Supersymmetric diagrams and the dynamical fishnet

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Recent and Current Work on Integrable Fishnet QFTs: Eclectic Spin Chains, Partition Functions, Supergraphs

- Earlier work with Asger Ipsen and Leo Zippelius on the one-loop dilatation operator of dynamical fishnet models, arXiv:1812.08794.

- Work with Changrim Ahn and Luke Corcoran on "eclectic" spin chains, arXiv:2010.14515, arXiv:2112.04506, arXiv:2207.02885.

- Work with Moritz Kade on brick wall, arXiv:2309.16640, and super brick wall, arXiv:2408.05805, models (this talk).

- Ongoing work by Moritz on super fishnet ABJM models (this talk).

- Ongoing work w. Changrim Ahn and Moritz Kade on fishnet boundaries.

Motivation

There has been some recent interest in strongly twisted planar $\mathcal{N}=4$ Super Yang-Mills Theory. This is a non-unitary yet still conformal and integrable quantum field theory. It was proposed that the model is simpler than the undeformed or finitely twisted theory, and that its integrability can be more easily understood: A kind of toy model for the $\mathcal{N}=4$ toy model. The crucial feature is a vastly reduced set of Feynman diagrams of “fishnet type”. Many exact computations are possible. Conceptually, it yields a rather clear picture of the previously rather mysterious “mirror particles” in the TBA approach.

However, so far it has not yet much helped to better understand $\mathcal{N}=4$ SYM integrability as such. Arguably this is due to the fact the Feynman graph expansion is now too simple, and it is not clear how to move back to the “real thing”. A first step would be to understand the “dynamical” fishnet models, where the rigid lattice structures are starting to “melt”.

Integrable Textbook Quantum Field Theories, I

ϕ^4-theory $\to$ biscalar fishnet model: [A. Zamolodchikov '80; O. Gürdoğan, V. Kazakov '15]

$$\mathcal{L}^{\mathrm{FN}} \;=\; \mathrm{N}\,\mathrm{tr}\left[\frac{1}{2}\sum_{j=1}^{2}\partial^{\mu}\phi_j^{\dagger}\partial_{\mu}\phi_j+\xi^2\phi_1^{\dagger}\phi_2^{\dagger}\phi_1\phi_2\right].$$

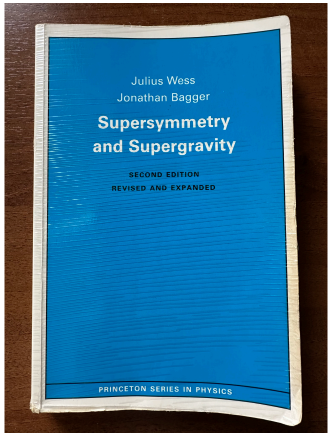
Yukawa theory $\to$ brick wall model: [A. Pittelli, M. Preti '18; MS + MK '23]

$$\mathcal{L}^{\mathrm{BW}} \;=\; \mathrm{N}\,\mathrm{tr}\left[-\frac{1}{2}\partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi+i\sum_{k=1}^{2}\bar{\psi}_k\partial\!\!\!/\psi_k+\rho\left(\psi_1\phi\psi_2+\bar{\psi}_1\phi^{\dagger}\bar{\psi}_2\right)\right].$$

Cooking recipe: Take a massless textbook QFT, turn fields into matrices, take the planar limit, make a chiral projection $\to$
Obtain a non-unitary, conformal, integrable QFT.

Integrable Textbook Quantum Field Theories, II

Apply to the following text book model:



Wess-Zumino model $\rightarrow$ super brick wall model: [MS + MK '23]

$$\mathcal{L}^{\mathrm{SBW}} = \mathrm{N}\,\mathrm{tr}\left(\sum_{i=1}^{3} \Phi_i^\dagger \Phi_i + i\xi\,\bar{\theta}^2\,\Phi_1\Phi_2\Phi_3 + i\xi\,\theta^2\,\Phi_1^\dagger\Phi_2^\dagger\Phi_3^\dagger\right).$$

Here the $\mathcal{N}=1$ superspace action for the three superfields Φ_j is

$$S^{\mathrm{SBW}} = \int \mathrm{d}^4x\;\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\,\mathcal{L}^{\mathrm{SBW}}.$$

Should be non-unitary, conformal, and integrable according to the recipe.

Relation to Strongly Twisted $\mathcal{N}$=4 SYM, I

Start from planar, integrable, three-parameter γ-deformed $\mathcal{N}$=4 SYM.

Perform double-scaling limit [O. Gürdoğan, V. Kazakov '15; Sieg, Wilhelm '16; Kazakov et.al. '18].

$$g = \frac{\sqrt{\lambda}}{4\pi} \longrightarrow 0 \quad \text{and} \quad q_j = e^{-i\gamma_j/2} \longrightarrow \infty$$

Write $q_j := g^{-1}\,\xi_j$ and take g to zero. This yields:

$$\begin{aligned} \mathcal{L}_{\text{int}} \;\; = \;\; & \mathrm{N}\,\mathrm{tr}\left(\xi_3^2\,\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2 + \xi_2^2\,\phi_3^\dagger\phi_1^\dagger\phi_3\phi_1 + \xi_1^2\,\phi_2^\dagger\phi_3^\dagger\phi_2\phi_3\right) \\ & + \mathrm{N}\,\mathrm{tr}\left(i\,\sqrt{\xi_2\xi_3}(\psi_3\phi_1\psi_2 + \bar{\psi}_3\phi_1^\dagger\bar{\psi}_2) + \text{cyclic}\right) \end{aligned}$$

Gauge fields "decouple". Biscalar fishnet model: $\xi_1 = \xi_2 = 0, \xi_3 = \xi$.
(The brick wall model cannot quite be obtained like that. One may derive it from a double-scaling limit of $\mathcal{N}$=2 SCFT.)

Relation to Strongly Twisted $\mathcal{N}$=4 SYM, II

The full three-parameter model is much richer than the fishnet model, it has been dubbed ,,dynamical fishnet" model. [V. Kazakov, E. Olivuccci, M. Preti '19]

While also integrable, its integrability is much harder to see and use. So far no graph-building operator, no star-triangle relation, no Yang-Baxter equation, no R-matrix, ...

