Staggered Fishnets

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*Bethe Forum "**Fishnets: CFTs and Feynman Graphs**"*

PERIMETER INSTITUTE

UNIVERSITÄT BONN

bctp

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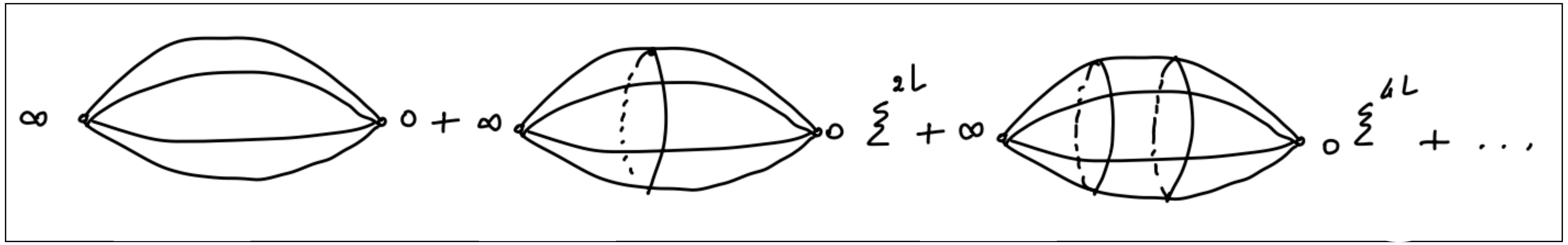
Intro

- *One original purpose of Fishnet Theory: direct derivation (aka proof) of Integrability properties at all-loops, starting from textbook QFT methods (Feynman diagrams and Bethe-Salpeter equation).*
- *The 1/2-BPS state of N=4 SYM theory is not protected after breaking SUSY. The spectrum of their anomalous dimensions (and of states obtained by insertion of derivatives) is related to the spectrum of a transfer-matrix of a non-compact, periodic, Heisenberg magnet with symmetry group SO(1,1+D) in the D-dimensional theory.*

$$\mathrm{Tr}\,(n\cdot\Phi)^L\,,\;\; n\cdot n=0\,,\; \Delta=L \quad \text{for } \mathcal{N}=4$$

$$\langle \mathrm{Tr}\,X^L(\infty)\,\mathrm{Tr}\,\bar{X}^L(0)\rangle \;=$$



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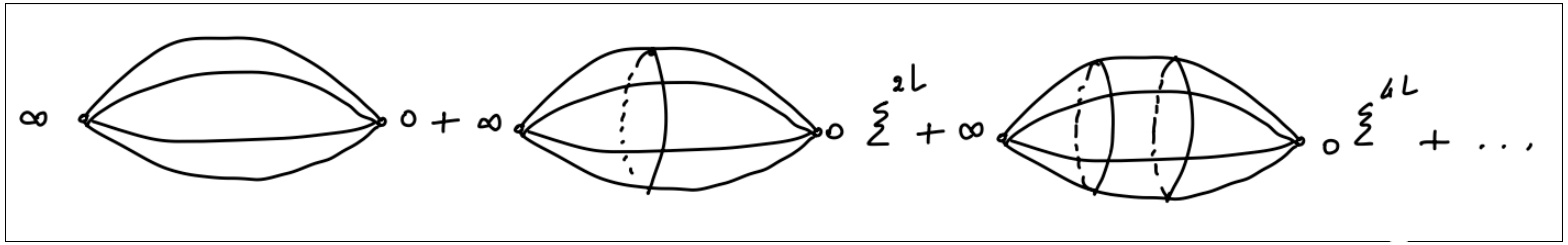

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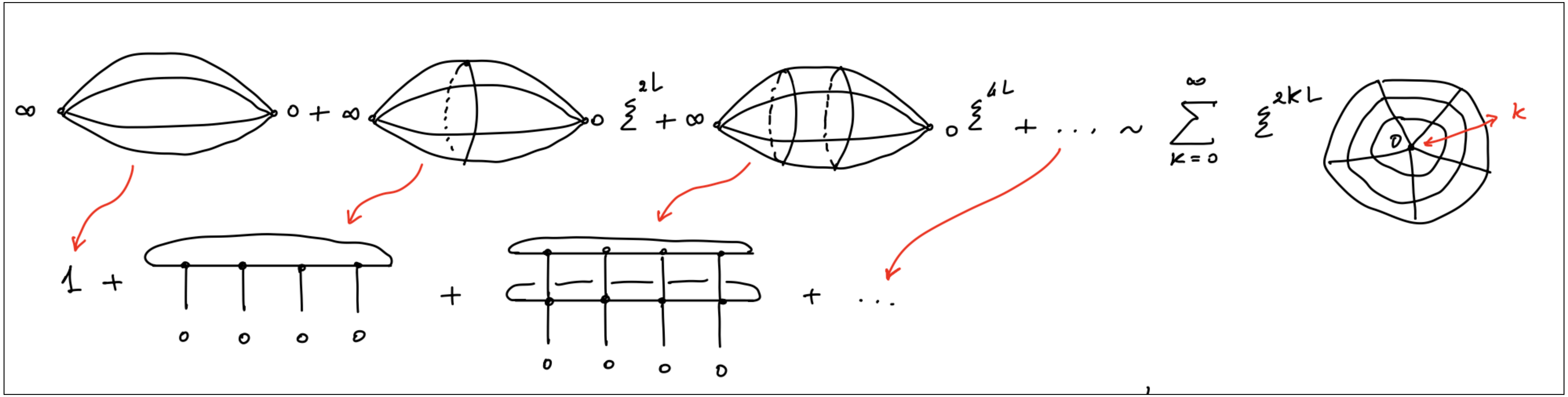

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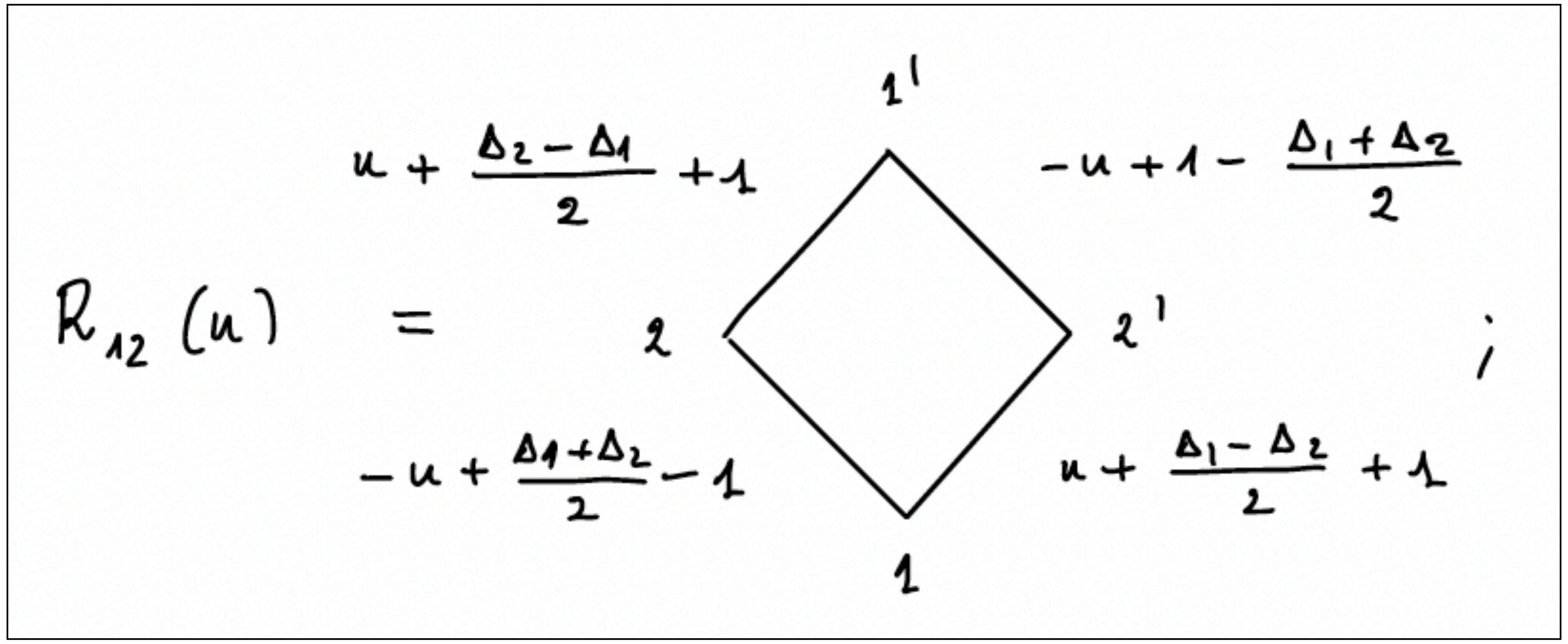

