Exploring Yangian invariant Feynman integrals

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Fishnets: Conformal Field Theories and Feynman Graphs

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Based on 2304.04654 with V.Kazakov, F. Levkovich-Maslyuk
&
to appear with F. Levkovich-Maslyuk

Introduction

- Fishnet Feynman integrals [A. Zamolodchikov], [O. Gürdoğan, V. Kazakov] of various shapes are known to be Yangian invariant [Chicherin,Kazakov, Loebbert, Müller, Zhong]

- Yangian symmetry appears to be powerful tool for bootstrapping integrals [F.Loebbert, D.Müller, H.Münkler] and intimately related to geometry [C. Duhr, A. Klemm, F. Loebbert, C. Nega, F. Porkert] , recall Christoph's talk .

- It naturally prompts to look for the most general graphs that would have Yangian symmetry and study the general structure of the differential equations

- Reminder: what it means to be Yangian invariant
- Part 1: Yangian symmetry of Loom graph integrals
- Part 2: General Yangian differential equations

Yangian symmetry

Conformal invariance

- Conformal $\mathfrak{so}(D,2)$ symmetry can be represented as:

$$P_j^\mu = -i\partial_{x_j^\mu}, \quad D_j = x_j^\mu \partial_{x_j^\mu} - i\Delta_j, \quad L_j^{\mu\nu} = \ldots, \quad K_j^\mu = \ldots$$

$$P^\mu I_\Gamma\left(D, \Delta_i | x_i\right) = \sum_j P_j^\mu I_\Gamma\left(D, \Delta_i | x_i\right) = 0$$

where the sum goes over all external vertices.

- Massless integrals are conformal if the sum of propagator dimensions in each vertex is D

Yangian invariance

Additional symmetry :

$$\hat{P}^{\mu} I_{\Gamma}(D, \Delta | x) = 0$$

with

$$\widehat{P}^{\mu} = -\frac{i}{2} \sum_{j<k} [(L_j^{\mu\nu} + g^{\mu\nu} D_j) P_{k,\nu} - (j \leftrightarrow k)] + \sum_j s_j(\Gamma) P_j^{\mu}$$

where P_j^{μ} etc. act on the j'th external leg and parameters $s_j(\Gamma)$ depend on the graph.

Yangian invariance

For any Lie algebra $\mathfrak{g}$ one can construct an infinite algebra (not a Lie algebra) $Y(\mathfrak{g})$ generated by: J^a -level zero, $\hat{J}^a$ - level one generators, with relations [V. Drinfeld] :

$$[J^a, J^b] = f_c^{ab} J^c\,, \quad [J^a, \hat{J}^b] = f_c^{ab} \hat{J}^c\,,$$

$$\begin{aligned}[J^a,[\hat{J}^b,J^c]]+[J^b,[\hat{J}^c,J^a]]+[\hat{J}^c,[J^a,J^b]] &= \\ &= f^a{}_{pd} f^b{}_{qx} f^c{}_{ry} f^{xyd}\,\mathrm{Sym}\,(J^p,J^q,J^r) \\ &+ \dots\end{aligned}$$

- It can be realized as:

$$J^a = \sum_{j=1}^{n} J_j^a\,, \quad \hat{J}^a = \sum_{j<k=1}^{n} f^a_{bc} J_j^b J_k^c + \sum_{j=1}^{n} u_j J_j$$

where J_i^a form n copies of the initial algebra.

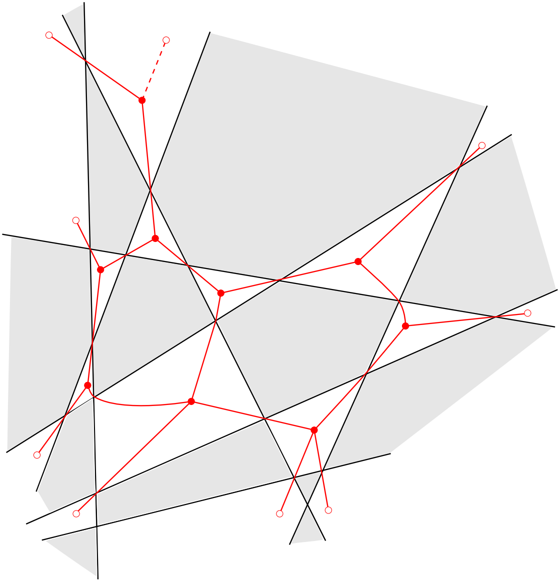
- In terms of spin chain n is the length of the chain. In terms of diagrams - the number of external legs.
- A single level one generator is enough to generate the Yangian - in our case the $\widehat{P}^\mu$