However, there is a special point in parameter space, $\xi_1 = \xi_2 = \xi_3 := \xi$:

$$\begin{aligned} \mathcal{L}_{\text{int}} \;&=\; \mathrm{N}\,\mathrm{tr}\,\xi^2 \left(\phi_1^\dagger \phi_2^\dagger \phi_1 \phi_2 + \phi_3^\dagger \phi_1^\dagger \phi_3 \phi_1 + \phi_2^\dagger \phi_3^\dagger \phi_2 \phi_3\right) \\ &\quad + \mathrm{N}\,\mathrm{tr}\, i\xi \left((\psi_3 \phi_1 \psi_2 + \bar{\psi}_3 \phi_1^\dagger \bar{\psi}_2) + \text{cyclic} \right) \end{aligned}$$

Obtained from the double scaling limit of β-deformed $\mathcal{N}$=4 SYM .
Crucially for us, it keeps its $\mathcal{N}$=1 supersymmetry!
Idea: Use $\mathcal{N}$=1 supergraphs.

$\mathcal{N} = 1$ supersymmetric β-deformation of $\mathcal{N} = 4$ SYM

Written in $\mathcal{N} = 1$ superspace, with $q = e^{i\beta}$, it reads [Jin,Roiban '12]

$$S = \int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \sum_{i=1}^{3} \mathrm{tr}\left[e^{-gV}\Phi_i^\dagger e^{gV}\Phi_i\right] + \frac{1}{2g^2}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{tr}\left[W^\alpha W_\alpha\right]$$

$$+\, ig\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{tr}\left[q\, \Phi_1\Phi_2\Phi_3 - q^{-1}\Phi_1\Phi_3\Phi_2\right] + \mathrm{h.c.}\,,$$

where V = vector superfield (encoding the gauge field), Φ_i are three chiral superfields, and $W_\alpha = i\bar{D}^2\left(e^{-gV}D_\alpha e^{gV}\right)$, $D, \bar{D}$ super derivatives. After double scaling, we find the announced super brick wall model action

$$S^{\mathrm{SBW}} = \mathrm{N}\,\mathrm{tr}\int \mathrm{d}^4x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\left[\sum_{i=1}^{3}\Phi_i^\dagger\Phi_i + i\xi\, \bar{\theta}^2\, \Phi_1\Phi_2\Phi_3 + i\xi\, \theta^2\, \Phi_1^\dagger\Phi_2^\dagger\Phi_3^\dagger\right].$$

Super Fields and Super Feynman Rules

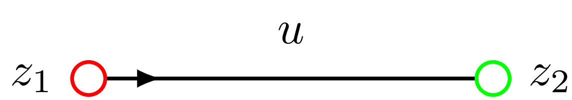
Chiral superfield at point $z = (x, \theta, \bar{\theta})$ in superspace:

$$\Phi_i(z) = e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu}\left[\phi_i(x) + \sqrt{2}\,\theta\psi_i(x) + \theta^2 F_i(x)\right]$$

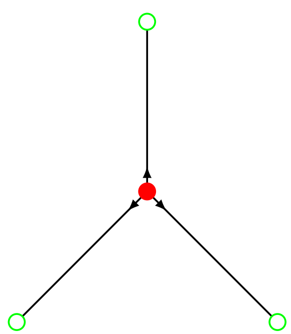
Here ϕ_i= scalars, ψ_i= fermions, F_i= auxiliary fields.

Generalized Superfield propagator w. $x^\mu_{1\bar{2}} := x^\mu_1 - x^\mu_2 + i\left[\theta_1\sigma^\mu\bar{\theta}_1 + \theta_2\sigma^\mu\bar{\theta}_2 - 2\theta_1\sigma^\mu\bar{\theta}_2\right]$

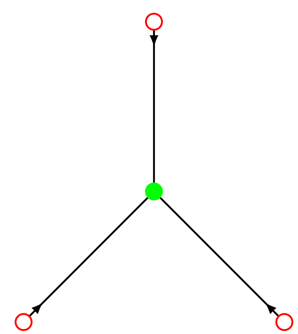
$$\left\langle \Phi_i(z_1)\Phi^\dagger_j(z_2)\right\rangle_u = e^{i\left[\theta_1\sigma^\mu\bar{\theta}_1 + \theta_2\sigma^\mu\bar{\theta}_2 - 2\theta_1\sigma^\mu\bar{\theta}_2\right]\partial_{1,\mu}}\frac{\delta_{ij}}{\left[x^2_{12}\right]^u} = \frac{\delta_{ij}}{\left[x^2_{1\bar{2}}\right]^u}$$

$=$ 

Super vertices (chiral and anti-chiral):



$\sim -\xi \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; \delta^{(2)}(\bar{\theta})\,,$



$\sim -\xi \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; \delta^{(2)}(\theta)\,.$

A Superconformal Near-Star-Triangle Relation

The success with finding a "spin chain" encoding the Feynman graphs of the biscalar fishnet model rests on the existence of a star-triangle relation, which then implies an R-matrix satisfying a Yang-Baxter-equation.

We could not find it yet. (Maybe it does not exist?)

However, we found the following relation due to Osborn: [Osborn '98; Dolan, Osborn '00]

$$i \int \mathrm{d}^4 x_0 \, \mathrm{d}^2\theta_0 \, \mathrm{d}^2\bar{\theta}_0 \, \delta^{(2)}(\theta_0) \, \frac{1}{\left[x_{1\bar{0}}^2\right]^{u_1}} \frac{1}{\left[x_{2\bar{0}}^2\right]^{u_2}} \frac{1}{\left[x_{3\bar{0}}^2\right]^{u_3}}$$

$$= -4\, r(u_1,u_2,u_3)\, \frac{(\theta_{12}\theta_{13})\, x_{23,+}^2 + (\theta_{23}\theta_{21})\, x_{31,+}^2 + (\theta_{31}\theta_{32})\, x_{12,+}^2}{[x_{12,+}^2]^{2-u_3}[x_{23,+}^2]^{2-u_1}[x_{31,+}^2]^{2-u_2}}$$

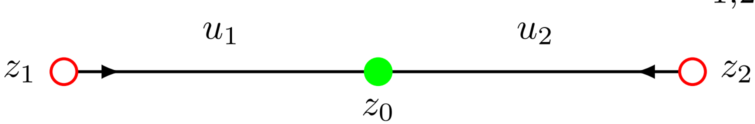
with $u_1+u_2+u_3=3$ and where $x_{ij,+}^\mu = x_{i,+}^\mu - x_{j,+}^\mu$ and $x_\pm^\mu = x^\mu \pm i\theta\sigma^\mu\bar{\theta}$, $\theta_{ij}=\theta_i-\theta_j$ and $r(u_1,u_2,u_3)=\pi^2 a(u_1)a(u_2)a(u_3)$ with $a(u)=\frac{\Gamma(2-u)}{\Gamma(u)}$.

