Chiral Phase Transition in QCD and in Solvable Models

Konstantin Zarembo
(Nordita & NBI)

NORDITA

Bethe Colloquium, BCTP, Bonn, 5.9.24

Zur Theorie der Metalle.

I. Eigenwerte und Eigenfunktionen der linearen Atomkette.

Von **H. Bethe** in Rom.

(Eingegangen am 17. Juni 1931.)

Es wird eine Methode angegeben, um die Eigenfunktionen nullter und Eigenwerte erster Näherung (im Sinne des Approximationsverfahrens von London und Heitler) für ein „eindimensionales Metall" zu berechnen, bestehend aus einer linearen Kette von sehr vielen Atomen, von denen jedes außer abgeschlossenen Schalen ein *s*-Elektron mit Spin besitzt. Neben den „Spinwellen" von Bloch treten Eigenfunktionen auf, bei denen die nach einer Richtung weisenden Spins möglichst an dicht benachbarten Atomen zu sitzen suchen; diese dürften für die Theorie des Ferromagnetismus von Wichtigkeit sein.

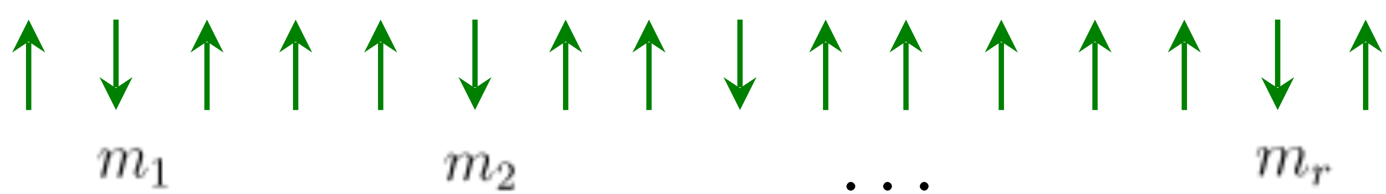
symmetrisch ist, sei $\psi(m_1 m_2 \ldots m_r)$. Die richtigen Eigenfunktionen nullter Näherung werden sich dann in der Form

$$\Psi = \sum_{m_1 m_2 \ldots m_r} a(m_1 m_2 \ldots m_r)\, \psi(m_1 \ldots m_r)$$

darstellen, wobei jede der Zahlen $m_1, m_2, \ldots, m_r$ die Werte von 1 bis N durchläuft. Wir setzen dabei fest, daß

$$m_1 < m_2 < \ldots < m_r.$$

[1]) W. Heisenberg, ZS. f. Phys. **49**, 619, 1928.



Mit Hilfe der Matrixelemente der Wechselwirkungsenergie erhält man die folgenden Gleichungen zwischen den Koeffizienten $a\,(m_1 \ldots m_r)$ der gesuchten Eigenfunktion Ψ

$$2\,\varepsilon\,a\,(m_1 \ldots m_r) + \sum_{m_1' \ldots m_r'} a\,(m_1' \ldots m_r') - a\,(m_1 \ldots m_r) = 0. \qquad (1)$$

$$H = \sum_{i=1}^{N} (1 - P_{i\;i+1})$$

permutes i-th and $(i+1)$-th spins

What is the probability to find quantum anti-ferromagnet in Néel state?

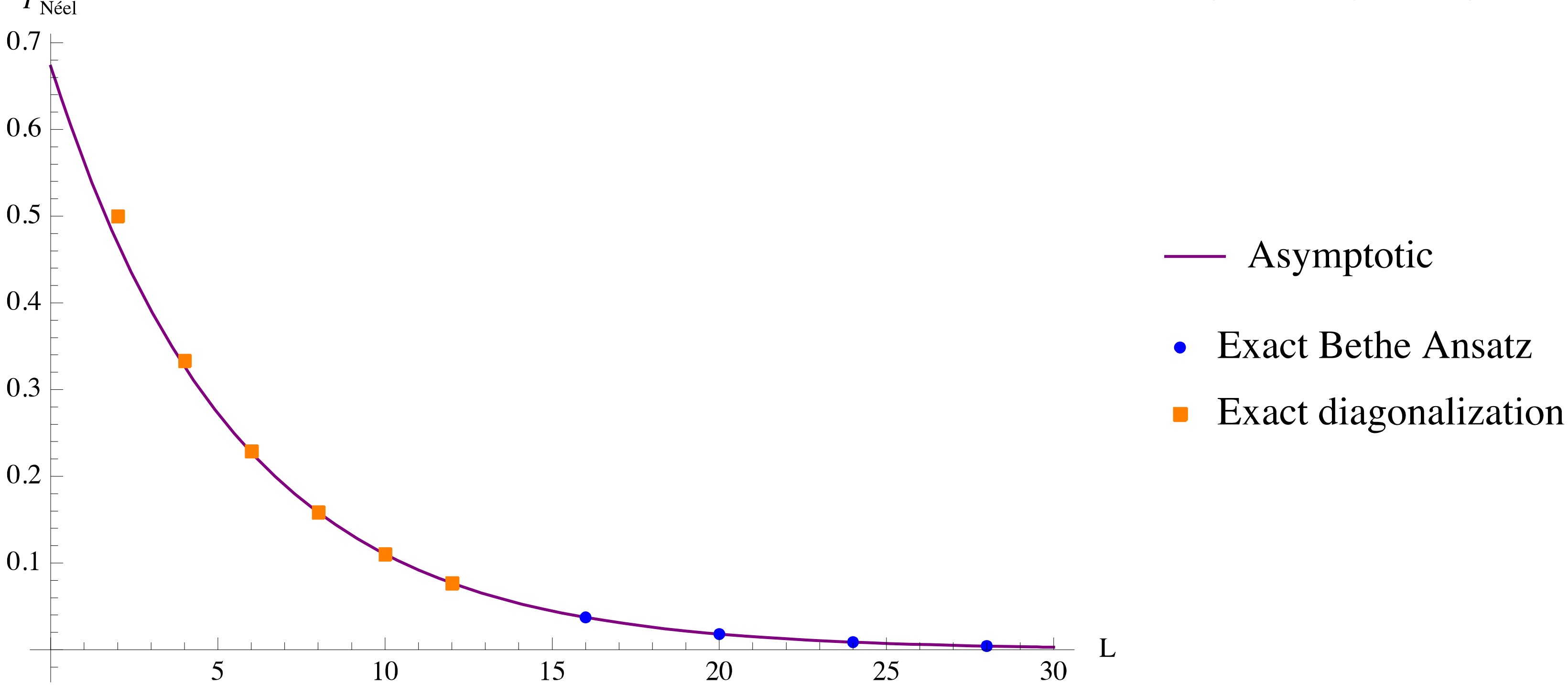
$$|\text{Néel}\rangle = |\uparrow\downarrow\uparrow\downarrow\ldots\uparrow\downarrow\rangle$$

$$\mathcal{P}_{\text{Néel}} = \frac{\langle \text{Néel}|0\rangle^2}{\langle 0|0\rangle}$$

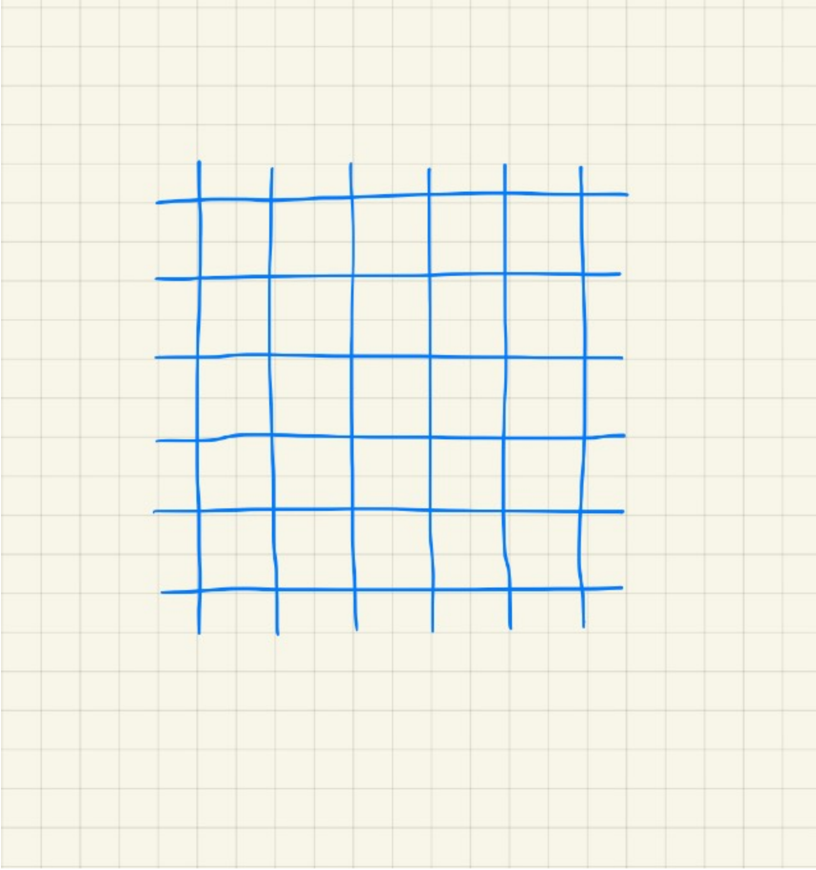
$$P_{\text{Néel}} = C e^{-\text{⌫}}$$

$$\text{⌫} = \ln 2 - \int_{-\infty}^{+\infty} \frac{du}{2\cosh \text{⇡} u} \ln \frac{u^2 + \frac{1}{4}}{u^2} = 0.181\ldots$$

Brockmann, De Nardis, Wouters, Caux’14



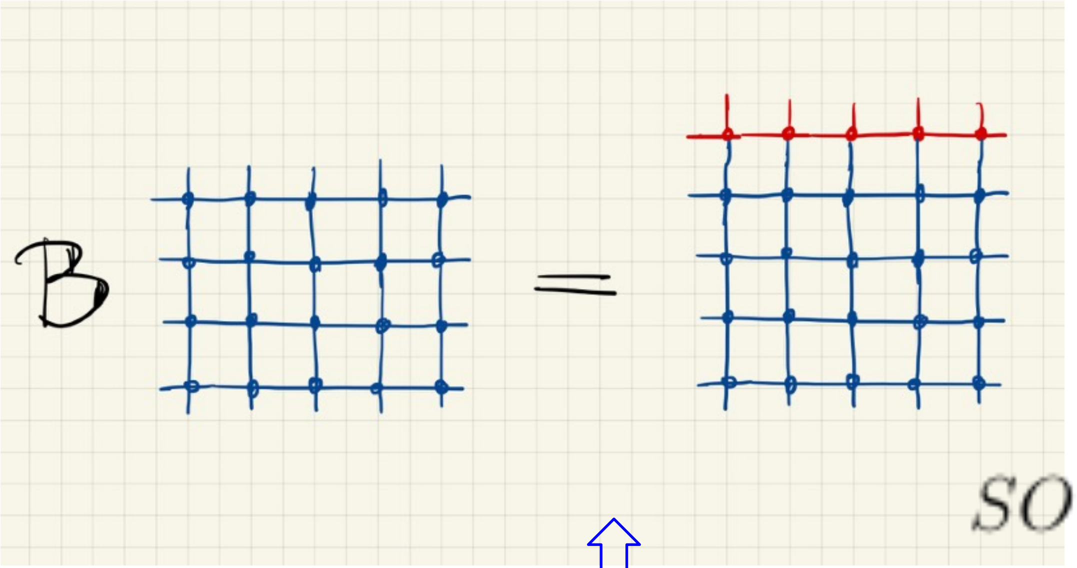
Fishnet diagrams



- Also solvable by Bethe Ansatz!