Part 1: Yangian invariance of Loom graph integrals

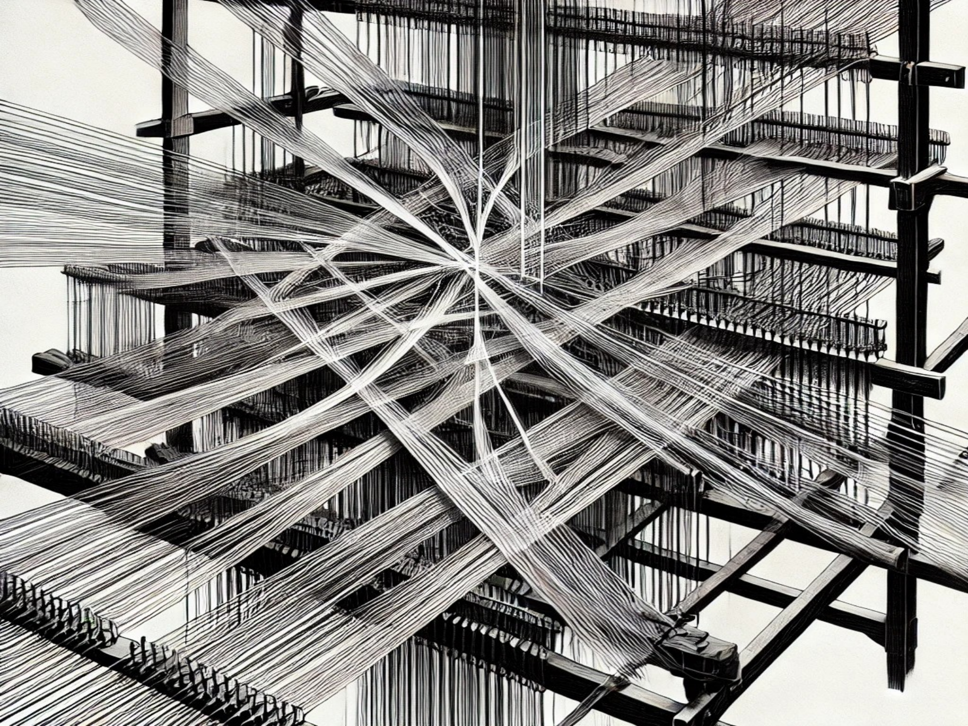
[2304.04654, V.Kazakov, F.Levkovich-Maslyuk,V.M.]

- Fishnets graphs are the only graphs present in the planar fishnet theory [O. Gürdoğan, V. Kazakov]
- [V. Kazakov,E.Olivucci] have constructed a generalization - the so called Loom CFT's in any dimensional with more generic graphs that dominate the planar limit.
- In our case we focus on single graphs and show that Loom graph integrals are Yangian invariant

The graphs



Graph drawn on a Baxter lattice



The graphs

- Vertices on the faces of a lattice made up from *straight lines*
- Internal vertex (the one integrated over) should necessarily connect to all neighbors
- External legs can also go in an "open" face
- Propagators

$$\frac{1}{|x_1 - x_2|^{2\Delta}} \ ,$$

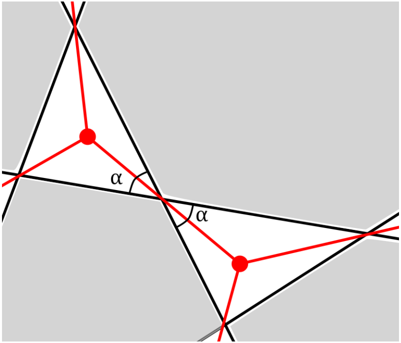
- where Δ is determined by the angle of the polygon through which the propagator passes as

$$\Delta_\alpha = D\frac{\pi - \alpha}{2\pi}.$$

- For a graph Γ we have:

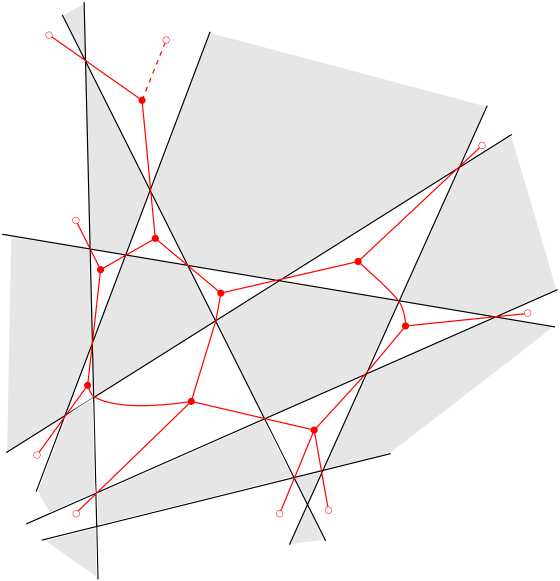
$$I_\Gamma\left(D, \Delta|x\right) = \int \prod_{k\in\text{internal}} d^D x_k \prod_{\langle i,j\rangle} \frac{1}{\left(x_i - x_j\right)^{2\Delta_{ij}}}$$

The graphs



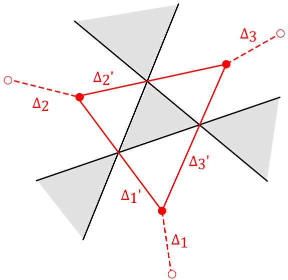
Fragment of a Feynman graph. The propagator between points x_1, x_2 is given by $(x_1 - x_2)^{-2\Delta}$ with the power determined by the angle α through which it passes according to $\Delta = D\frac{\pi-\alpha}{2\pi}$.