Clearly the l.h.s. is a star, but r.h.s. does not quite factor into a triangle.

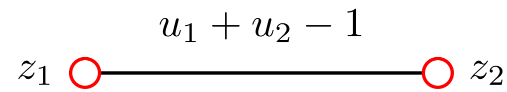
Super Chain Relations

As in the fishnet case, we may derive the following chain relation

$$\left[i \int \mathrm{d}^4 x_0 \, \mathrm{d}^2 \bar{\theta}_0 \frac{1}{\left[x_{1\bar{0}}^2\right]^{u_1}} \frac{1}{\left[x_{2\bar{0}}^2\right]^{u_2}} \right]_{\substack{\theta_0 = 0 \\ \bar{\theta}_{1,2}=0}} = -4\, r(3-u_1-u_2, u_1, u_2)\, \frac{\theta_{12}^2}{\left[x_{12}^2\right]^{u_1+u_2-1}} \,,$$

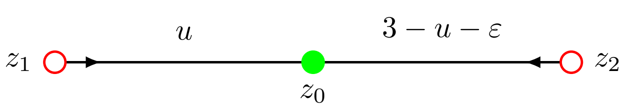


$$= -4\, r(3-u_1-u_2, u_1, u_2)$$



as well as its chiral analogue. A crucial special case is

$$\lim_{\varepsilon \to 0}$$



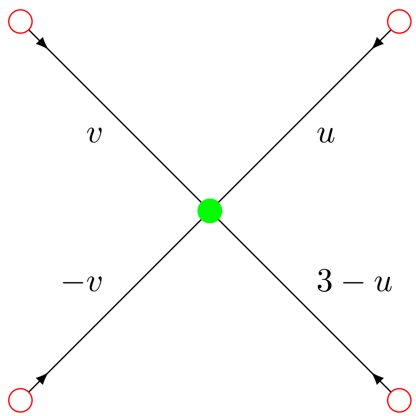
$$= -4\, \pi^4 \cdot a(u)\, a(3-u)$$

$$= -4\, \pi^4 \cdot a(u)\, a(3-u)\, \delta^{(2)}(\theta_{12})\, \delta^{(4)}(x_{12})$$

plus is chiral counterpart.

Super X-Unity Relations

Intriguingly, for our super brick wall model an x-unity relation also exists. It may be derived from the near-star-triange relation. We found



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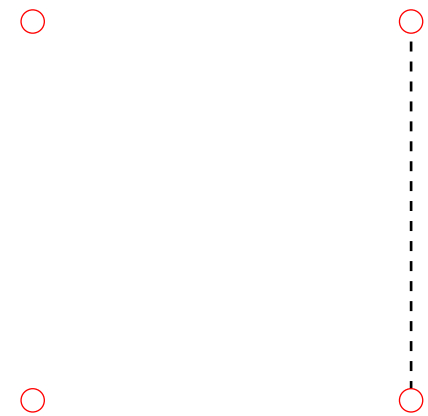

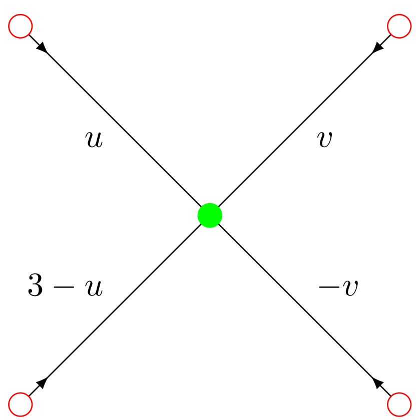

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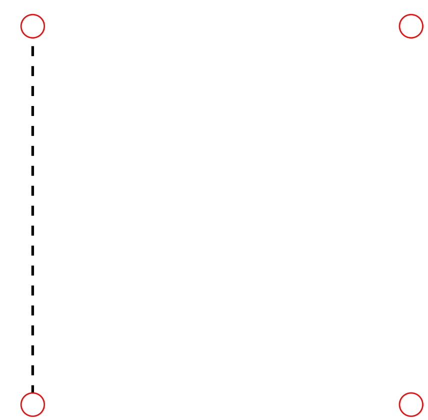
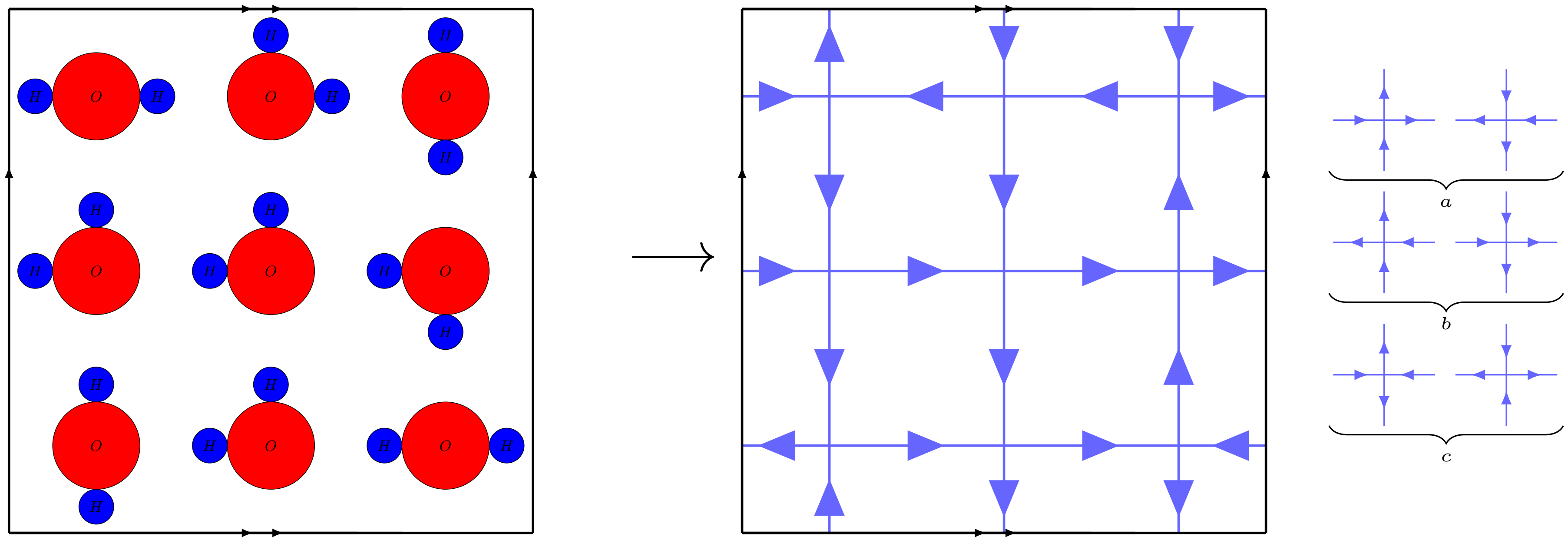


as well as their two chiral analogues.