Zamolodchikov'80
Gürdoğan, Kazakov'15

Graph-building operator:



$SO(4,2)$ $\quad L^2(\mathbb{R}^{3,1})$

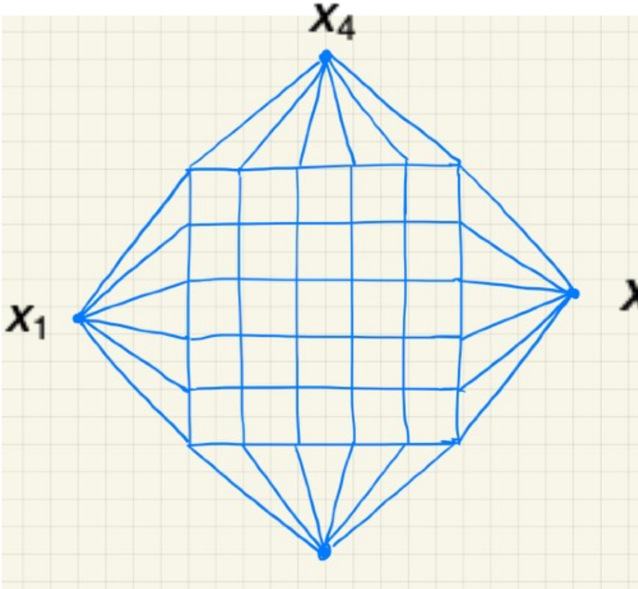
$\Updownarrow$

$$H = \sum_{i=1}^{N} (1 - P_{i\, i+1})$$

$SU(2)$ $\quad |\uparrow\rangle, |\downarrow\rangle$

Gromov, Kazakov, Korchemsky, Negro, Sizov'17

Basso, Dixon'17



$$= \frac{1}{x_{12}^{2n} x_{34}^{m}} \left[\frac{(1-z)(1-\bar{z})}{z-\bar{z}} \right]^m \det_{ij} M_{i+j+n-m-1}$$

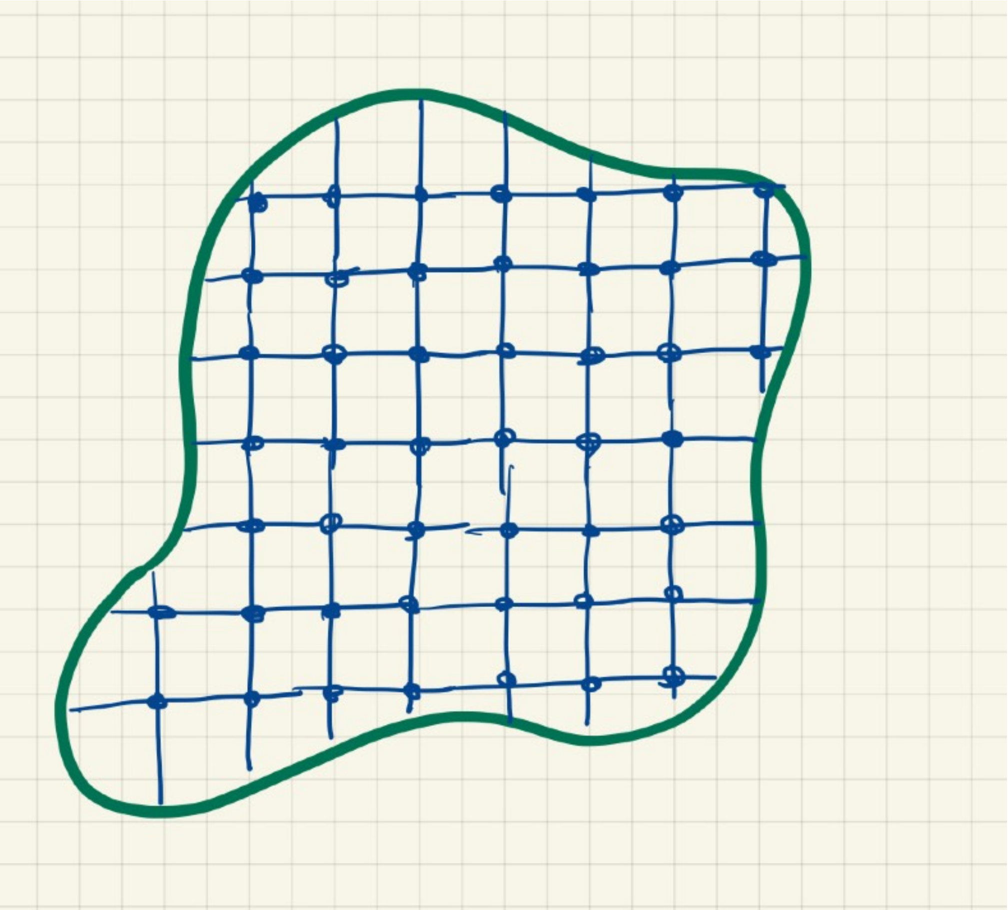
$$M_p = p!(p-1)!L_p(z,\bar{z})$$

$$z\bar{z} = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

$$\frac{(1-z)(1-\bar{z})}{z\bar{z}} = \frac{x_{12}x_{34}}{x_{14}x_{23}}$$

$$L_p(z,\bar{z}) = \sum_{j=p}^{2p} \frac{j!}{p!(j-p)!(2p-j)!} (-\ln z - \ln \bar{z})^{2p-j} (\mathrm{Li}_j(z) - \mathrm{Li}_j(\bar{z}))$$

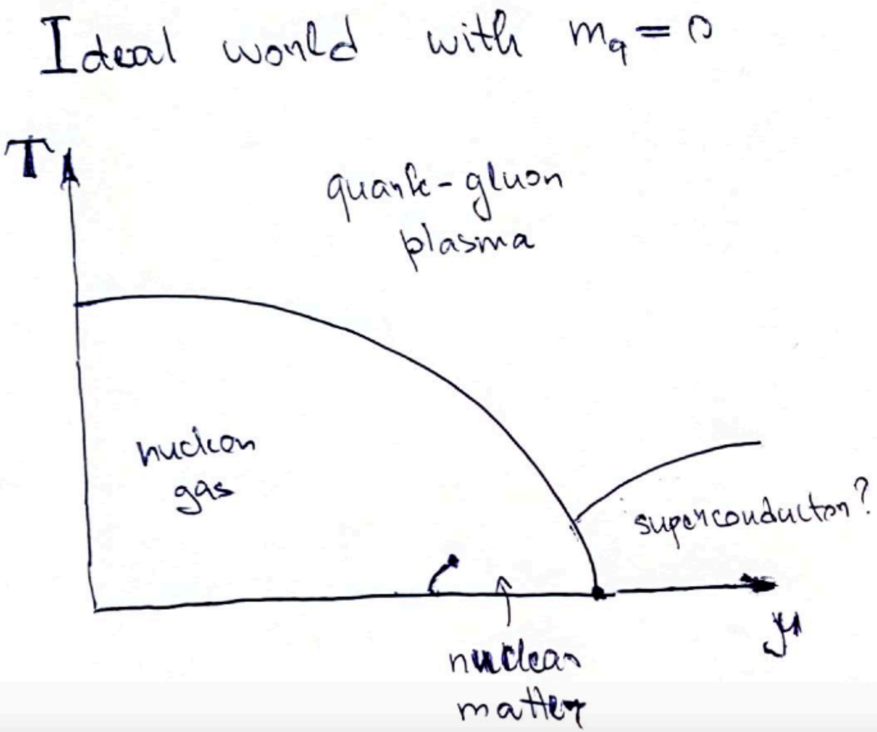
ladder integral

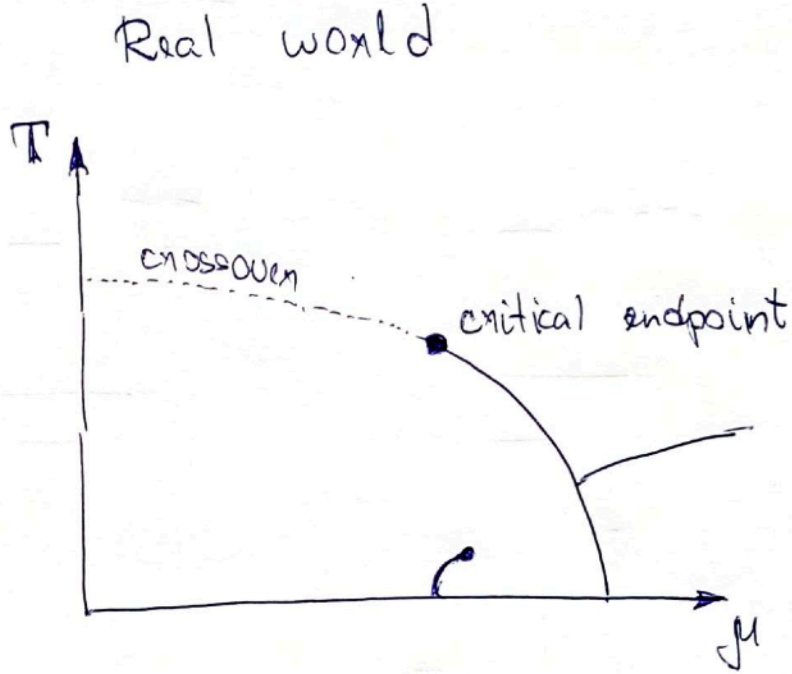


Yangian invariant and (in principle) calculable for any shape

Chicherin, Kazakov, Loebbert, Müller, Zhong'17
Duhr, Klemm, Loebbert, Nega, Porkert'23
Loebbert, Stawinski'24

QCD at finite temperature and density





Chiral symmetry

$$-\frac{1}{2g^2}\,\mathrm{tr}\,F^2_{\mu\nu} + \sum_f \bar{\psi}_f\,(i\gamma^\mu D_\mu - m_f)\,\psi_f$$

baryon number (exact)

$$U(1)_B \times SU(2)_L \times SU(2)_R$$

approximate, at $m_f \neq 0$, apart from isospin $U(1)_I$

Chiral symmetry breaking

$$SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$$

Order parameter: $\langle \ \bar{\ } \ \ \rangle \neq 0$

Goldstone bosons: triplet of pions π^a

Constituent quark mass: $\mathcal{L}_{\rm eff} \supset -M \bar{\psi}\, {\rm e}^{i\gamma^5 T_a \pi^a} \psi$

Hierarchy of scales in strong interactions

Hadron masses:

$$m_p = 940\,\text{MeV} \approx 3M$$

$$m_\rho = 770\,\text{MeV} \approx 2M$$

$\Longrightarrow$ $M = 300 \div 350\,\text{MeV}$

Pion mass:

$$m_\pi^2 = (140\,\text{MeV})^2 \sim M m_q$$

$$\text{nuclear density} = (98\,\text{MeV})^3 \sim m_\pi^3$$

Nuclear binding energies:

$$E_{\text{binding}} \sim \frac{p^2}{m_p} \sim \frac{m_\pi^2}{m_p} \sim m_q \sim \text{few MeV}$$

