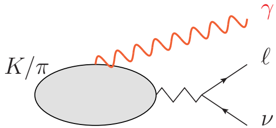
Real Photon emission for $K \to \ell_2$
...and other stuff



Istituto Nazionale di Fisica Nucleare

Extended Twisted Mass Collaboration

Francesco Sanfilippo, INFN, Roma Tre

Bonn ETMC meeting, 24 September 2019

Summary

Introduction

- Motivation to include QED in QCD
- Why lattice QCD+QED
- If Lattice QCD is though, including QED is even harder!
- Include QED: the perturbative approach

Hadron decay

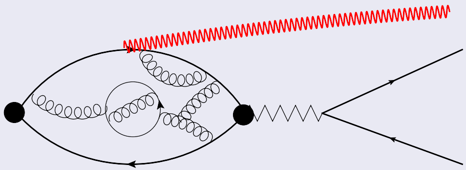
1. Infrared divergence
2. Virtual QED corrections to $K \to \ell_2$
3. Real emission of a photon

Some final words

- Work in progress
- Future developments

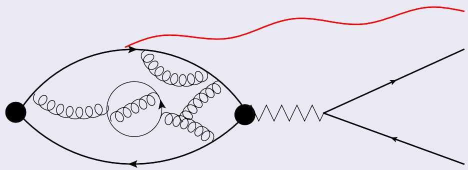
Dealing with photons

Hard photons - $E \sim$ many GeV



Perturbation theory

Ultrasoft photons - $E \sim$ few MeV



Point-like hadrons

Dealing with photons

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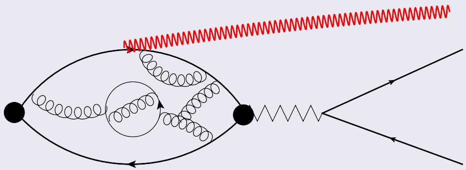

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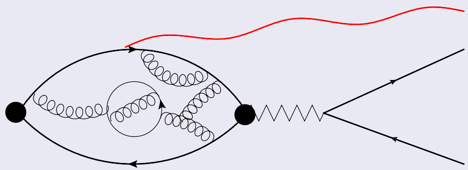
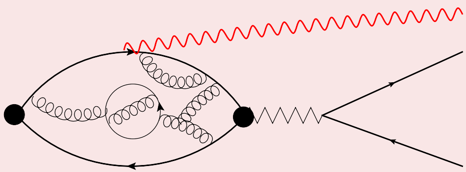
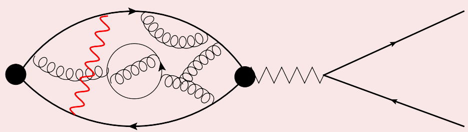


Point-like hadrons

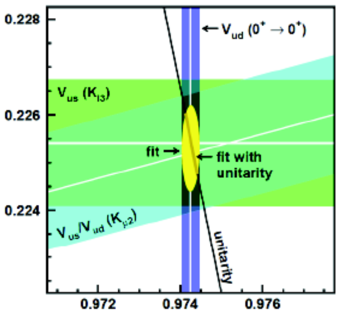
What to do with soft photons?

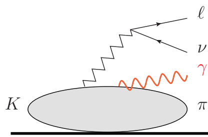




...Here lattice come to the rescue...

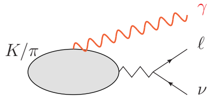
Example: CKM matrix elements from semileptonic and leptonic K and π decays





Semileptonic

$$\underbrace{\Gamma_{K\to\pi l\bar{\nu}(\gamma)}}_{experiments} \propto |V_{us}|^2 \underbrace{\left|f_+^{K\pi}(0)\right|^2}_{QCD}$$



Leptonic

$$\underbrace{\frac{\Gamma_{K\to\ell\bar{\nu}(\gamma)}}{\Gamma_{\pi\to\ell\bar{\nu}(\gamma)}}}_{experiments} \propto \frac{|V_{us}|^2}{|V_{ud}|^2} \underbrace{\left(\frac{f_K}{f_\pi}\right)^2}_{QCD}$$

Hadronic matrix elements, lattice results

$$\begin{aligned} f_+^{K\pi}(0) &= 0.956\,(8) \\ f_K/f_\pi &= 1.193\,(5) \end{aligned}$$

in the **isospin symmetric** limit.

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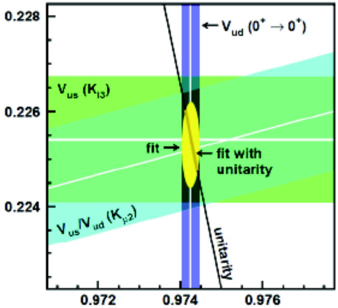

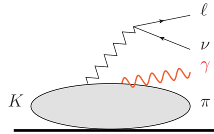

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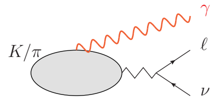

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→ At current precision (0.5–1%), IB corrections **not negligible** ←

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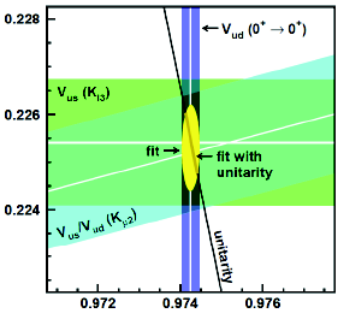

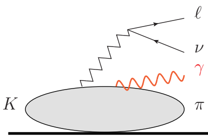

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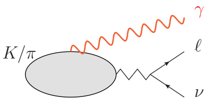

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$\rightarrow$ At current precision (0.5–1%), IB corrections **not negligible** $\leftarrow$

Indeed ChPT estimates of these effects are:

$\left(f_+^{K^+\pi^0}/f_+^{K^-\pi^+}-1\right)^{QCD}=2.9(4)\%$	$\left(\frac{f_{K^+}/f_{\pi^+}}{f_K/f_\pi}-1\right)^{QCD}=-0.22(6)\%$
A. Kastner, H. Neufeld (EPJ C57, 2008)	V. Cirigliano, H. Neufeld (Phys.Lett.B700, 2011)

More complications from QED

The target: Fully unquenched QCD + QED

$$\mathcal{L} = \sum_i \bar{\psi}_i \left[m_i - i\not{D}_i\right] \psi_i + \mathcal{L}_{gluons} + \mathcal{L}_{photon}, \quad D_{i,\mu} = \partial_\mu + igA^a_\mu T^a + ie_i A_\mu$$

Simulate each quark with its physical mass and charge

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Introducing photons

Power-like Finite Volume Effects due to long range interaction

Zero mode from photon propagator: $\int \frac{\delta_{\mu\nu}}{k^2} d^4k \;\rightarrow\; \sum_k \frac{\delta_{\mu\nu}}{k^2}$
massive photons, removal of zero mode, C^* boundary conditions...

Renormalization pattern gets more complicated

Additional divergencies arises!

UV completeness: Nobody knows how to tame QED to all orders!

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Practical problem

- Traditionally, gauge configuration datasets include only gluons
- Dedicated simulations with huge cost
- Even greater cost due to additional zero modes.

