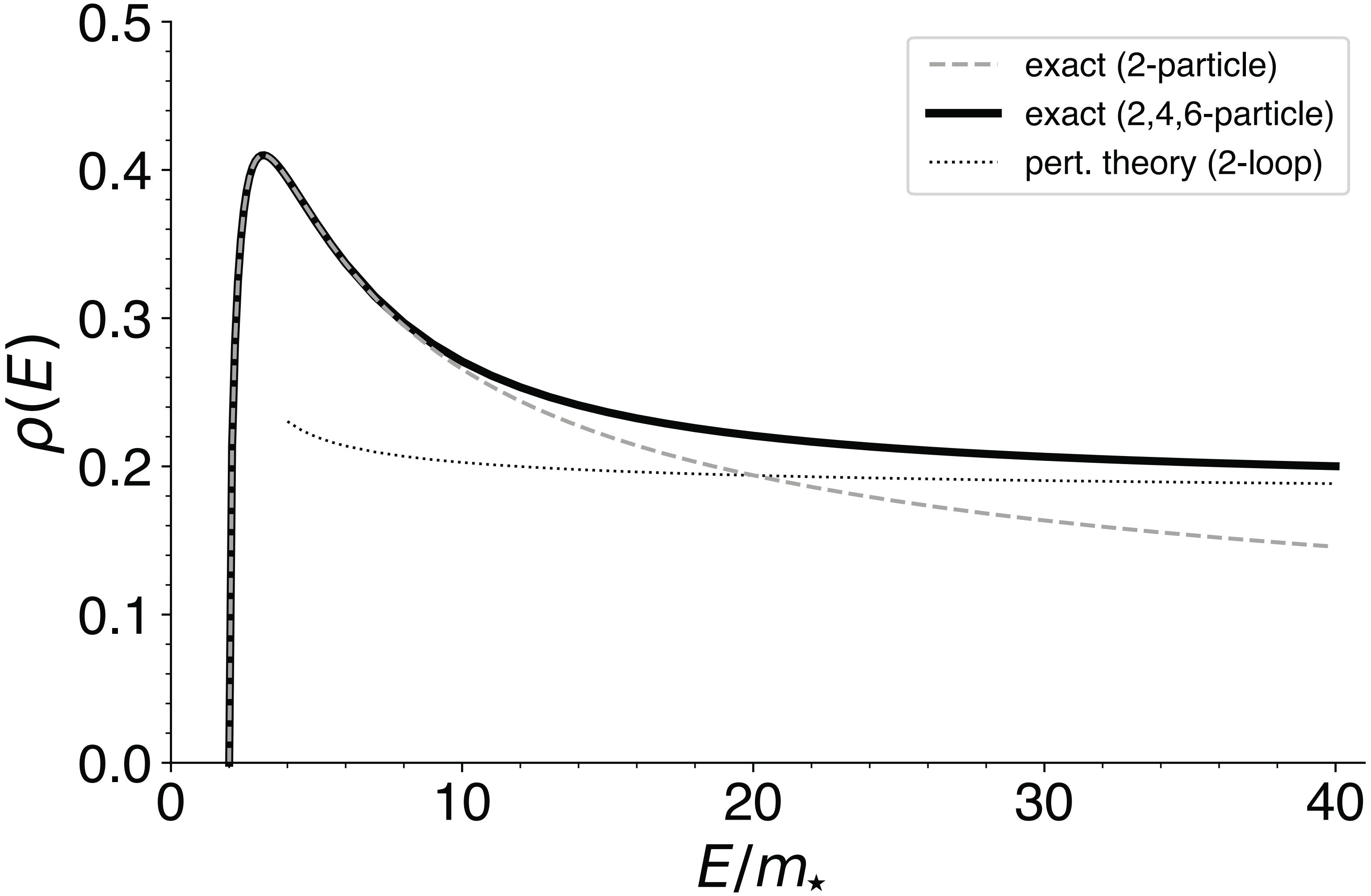
1+1 O(3) Model

- *Integrable* theory with some nice similarities to QCD
  - Asymptotically free
  - Dynamically generated mass gap
  - Iso-spin like symmetry
  - Conserved iso-vector vector current