Free energy in the thermodynamic limit of water ice



Experimental residual entropy: 1.540 ± 0.001

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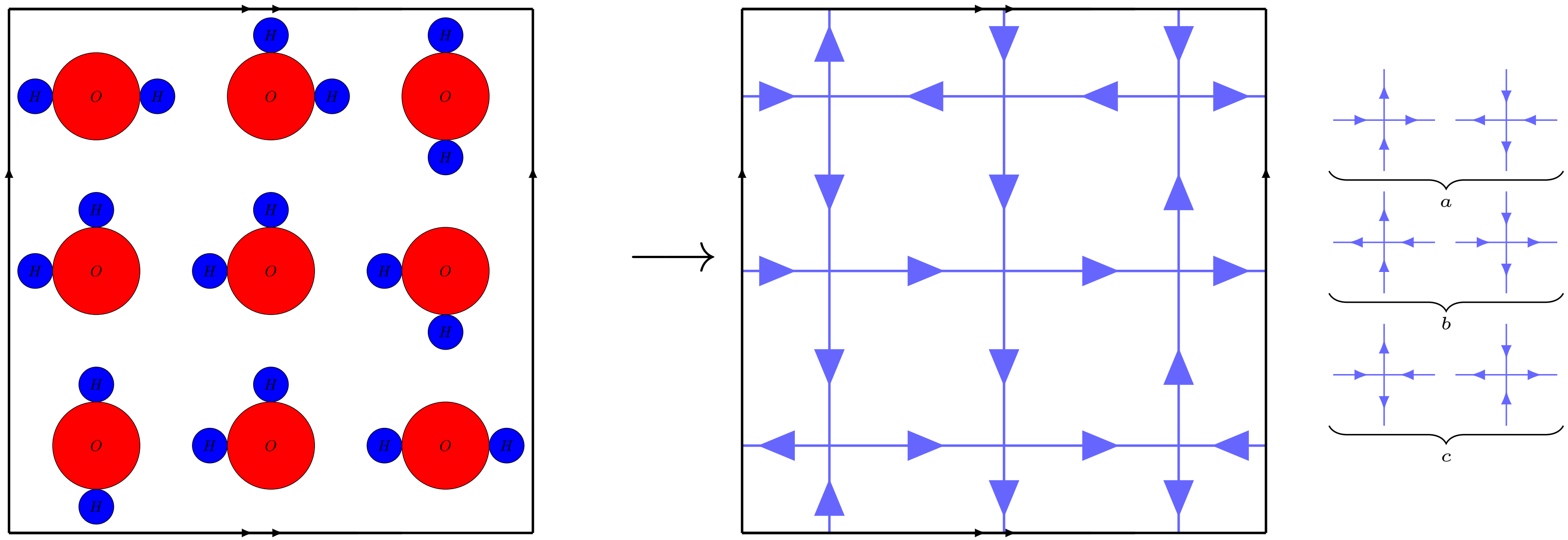


Experimental residual entropy: 1.540 ± 0.001

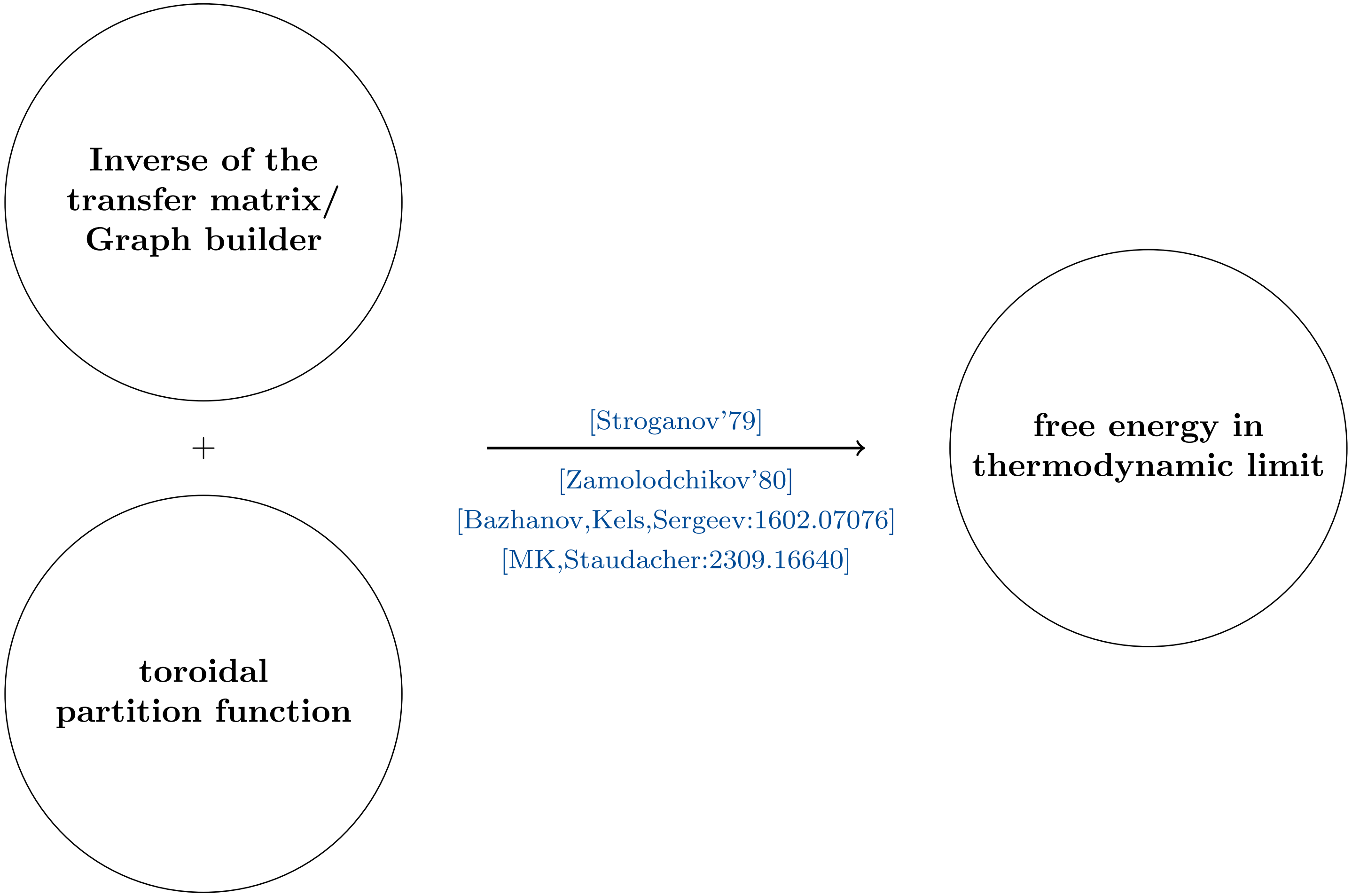
$$Z_{MN} = \sum_{\Omega \in \Lambda_{MN}} a^{n_1+n_2} b^{n_3+n_4} c^{n_5+n_6}$$