The Roman approach - RM123 collaboration

Pioneering papers

- "*Isospin breaking effects due to the up-down mass difference in Lattice QCD*", [JHEP 1204 (2012)]
- "*Leading isospin breaking effects on the lattice*", [PRD87 (2013)]

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3) Roma Tre

D.Giusti,
V.Lubicz,
S.Romiti,
F.S,
S.Simula,
C.Tarantino

1) La Sapienza

M.Di Carlo,
G.Martinelli

2) Tor Vergata

G.deDivitiis,
R.Frezzotti,
N.Tantalo

⋆ **Guest Star** from Southampton University: C.T.Sachrajda

The Roman approach - RM123 collaboration

Perturbative expansion

Work on top of the isospin symmetric theory $\boldsymbol{\mathcal{L} = \mathcal{L}_{Iso\,symm} + \mathcal{L}_{Iso\,break}}$

$$\mathcal{L}_{Iso\,break} = e\mathcal{L}_{QED} + \delta m \mathcal{L}_{mass}, \quad e^2 = \frac{4\pi}{137.04}, \quad \delta m = (m_d - m_u)/2$$

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Cleaner: Factorize small parameters e and δm, introduce QED only when needed

Cheaper: No need to generate new QCD gauge field backgrounds (and, newly generated ones are general purpose).

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Cons

- More vertex and correlations functions to be computed
- Corrections to be computed separately for each observable
- Including charge effects in the sea is costly (fermionically disconnected diagrams).

The perturbative expansion in e^2

Keep QCD to all orders and QED to $\mathcal{O}\left(e^2\right)$

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int D\left[A_\mu, U^{QCD}, \psi, \bar{\psi}\right] O \; \left(1 - e^2 S_1 + \mathcal{O}\left(e^4\right)\right) \exp\left[-S_0\right]$$

N.B: $\mathcal{O}\left(e\right)$ vanishes due to charge symmetry.

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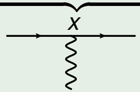
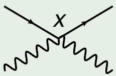
Which on the lattice means...

$$S_1 = \underbrace{\left[\int dx\, V_\mu(x)\, A_\mu(x)\right]^2}_{} + \underbrace{\int dx\, T_\mu(x)\, A_\mu^2(x)}_{}$$





- V^2: Two photon-fermion-fermion vertices (as in the continuum)
- T: One photon-photon-fermion-fermion vertex (tadpole: lattice special).

The case of the pion

Basic correlation function

$$C(t) = \sum_{\vec{x}} \left\langle P(\vec{x}, t)\, P^{\dagger}(0) \right\rangle_{QCD+QED}, \qquad P = \bar{\psi}\gamma_5\psi$$

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Functional integral

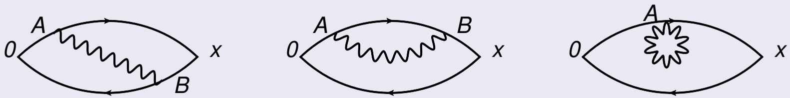
$$C(t) = C_0(t) + C_1(t) =$$

$$\left\langle P(\vec{x}, t)\, P^{\dagger}(0) \right\rangle_{QCD} - e^2 \left\langle P(\vec{x}, t) \sum_y S_1(y) P^{\dagger}(0) \right\rangle_{QCD}$$

Now take all Wick contractions...

Diagrams

Fermionically connected - easy part (so to say)



(gluons not drawn, connecting fermion lines in all possible ways)

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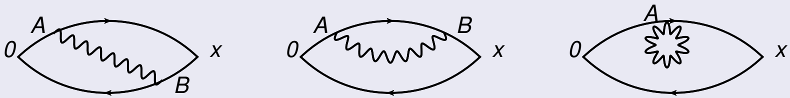
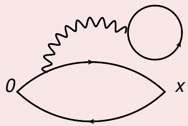
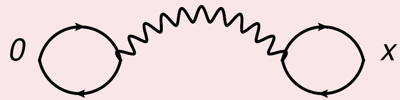


(gluons not drawn, connecting fermion lines in all possible ways)

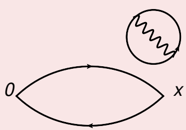
Disconnected - various degree of nastiness - work is in progress to include



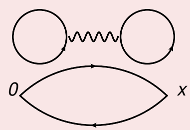
"monocle"



"handcuffs"

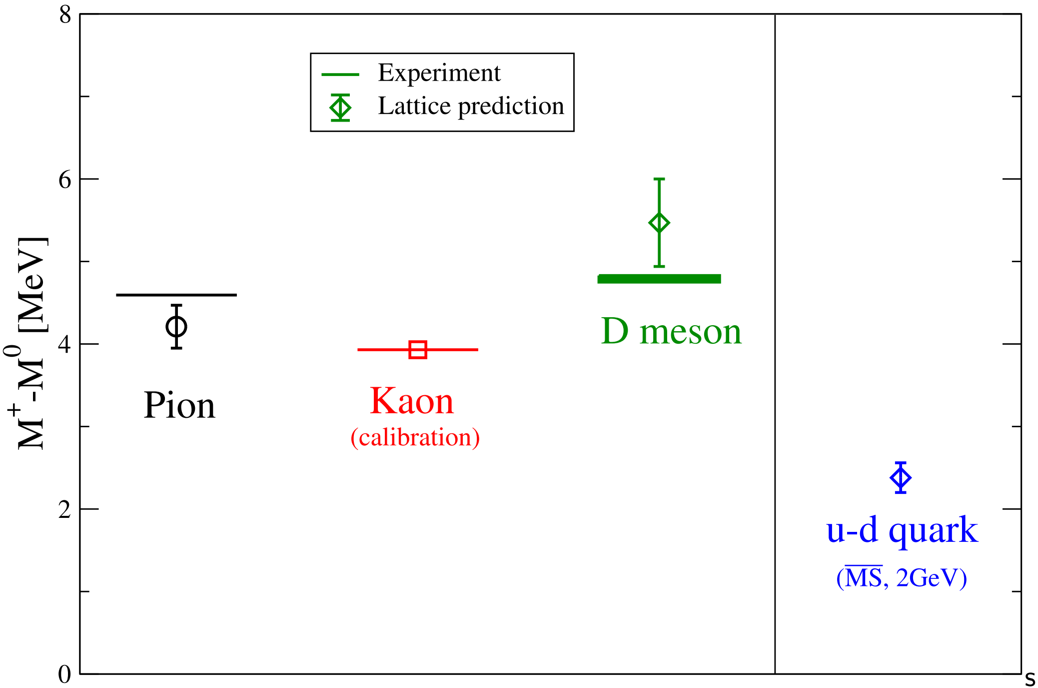


"blinking"



"laughing"

Some results, meson mass (perturbative expansion)



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Matrix elements

More problems

- In general the amplitudes, are **infrared divergent**
- On the lattice, a natural infrared cutoff is provided by the **finite volume**
- But physically, only combinations of **Real** + **Virtual** contribution is finite.

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- But physically, only combinations of **Real** + **Virtual** contribution is finite.