The graphs



Graph drawn on a Baxter lattice

The graphs



Graph with external legs on open faces

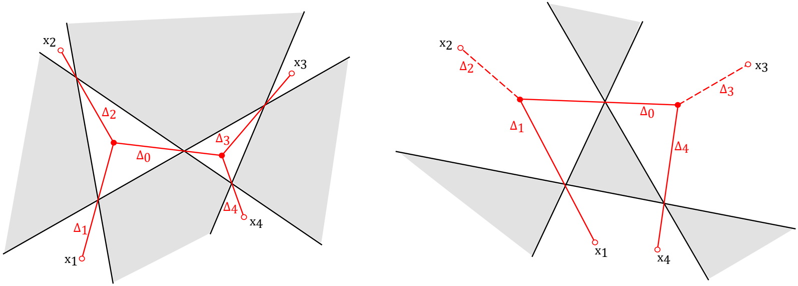
Notice that in addition to constraints coming from each vertex:

$$\Delta_1 + \Delta_1' + \Delta_3' = D \ , \quad \Delta_2 + \Delta_1' + \Delta_2' = D \ , \quad \Delta_3 + \Delta_2' + \Delta_3' = D \ ,$$

we have an extra 'non-local' (i.e. not associated to a single internal vertex) constraint

$$\Delta_1' + \Delta_2' + \Delta_3' = D/2 \ .$$

The graphs



The same Feynman graph drawn on two different looms.

Conformal invariance

- The construction ensures that

$$\sum_{i=1}^{n} \Delta_{\alpha_i} = D \sum_{i=1}^{n} \frac{\pi - \alpha_i}{2\pi} = D$$

- The star-triangle transform can be utilized and corresponds to moving the lines of the Baxter lattice.

The graphs

There are two main constraints on the integral that are considered

- The graph is necessarily drawn on the Baxter lattice
- The propagator powers are determined by the respective angles.

The combination of the two constrains leads to "non-local" relations between propagator powers of the graph.

Lasso method

- Usual origin of Yangian symmetry in spin-chains/integrable field theories is the RLL/RTT relation Enrico's talk .

$$\text{Lax operator (RLL)} : L(u)$$
$$\Downarrow$$
$$\text{Monoromy matrix (RTT)} : T(u) = \prod_{i \in \text{ext.}} L_i(u + \delta_i)$$

Conformal Lax operator

- In the case at hand, the Lax operator explicitly given by [D. Chicherin, S. Derkachov, A.P. Isaev]:

$$L_{\alpha\beta}\left(u_+ = u + \delta^+, u_- = u + \delta^-\right) = \begin{pmatrix} u_+ - \mathbf{p}\mathbf{x} & \mathbf{p} \\ \mathbf{x}(u_+ - u_-) - \mathbf{x}\mathbf{p}\mathbf{x} & \mathbf{x}\mathbf{p} + u_- \end{pmatrix}_{\alpha\beta} .$$

  with

$$\mathbf{x} = -i\bar{\sigma}^\mu x_\mu \ , \quad \mathbf{p} = -\frac{i}{2}\sigma^\mu \partial_{x_\mu} \ ,$$

- Spinor representation in the auxiliary space, and an infinite dimensional representation in the physical space

Yangian symmetry

- One shows that the Feynman graph is an eigenvector of the whole monodromy matrix [D.Chicherin, V. Kazakov, ,F.Loebbert, D. Müller,D.-l. Zhong] :

$$T_{\alpha\beta}(u,\vec{\delta})I_\Gamma = \lambda_\Gamma(u,\vec{\delta})\delta_{\alpha\beta}I_\Gamma$$

- As usually, the monodromy is constructed via a chain of Lax operators:

$$T(u,\vec{\delta}) = \prod_{i=1}^{n_{ext}} L_i(u+\delta_i^+, u+\delta_i^-) =$$
$$= \prod_{i=1}^{n_{ext}} L[\delta_i^+, \delta_i^-]$$

over all external lines.

Yangian symmetry

- According to the general prescription :

$$T(u,\vec{\delta}) \sim u^n \left(\mathbb{1} + \frac{1}{u} J + \frac{1}{u^2} \hat{J}(\vec{\delta}) + \dots \right)$$

- Shift $\delta_i \quad \Rightarrow \quad s_j(\Gamma)$ in the equations.

The Lasso method

- To prove Yangian symmetry (and determine the s_j parameters) we generalize the "Lasso method" [D.Chicherin, V. Kazakov, ,F.Loebbert, D. Müller,D.-l. Zhong] , which relies on the intertwining relations:

$$L_1(u+\Delta, u')L_2(v,u)\frac{1}{x_{12}^{2\Delta}} = \frac{1}{x_{12}^{2\Delta}}L_1(u,u')L_2(v,u+\Delta) \ ,$$

and action at special points:

$$L_{\alpha\beta}(u, u+D/2)\cdot 1 = (u+D/2)\delta_{\alpha\beta} \ ,$$

- Consistency of using consequent intertwining relations $\Leftrightarrow$ choice of $\delta^{\pm}$ on each leg.