$$K(a,b,c) = \lim_{M,N \to \infty} (Z_{MN})^{1/MN}$$

$$K(1,1,1) = \left(\frac{4}{3}\right)^{3/2} \approx 1.5396$$ [Lieb '67]

Also obtained by **method of inversion relations** [Stroganov '79]

Method of inversion relations

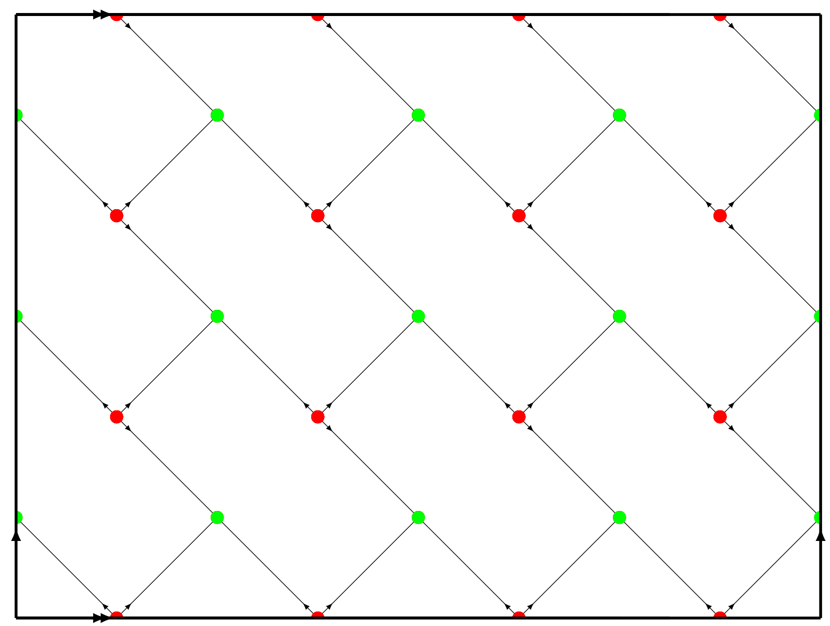


Applicable to **integrable QFTs** by interpreting them as integrable lattice models with **generalized propagators** as weights, results in **exact value for critical coupling**.

Vacuum superdiagrams

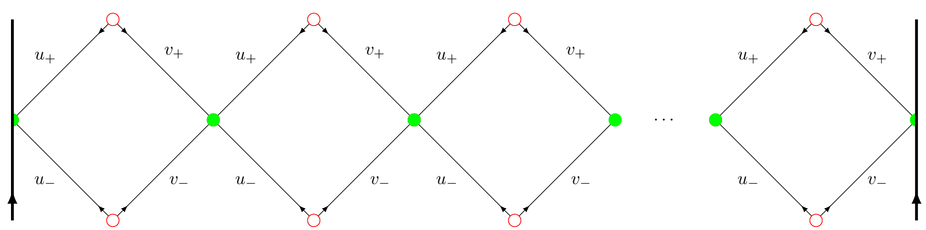
Row-matrices are stacked up and periodically identified to form a **toroidal vacuum superdiagrams** of double-scaled β-deformation of $\mathcal{N} = 4$ SYM:

$$Z_{3,4}\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right) = \mathrm{Tr}\left[T_4\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)^3\right] =$$



with generalized row-matrix

$$T_N\left(\begin{smallmatrix}u_+&v_+\\u_-&v_-\end{smallmatrix}\right) =$$



Generalized vacuum diagrams

The generalized free energy

$$Z\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right) = \sum_{M,N=1}^{\infty} Z_{MN}\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right)(-\xi)^{2M\cdot N}$$

possesses the **critical coupling = (radius of convergence)$^{-1}$**

$$K\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right) = \lim_{M,N\to\infty}\left|Z_{MN}\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right)\right|^{\frac{1}{MN}}$$

$$= \lim_{N\to\infty}\left|\Lambda_{\mathrm{max},N}\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right)\right|^{\frac{1}{N}}$$