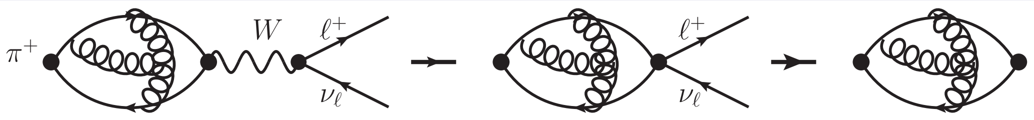
To be specific

- We consider the leptonic decay of a **charged pion**
- The method is general

$N_f = 2 + 1 + 1$ ETM collaboration configurations

Leptonic decays of mesons (at tree level in QED: $e = 0$)

Full process — Eff. weak hamiltonian — QCD side



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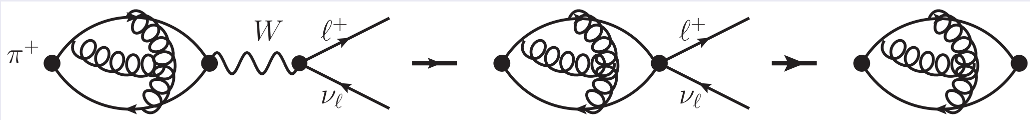


Two point correlation functions

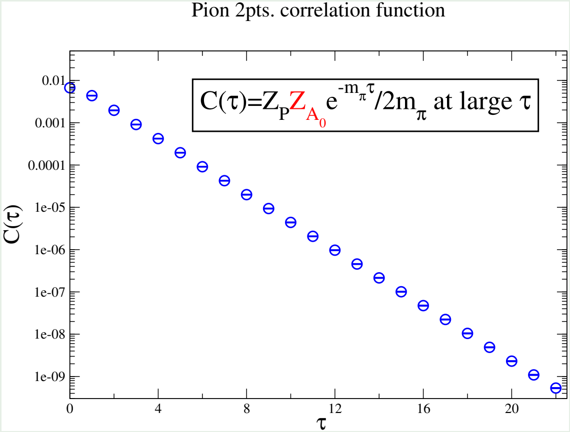
$$\Gamma_{\pi\to\ell\bar\nu} = \underbrace{|V_{xy}|^2}_{\text{CKM}} \underbrace{\mathcal{K}(m_\ell, m_M)}_{\text{kinematics}} \Big| \underbrace{f_\pi}_{\text{dec. constant}} \Big|^2$$

$$f_\pi = \frac{Z_A}{m_\pi} = \frac{\langle 0|A_0|\pi\rangle}{m_\pi}$$

Z: coupling of current inducing decay

From lattice, 2 point correlation functions:

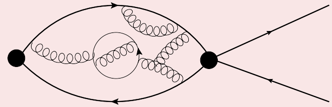
$$C(\tau) = \langle O^\dagger_{A_0}(\tau)\, O_P(0)\rangle, \; O = \bar\psi\Gamma\psi$$

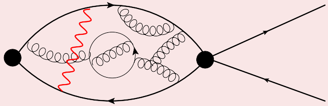
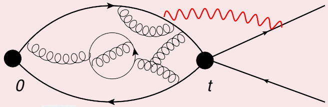


Leptonic decays of mesons (with QED)

Zero photons in the final state, $\mathcal{O}\left(e^2\right)$

$$\Gamma^{0ph}_{\pi^+\to\ell^+\nu} = \left|A^0\right|^2 + 2e^2\left|A^0 A^1\right| + \mathcal{O}\left(e^4\right)$$

$A_0 =$ 

$A_1 =$  $+$ 

$+ \ldots$

IR DIVERGENT 

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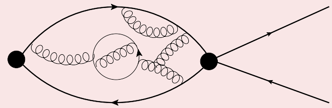
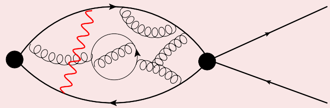
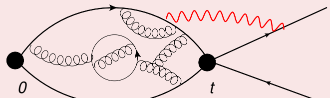
$A_0 =$ 

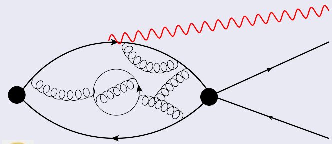
$A_1 =$  $+$ 

$+ \ldots$

IR DIVERGENT

One photon in the final state, $\mathcal{O}\left(e^2\right)$

$$\Gamma^{1ph}_{\pi^+\to\ell^+\nu\gamma} = e^2\left|B^0\right|^2$$

$B_0 =$ 

Again, IR DIVERGENT

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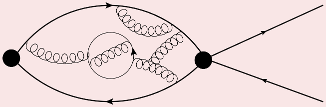
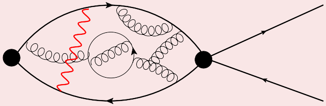
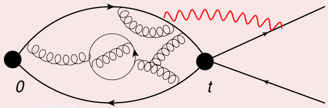
$A_0 =$ 

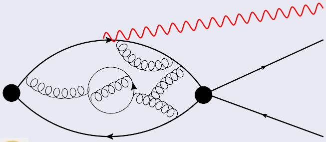
$A_1 =$  $+$ 

$+ \ldots$

IR DIVERGENT

One photon in the final state, $\mathcal{O}\left(e^2\right)$

$$\Gamma^{1ph}_{\pi^+\to\ell^+\nu\gamma} = e^2\left|B^0\right|^2$$

$B_0 =$ 

Again, IR DIVERGENT

Solution

[Bloch and Nordsieck, PR52 (1937)]

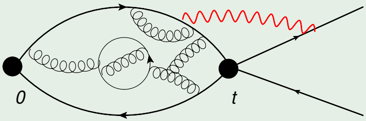
$\Gamma = \Gamma^{0ph} + \Gamma^{1ph}$ is finite

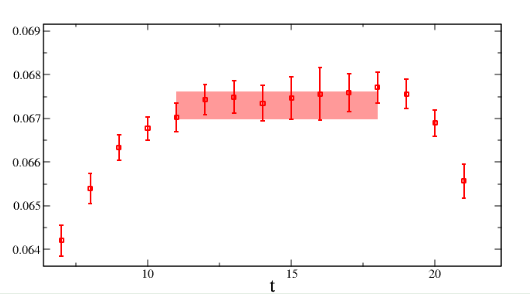
The strategy

[N.Carrasco et al., PRD91 (2015)]

Virtual photon

- Needs to implement leptons
- Not too demanding numerically.





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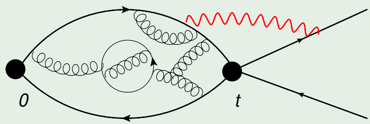
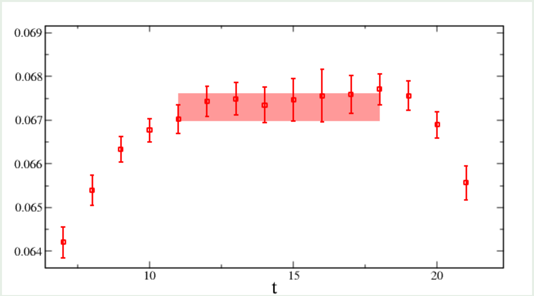
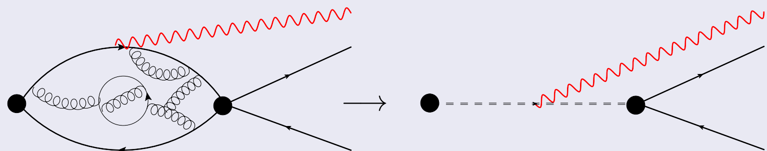




Real photon

- Slightly more numerically demanding/different process
- Initially, we used point-like approximation and consider $E_\gamma < 20$ MeV

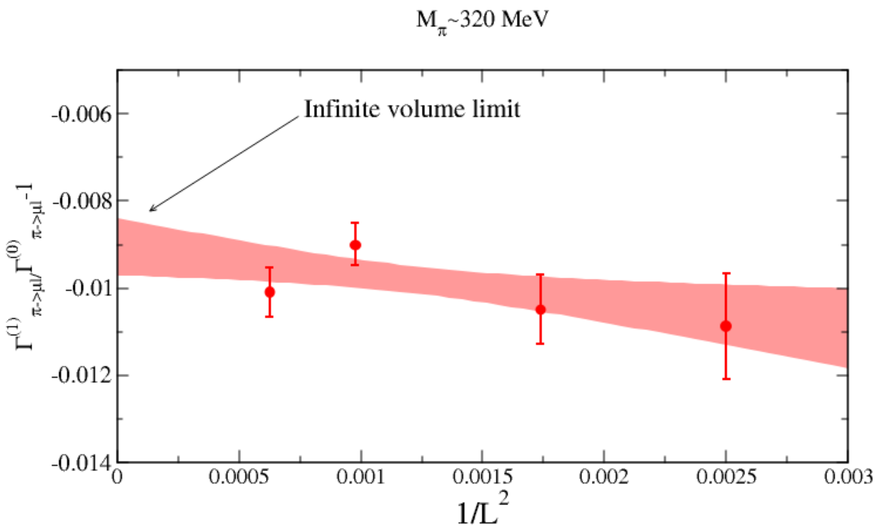
$B_0 =$ 

Cut-off appropriate experimentally (γ detector sensitivity) and theoretically (π inner structure)