- After that one has to still show that the relations between distant legs are consistent

Example: cross

- Consider the cross integral (all $\Delta_i = 1, D = 4$):

$$I_+(x_1, \dots x_4) = \int d^4 x_0 \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}$$

- The Lax chain is given by:

$$L_4[4,5] L_3[3,4] L_2[2,3] L_1[1,2] I_+ = [3]4][5][4] I_+$$

- Notation $[\delta] \equiv u + \delta$

Example: cross

- First we insert a total derivative

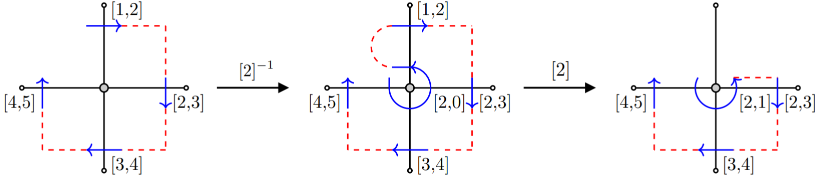
$$I_+(x_1, \dots x_4) = \int d^4x_0 \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \left(\frac{L_0^T[2,0] \cdot 1}{[2]} \right) =$$
$$= \frac{1}{[2]} \int d^4x_0 L_0[2,0] \frac{1}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2} \; .$$

- Then one pushes the Lax operators through the propagators:

$$L_1[\Delta_1, 2] L_0[2,0] \frac{1}{x_{10}^2} = \frac{1}{x_{10}^2} L_1[0,2] L_0[2,1]$$

The parameters are suited in such way that we obtain $L_1[0,2] \cdot 1 = [2]\mathbb{1} \cdot 1$

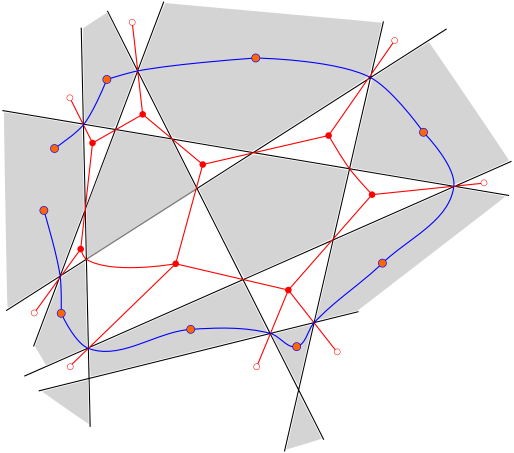
The Lasso method



From [D.Chicherin, V. Kazakov, ,F.Loebbert, D. Müller,D.-I. Zhong] .
Lasso for square 4D fishnets

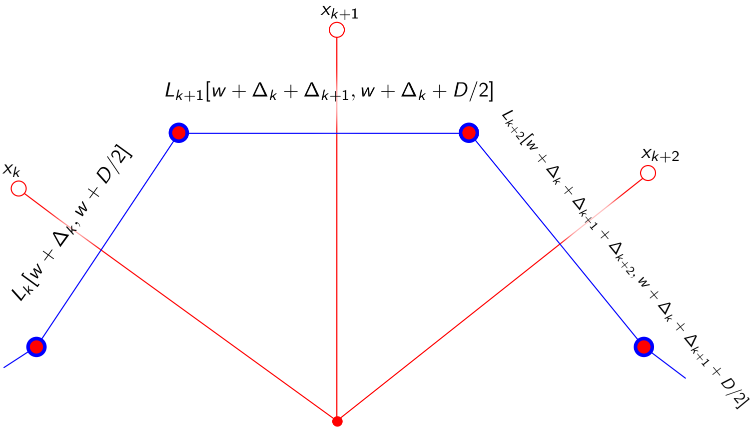
The Lasso method

Represent the Lax chain graphically:



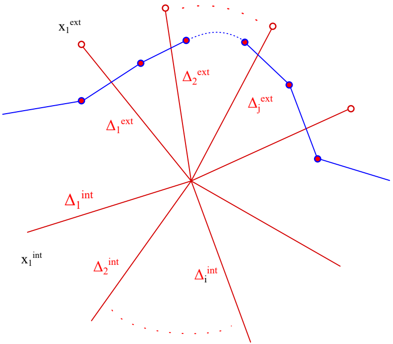
The blue "interval" represent Lax operators acting on the propagator which they cross. The "Lasso" represents their product, aka the monodromy matrix.

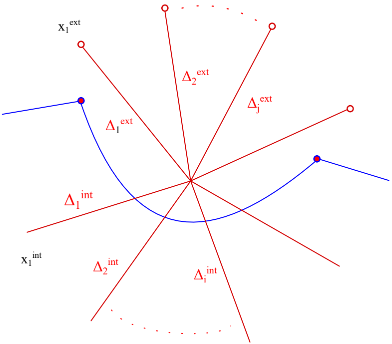
The Lasso method



Labels for consecutive Lax operators, that share a common vertex. Three of the vertices are depicted with the corresponding Lax operators

The Lasso method





The Lasso method

The move corresponds to the following transformation:

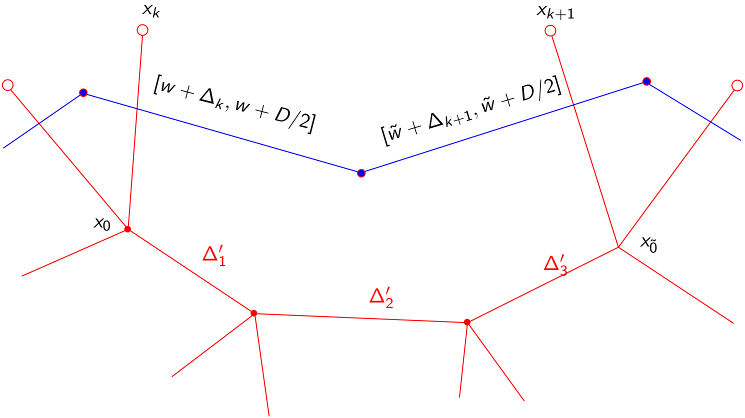
$$\int d^D x_0 \left(\prod_{i=1}^{n_{\rm ext}} L_i \left[w + \sum_{k=1}^{i} \Delta_k^{\rm ext}, w + \frac{D}{2} + \sum_{k=1}^{i-1} \Delta_k^{\rm ext} \right] \right) \times$$