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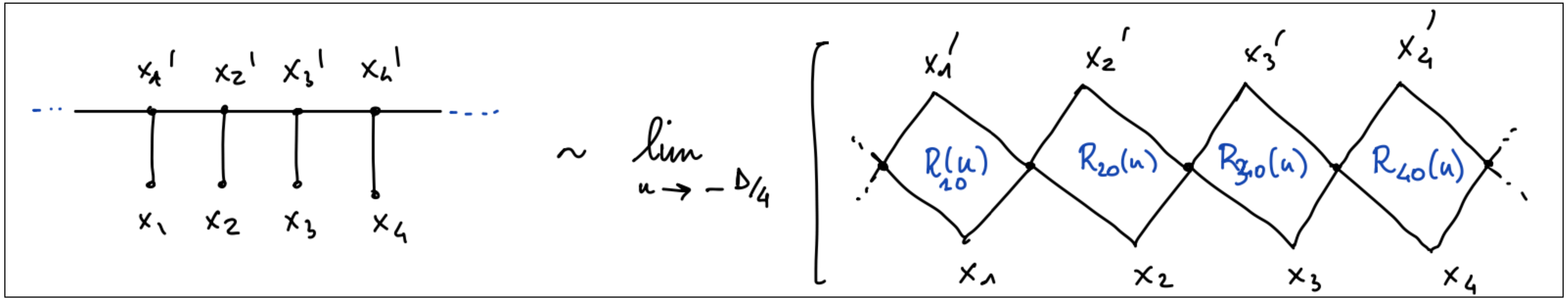


$$\hat{B}(x_j|x_j') = \lim_{u\to -D/4} \mathrm{Tr}_0\left[R_{10}(u)R_{20}(u)R_{30}(u)R_{40}(u)\right]\ ,\ \Delta_1 = \Delta_2 = D/4\,.$$

Conformal Spin-Chain

- *The relation between spin-chain and spectrum of anomalous dimension follows from Bethe-Salpeter method and from the conformal Operator-Product-Expansion (OPE) of certain point-split four-point correlator.*

$$R_{12}(u;\Delta_1,\Delta_2) = \mathbb{P}_{12}\, x_{12}^{2(u-\Delta_+)} p_1^{2(u+\Delta_-)} p_2^{2(u-\Delta_-)} x_{1'2'}^{2(u+\Delta_+)}\,,\;\; \Delta_+ = (\Delta_1+\Delta_2)/2 - 2\,, \Delta_- = (\Delta_1-\Delta_2)/2\,.$$

- *Yang-Baxter equation*

$$R_{12}(u-v)R_{13}(u-w)R_{23}(w-v) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v)$$

- *Unitarity*

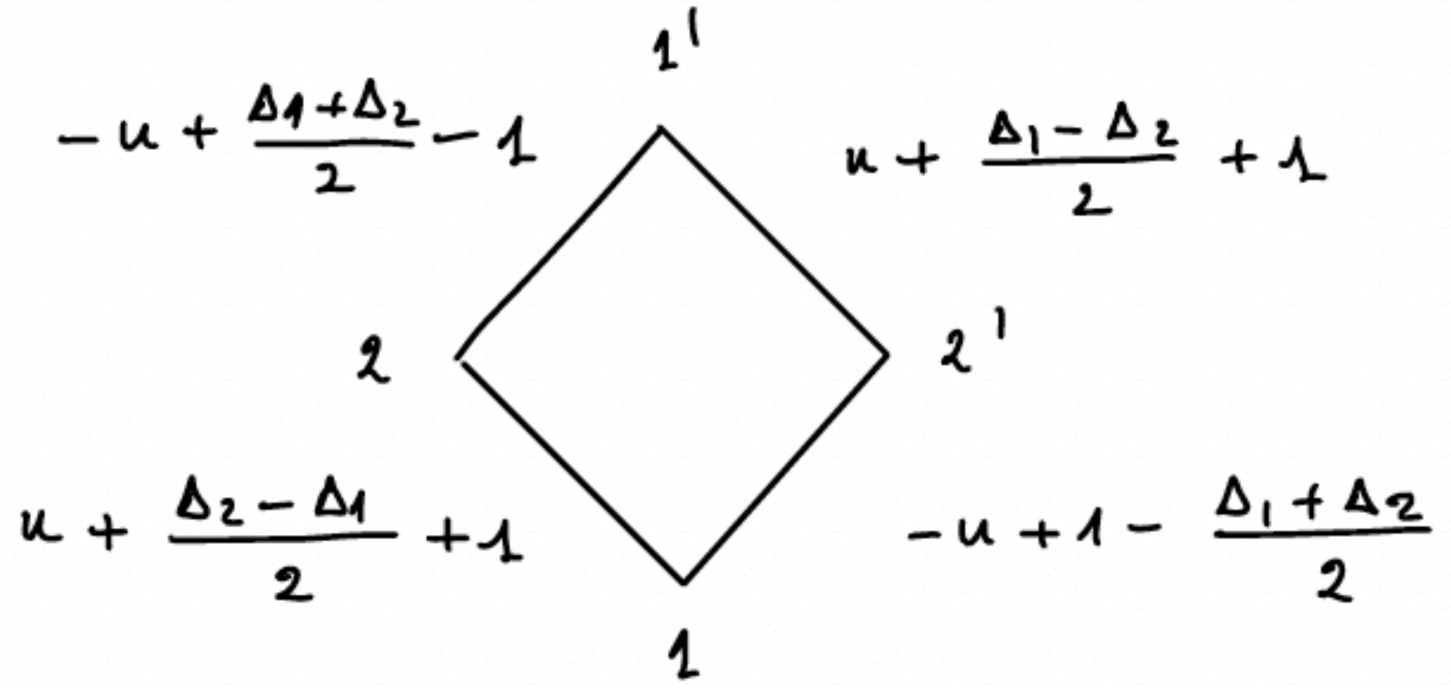
$$R_{12}(u-v)R_{12}(v-u) = 1$$

- *Crossing-symmetry*

$$R_{12}(u)^{t_1} = f_{12}(u)\,(\Sigma_1\otimes 1)\,R_{12}(-u-1)\,(\bar{\Sigma}_1\otimes 1) \; = R_{12}(u)^{t_2} = f_{12}(u)\,(1\otimes\Sigma_2)\,R_{12}(-u-1)\,(1\otimes\bar{\Sigma}_2)$$