Infinite volume extrapolation

Volume dependence

- IR divergences $\propto \log L$ cancel in the difference $\Gamma^{0ph}(L) - \Gamma^{0}_{pt}(L)$
- $1/L$ cancel as well (Ward identity)
- Best fit with $1/L^2$ (and $1/L^3$) and extrapolate to $L \to \infty$



Let's start from a slightly simpler quantity

QED contribution to ratio of decay width of Kaon and Pion

$$\frac{\Gamma_{K^+\to\ell^+\nu(\gamma)}}{\Gamma_{\pi^+\to\ell^+\nu(\gamma)}} = \frac{\Gamma_{K^+\to\ell^+\nu}}{\Gamma_{\pi^+\to\ell^+\nu}}\left(1+\delta R_{K\pi}\right), \qquad \delta R_{K\pi} = \delta R_K - \delta R_\pi$$

- Reduction of noise
- Large cancellation of renormalization correction
- Suppression of finite volume dependence.

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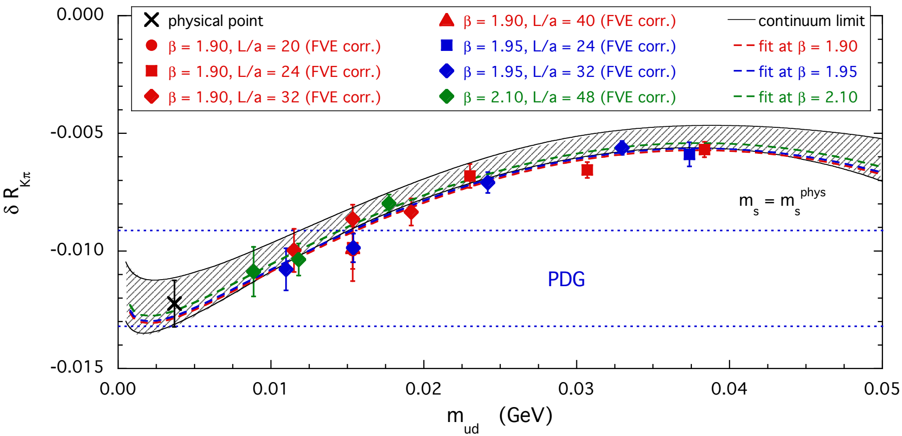
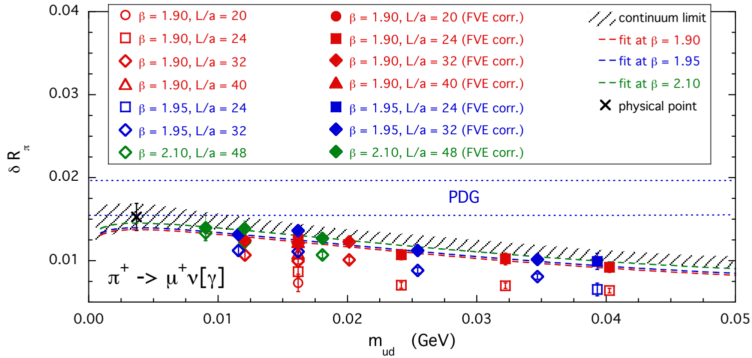
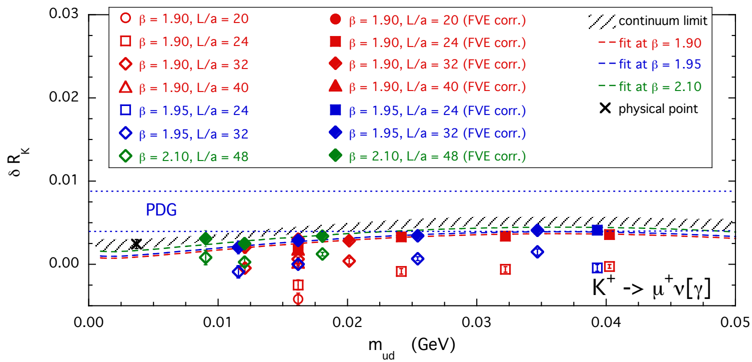


[D.Giusti et al., Phys. Rev. Lett. 120, 072001 (2018)]

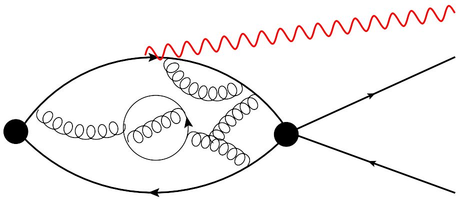
Separate Pion and Kaon corrections





[M. Di Carlo et al., Phys.Rev. D100 (2019) no.3, 034514]

Real photon on the lattice



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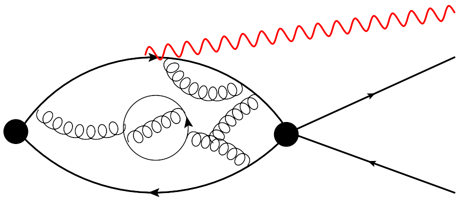
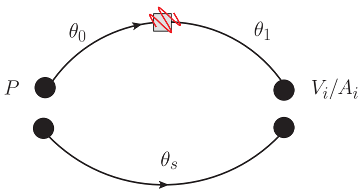


$\rightarrow$



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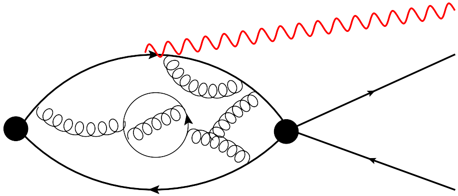

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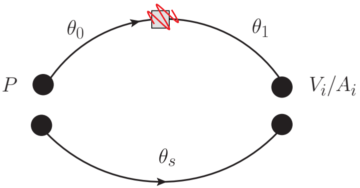

Remarks

 = projection on photon of momentum k (2 physical helicities)

no photon is actually present (no power volume corrections)

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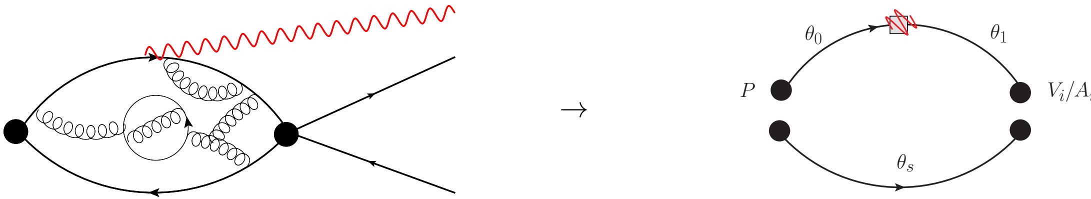

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$$p = \frac{2\pi}{L}(\theta_0 - \theta_s)\,, \qquad \text{pion momentum}$$