$$\times \prod_{i=1}^{n_{\rm ext}} \frac{1}{(x_{0i}^{\rm ext})^{2\Delta_i^{\rm ext}}} \cdot \prod_{j=1}^{n_{\rm int}} \frac{1}{(x_{0j}^{\rm int})^{2\Delta_j^{\rm int}}} =$$

$$= \prod_{i=1}^{n_{\rm ext}} \left[w + \Delta_j^{\rm ext} + \frac{D}{2} \right] \times$$

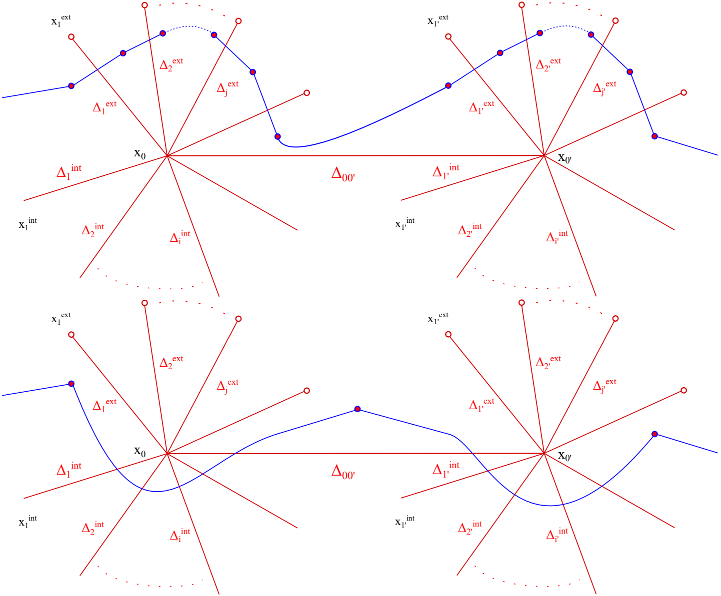
$$\times \quad \int d^D x_0 \prod_{i=1}^{n_{\rm ext}} \frac{1}{(x_{0i}^{\rm ext})^{2\Delta_i^{\rm ext}}} \cdot L_0 \left[w + \frac{D}{2}, w + \sum_{j=1}^{n_{\rm ext}} \Delta_j^{\rm ext} \right] \cdot \prod_{j=1}^{n_{\rm int}} \frac{1}{(x_{0j}^{\rm int})^{2\Delta_j^{\rm int}}} \ .$$

The Lasso method

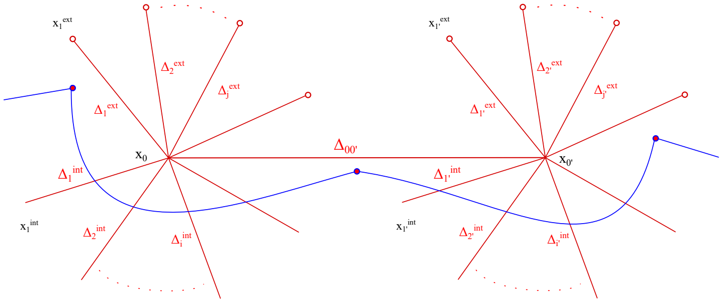


Prescription of labels for the Lax operator for consecutive external legs
Here $\tilde{w}$ is given by $\tilde{w} = w + \Delta_k + \sum_{i=1}^{p}(\Delta'_i - D/2)$ with $p = 3$.

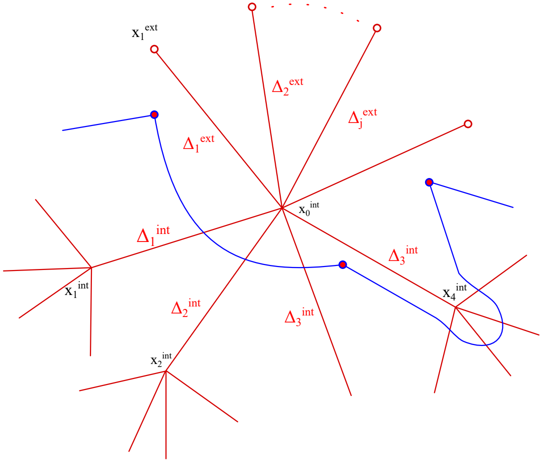
The Lasso method



The Lasso method



The Lasso method



The Lax chain of the lasso after transformation has been applied to x_0 and $x^{\mathrm{int}}_{n_{\mathrm{int}}}$, for $n_{\mathrm{int}} = 4$. As clearly seen from the picture, after applying it to all vertices x^{int}_j the lasso won't act on the the coordinate x_0 anymore.