$$S[\sigma] = \frac{1}{2g^2} \int d^2x\, \partial_\mu \sigma(x) \cdot \partial_\mu \sigma(x)$$

$$j^c_\mu(x) = \frac{1}{g^2} \epsilon^{abc} \sigma^a(x) \partial_\mu \sigma^b(x)$$

conserved current



$$\rho(E) = 2\pi \langle \Omega | \hat{j}^a_1(0)\, \delta^2(\hat{P} - p)\, \hat{j}^a_1(0) | \Omega \rangle$$

spectral function

$$\rho^{(2)}(E) = \frac{3\pi^3}{8\theta^2} \frac{\theta^2 + \pi^2}{\theta^2 + 4\pi^2} \tanh^3 \frac{\theta}{2}$$

$\theta = 2 \cosh^{-1} \frac{E}{2m}$

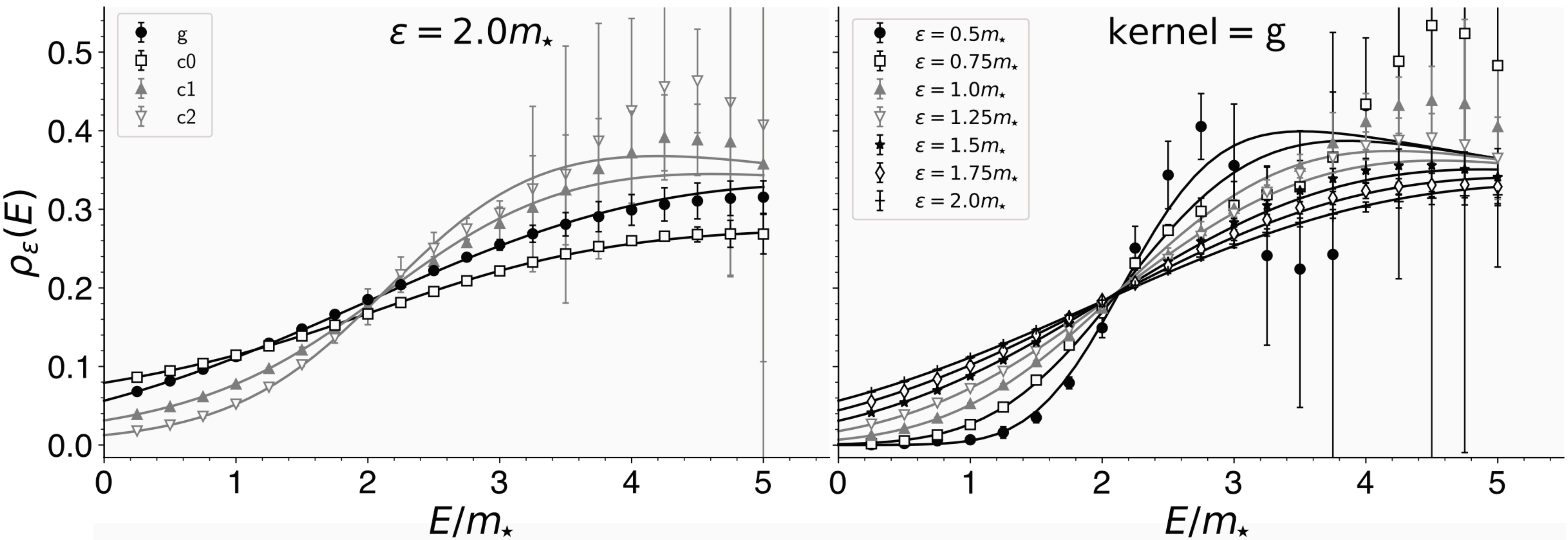
Smeared spectral function vs analytic result