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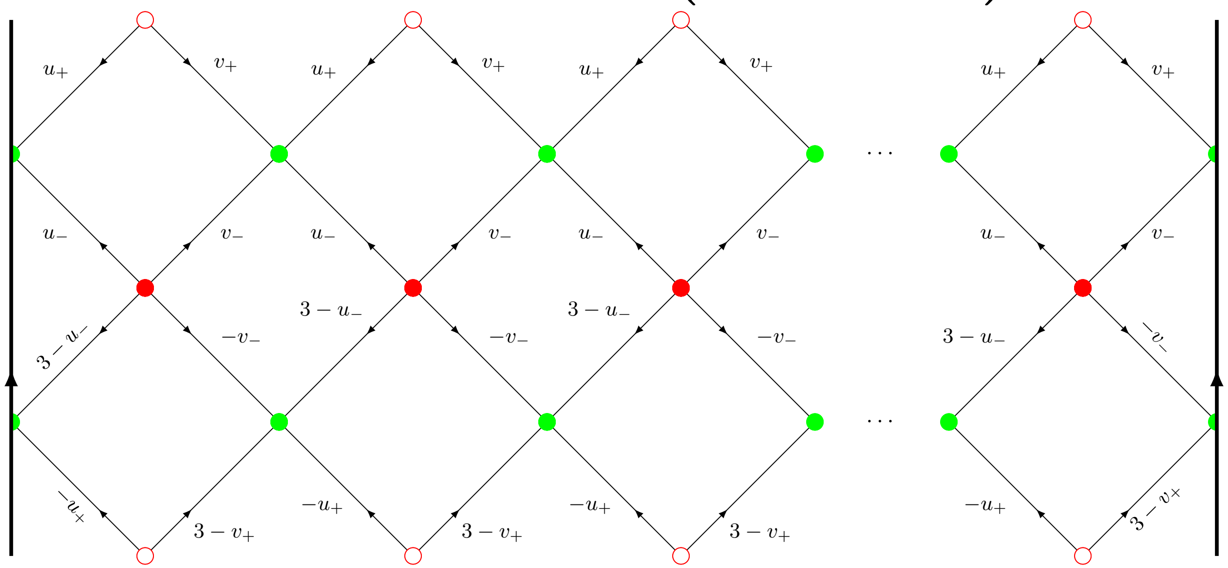
$$= \lim_{N\to\infty} \left|\Lambda_{\mathrm{max},N}\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right)\right|^{\frac{1}{N}}$$

Goal: Find the inverse of the row-matrix, which should be itself at different spectral parameter point. Then let the product act on the eigenvector corresponding to the maximal eigenvalue $\Lambda_{\mathrm{max},N}$ to obtain **functional relations**.

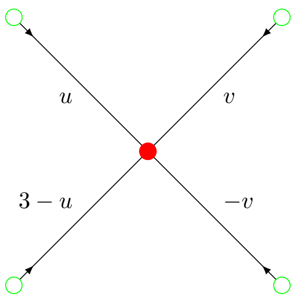
Inversion relations

There are **four representations of the inverse** of $T_N \begin{pmatrix} u_+ & v_+ \\ u_- & v_- \end{pmatrix}$.

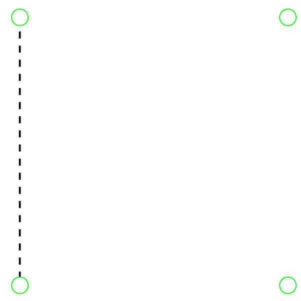
E.g.: $T_N \begin{pmatrix} u_+ & v_+ \\ u_- & v_- \end{pmatrix} \circ T_N \begin{pmatrix} 3-u_- & -v_- \\ -u_+ & 3-v_+ \end{pmatrix} \sim$



Auxiliary relation:



$= -4\,\pi^4 a(u)\, a(3-u)$



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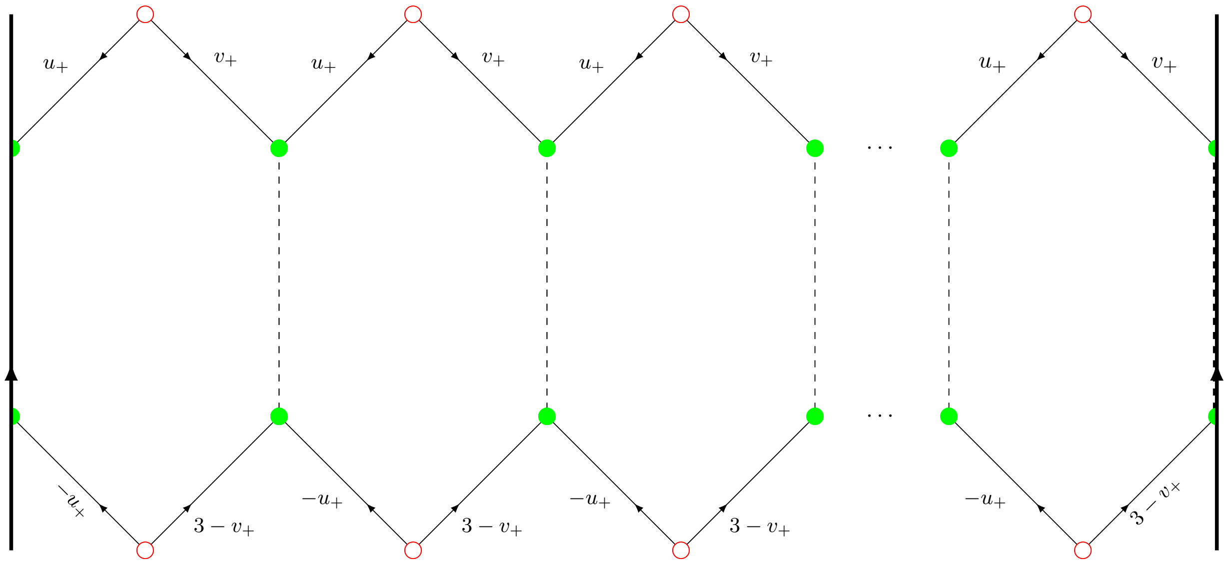

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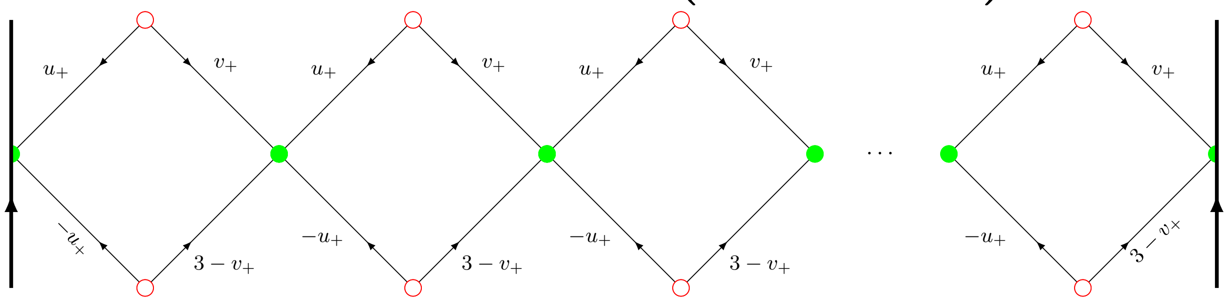