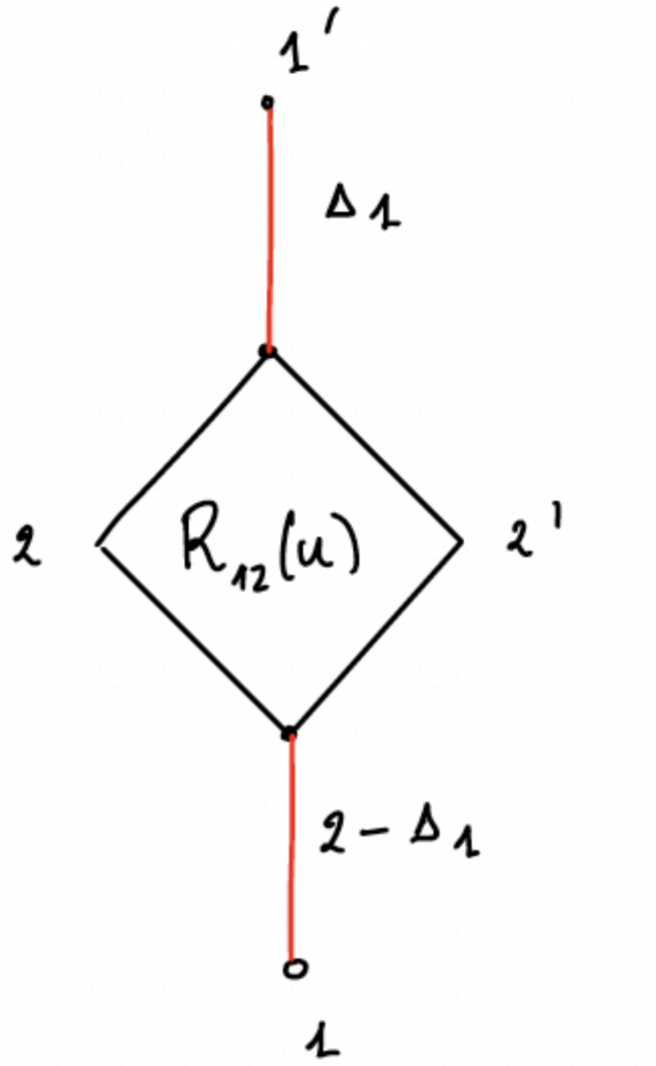
Crossing-symmetry in inf-dimensions

$R_{12}(u)^{t_1} =$

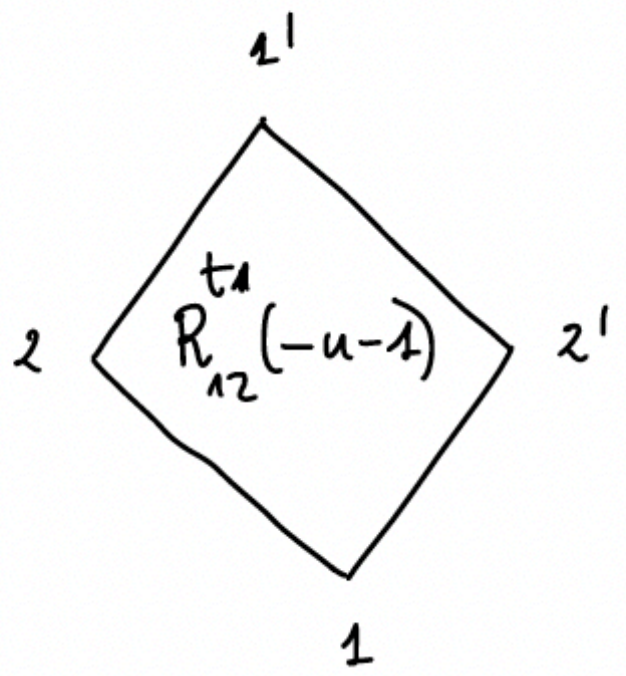


$$R_{12}^{t_1}(u) = \left(\Sigma_1^{-1} R_{12}(-u-1) \Sigma_1\right) \times f(\Delta_1, \Delta_2, -u-1)$$

$$\text{if } \Delta_k = \frac{D}{2} + i\nu, \quad \nu \in \mathbb{R} \Rightarrow \Sigma_1^{-1} = \Sigma_1^{\dagger}$$



$= \Sigma_1 R_{12}(u) \Sigma_1^{-1} =$



$\times f(\Delta_1, \Delta_2, u)$

$$f(\Delta_1, \Delta_2, u) = \pi^2 \frac{\Gamma\left(2+u-\frac{\Delta_1+\Delta_2}{2}\right)\Gamma\left(-u+\frac{\Delta_{21}}{2}\right)\Gamma\left(-u+\frac{\Delta_{12}}{2}\right)\Gamma\left(u+\frac{\Delta_1+\Delta_2}{2}\right)}{\Gamma\left(-u+\frac{\Delta_1+\Delta_2}{2}-1\right)\Gamma\left(u+\frac{\Delta_{12}}{2}+1\right)\Gamma\left(u+\frac{\Delta_{21}}{2}+1\right)\Gamma\left(-u+1-\frac{\Delta_1+\Delta_2}{2}\right)};$$

Conformal Spin-Chain

- *The relation between spin-chain and spectrum of anomalous dimension follows from Bethe-Salpeter method and from the conformal partial waves (CPW) expansion of certain point-split correlator.*

$$T_0(u) = R_{10}(u)R_{20}(u)\cdots R_{L0}(u)\,; \qquad R_{00'}(u-v)T_0(u)T_{0'}(v) = T_{0'}(v)T_0(u)R_{00'}(u-v)\,, \quad [\mathrm{tr}_0\, T_0(u), \mathrm{tr}_{0'}\, T_{0'}(v)] = 0$$

$$\hat{B} \sim \mathrm{tr}_0\, T_0(u = u_*)$$

$$\sum_{k=1}^{\infty} \xi^{2(k-1)L} \langle \Psi \,|\, (\hat{B})^k \,|\, \Psi' \rangle = \langle \Psi \,|\, \frac{1}{\xi^{2L} - \hat{B}^{-1}} \,|\, \Psi' \rangle \sim \sum_{\Delta, S, \{q_3, q_4, \ldots\}} \langle \Psi \,|\, \Phi_{\Delta,S,\mathbf{q}} \rangle \frac{1}{\xi^{2L} - 1/b(\Phi_{\Delta,S,\mathbf{q}})} \langle \Phi_{\Delta,S,\mathbf{q}} \,|\, \Psi' \rangle$$

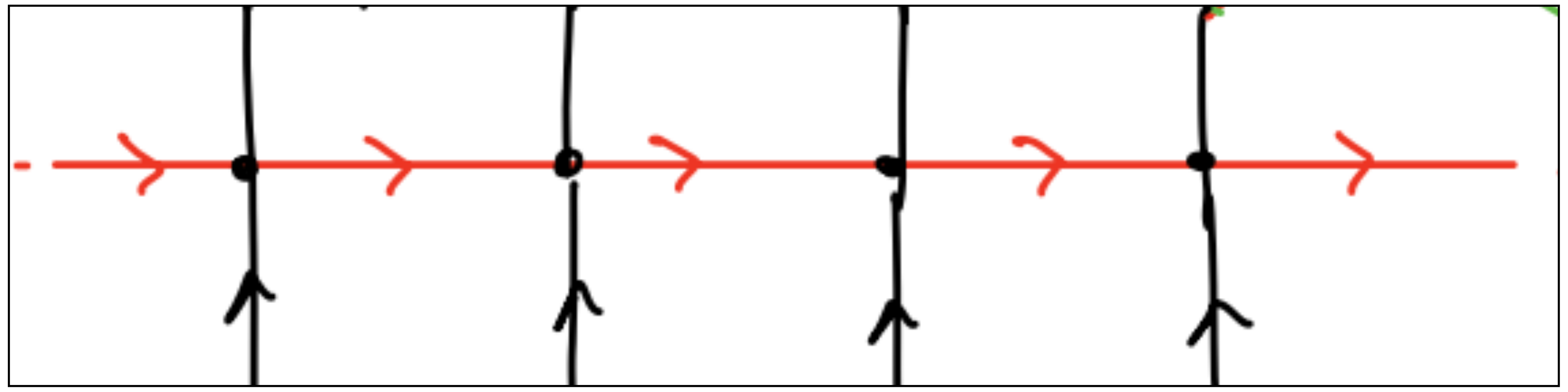
$$\sum_{k=1}^{\infty} \xi^{2(k-1)L} \langle x_1, x_2 \,|\, (\hat{B})^k \,|\, x_3, x_4 \rangle = \langle x_1, x_2 \,|\, \frac{1}{\xi^{2L} - \hat{B}^{-1}} \,|\, x_3, x_4 \rangle \sim \sum_{\Delta, S} \langle x_1, x_2 \,|\, \Phi_{\Delta,S} \rangle \frac{1}{\xi^{2L} - 1/b(\Phi_{\Delta,S})} \langle \Phi_{\Delta,S} \,|\, x_3, x_4 \rangle$$