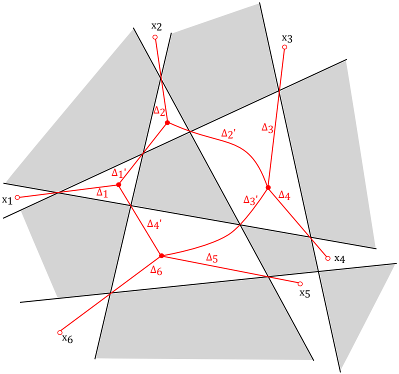
The Lasso method

- A minor modification from [Chicherin,Kazakov, Loebbert, Müller, Zhong] allows also to find eigenvalues and hence obtain the $s_j(\Gamma)$ parameters:

$$s_j = \frac{1}{2} \sum_{j \neq j} (\delta_k^+ + \delta_k^- + D/2)$$

- At least locally we prove that the Lasso method works, hence generic Loom fishnets are Yangian invariant
- We get the shift prescriptions for Lax for generic loom graphs $\Rightarrow$ explicit $s_i(\Gamma)$ in Yangian equations from the graph

The Lasso method: example



Square with 6 legs.

In addition to local relations between dimension for each vertex we have:

$$\Delta_6 + \Delta_2 + \Delta_5 = D \quad \Rightarrow \quad \text{5 independent parameters.}$$

The Lasso method: example

- The monodromy matrix:

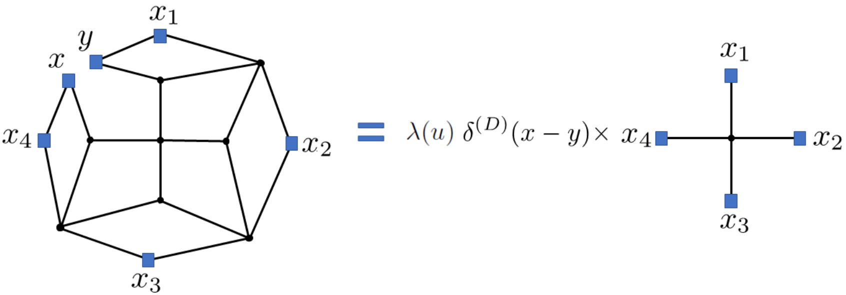
$$L_6[\Delta_{(11')} + D/2, \Delta_{(121'5)}]L_5[\Delta_{(121'5)} - D/2, \Delta_{(121')}]\times$$
$$\times\, L_4[D, \Delta_{(13)} + D/2]L_3[\Delta_{(13)}, \Delta_1 + D/2]\times$$
$$\times\, L_2[\Delta_{(121')} - D/2, \Delta_{(11')}]L_1[\Delta_1, D/2]$$

- The parameters

$$s_j = \Big\{0, -\Delta_1' - \frac{\Delta_1}{2} - \frac{\Delta_2}{2} + D/2, -\frac{\Delta_1}{2} - \frac{\Delta_3}{2}, -\frac{\Delta_3}{2} - D/2,$$
$$-\Delta_1' - \frac{\Delta_1}{2} - \Delta_2 - \frac{\Delta_5}{2} + D/2, -\Delta_1' - \frac{\Delta_1}{2} - \frac{\Delta_2}{2} - \frac{\Delta_5}{2}\Big\}$$

Infinite-dimensional auxiliary space

- One can take the auxiliary space to be infinite dimensional.



Part 2: Studying Yangian invariant integrals

[to appear, F.Levkovich-Maslyuk,V.M.]

- Which kinds of graphs are in the class of Loom graphs and hence Yangian symmetric?
  - For a given graph, how many free parameters?
  - Is the Loom equivalent to dual conformal symmetry?
  - What if we look for Yangian or Yangian like symmetry - will we recover the Loom?

- What is the structure of the Yangian equations?
  - How many independent constraints?
  - Are they always consistent?
  - What is the space of solutions?

Question where analyzed in [F.Loebbert, D.Müller, H.Münkler]

Structure of the equations

- Explicit form of the level one momentum generators:

$$\hat{P}^{\mu} = \frac{1}{2}\sum_{j<k}\left(\delta^{\mu\alpha}\delta^{\lambda\nu} - \delta^{\nu\alpha}\delta^{\mu\lambda} - \delta^{\mu\nu}\delta^{\alpha\lambda}\right)(x_j - x_k)^{\alpha}\frac{\partial^2}{\partial x_j^{\lambda}\partial x_k^{\nu}} +$$

$$+\sum_j s_j \frac{\partial}{\partial x_j^{\mu}}$$

- Conformal symmetry implies that:

$$I_\Gamma(x) = \prod_{i<j} x_{ij}^{2\beta_{ij}} I_\Gamma^{(0)}(\xi^A)$$

Cross ratios:

$$\xi^A = \prod_{i<j} x_{ij}^{2\alpha_{ij}}$$

with

$$\alpha_{ij}^A = \alpha_{ji}^A\,,\ \alpha_{ii}^A = 0\,,\ \sum_i \alpha_{ij}^A = 0$$

Where A labels different $\frac{N(N-3)}{2}$ cross ratios $\left(ND - \frac{(D+1)(D+2)}{2}\right)$