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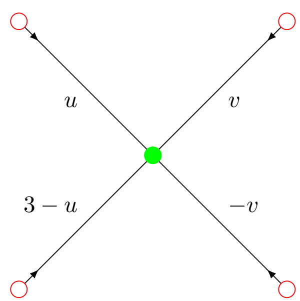

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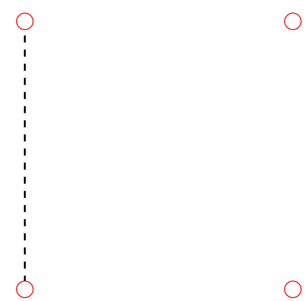

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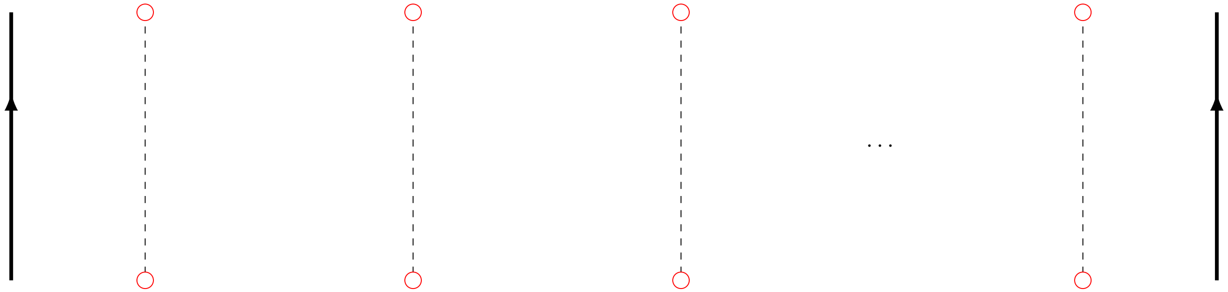

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$$T_N\begin{pmatrix} u_+ & v_+ \\ u_- & v_- \end{pmatrix} \circ T_N\begin{pmatrix} -u_- & 3-v_- \\ 3-u_+ & -v_+ \end{pmatrix} = \left[16\pi^8\, a(u_+)\, a(3-u_+)\, a(v_-)\, a(3-v_-)\right]^N \cdot \mathbb{1}_N$$

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Inversion relations

There are **four representations of the inverse** of $T_N \left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix} \right)$. E.g.:

$$K \left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix} \right) \circ K \begin{pmatrix} -u_- & 3-v_- \\ 3-u_+ & -v_+ \end{pmatrix} = 16\pi^8 \, a(u_+) \, a(3-u_+) \, a(v_-) \, a(3-v_-)$$

$$K \left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix} \right) \circ K \begin{pmatrix} -u_- & 3-v_- \\ -u_+ & 3-v_+ \end{pmatrix} = 16\pi^8 \, a(v_+) \, a(3-v_+) \, a(v_-) \, a(3-v_-)$$

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Result for the critical coupling

Solve **functional relations** with ansatz
$K\left(\begin{smallmatrix} u_+ & v_+ \\ u_- & v_- \end{smallmatrix}\right) = \kappa(u_+)\kappa(u_-)\kappa(v_+)\kappa(v_-)$ satisfying

$$\kappa(u)\kappa(-u) \;=\; 1\,,$$

$$\kappa(u)\kappa(3-u) = 4\pi^4\; a(u)\, a(3-u) = 4\pi^4 \frac{\Gamma(2-u)\Gamma(u-1)}{\Gamma(u)\Gamma(3-u)}$$

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The solution with the **right analytic properties** is

[Bazhanov,Kels,Sergeev'16;Zamolodchikov'77;Shankar,Witten'78;Bombardelli:1606.02949]

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The solution with the **right analytic properties** is

[Bazhanov,Kels,Sergeev'16;Zamolodchikov'77;Shankar,Witten'78;Bombardelli:1606.02949]

$$\kappa(u) = 12^{\frac{u}{3}} \pi^{\frac{4u}{3}} \frac{\Gamma\left(\frac{u+1}{3}\right)\Gamma(2-u)}{\Gamma\left(\frac{1}{3}\right)} \prod_{k=1}^{\infty} \frac{\Gamma(3k-u+2)\Gamma(3k+u)\Gamma(3k-2)}{\Gamma(3k+u-2)\Gamma(3k-u)\Gamma(3k+2)}$$

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$$\kappa(u) = 2^{\frac{2u}{3}}\,3^{\frac{4u}{3}-2}\pi^{\frac{4u}{3}}\,\frac{\Gamma(2-u)\Gamma\left(\frac{u}{3}\right)\Gamma\left(\frac{u+1}{3}\right)}{\Gamma\left(1-\frac{u}{3}\right)\Gamma\left(\frac{4}{3}-\frac{u}{3}\right)\Gamma(u)}$$

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The **critical coupling** of double-scaled β-deformed $\mathcal{N}=4$ is

[MK,Staudacher:2408.05805]

$$\xi_{\mathrm{cr}} = \left[\mathbb{K}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)\right]^{-1/2} = \kappa\,(1)^{-3/2} = \frac{3}{2\pi^2\,\Gamma(\frac{1}{3})^{3/2}} = \frac{3^{9/8}\mathrm{e}^{-\mathrm{i}\pi/24}}{4\pi^3\cdot\eta(\mathrm{e}^{2\pi\mathrm{i}/3})}$$

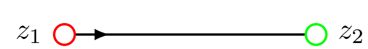
Superfishnet theory

Double-scaled β-deformation of $\mathrm{U(N)}^2$ ABJM (3D $\mathcal{N}=2$)
[Caetano,Gürdogan,Kazakov:1612.05895] in $\mathcal{N}=1$ formalism [MK, to appear]

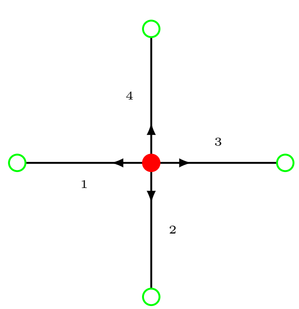
$$S = \mathrm{N} \int \mathrm{d}^3x\, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \left\{ -\sum_{i=1}^{4} \mathrm{tr}\left[\Phi_i^\dagger \Phi_i\right] + \mathrm{i}\xi \cdot \bar{\theta}^2\, \mathrm{tr}\left[\Phi_1\Phi_2\Phi_3\Phi_4\right] + \mathrm{i}\xi \cdot \theta^2\, \mathrm{tr}\left[\Phi_1^\dagger\Phi_2^\dagger\Phi_3^\dagger\Phi_4^\dagger\right] \right\}$$