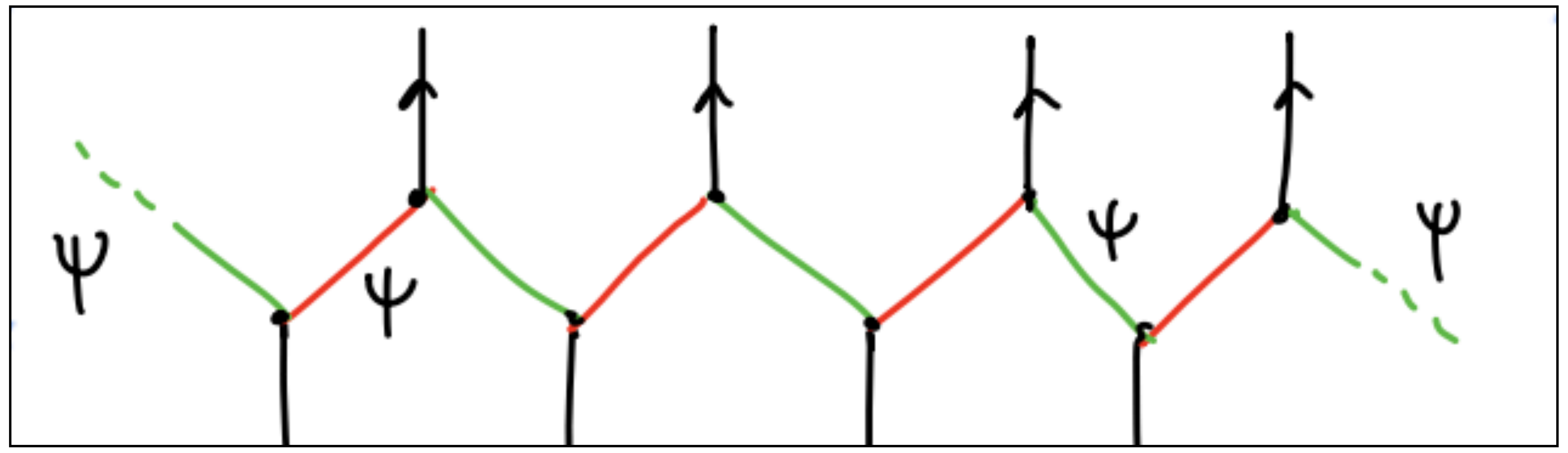
Conformal Spin-Chain

- *Analogue considerations hold for other single-trace operators in the bi-scalar Fishnet Theory. The general rule is to play with the physical space representations, inhomogeneities in the spin-chain sites and boundary operators [see Fishchain papers].*

Conformal Spin-Chain

- *The picture holds for "generalized" Fishnets in the DS limit of N=4 SYM, by addition of fermionic loops.*





- *Same spin-chain (physical spaces), different auxiliary space representation.*

- *The spin-chain pictures in Fishnets, as opposed to N=4 SYM, constitutes an all-loop proof of integrability. The conformal Heisenberg magnet is a 1D model of QM but captures not only one-loop nearest-neighbour interactions, but all the perturbative expansion including finite-size effects.*

Conformal Spin-Chain

- *The spin-chain pictures in Fishnets, as opposed to N=4 SYM, constitutes an all-loop proof of integrability.*

- ***Baxter equations*** *which match with those obtained by explicit double-scaling deformation of the Quantum Spectral Curve (integrability by Kolya Gromov et al. ; [1706.04167]).*

$$\left(\frac{\Delta(\Delta-2)}{4u^2}-2\right)q(u)+q(u+i)+q(u-i)=0 \qquad L=2 \qquad \Delta, S=0$$

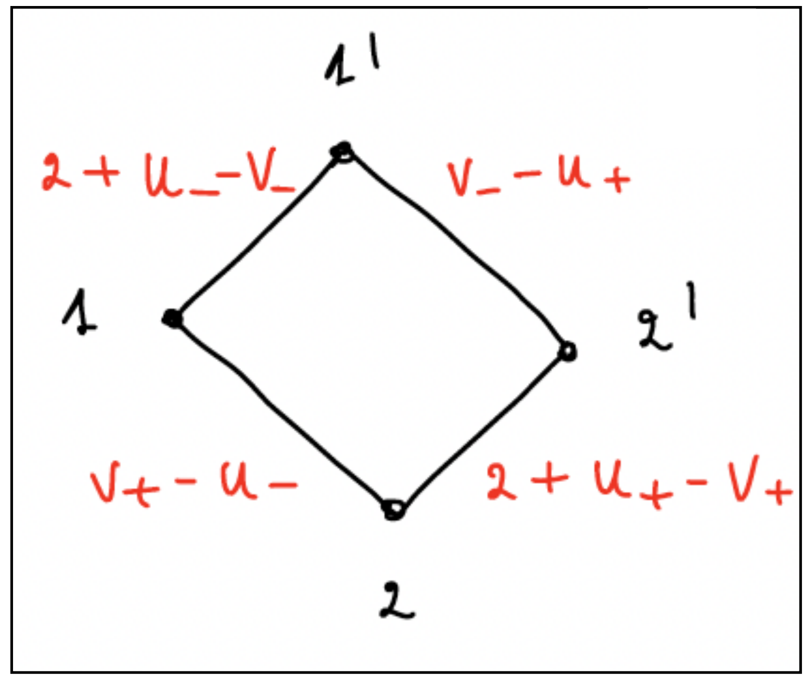
$$\left(\frac{(\Delta-1)(\Delta-3)}{4u^2}-\frac{m}{u^3}-2\right)q(u)+q(u+i)+q(u-i)=0 \qquad L=3 \qquad \Delta, S=0, m=\mathbf{q}_3$$

- *Quantization conditions of charges from analyticity requirements.*

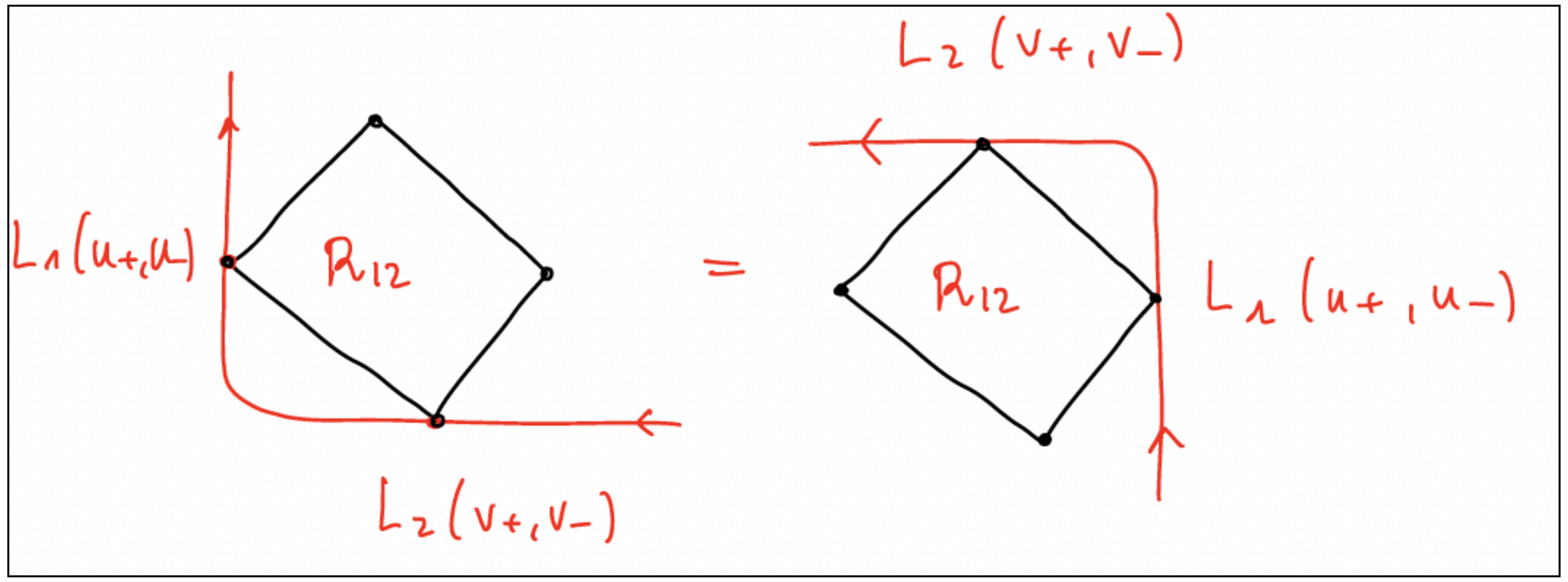
Conformal Spin-Chain

- *The spin-chain pictures in Fishnets, as opposed to N=4 SYM, constitutes an all-loop proof of integrability.*
- ***Yangian symmetry** equations for the general Fishnet Feynman graph with point-split boundary conditions (bootstrap program by Florian Loebbert et al.; [1708.00007])*

$$R_{12}(u-v) = R_{12}(u_\pm, v_\pm)$$



$$R_{12}(u_\pm, v_\pm)\, L_1(u_+, u_-)\, L_2(v_+, v_-) = L_2(v_+, v_-)\, L_1(u_+, u_-)\, R_{12}(u_\pm, v_\pm)$$