- Construct different smearings of $\rho(\omega)$

$$\rho_\epsilon^\lambda(E) = \int_0^\infty d\omega\, \delta_\epsilon^\lambda(E,\omega)\, \rho(\omega)$$

$$\delta_\epsilon^{\rm g}(x) = \frac{1}{\sqrt{2\pi}\epsilon} \exp\left[-\frac{x^2}{2\epsilon^2}\right], \qquad \delta_\epsilon^{\rm c0}(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2},$$

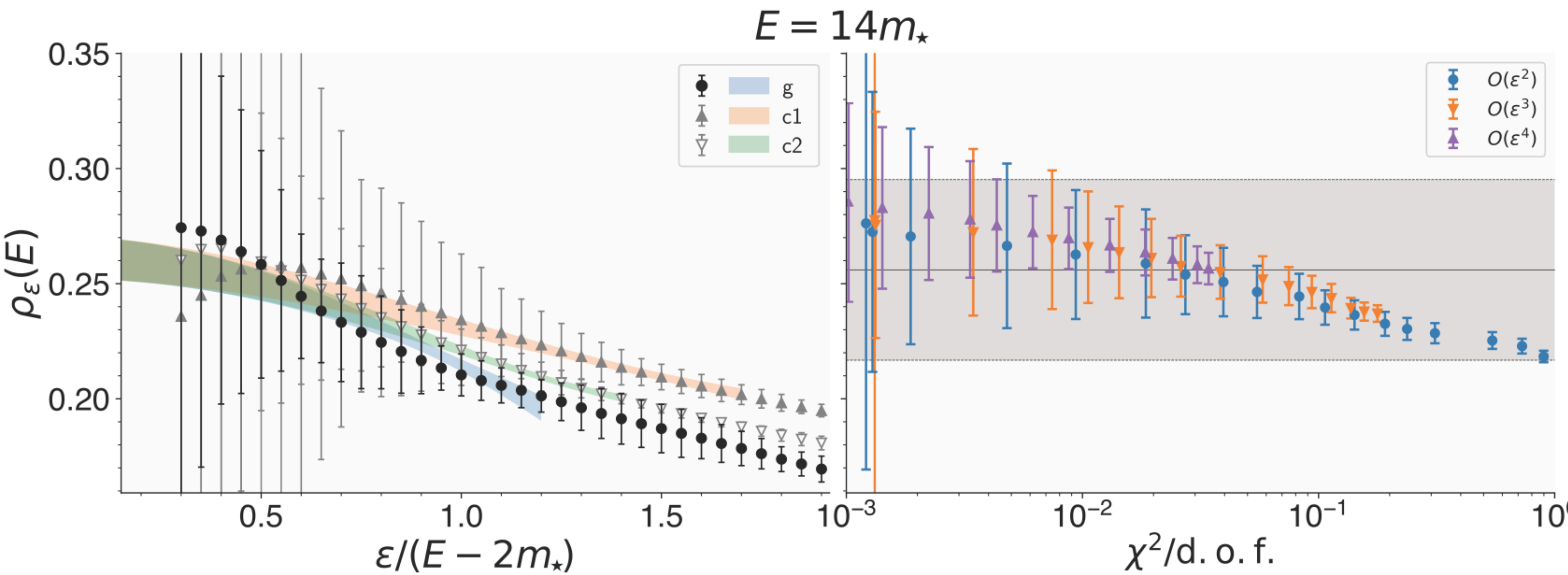
$$\delta_\epsilon^{\rm c1}(x) = \frac{2}{\pi}\frac{\epsilon^3}{(x^2+\epsilon^2)^2}, \qquad \delta_\epsilon^{\rm c2}(x) = \frac{8}{3\pi}\frac{\epsilon^5}{(x^2+\epsilon^2)^3}.$$



Bulava, MTH, Hansen, Patella, Tantalo (2021)