QCD at finite temperature and density

baryon charge

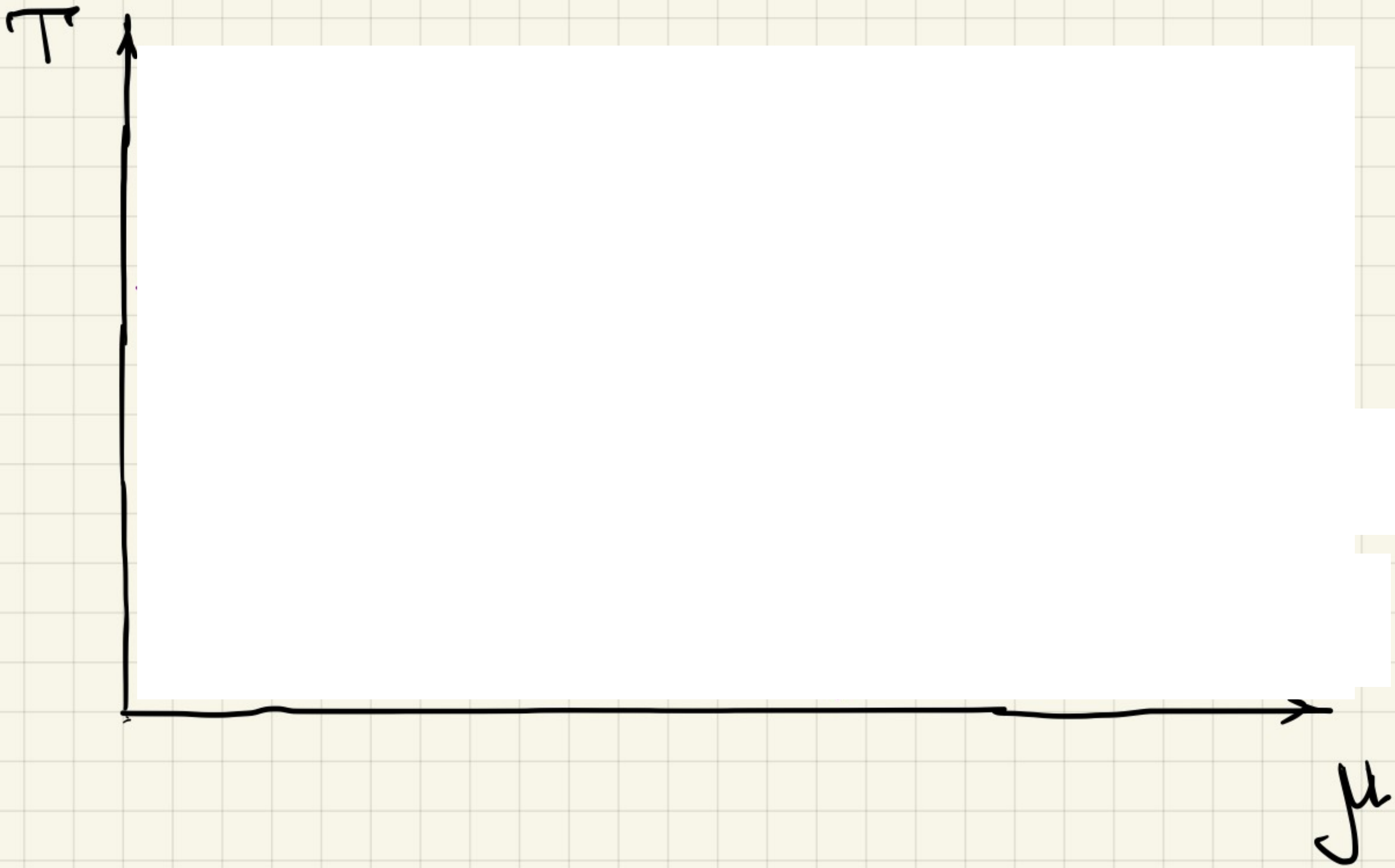
$$Z = \mathrm{tr}\; \mathrm{e}^{-\frac{H-\mu B}{T}}$$

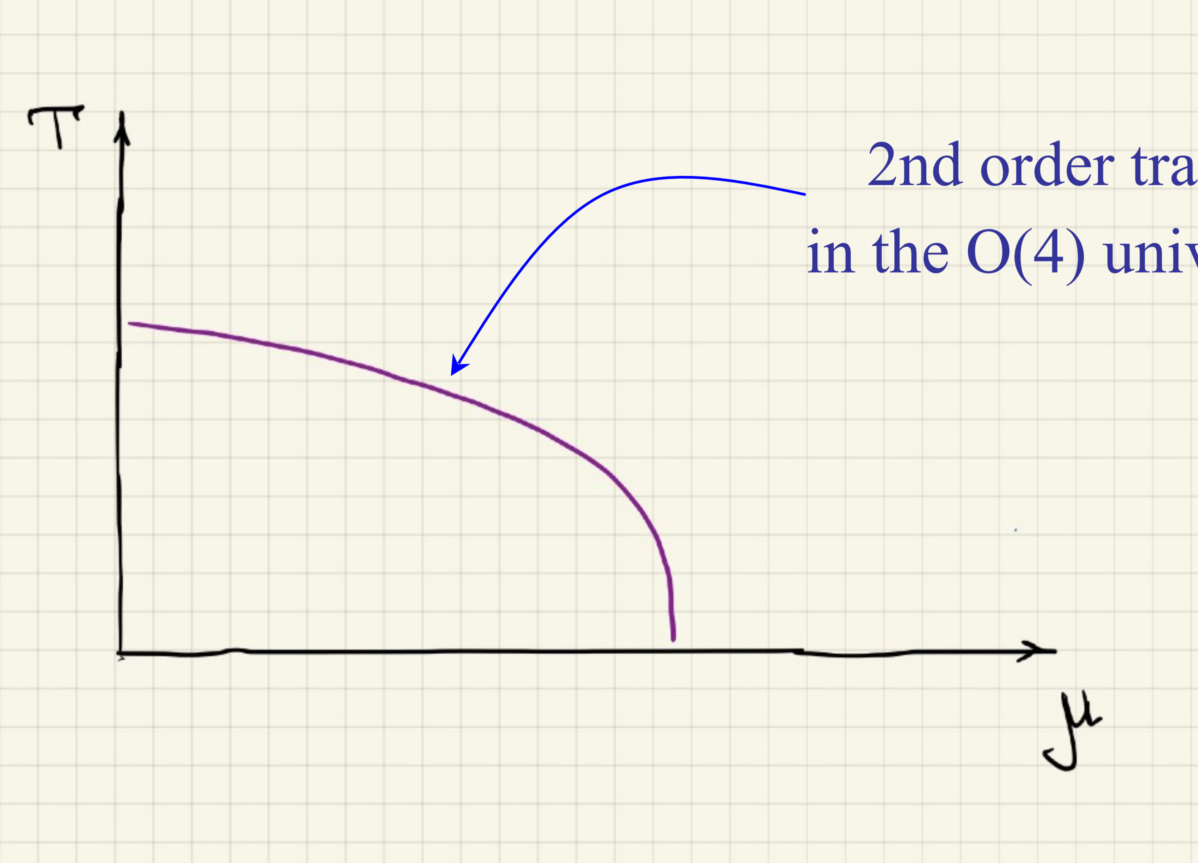
Asymptotic freedom:

$T \to \infty$ or $\mu \to \infty$

⇓

free quarks and gluons


T
μ



2nd order transition line
in the O(4) universality class

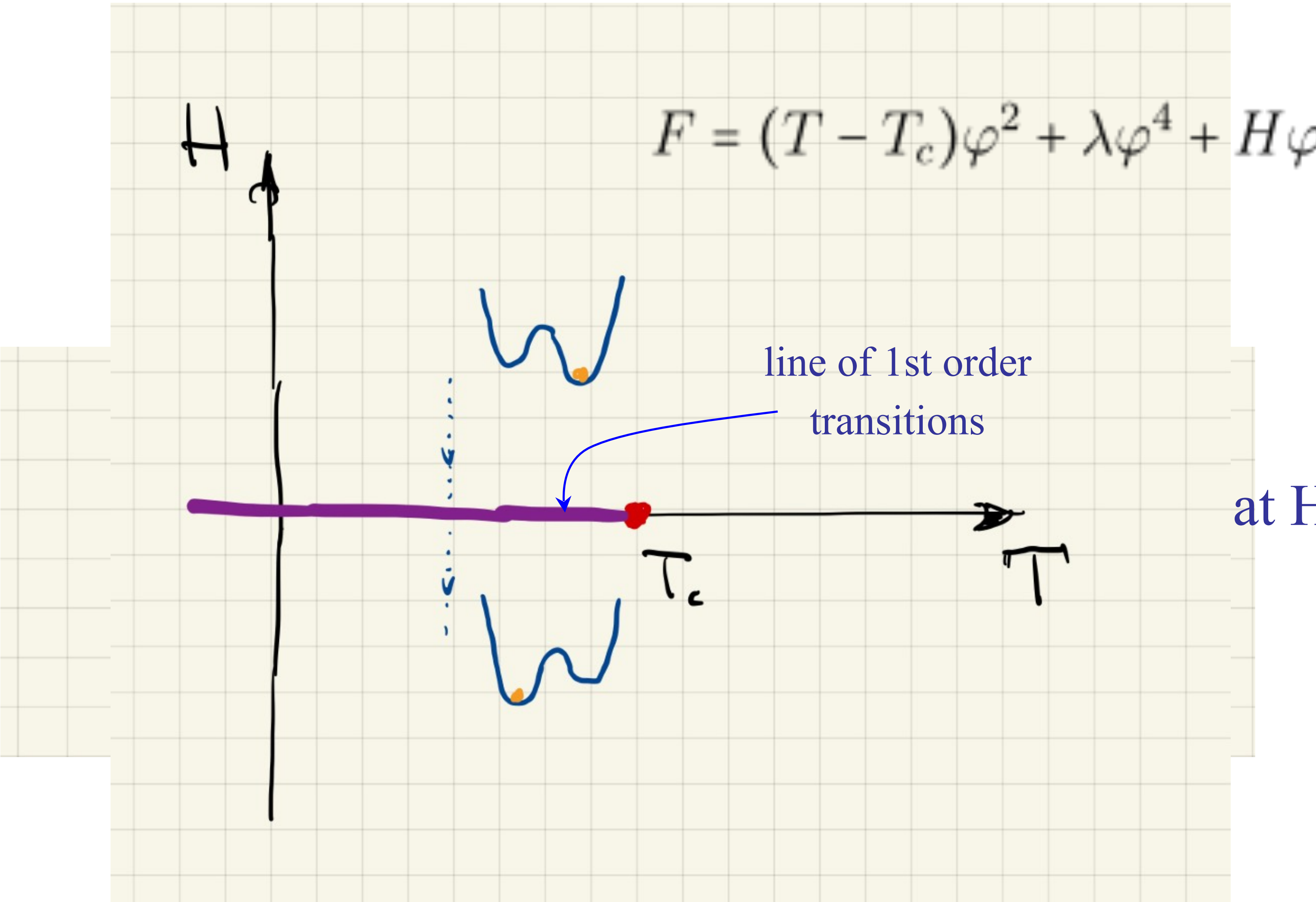
Pisarski, Wilczek'84

$$SU(2) \times SU(2) \longrightarrow SU(2)_{\text{diag}}$$

is the same as

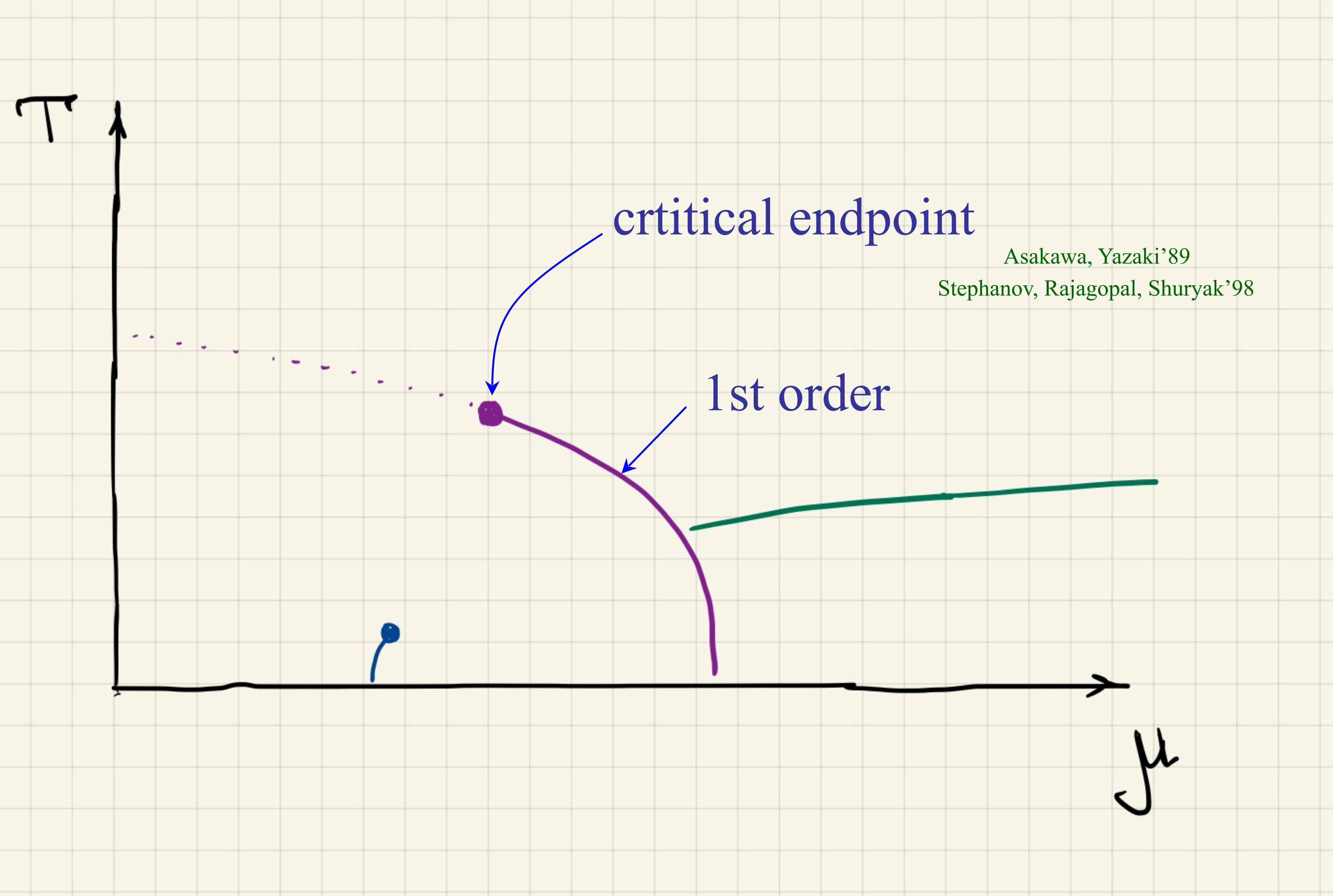
$$O(4) \longrightarrow O(3)$$

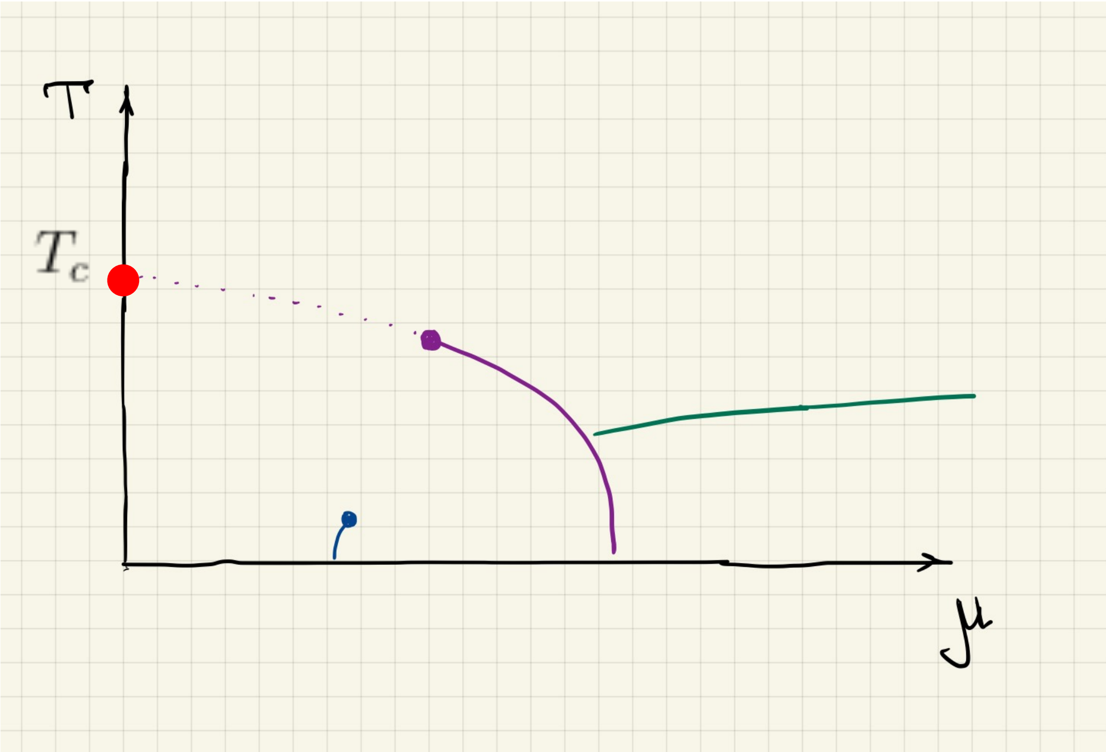
Breaking approximate symmetry: an example
Ising model in a magnetic field



at H=0

$$m_q \neq 0$$


T
crtitical endpoint
Asakawa, Yazaki'89
Stephanov, Rajagopal, Shuryak'98
1st order
μ