Conformal Spin-Chain

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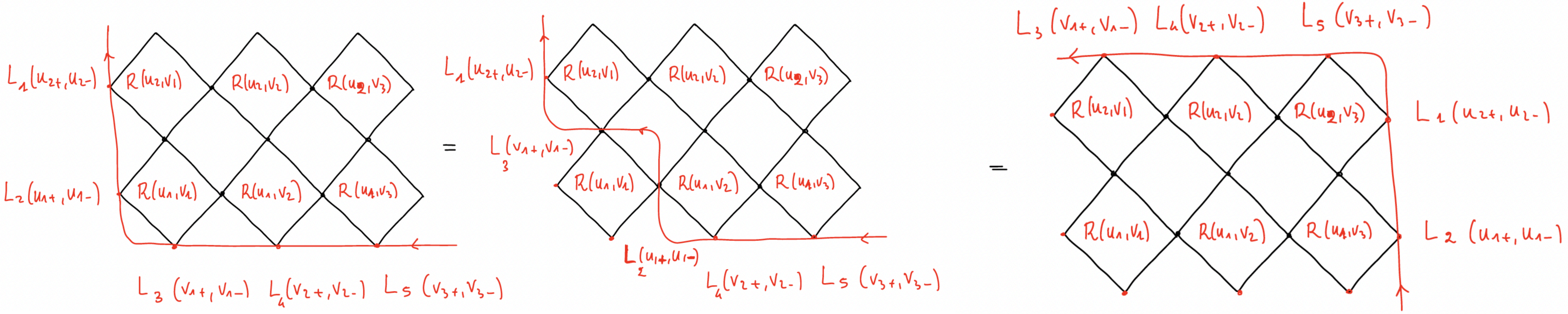
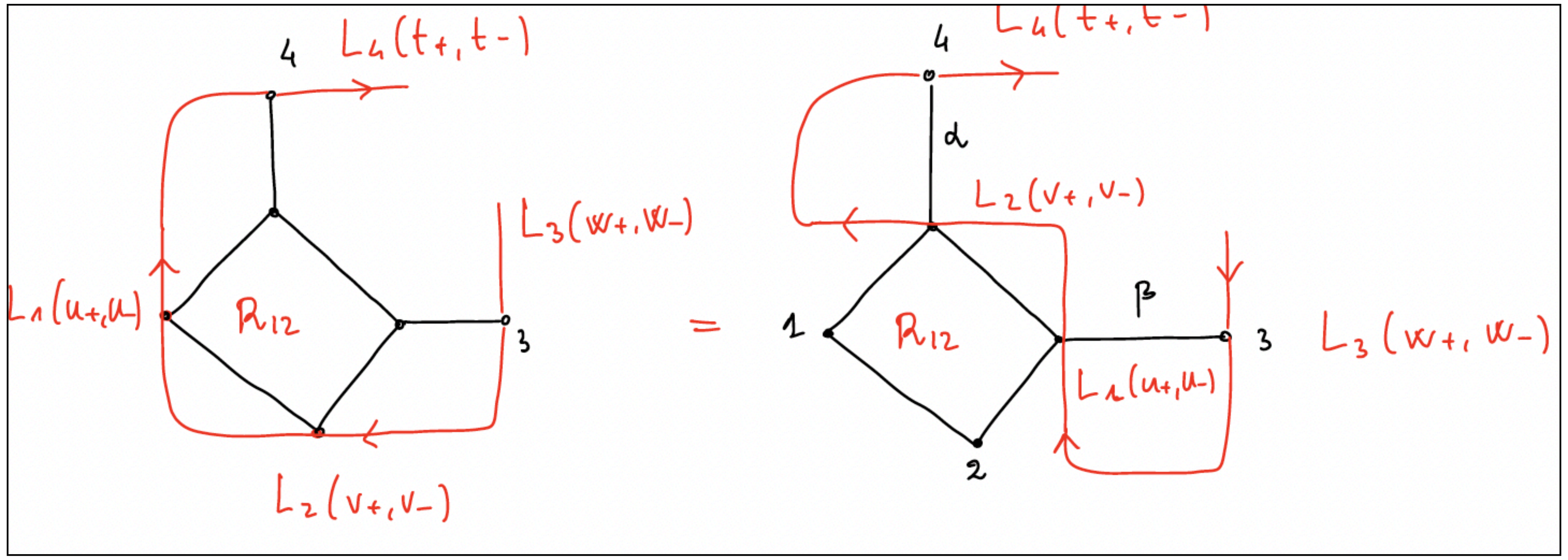
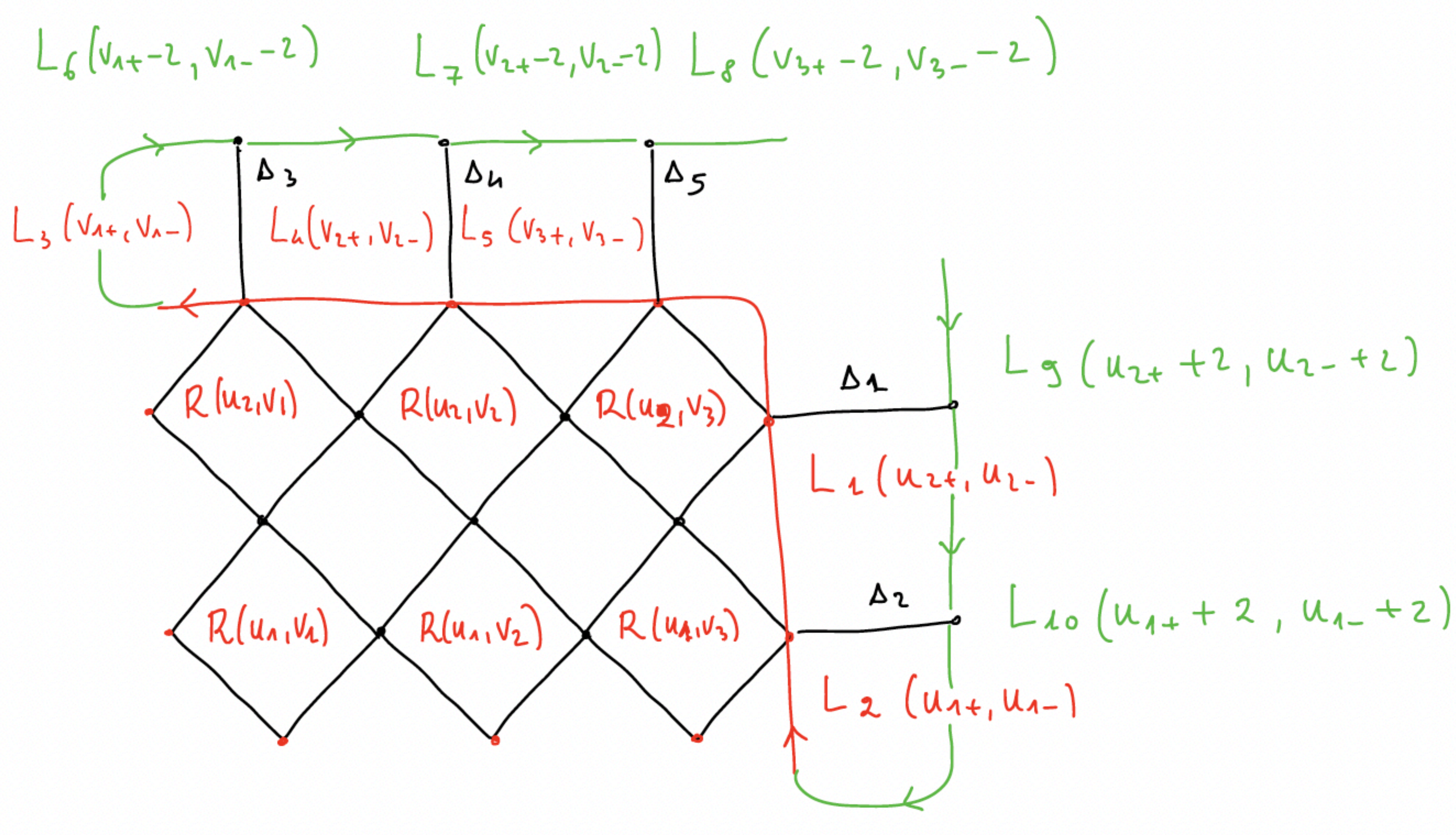


[how to construct a Yangian-symmetric integral?]