- Conformal weights satisfy

$$\beta_{ij} = \beta_{ji}\ \ \beta_{ii} = 0\ \sum_i \beta_{ij} = -\Delta_i$$

- The level one generator then rewrites in terms of cross ratios as [F.Loebbert, D.Müller, H.Münkler]:

$$\hat{P}^{\mu} = \sum_{jk} \frac{x_{jk}^{\mu}}{x_{jk}^{2}} \mathrm{PDE}_{jk}$$

- Equations PDE_{jk} are purely in terms of cross ratios

Example: The cross

- Following [F.Loebbert, D.Müller, H.Münkle]

$$I_+(x) = \int \frac{d^D x_0}{x_{10}^{2\Delta_1} x_{20}^{2\Delta_2} x_{30}^{2\Delta_3} x_{40}^{2\Delta_4}} =$$

$$= x_{14}^{2\Delta_2+2\Delta_3-D} x_{13}^{2\Delta_4-D} x_{34}^{-2\Delta_3-2\Delta_4+D} x_{24}^{-2\Delta_2} I_+^{(0)}(u,v)$$

- Cross ratios are chosen as:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$$

$$\alpha_{12}^1 = \alpha_{34}^1 = -\alpha_{13}^1 = -\alpha_{24}^1 = 1$$
$$\alpha_{14}^2 = \alpha_{23}^1 = -\alpha_{13}^1 = -\alpha_{24}^1 = 1$$

- Out of $4 \cdot 3/2 = 6\ \mathrm{PDE}_{ik}$ only 2 are independent

$$
\begin{aligned}
0 = \big(&\alpha\beta + (\alpha+\beta)(u\partial_u + v\partial_v) + \\
&+(u\partial_u + v\partial_v)^2 - u\partial_u^2 - \gamma\partial_u\big)\, I_+^{(0)}(u,v) \\
0 = \big(&\alpha\beta + (\alpha+\beta)(u\partial_u + v\partial_v) + \\
&+(u\partial_u + v\partial_v)^2 - v\partial_v^2 - \gamma'\partial_v\big)\, I_+^{(0)}(u,v).
\end{aligned}
$$

- 4-dim solution space - Appel F_4 functions
- + choice of convergence region + symmetries + boundary conditions [F.Loebbert, D.Müller, H.Münkler]

- Invert logic - suppose we have a Yangian invariant function, i.e. we don't know the graph and the Δ_i, s_i are generic.

  In the previous example s_i are fixed and Δ_i satisfy the conformal condition.

- Are the equations consistent?

- How do equation look for generic number of points and choice of cross ratios.

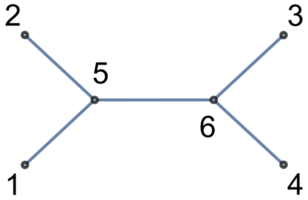
- Start with 4-point again. The system PDE_{ik} is overdetermined in general.
- In particular a linear combination of equations produces

$$\frac{\partial}{\partial u} I^{(0)}(u, v) = 0$$

- Demand that there are no equations of first order, this is possible only in three cases :

$$\sum_{i=1}^{4} \Delta_i = D\,,$$
$$\Delta_1 + \Delta_2 = \Delta_3 + \Delta_4$$
$$\Delta_1 + \Delta_4 = \Delta_2 + \Delta_3$$

  and s_i parameters are fixed.
- We obtain the same conditions if we look for a non-zero series solution.


2
3
5
6
1
4

Higher points

- Higher points - long equations, many cross ratios. We need general form of PDE_{ik}.
- Six-points - 9 cross ratios, 15 PDE_{ik}. [F.Loebbert, D.Müller, H.Münkler]

General equation

- General form of the equation

$$\mathrm{PDE}_{ik} = 2\left(\sum_{l>j>i} - \sum_{l<j<i} + \sum_{l<k<i,j} - \sum_{l>k>i,j}\right)\chi_{iklj}\theta_{il}\theta_{jk} +$$

$$+\sum_{j\neq i}(\delta_{j>i} - \delta_{j<i})\theta_{ik}\theta_{ij} + \delta_{i>k}\left(2\sum_{j=k+1}^{i-1}\Delta_j + \Delta_i + D\right)\theta_{ik} -$$

$$-\delta_{i<k}\left(2\sum_{j=i+1}^{k-1}\Delta_j + \Delta_i + D\right)\theta_{ik} + 2(s_k - s_i)\theta_{ik}$$

where:

$$\chi_{iklj} = \frac{x_{ik}^2 x_{lj}^2}{x_{il}^2 x_{kj}^2}\,, \quad \theta_{ij} = \sum_A \alpha_{ij}^A \xi^A \frac{\partial}{\partial \xi^A} + \beta_{ij}$$

- Given any specific choice of cross-ratios the equations above are immediately rewritten. The equations itself is written in an invariant form. Reproduce 15 eq-s of [F.Loebbert, D.Müller, H.Münkler]

- Asking for consistency conditions at higher N is still hard.

- Instead notice:

$$\mathcal{L}_{iklj} = \theta_{ik}\theta_{lj} - \chi_{iklj}\theta_{il}\theta_{kj}$$

Which are just:

$$L_{iklj} = \frac{\partial^2}{\partial(x_{ik}^2)\partial(x_{lj}^2)} - \frac{\partial^2}{\partial(x_{il}^2)\partial(x_{kj}^2)}$$

- L_{iklj} lie in the GKZ [I.Gel'fand, A. Zelevinskii, M. Kapranov (1989)] differential ideal (conjecturaly equivalent) [A.Pal,K.Ray]

- GKZ equations have known solutions in therms of $\mathcal{A}$-hypergeometric functions and can be treated with hypergeometric methods

with integral representations. We only note that among the Euler type integrals associated with systems of the form (0.2) there are the integrals $\int \Pi P_i (t_1, \ldots, t_n)^{\alpha_i} t_1^{\beta_1} \ldots t_n^{\beta_n} dt_1 \ldots dt_n$, where P_i are polynomials, i.e., practically all integrals which arise in quantum field theory. A separate paper will be devoted to these integrals.