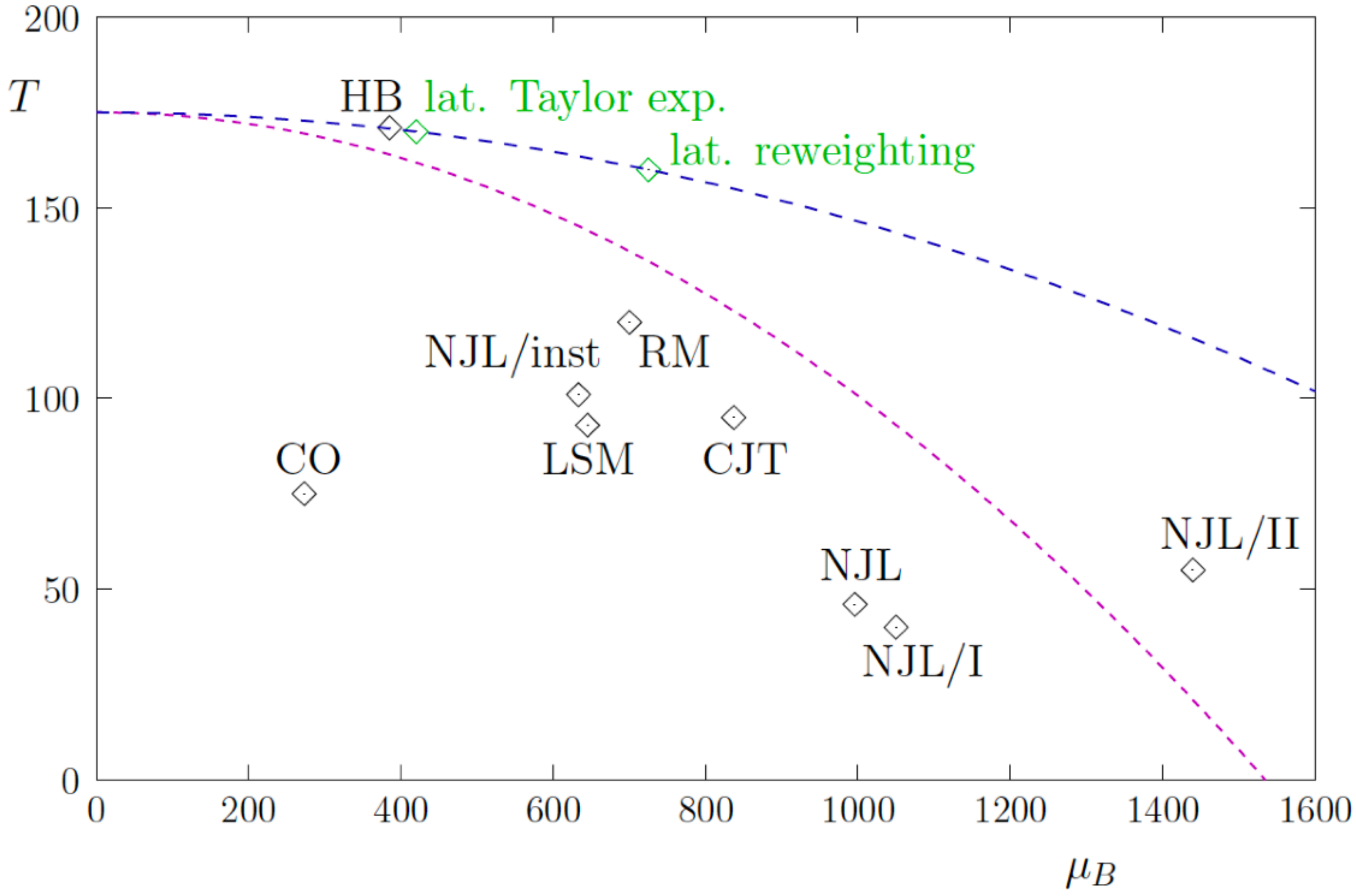


$T_c \simeq 157\,\mathrm{MeV}$ (lattice) Bazavov et al'18

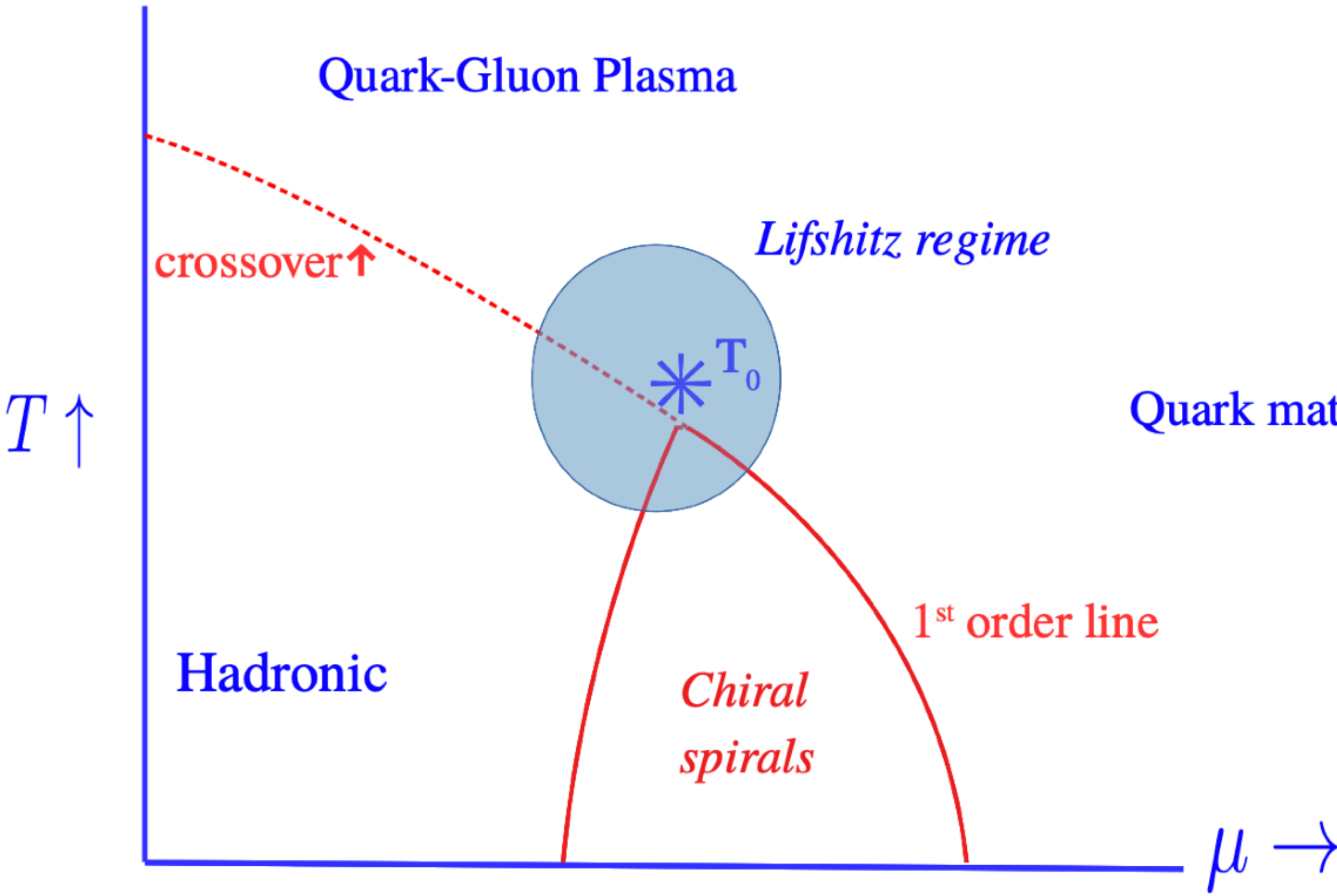
$T_c \approx \sqrt{3}\,F_\pi = 159\,\mathrm{MeV}$ (pion EFT, mean-field) Bochkarev, Kapusta'96

- lattice simulations too difficult at $\mu > 0$

Model estimates of CEP



from M. Stephanov, hep-ph/0402115



from Pisarski, Skokov, Tsvelik, 2202.01036

Inhomogeneous phases

Chiral density wave in large-N_c QCD

Deryagin, Grigoriev, Rubakov'92

Chiral spiral:

$$\langle \bar{\psi}\psi \rangle = \mathrm{e}^{\,2i\mu\gamma^0\gamma^z z}$$

Kojo, Hidaka, McLerran, Pisarski'09

likely disordered by transverse fluctuations:

- Quantum Pion Liquid

Pisarski, Tsvelik, Valgushev'20
Winstel, Valgushev'24

Gross-Neveu model

N interacting Dirac fermions in (1+1)d:

$$\mathcal{L} = i\bar{\psi}_i \gamma^\mu \partial_\mu \psi_i + \frac{g^2}{2}\left(\bar{\psi}_i \psi_i\right)^2$$

- Asymptotically free: $\beta = -\frac{N-1}{2\pi} g^3 + \ldots$ Anselm'59

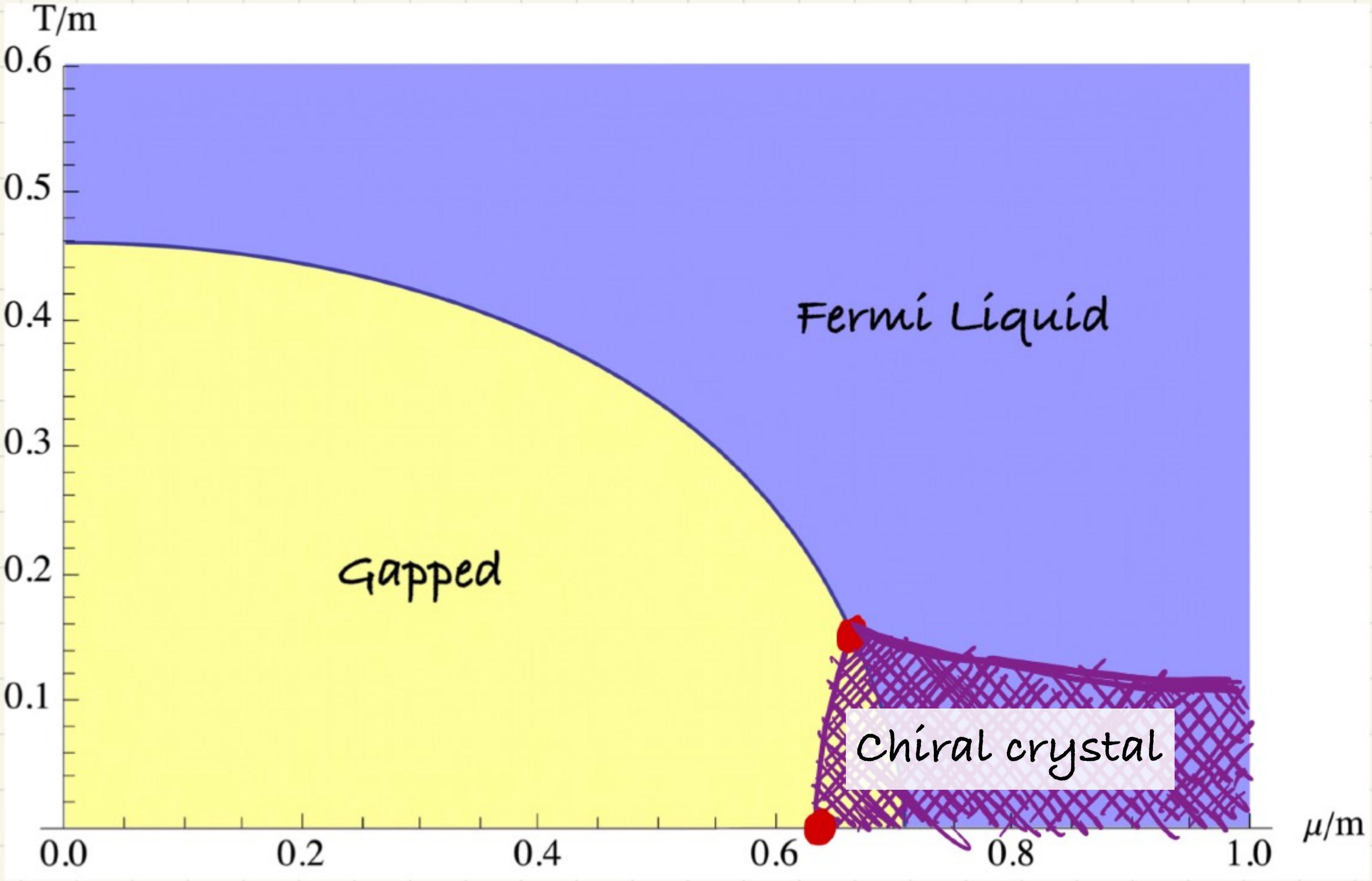
- Dimensional transmutation:

$$m = \Lambda\, e^{-\frac{\pi}{\lambda}} \qquad \lambda = g^2 N$$

- Chiral symmetry breaking: Gross, Neveu'74

$$\langle \bar{\psi}\psi \rangle = \text{const} \cdot m$$

Finite temperature and density



$$\mu_c = \frac{2}{\pi} m$$

Thies, Ulrichs'03

Crystalline phase

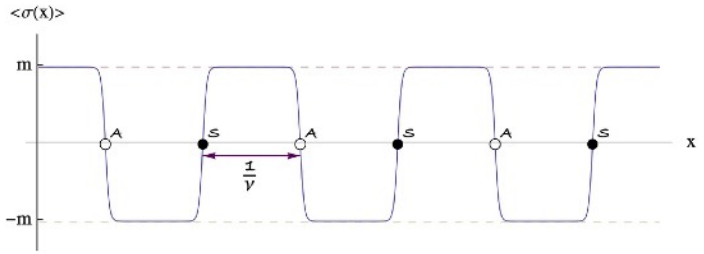
Thies, Ulrichs'03
Schnetz, Thies, Ulrichs'04
Thies'06
Başar, Dunne, Thies'09

$\mu < \mu_c$:

$$\langle \bar{\psi}\psi \rangle = \alpha m$$

$\mu > \mu_c$:

$$\langle \bar{\psi}\psi \rangle = \mathcal{A}\, \mathrm{sn}(x;k)$$



cnoidal wave

Brazovskii, Kirova'81
Başar, Dunne, Thies'09

Large-N solution

Gross, Neveu'74

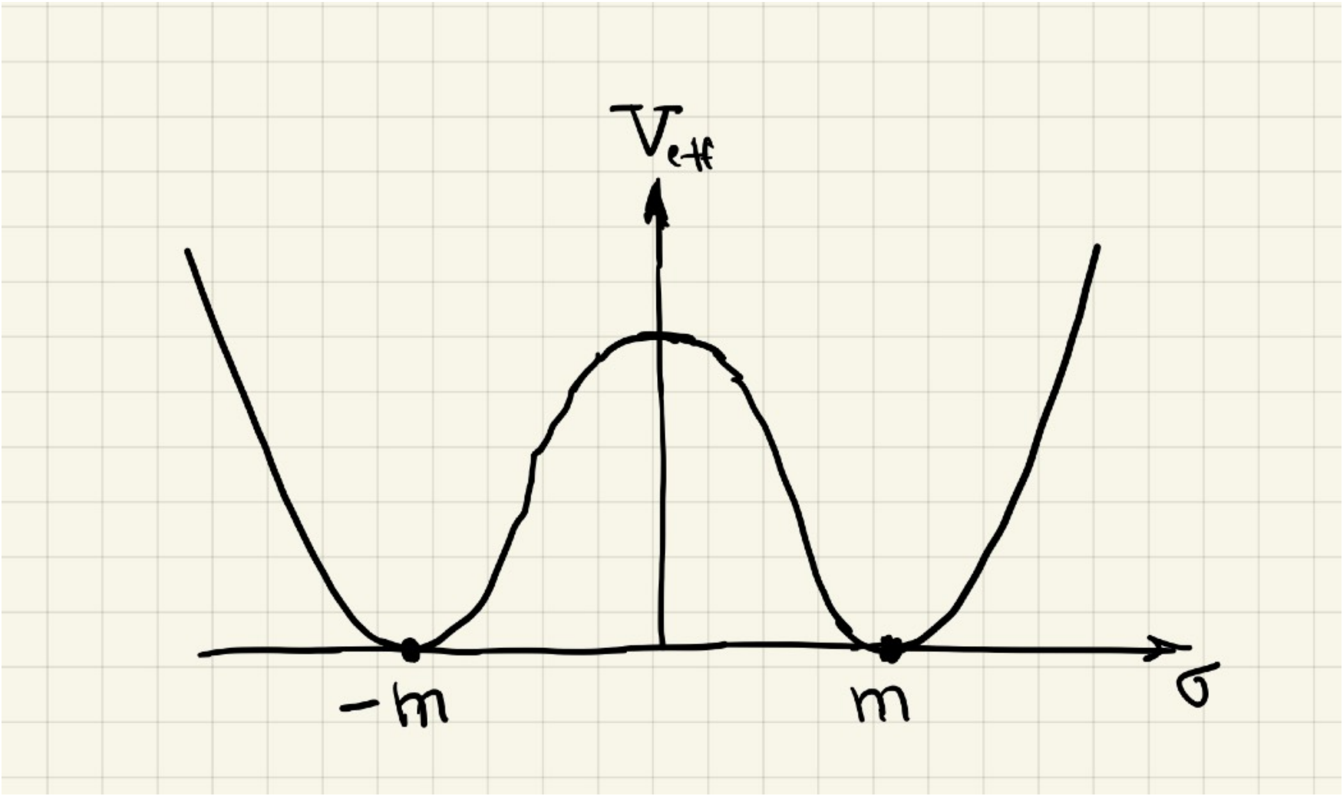
$$\mathcal{L} = i\bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \sigma \bar{\psi}_i \psi_i - \frac{N}{2\lambda} \sigma^2$$

integrate out fermions:

$$S_{\text{eff}} = -N \left[\frac{1}{2\lambda} \int d^2x\, \sigma^2 + i \ln \det \left(i\gamma^\mu \partial_\mu - \sigma \right) \right]$$