- *General technique: patch of Fishnet lattice with inhomogeineities + boundary conditions TBD.*



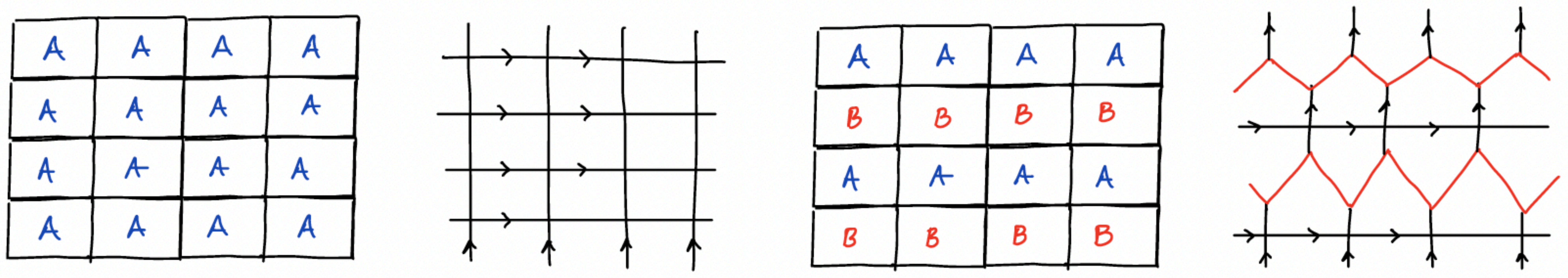


$$L_2(u, \star)L_1(\bullet, u+\Delta)\frac{1}{x_{12}^{2\Delta}} = \frac{1}{x_{12}^{2\Delta}} L_2(u+\Delta, \bullet)L_1(\star, u)$$

$$L(u, u+d/2)\cdot 1 \propto 1$$

Conformal Spin-Chain?

- *So far we considered Feynman diagrams with a simple structure of the bulk, eg. for 1/2-BPS states:*



- *Apparent staggered structure, but "A" and "B" rows define commuting transfer matrices (due to star-triangle symmetry).*
- *What about going beyond 1/2-BPS and beyond bi-scalar theory at the same time?*

Conformal Spin-Chain?

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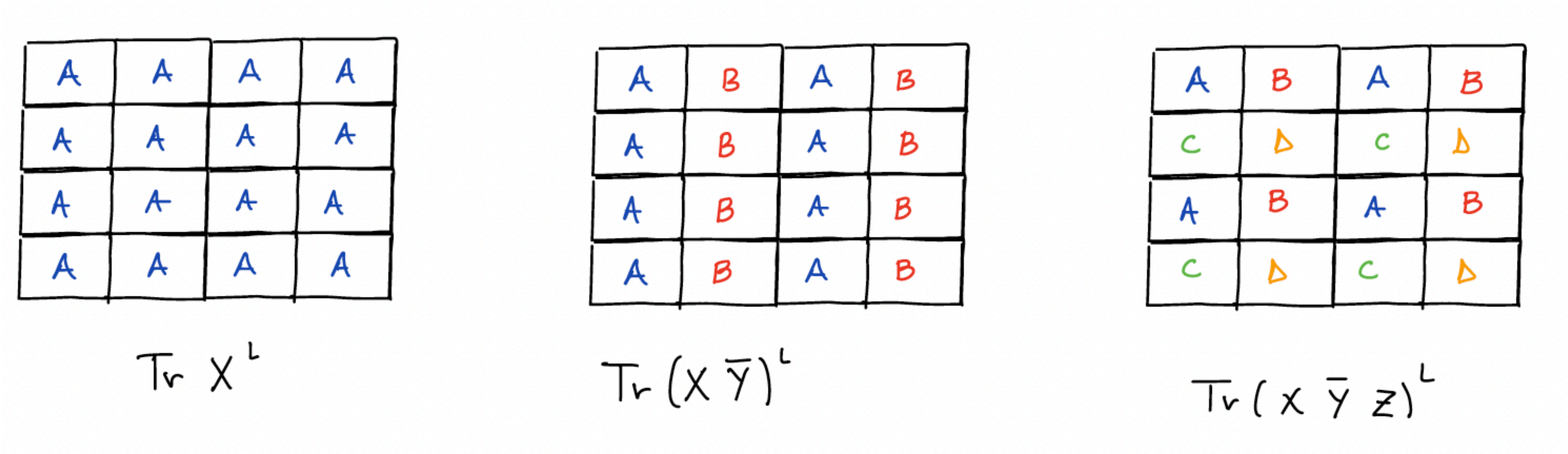
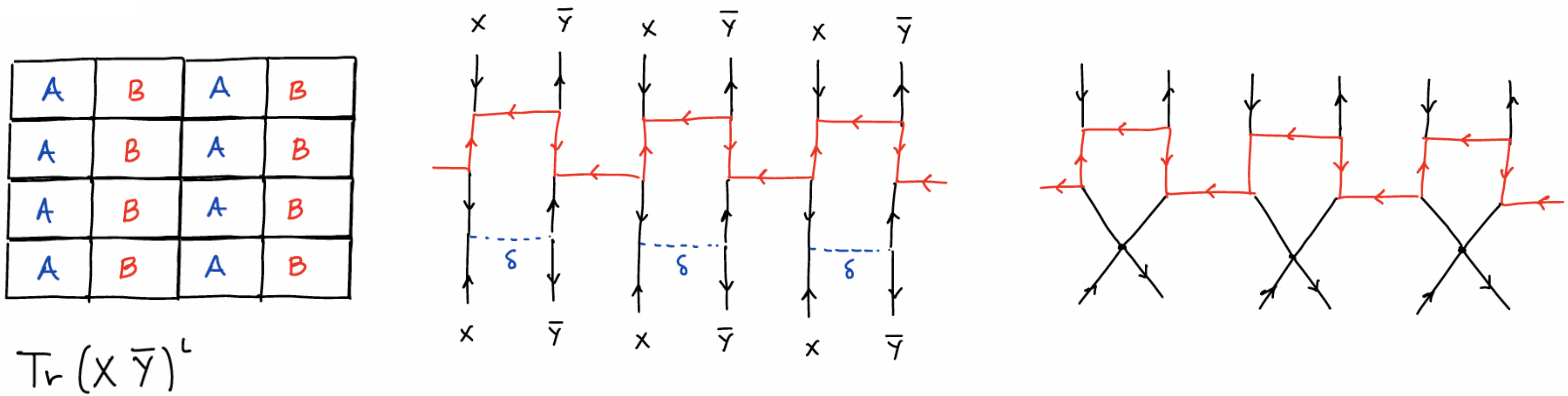