Extrapolation

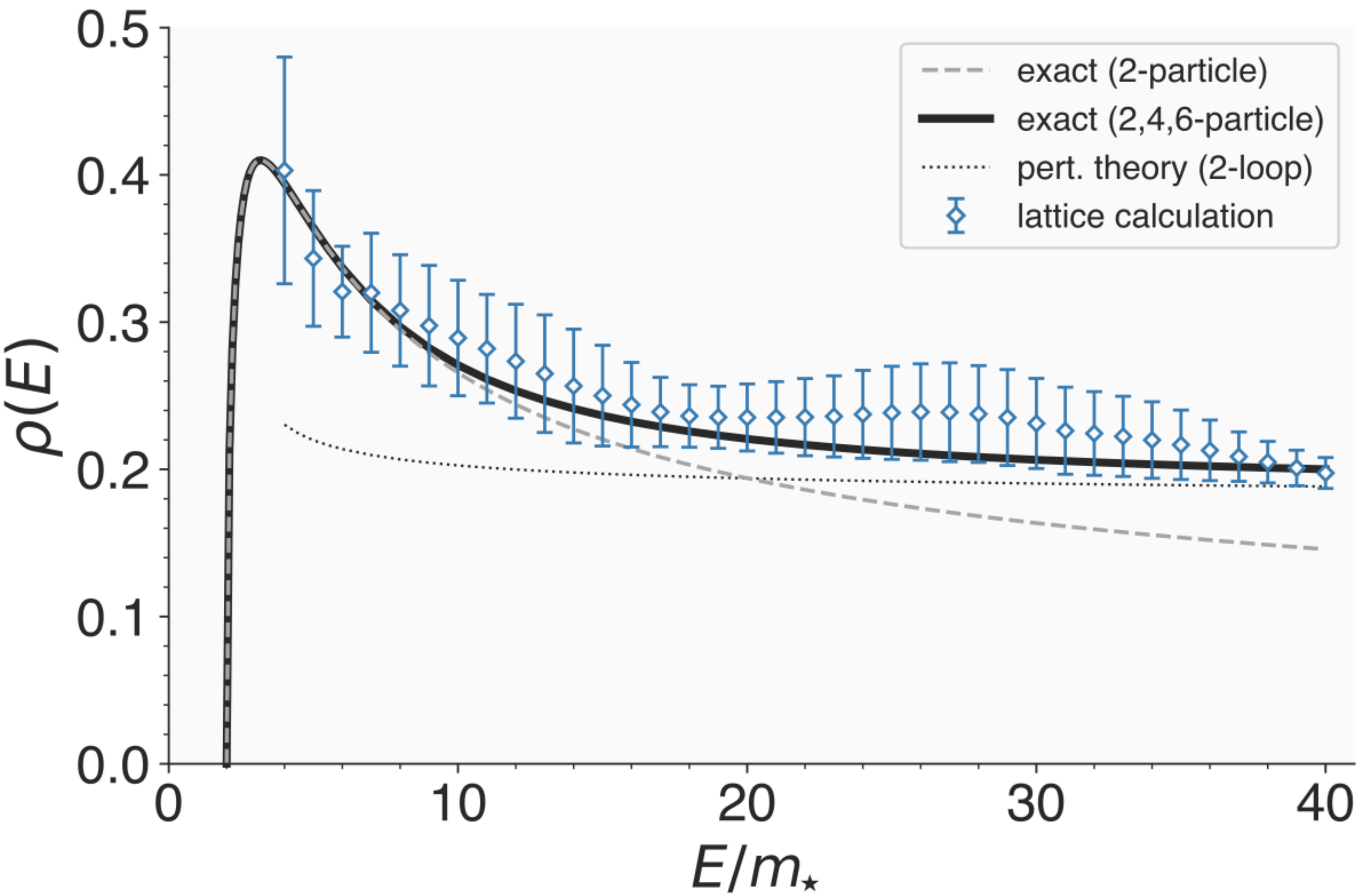
- Targeting $\rho(E)$ for $E = 14m$ here



- Use known relations between different smearing kernels

Bulava, MTH, Hansen, Patella, Tantalo (2021)

Result



Bulava, MTH, Hansen, Patella, Tantalo (2021)

Many QCD applications already published… see work by A. Barone, S. Hashimoto, A. Jüttner, T. Kaneko, R. Kellermann, R. Frezzotti, G. Gagliardi, V. Lubicz, F. Sanfilippo, S. Simula …