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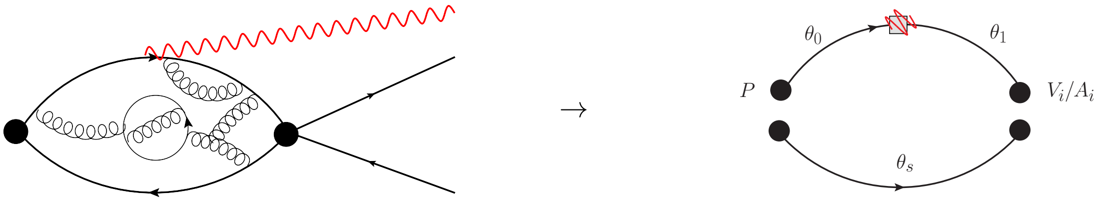

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Cost and reach

$2N_{\theta_0}N_{m_0}N_{\theta_1} + N_{\theta_s}N_{m_s}$ propagators involved for each mass/momenta

$N_{\theta_0}N_{\theta_1}N_{\theta_s}$ kinematic combination for each meson

Matrix element extraction

Correlators decomposition and kinematics

$$C_W^{i,r}(t;p,k) = H_W^{i,r}(p,k) K(p,k)\,,$$
$$K(p,k) = \frac{\langle P(p)|\, P\, |0\rangle}{4E_P E_\gamma} e^{-tE_P} e^{-(T/2-t)E_\gamma}$$

$E_\gamma = \text{Energy of photon}$

$E_P = \text{Energy of meson}$

$x_\gamma \equiv 2E_\gamma / E_P \text{ in meson rest frame}$

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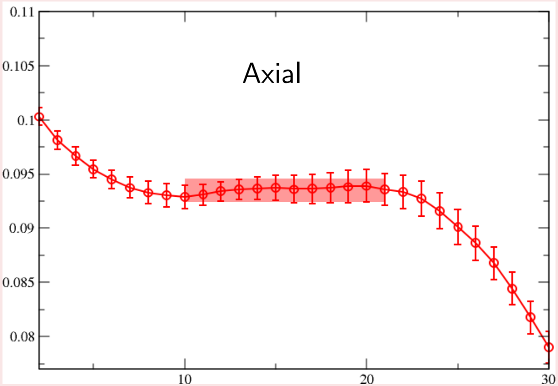
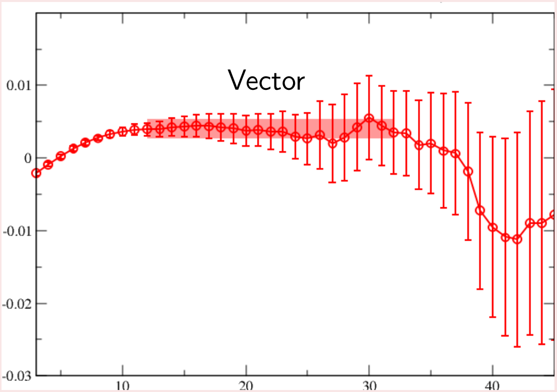
A30.32, K meson, H_A | D20.48, D meson, H_V



t/a



t/a

Form factors

With ON SHELL photon, polarizations ϵ_r

$$H_A^{i,r}(p,k) = \frac{\epsilon_r^i M_P}{2} x_\gamma \left[\boldsymbol{F_A(x_\gamma)} + \underbrace{\frac{2 f_P}{M_P x_\gamma}}_{IR\,divergence} \right], \qquad H_V^{i,r}(p,k) = \frac{[\epsilon_r \wedge (E_\gamma p - E_P k)]_i}{m_P} \boldsymbol{F_V(p,k)}$$

Axial matrix element is **divergent**, coefficient f_P exactly known (WI)

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TWO form factors, F_A and F_V

- Contain the structure-dependent part of the amplitude
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- Ch-PT prediction for light pseudoscalar meson
- Enhanced when excited states are close in energy (D, B mesons)

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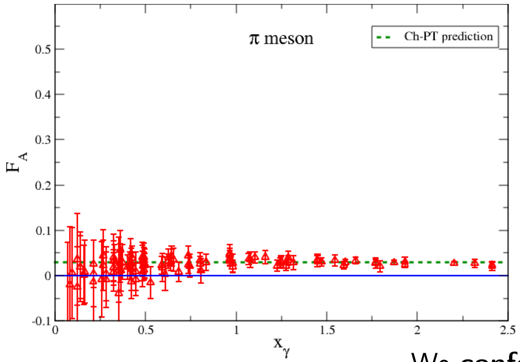
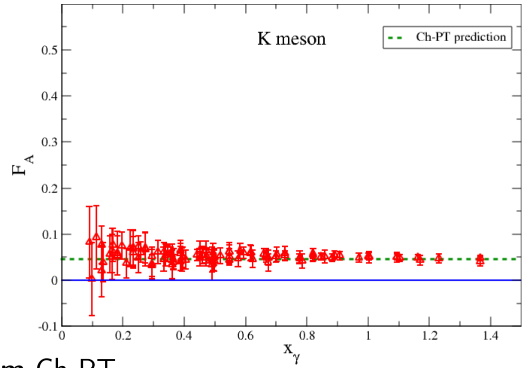
TWO MORE form factors out-shell

- ...and problems with analytic continuations
- but would allow to study additional processes...