$N \to \infty$ $\Longrightarrow$ Mean-field exact

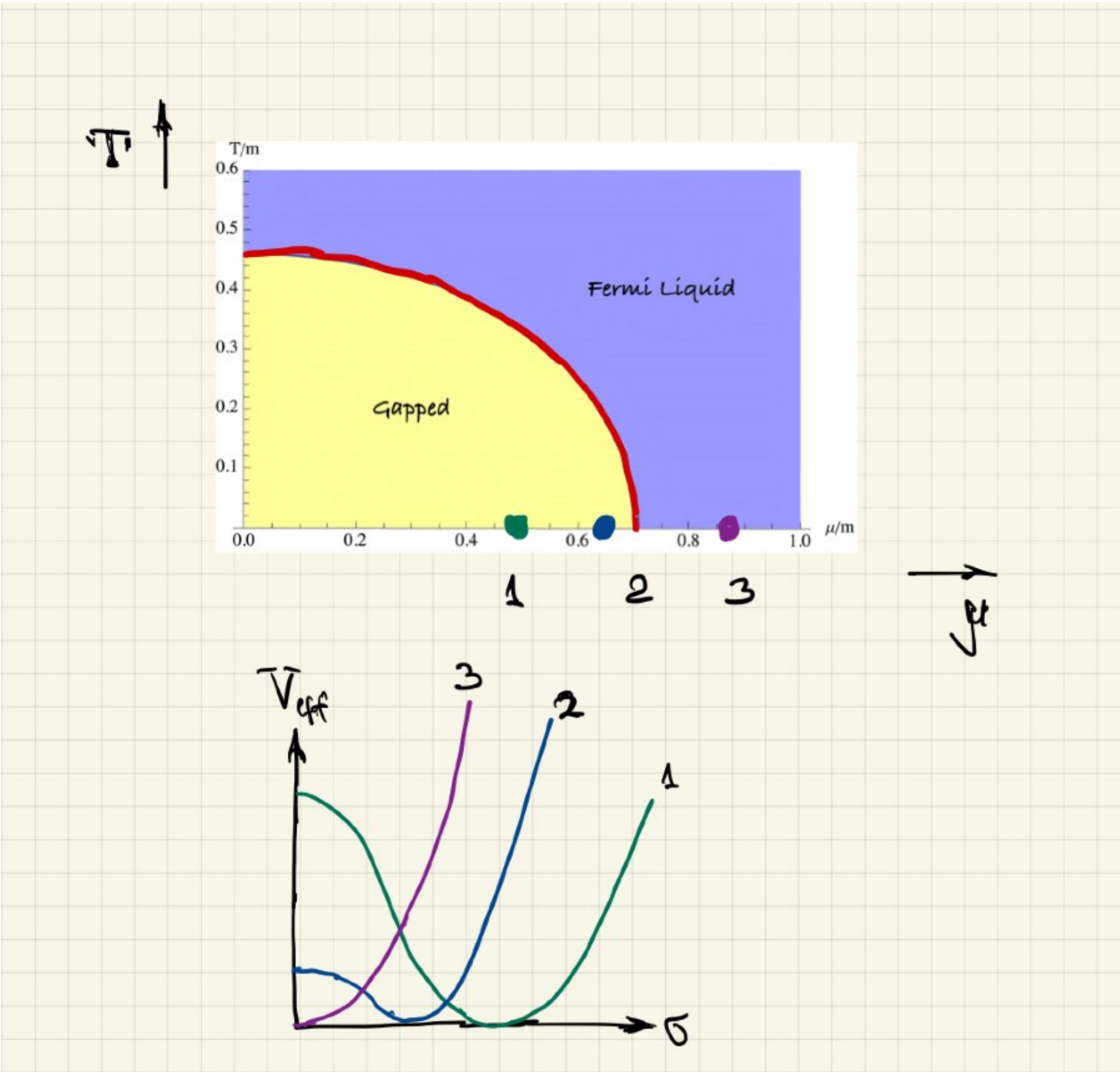
$\langle \sigma \rangle = m$ gives fermions a mass

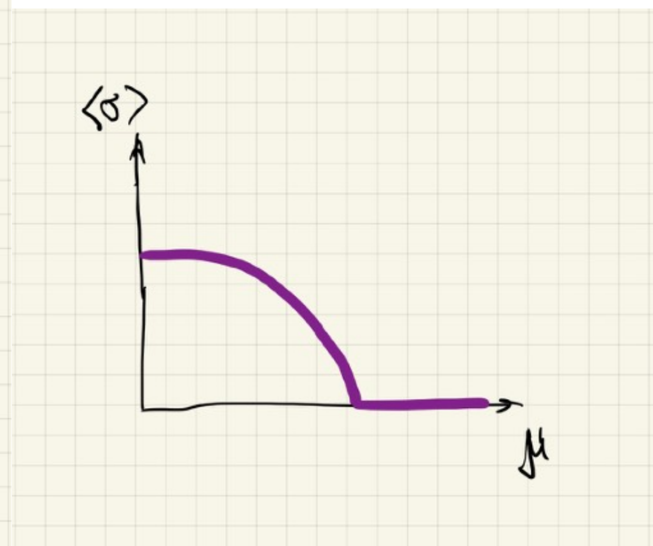


Gap equation:

$$\frac{1}{\lambda} = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + m^2} \quad \Longrightarrow \quad m = \Lambda\, e^{-\frac{\pi}{\lambda}}$$