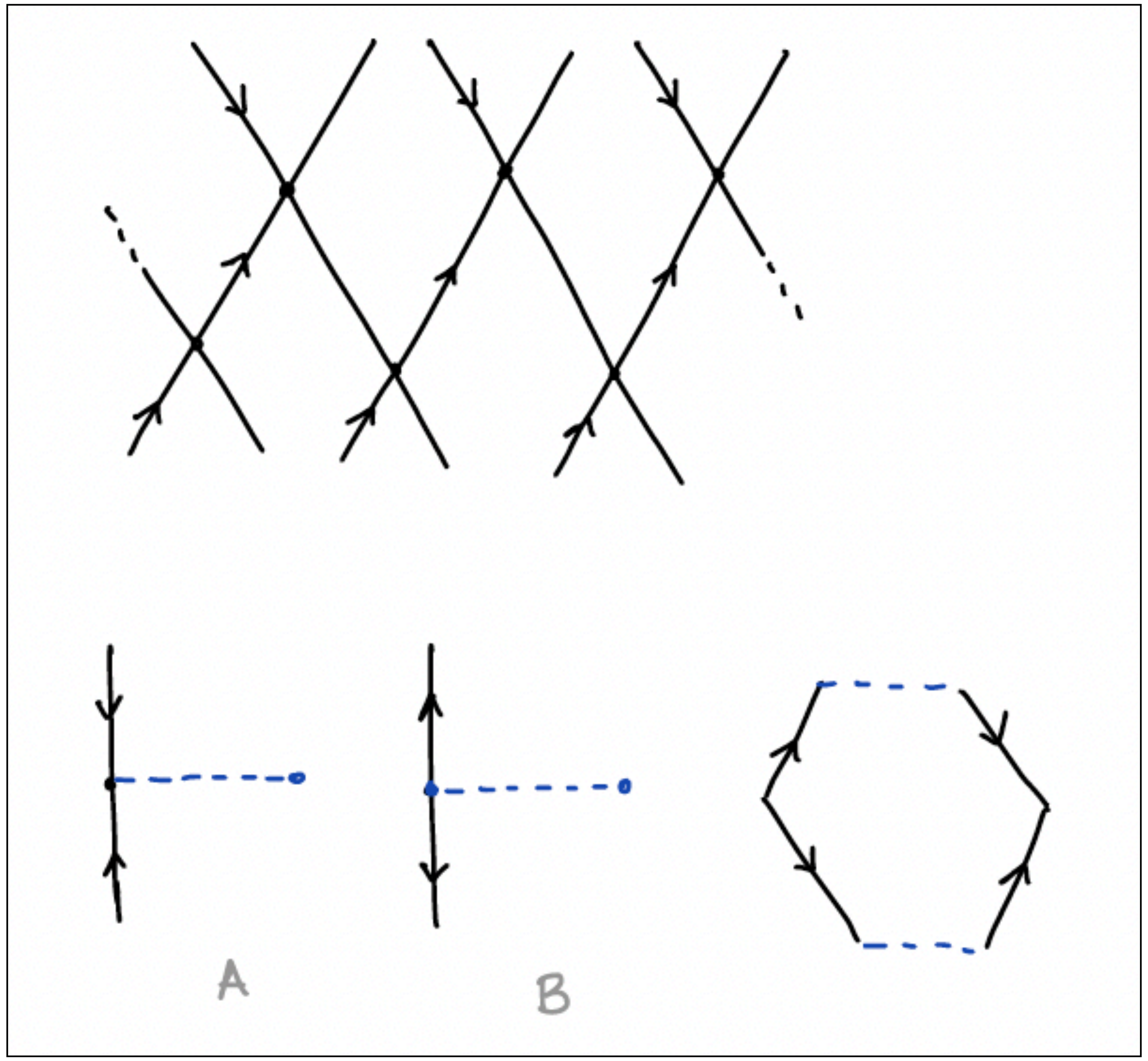
Staggering

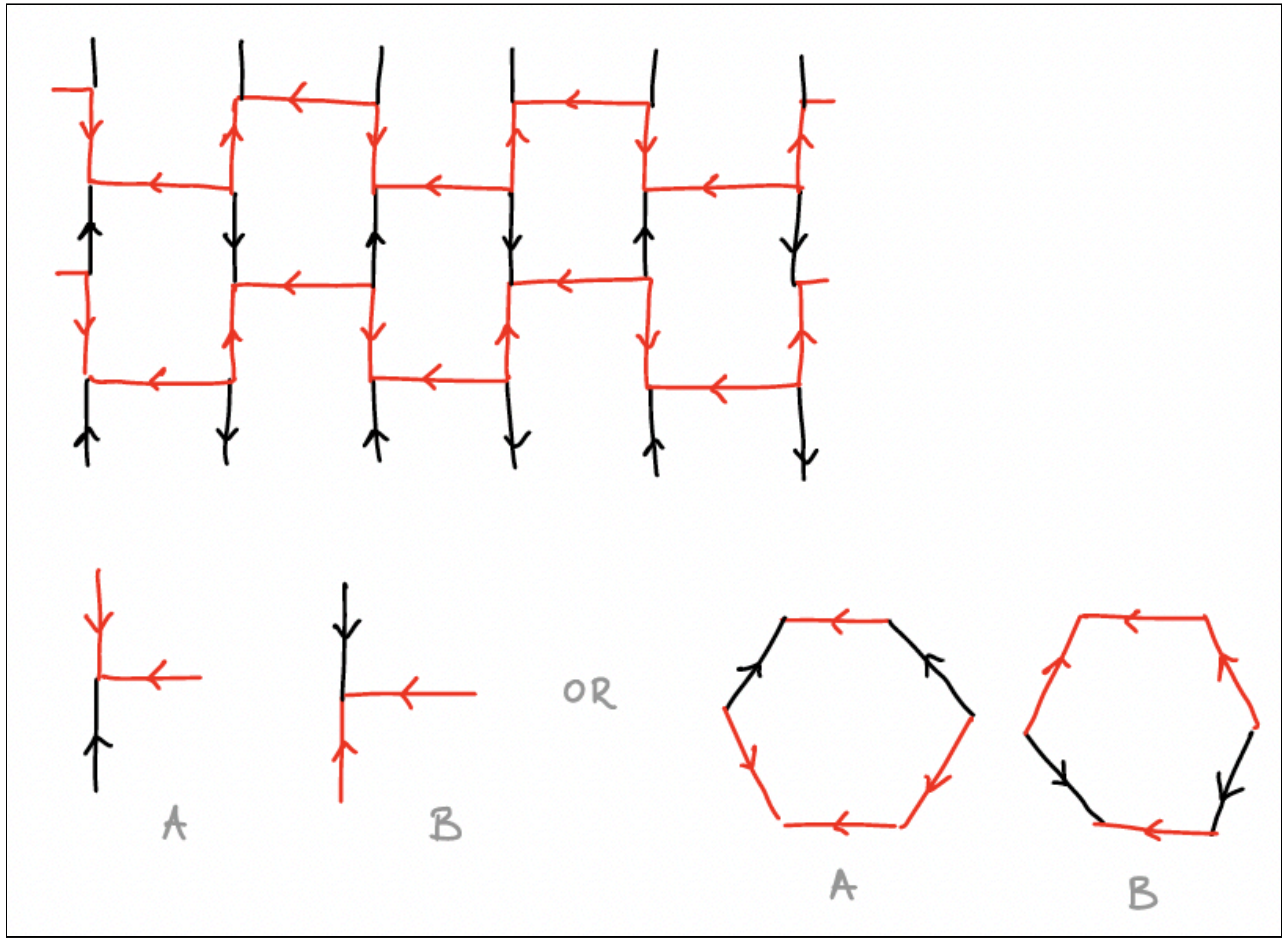
- *What about going beyond 1/2-BPS and beyond bi-scalar theory at the same time?*