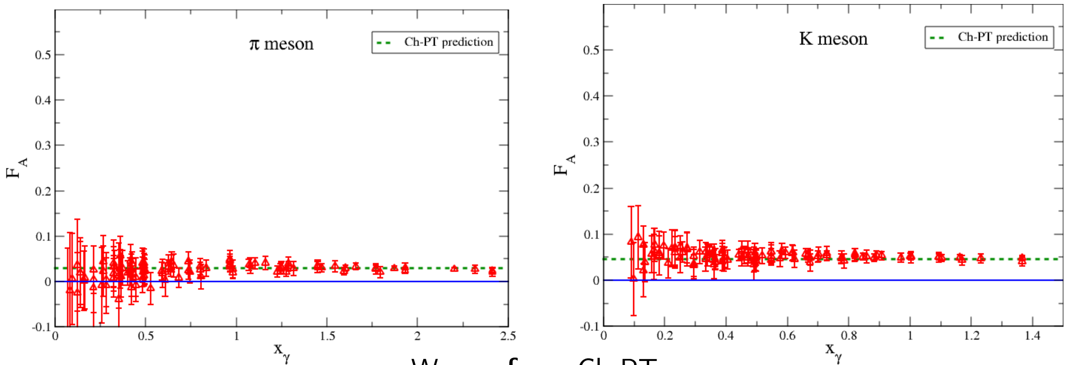
Axial form factor





We **conferm** Ch-PT

Axial form factor



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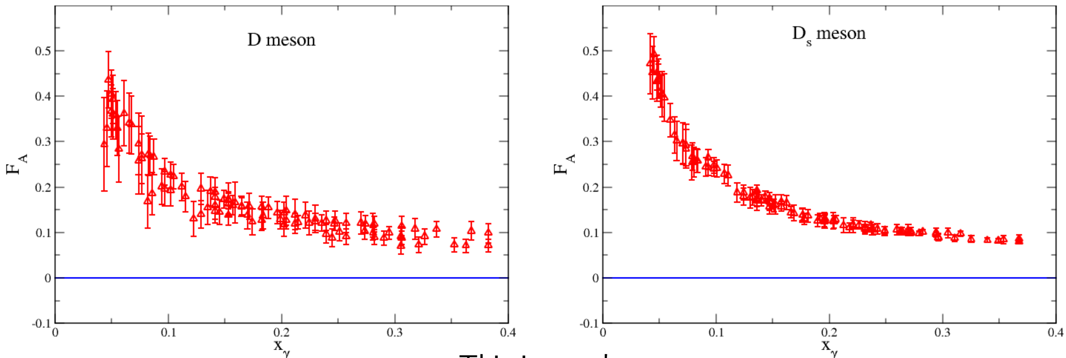
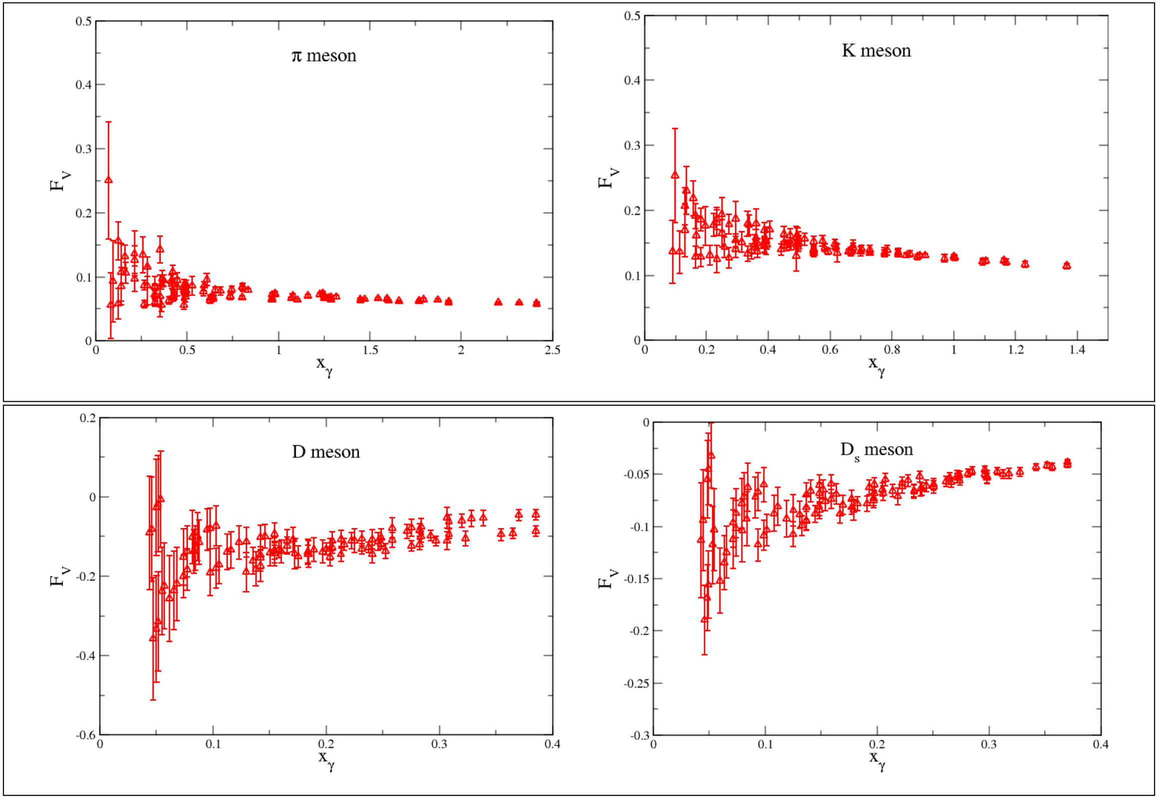


This is **new**!

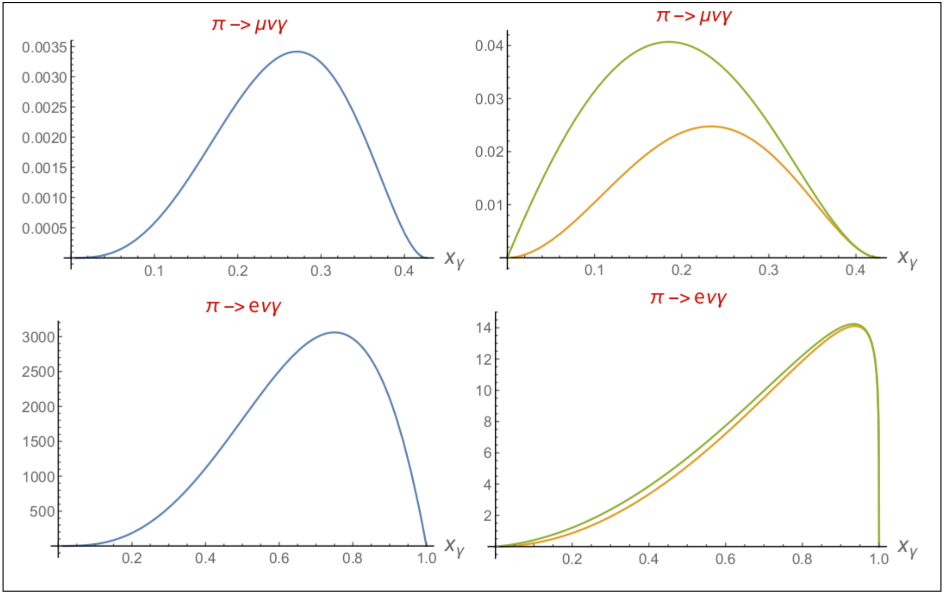
Vector form factor



Differential rate to be integrated

$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\frac{d\Gamma_1^{\rm SD}}{dx_\gamma} = \frac{m_P^2}{6 f_P^2\, r_\ell^2\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)^2 + F_A(x_\gamma)^2\right] f^{\rm SD}(x_\gamma)$$

$$\frac{4\pi}{\alpha\,\Gamma_0^{\rm tree}}\frac{d\Gamma_1^{\rm INT}}{dx_\gamma} = -\frac{2\, m_P}{f_P\,(1-r_\ell^2)^2}\left[F_V(x_\gamma)\, f_V^{\rm INT}(x_\gamma) + F_A(x_\gamma)\, f_A^{\rm INT}(x_\gamma)\right]$$



Next steps

Finalize the calculation

- Finalize the analysis with improved statistics
- Complete the chiral/continuum/infinite volume extrapolation
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- Disconnected diagrams...?
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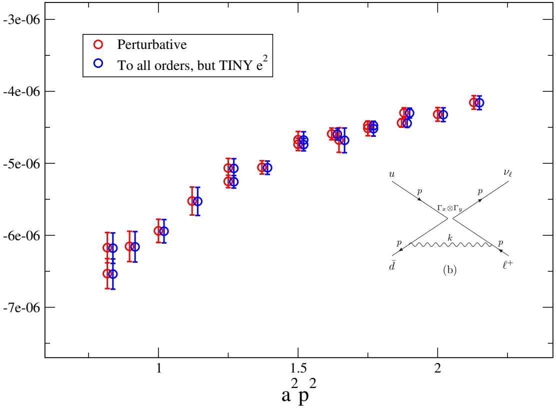
THANK YOU!

Backup slides

Can't we compute this with the "to all order" approach?

Stochastic = Put the photons in the links $U_{x,\mu}^{QCD} \to U_{x,\mu}^{QCD} \exp\left(ieA_{x,\mu}\right)$



In the quenched QED approximation

- Can be used to isolate $\propto e^2$ contribution
- $\frac{O(+e)+O(-e)}{2e^2} \xrightarrow{e\to 0} \partial_{e^2} O\left(e\right)\big|_{e=0}$
  "numerical calculation of derivative"
- strictly the same cost, for 2pts
- easier & cheaper for higher correlations!?
- needs some more investigations

What if you don't take $e \to 0$?