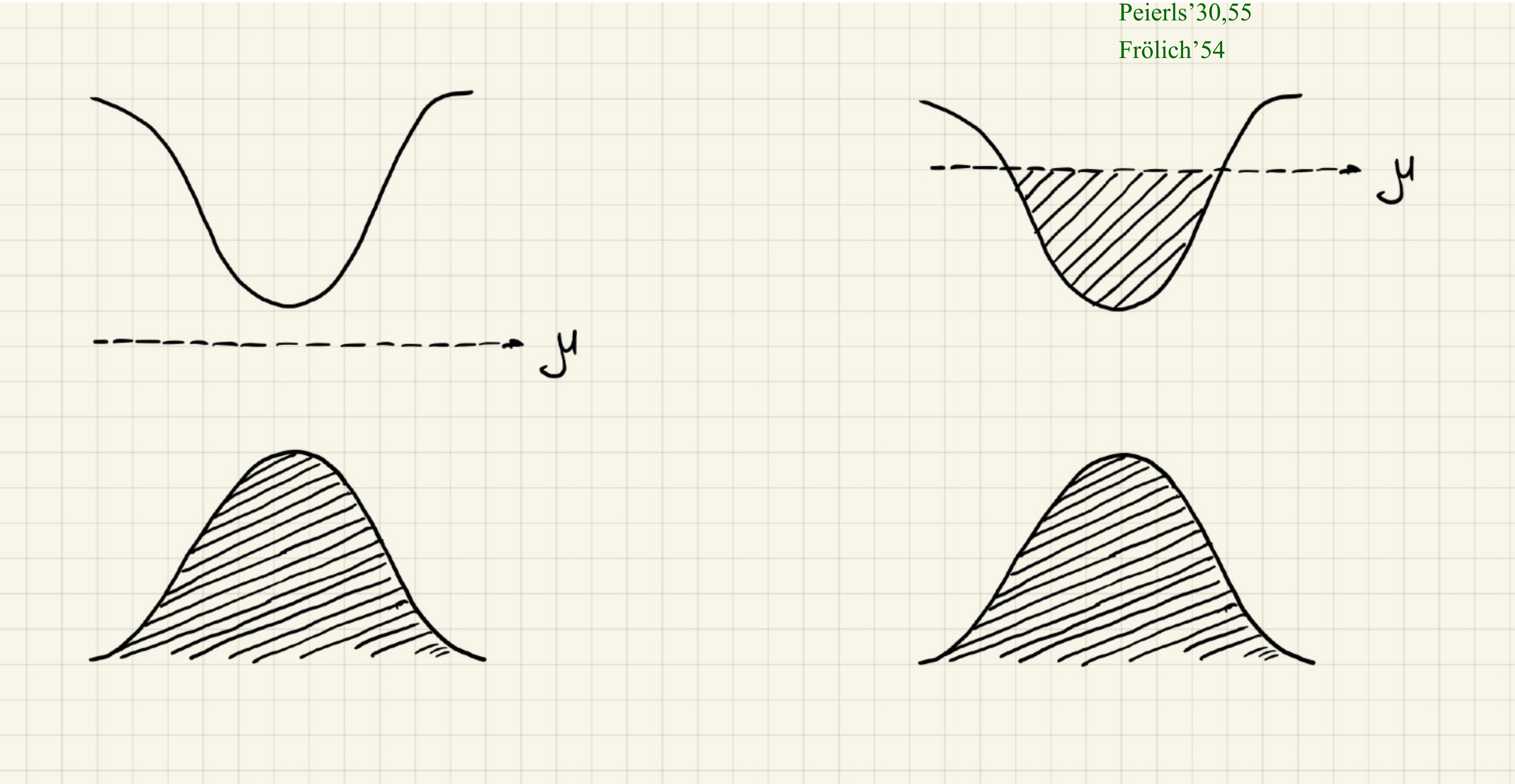
Finite density





For $\mu > \mu_c$ condensate melts

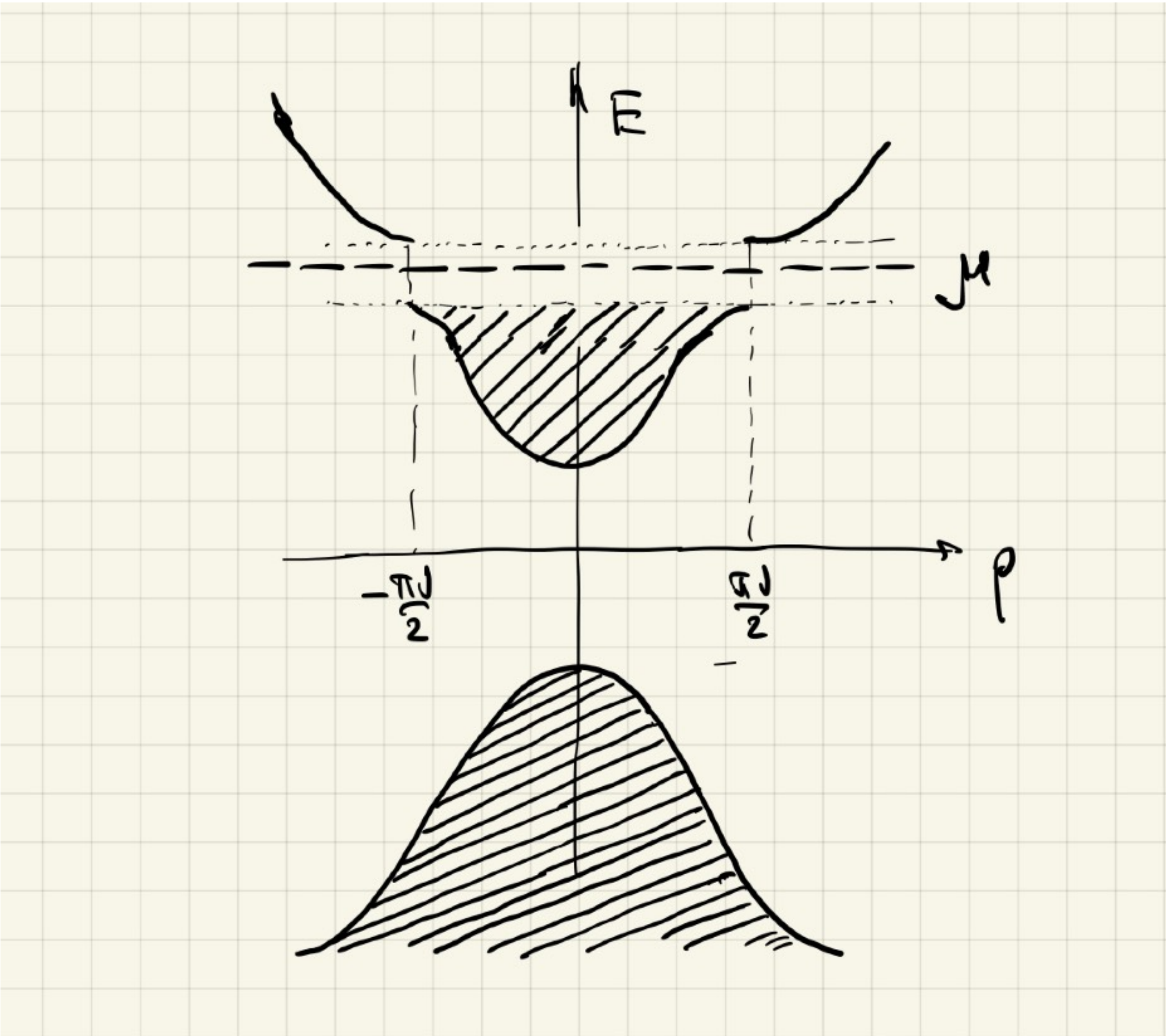
Peierls instability



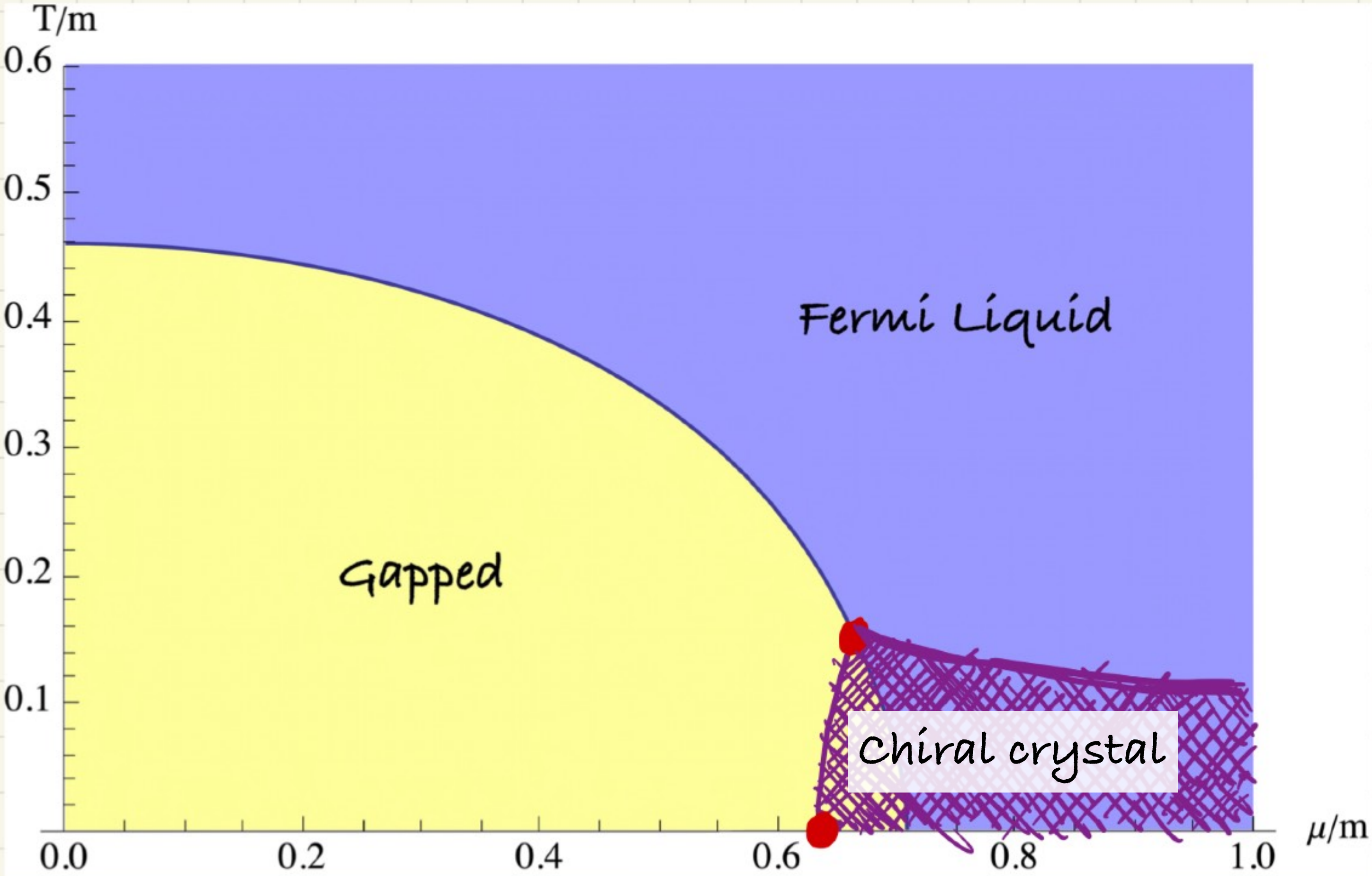
Very stable

Unstable

- instability is offset by periodic modulation of environment
- such that bandgap opens exactly at Fermi level
- and brings fermions back to half-filling



$$\langle \sigma(x) \rangle = \mathcal{A} \sin \pi \nu x$$



$$\mu_c = \frac{2}{\pi} m$$

Quasi-long-range order

$\mathbb{R} \times \mathbb{Z}_2 \to \mathbb{Z}$

- forbidden by CMW th.

  Mermin, Wagner'66
  Coleman'73

- 1d crystals do not exist

  Peierls'34
  Landau'38

Quasi-long-range order:

$$\langle \sigma(x)\, \sigma(0) \rangle \simeq \frac{\cos \text{[illegible]}}{|x|^{\frac{\text{[illegible]}}{N}}}$$

Cicconi, Di Pietro, Serone'22,23

- large-N and infinite-volume limits do not commute

  Witten'78

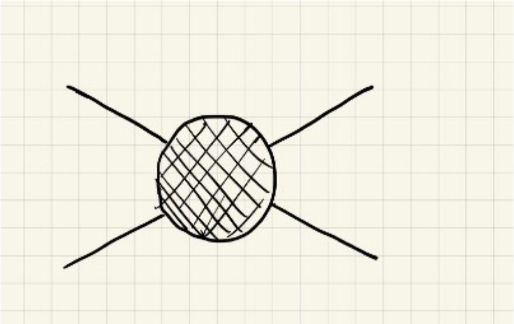
Beyond mean field

- Phase transition or crossover at finite N?
- Is there critical N_c such that no transition happens for $N<N_c$?
- Is spectrum gapless of small non-perturbative gap $O(e^{-N})$?
- How accurate is large-N approximation?

Integrability

GN model is integrable:

- ✓ exact spectrum
- ✓ exact S-matrix



Zamolodchikov, Zamolodchikov'79
Karowski, Thun'81

⇓

exact thermodynamics
from TBA (Thermodynamic Bethe Ansatz)

Yang, Yang'69
Zamolodchikov'91

Symmetries

$$\mathcal{L} = i\bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \sigma \bar{\psi}_i \psi_i - \frac{N}{2\lambda}\sigma^2$$

$$O(2N) \times \mathbb{Z}_2$$

rotates Re and Im components of $\boldsymbol{\psi}_i$

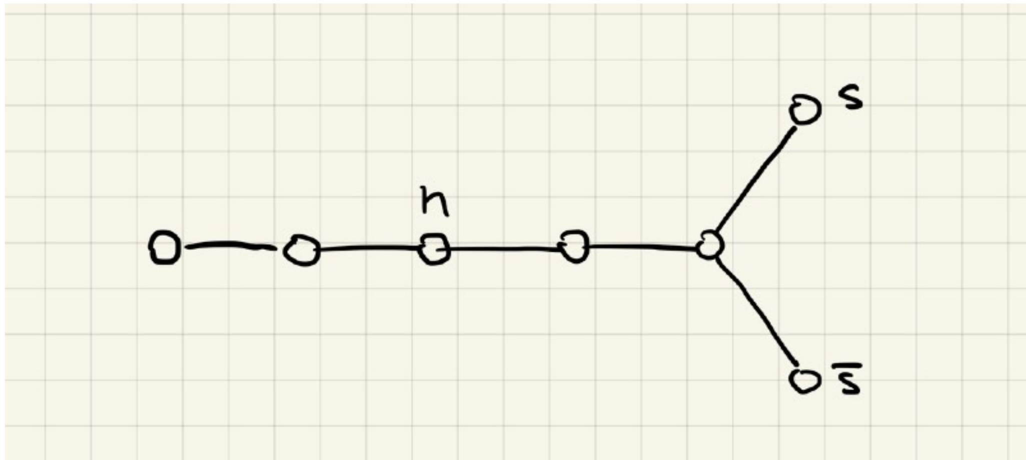
$$_i \to \gamma^3 \quad _i$$

$$\sigma \to -\sigma$$

N ≥ 2:

$$\beta = -\frac{N-1}{\uparrow N}\lambda^2 + \ldots$$

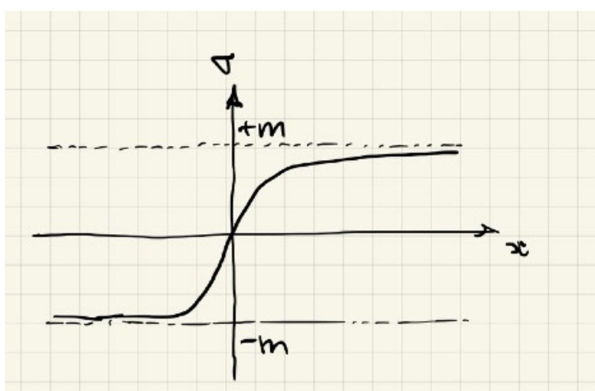
Spectrum



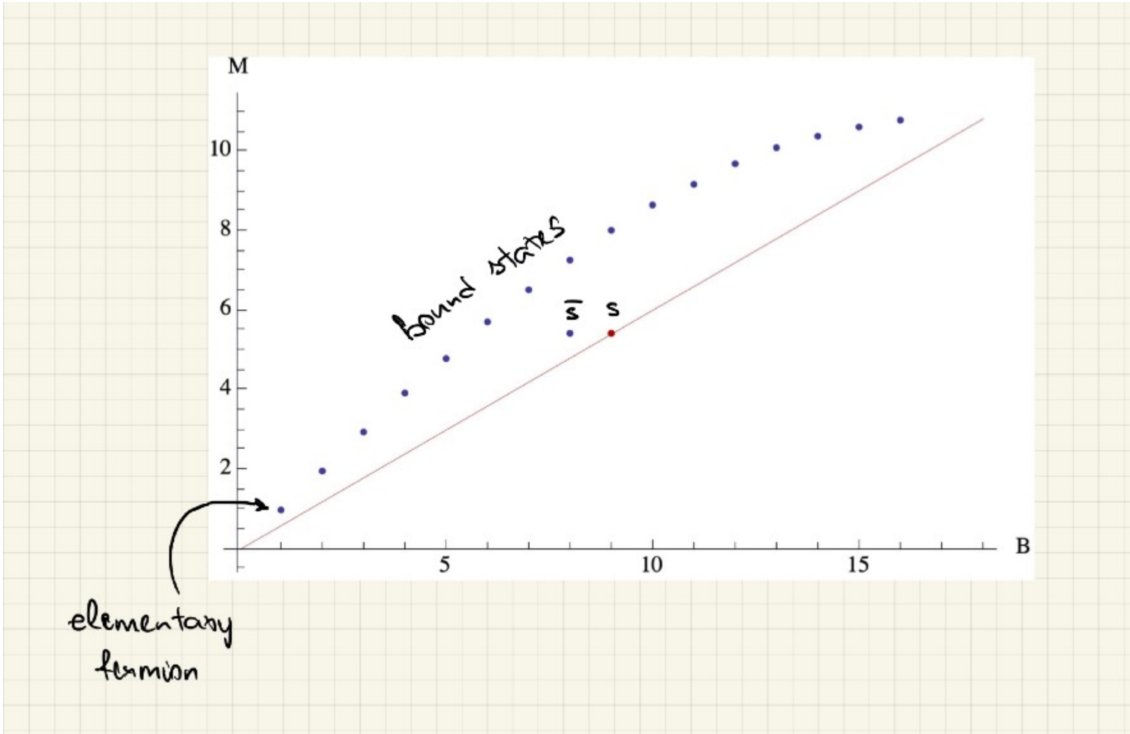
n – particle bound states:

$$m_n = m \frac{\sin \frac{\uparrow n}{2N-2}}{\sin \frac{\uparrow}{2N-2}}$$

Solitons:



$$m_s = \frac{m}{2 \sin \frac{\uparrow}{2N-2}}$$



Solitons carry baryon charge:

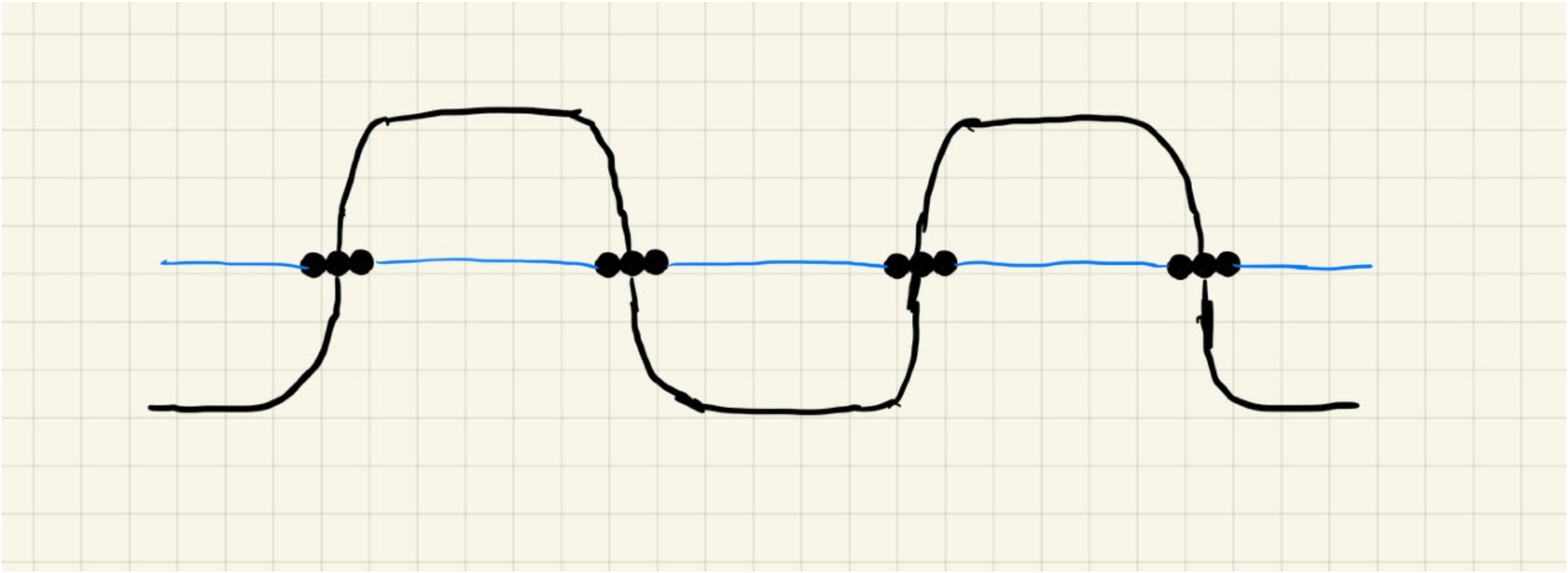
(due to fermion zero modes)

$$B_s = \frac{N}{2}$$

Solitons are most energy efficient! Have the smallest $\frac{m_a}{B_a}$

Soliton crystal

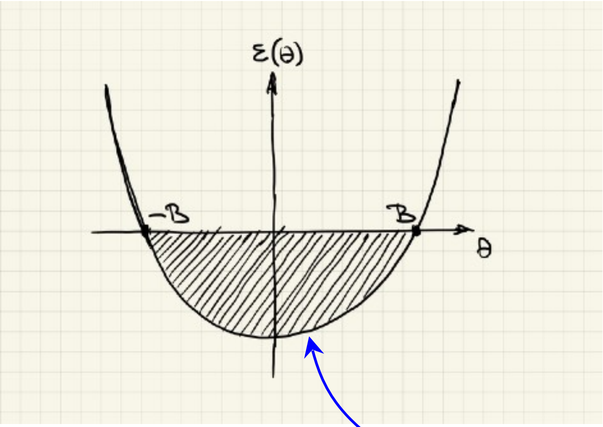
Ground state at finite baryon density:



TBA

$$''(\surd) - \int_{-B}^{B} d\surd\, K(\surd - \surd)\, ''(\surd) = m_s \cosh \surd - \mu B_s$$

$$K(\theta) = \frac{1}{2\pi i} \frac{d \ln S(\theta)}{d\theta}$$



soliton-soliton scattering phaseshift:

Fermi sea of solitons

$$K(\theta) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \mathrm{e}^{-i\omega\theta} \widetilde{K}(\omega),$$

$$\widetilde{K}(\omega) = 1 - \frac{\mathrm{e}^{\frac{\pi|\omega|}{2N-2}} \left(\tanh \frac{\pi\omega}{2} + \tanh \frac{\pi\omega}{2N-2}\right)}{4 \sinh \frac{\pi\omega}{2N-2}}$$

Karowski, Thun’81

Phase transition

$$\mu_c = \frac{m_s}{B_s} = \frac{m}{N \sin \frac{\pi}{2N-2}}$$

Agrees with mean field at large-N:

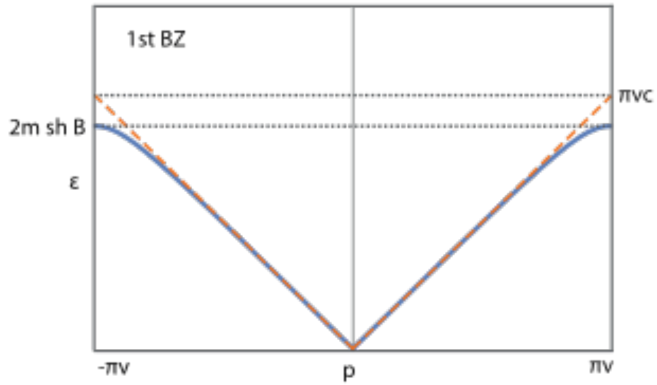
$$\mu_c \overset{N\to\infty}{\simeq} \frac{2}{\pi} m$$

- 2nd order transition (activation point), exists for any N ≥ 2

Solution at large-N

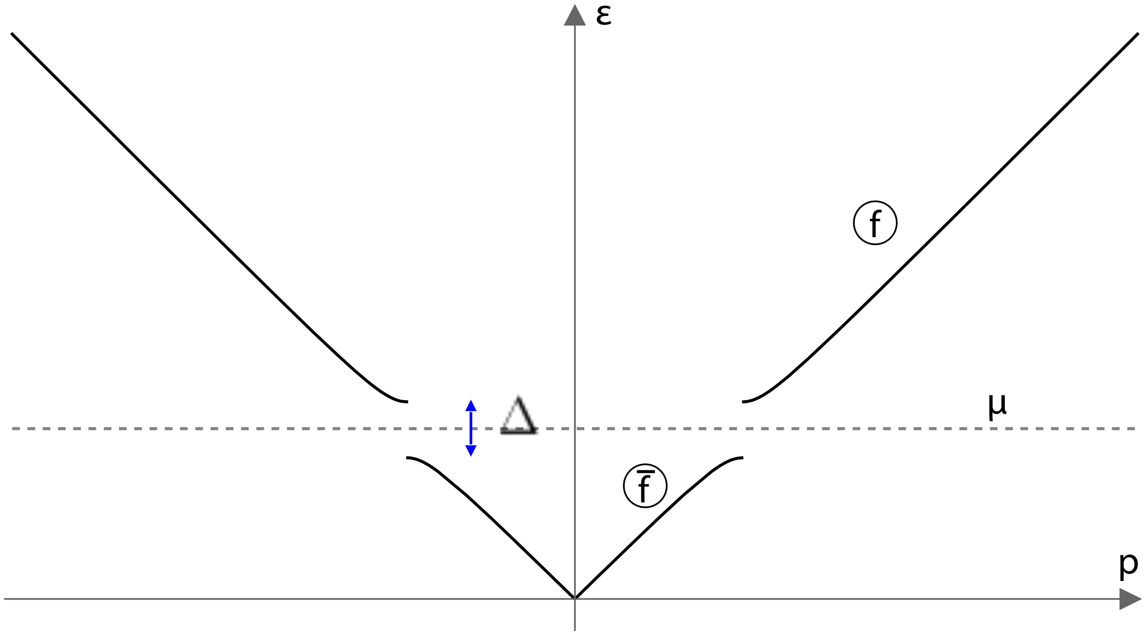
$$\varepsilon(\theta) = -2m\sqrt{\sinh^2 B - \sinh^2 \theta}\,.$$

Hole dispersion:



Phonon ⇔ Hole in Fermi sea of solitons

Fermion spectrum



$$\Delta \simeq \mu \, e^{-\frac{2\pi}{\lambda(\mu)}}$$

fully non-perturbative

Conclusions

- Bethe ansatz is one of the very few non-perturbative techniques in QFT
- Applicable mostly in (1+1)d
- but also in (3+1)d!
  - ✓ N=4 SYM & fishnets
- TBA can be used to study thermal phase transitions in (1+1)d integrable models