GKZ systems

- GKZ system of differential equations is defined with a $n \times N$ matrix $\mathcal{A}$ and a n vector b, such that vector $(1, \ldots, 1)$ is in the row span of $\mathcal{A}$.
- For all $\ell \in \mathbb{Z}^N$ and

$$\mathcal{A}\ell = 0$$

set:

$$\prod_{\ell_i > 0} \partial_{z_i}^{\ell_i} - \prod_{\ell_i < 0} \partial_{z_i}^{-\ell_i}$$

and

$$\sum_j \mathcal{A}_{ij} z_j \frac{\partial}{\partial z_j} - b_i$$

- GKZ system has a finite space of solutions, convergent series representations are determined from the data.
- Solutions come in form of generalized hypergeometric series:

$$\sum_{u \in \ker \mathcal{A}} \frac{1}{\prod_{i=1}^{N} \Gamma(\gamma_i + u_i + 1)} z_i^{u_i + \gamma_i}$$

where $\mathcal{A}\gamma = b^T$

- GKZ systems coming from Yangian correspond to matrices $\mathcal{A}$ of size $N \times \frac{N(N-1)}{2}$:

$$A_{i,jk} = \delta_{ik} + \delta_{ij}$$

- The equations look like [A.Pal,K.Ray]

$$\sum_{\substack{j,k=1 \\ j<k}}^{N} \mathcal{A}_{i,jk} x_{jk}^2 \partial_{jk} + \Delta_i, \quad \forall i$$

$$\prod_{\ell_{ij}^A>0} \partial_{ij}^{\ell_{ij}^A} - \prod_{\ell_{ij}^A<0} \partial_{ij}^{-\ell_{ij}^A}, A = 1, 2, \cdots, N_0$$

for

$$\sum_{\substack{(jk)|j<k \\ j,k=1,2,\cdots,N}} \mathcal{A}_{i,jk} \ell_{jk}^A = 0 \quad \Leftrightarrow \quad \ell_{jk}^A = \alpha_{jk}^A$$

Yangian vs GKZ

- Is PDE_{ik} just a sum of $\mathcal{L}_{ikjl}$?
- The expression

$$\mathrm{PDE}_{ik} - \left(2\left(\sum_{l>j>i} - \sum_{l<j<i} + \sum_{l<k<i,j} - \sum_{l>k>i,j}\right)\mathcal{L}_{iklj}\right)$$

  is a first order operator.
- The difference is identically zero iff:

$$\sum_{i=1}^{n} \Delta_i = D$$

  with s_i fixed. We recover the cross case [A.Pal,K.Ray]

- Introduce conjugated GKZ system operators

$$L^{\kappa}_{iklj} = \left(\prod_{ij} x_{ij}^{2\kappa_{ij}}\right)^{-1} L_{iklj} \left(\prod_{ij} x_{ij}^{2\kappa_{ij}}\right)$$

and its counterpart $\mathcal{L}^{\kappa}_{iklj}$, where:

$$\mathcal{L}^{\kappa}_{iklj} = \theta^{\kappa}_{ik}\theta^{\kappa}_{lj} - \chi_{iklj}\theta^{\kappa}_{il}\theta^{\kappa}_{kj}\,, \qquad \theta^{\kappa}_{ij} = \theta_{ij} + \kappa_{ij}$$

- Then ask to eliminate s_i and κ from:

$$\mathrm{PDE}_{ik} - \left(2\left(\sum_{l>j>i} - \sum_{l<j<i} + \sum_{l<k<i,j} - \sum_{l>k>i,j}\right)\mathcal{L}^{\kappa}_{iklj}\right) = 0$$

- At 4 points we get the conditions:

$$(\Delta_1 + \Delta_2 - \Delta_3 - \Delta_4)(\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4)(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 - D) = 0$$

- At 5 points we get many conditions like:

$$\begin{aligned} 2D &= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \\ D &= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 \\ \Delta_1 + \Delta_4 + \Delta_5 &= \Delta_2 + \Delta_3 \\ D + \Delta_3 &= \Delta_1 + \Delta_2 + \Delta_4 + \Delta_5 \\ &\vdots \end{aligned}$$

Conclusion & Speculations

Conclusion & Speculations

- Yangian symmetry proved for any Loom graphs. Explicit prescription to evaluate the s_i parameters
- The desired graphs are more common among planar graphs. However, often one needs unusual powers of propagators.

Conclusion & Speculations

- The general form of the cross ratios equations is given.
- Consistency of the equations produces all the possible cases, when a graph can be Yangian invariant
- In all consistent cases the Yangian system is equivalent to a GKZ systems with a very special type of toric matrices $\mathcal{A}$

Further directions

- Further study of the infinite dimensional auxiliary space.
- GKZ systems are related to Calabi-Yau geometry. Is our observation related to some D-dimensional deformation of P.F. equation for fishnet CY periods?
- Yangian invariant integrals in D dimensions are generalized hypergeometric functions. Relation to work of [C.Duhr, F.Porkert] in two-dimensions?