- Higher orders are kept in the calculation
- Can be fine if the observable is not pathological
- Extrapolating has little cost...

Unquenched QED

reweighting: can be used to compute disconnected diagrams

simulations: no easy way to to keep correlation of two independent runs

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$$\int \frac{\delta_{\mu\nu}}{k^2} d^4k \quad \rightarrow \quad \sum_k \frac{\delta_{\mu\nu}}{k^2}$$

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Remove "some" zero modes

4D zero mode only: $\sum_k \;\to\; \sum_{k\neq 0}$

✓ pole removed, irrelevant when $V \to \infty$

✗ nonlocal constraint, T/L^3 divergence

$\sim$ not tragic when working at fixed T/L.

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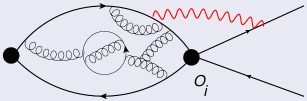
Use C^* Boundary conditions

- ✓ local
- ✗ needs dedicated simulations
- $\sim$ flavor violation across boundaries.

Matching to the "real world"

Correlation functions computed with bare operators



Needs renormalization: $O_i^{ren} = Z_{ij} O_j^{bare}$

$$O_{1,2} = (V \mp A)_q \otimes (V - A)_\ell$$
$$O_{3,4} = (S \mp P)_q \otimes (S - P)_\ell$$
$$O_5 = \left(T + \tilde{T}\right)_q \otimes \left(T + \tilde{T}\right)_\ell$$

RI-MOM (no QED)

- Compute amputated green functions:

$$\Lambda_O(p) = S^{-1}(p) \left\langle \sum_{x,y} e^{-ip(x-y)} \psi(x) O(0) \psi(y) \right\rangle S^{-1}(p)$$

- Impose RI-MOM condition at given p^2 (average all equivalent momenta)

$$\boldsymbol{Z_O} = \frac{Z_q}{\mathrm{Tr}\left[\Lambda_O(p) \Lambda_O^{tree}(p)^{-1}\right]}$$

- Chiral extrapolate $m \to 0$

Matching to the "real world" (continued)

RI-MOM with QED

- As a first step [D.Giusti et al., PRL '18]: RI-MOM for QCD + perturbation theory for QED
- In the coming-soon paper: RI-(S)MOM for QCD + QED

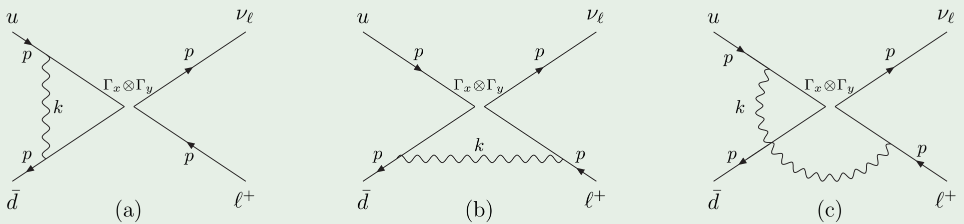
RI-MOM, perturbative expansion: ratio with QCD and QED

$$\frac{\boldsymbol{\delta Z_O^{QED+QCD}}}{\boldsymbol{Z_O^{QCD} Z_O^{QED}}} = \frac{\delta Z_q^{QCD+QED}}{Z_q^{QCD} Z_q^{QED}} - \frac{\mathrm{Tr}\left[\delta\Lambda_O^{QCD+QED}(p)\,\Lambda_O^{tree}(p)^{-1}\right]}{\mathrm{Tr}\left[\Lambda_O^{QCD}(p)\,\Lambda_O^{tree}(p)^{-1}\right]\mathrm{Tr}\left[\Lambda_O^{QED}(p)\,\Lambda_O^{tree}(p)^{-1}\right]}$$

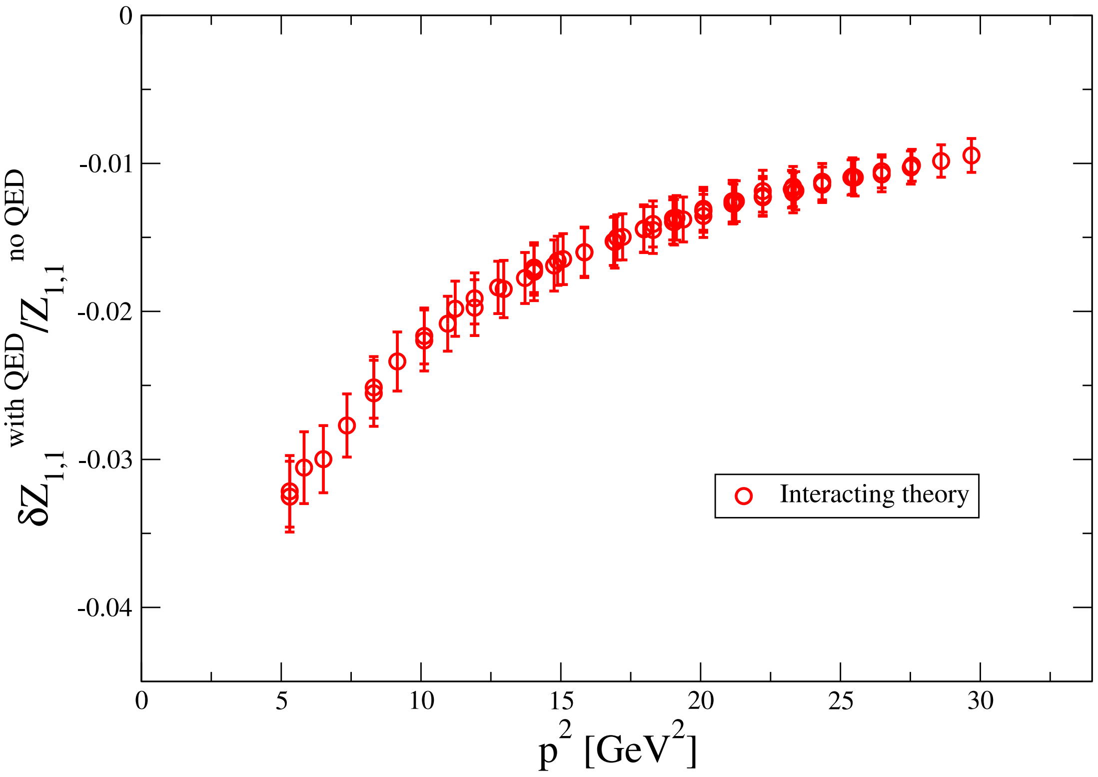
Large cancellation of cut-off effects, anomalous dimensions, noise, etc

Measure of the non-factorizability of the renormalization constants.