Superpropagator

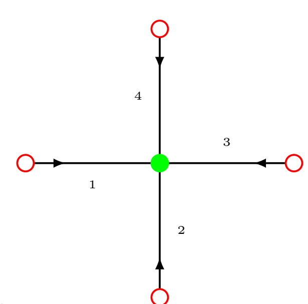
$$\left\langle \Phi_i(z_1)\Phi_j^\dagger(z_2)\right\rangle = \mathrm{e}^{\mathrm{i}\left[\theta_1\gamma^\mu\bar{\theta}_1+\theta_2\gamma^\mu\bar{\theta}_2-2\theta_1\gamma^\mu\bar{\theta}_2\right]\partial_{1,\mu}} \frac{\delta_{ij}}{\left[x_{12}^2\right]^{1/2}}$$

$=$ 

Vertices

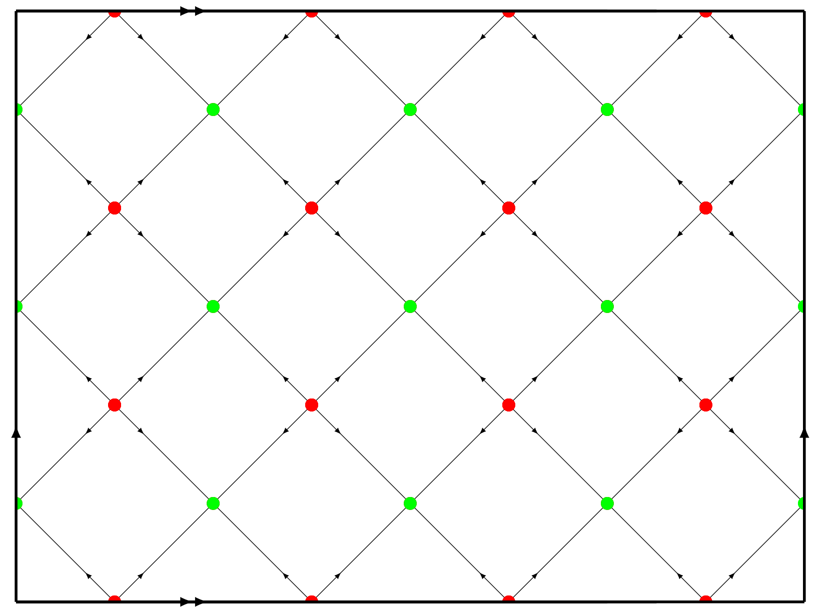


$\sim -\xi \int \mathrm{d}^3x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \delta^{(2)}(\bar{\theta})$,



$\sim -\xi \int \mathrm{d}^3x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \delta^{(2)}(\theta)$.

Vacuum diagrams and critical coupling

$$Z_{3,4}\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} =$$ 

Functional relations:

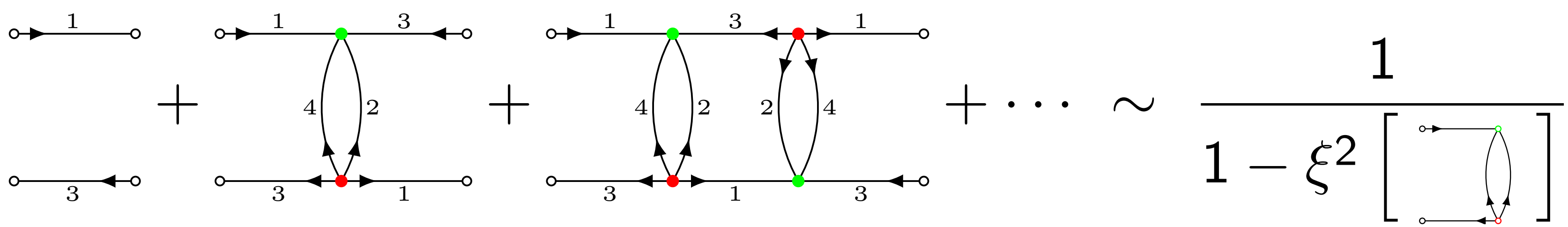
$$\kappa(u)\kappa(-u) \;=\; 1\,,$$

$$\kappa(u)\kappa(2-u) = 4\pi^3\, a(u)\, a(2-u) = 4\pi^3\, \frac{\Gamma(3/2-u)\Gamma(u-1/2)}{\Gamma(u)\Gamma(2-u)}$$

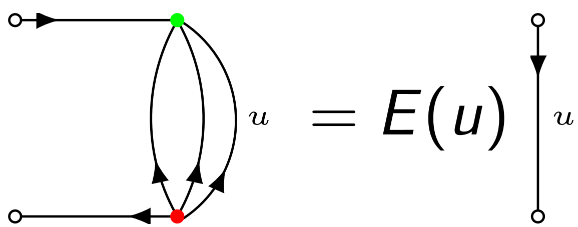
Critical coupling: $\xi_{\rm cr} = \sqrt{\frac{2}{\pi}}\,\frac{1}{\Gamma(1/4)^2} = \frac{1}{\sqrt{8\pi^4}}\frac{1}{\eta(\mathrm{i})^2}$

All-loop anomalous dimension

Extract **anomalous dimension** of $\mathrm{tr}\left[\Phi_1^\dagger \Phi_3\right]$ from 4pt.-function



Eigenvalue equation



with $E(u) = \dfrac{2\pi^2}{u(1-2u)}$

At the pole

$$1 = \xi^2 E(-\gamma/2) \quad \Rightarrow \quad \gamma = -\frac{1}{2} + \frac{1}{2}\sqrt{1-16\pi^2\xi^2}$$

[Caetano,Gürdogan,Kazakov:1612.05895] [Bak,Min,Rey:0911.0689]

Outlook

- Uncovering the complete integrable structure of the discussed super brick wall and super fishnet models would require the construction of non-compact superconformal $s\ell(4|1)$ and $osp(2|4)$ R-matrices, respectively. Are the near-star-triangle relations sufficient for this purpose?
- How to uplift the many exact results for non-dynamical fishnet models to super brick wall and super fishnet models?
- We showed that supergraphs can prevent the 'dynamical melting' of fishnets. Could similar ideas possibly work for general dynamical fishnets, or even for $\mathcal{N} = 4$ SYM?

Thanks for your attention!