Vertices (with or without gluons, not drawn)



An example: QED correction to $Z_{1,1}$



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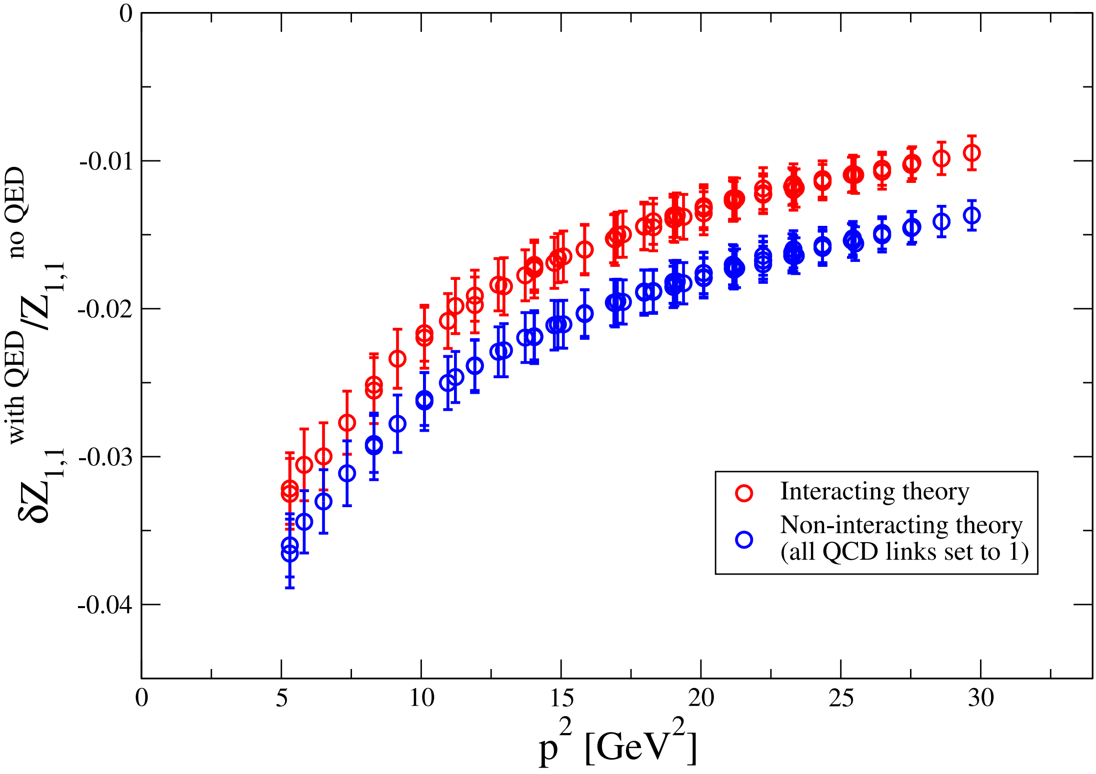

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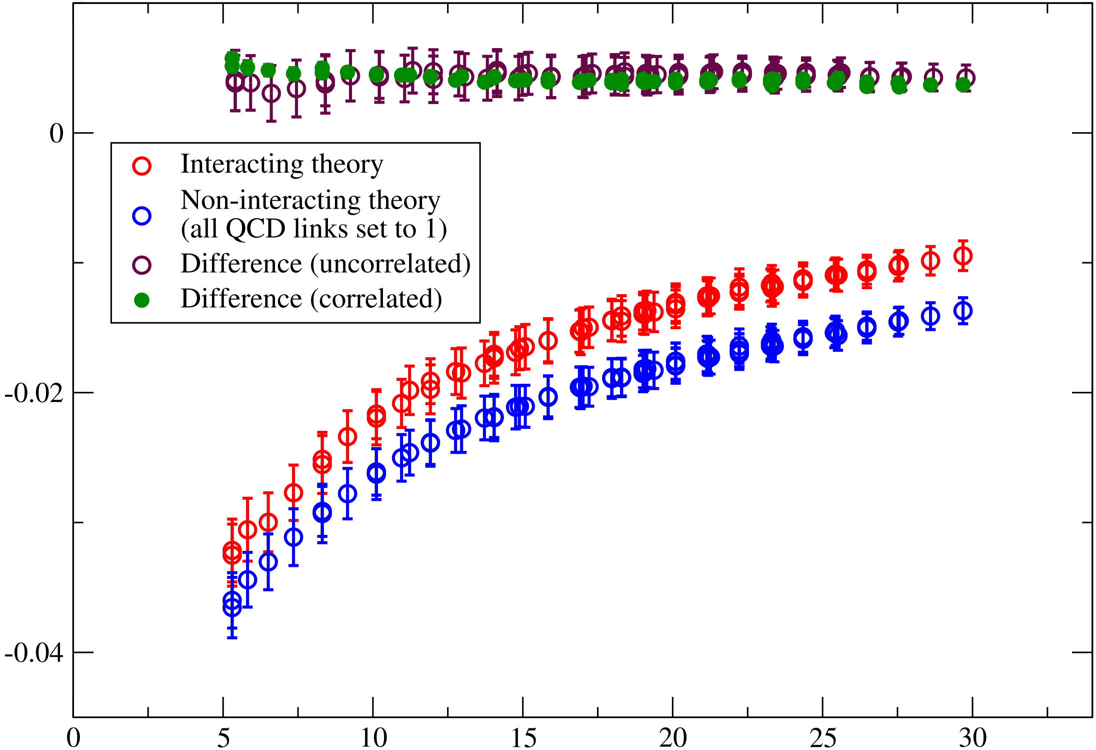

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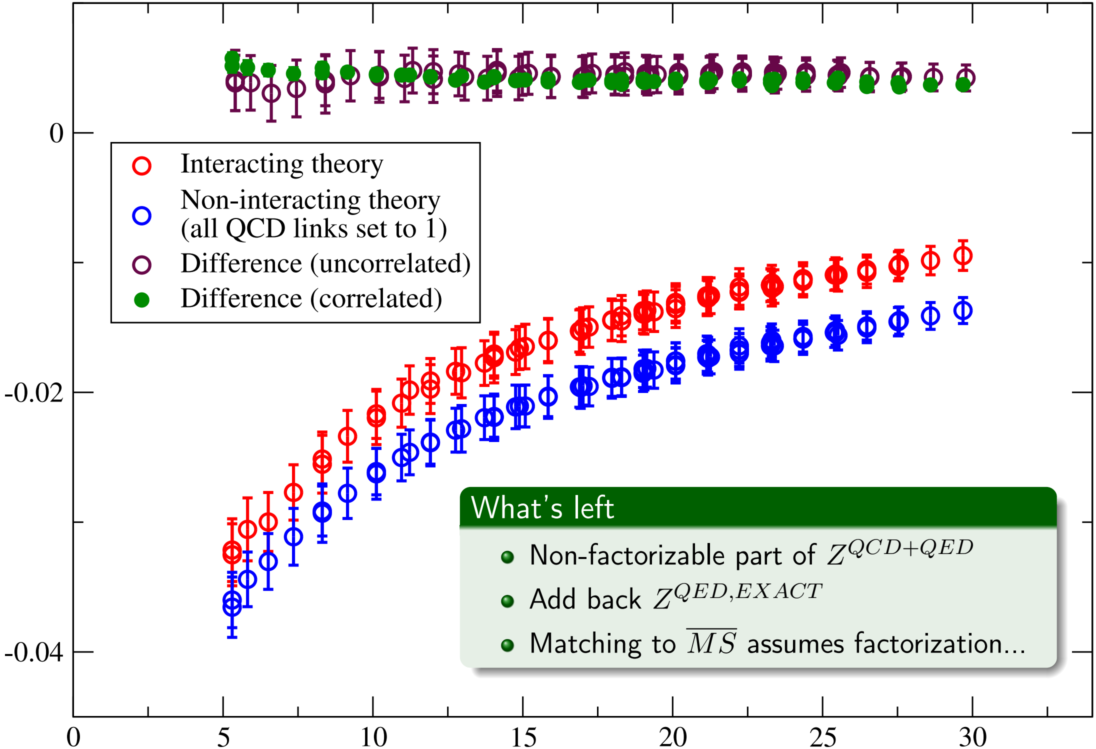

What's left

- Non-factorizable part of $Z^{QCD+QED}$
- Add back $Z^{QED,EXACT}$
- Matching to $\overline{MS}$ assumes factorization...