Staggering

- *Appearance of lattice structure with two alternating cells:*





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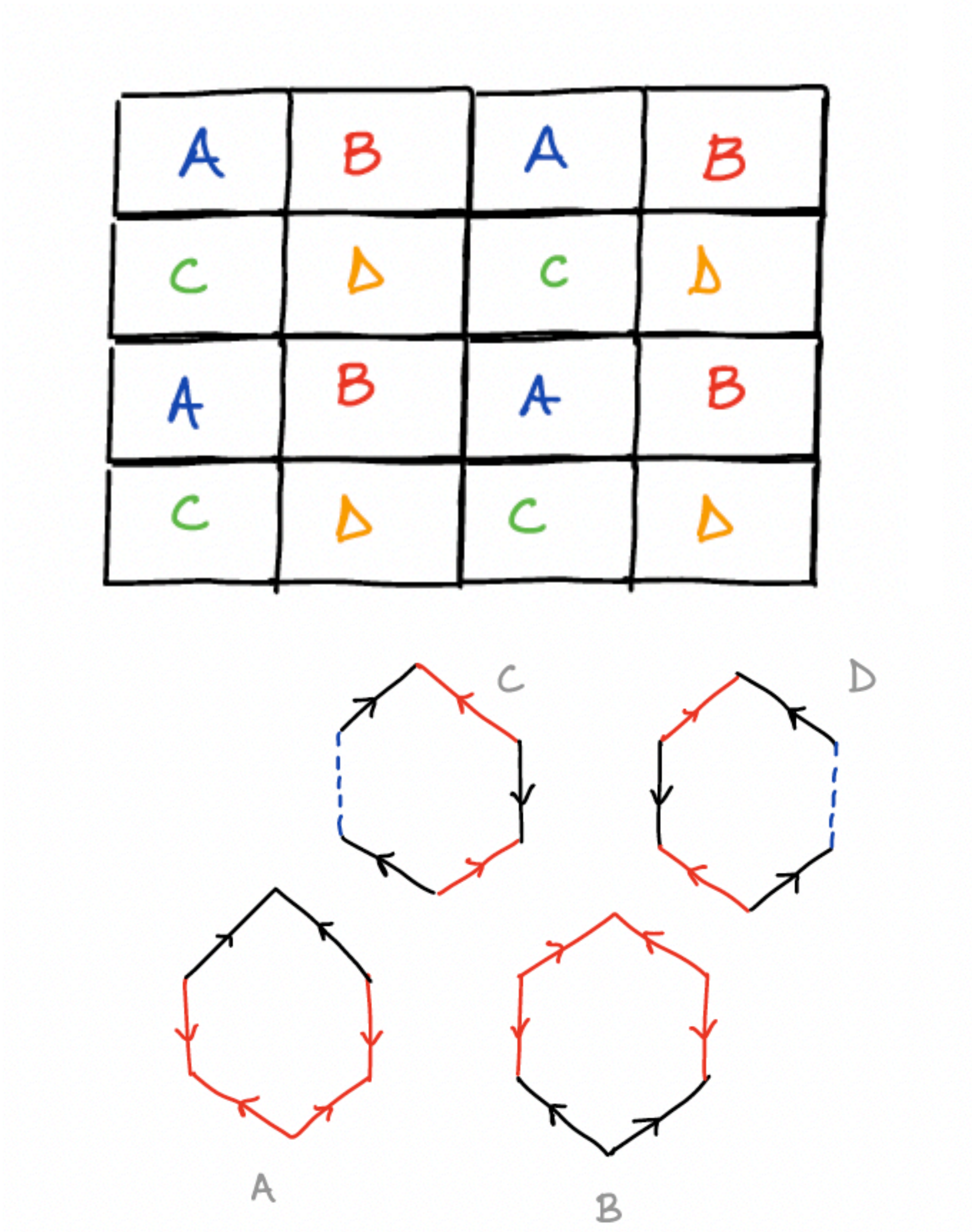
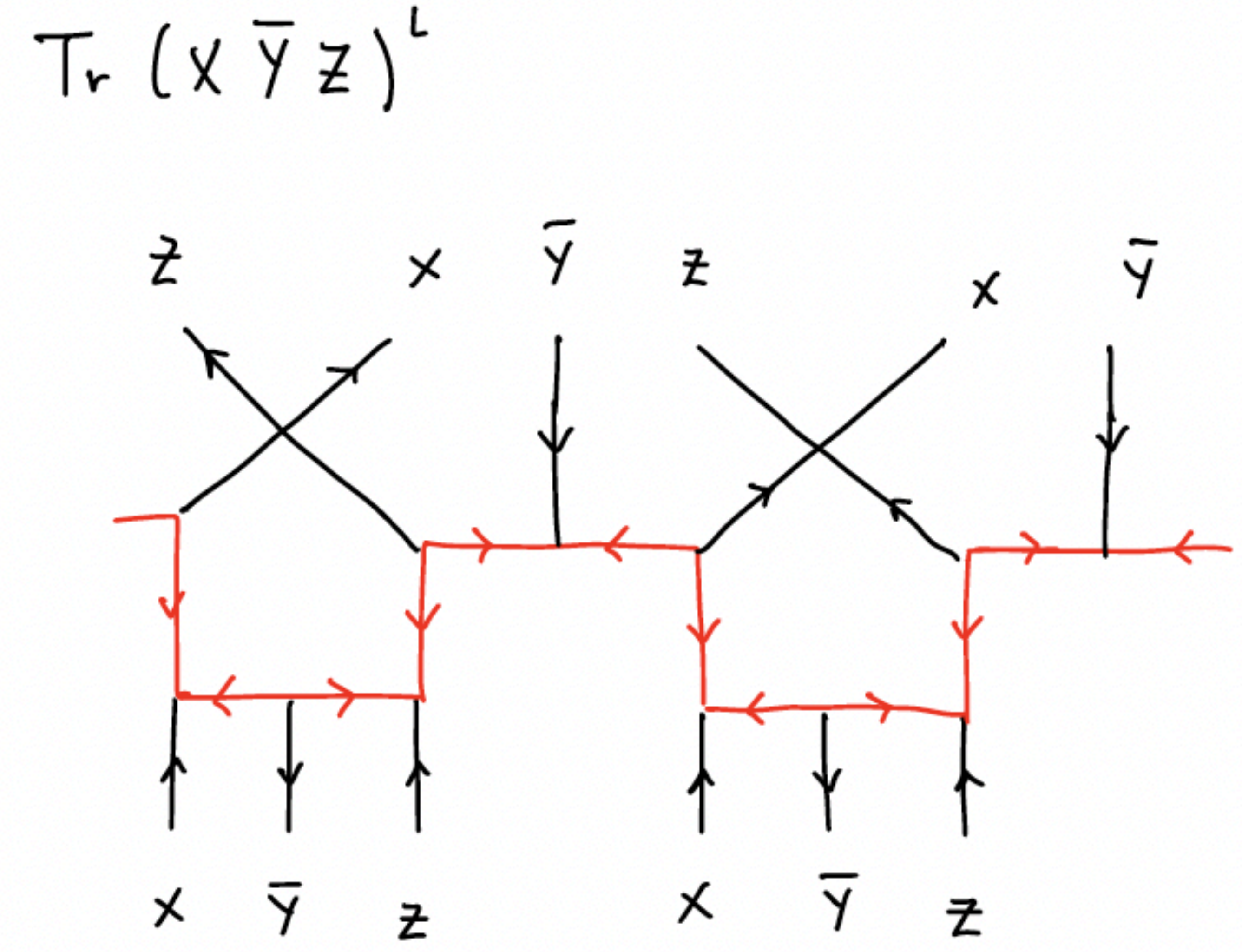
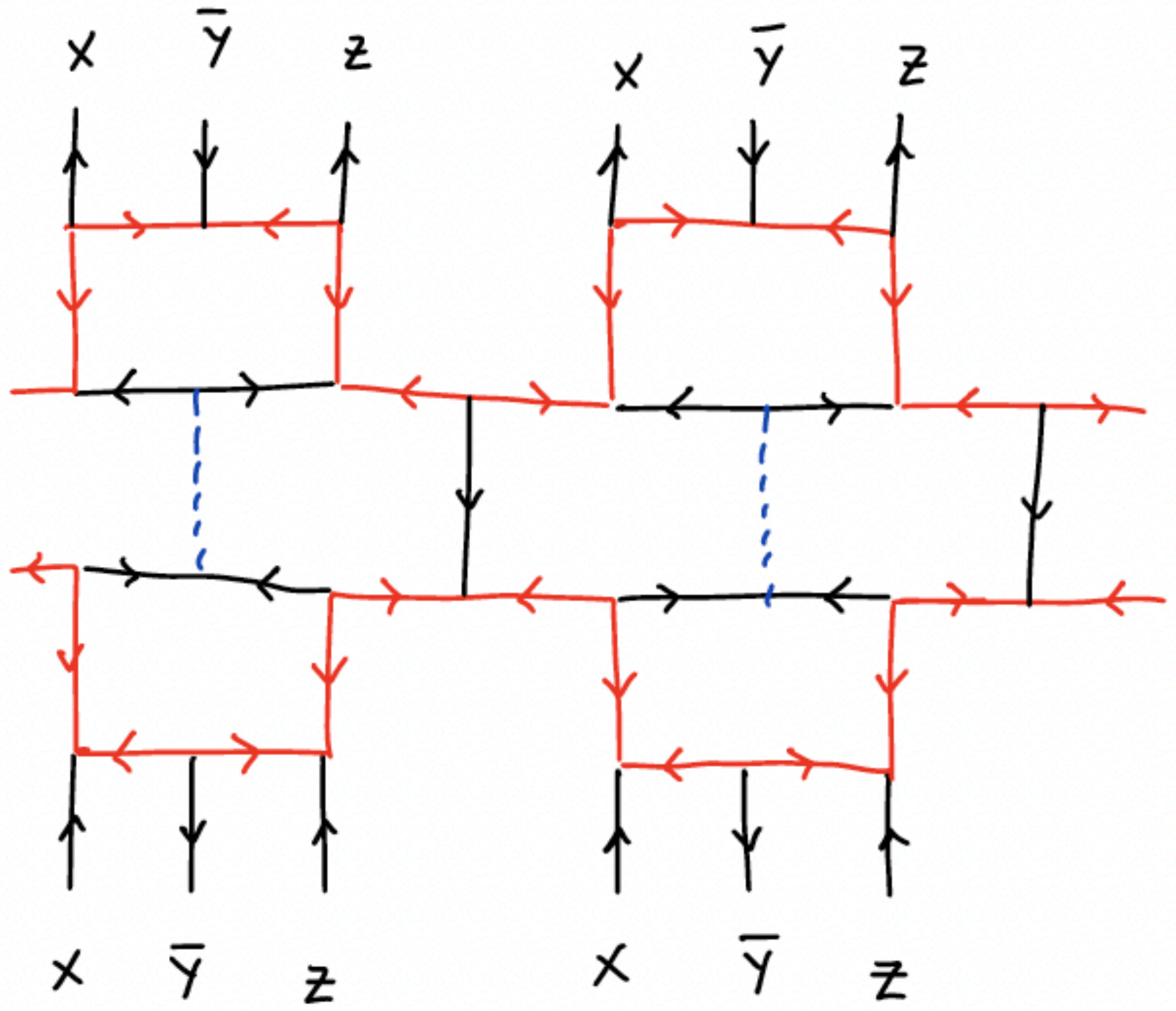


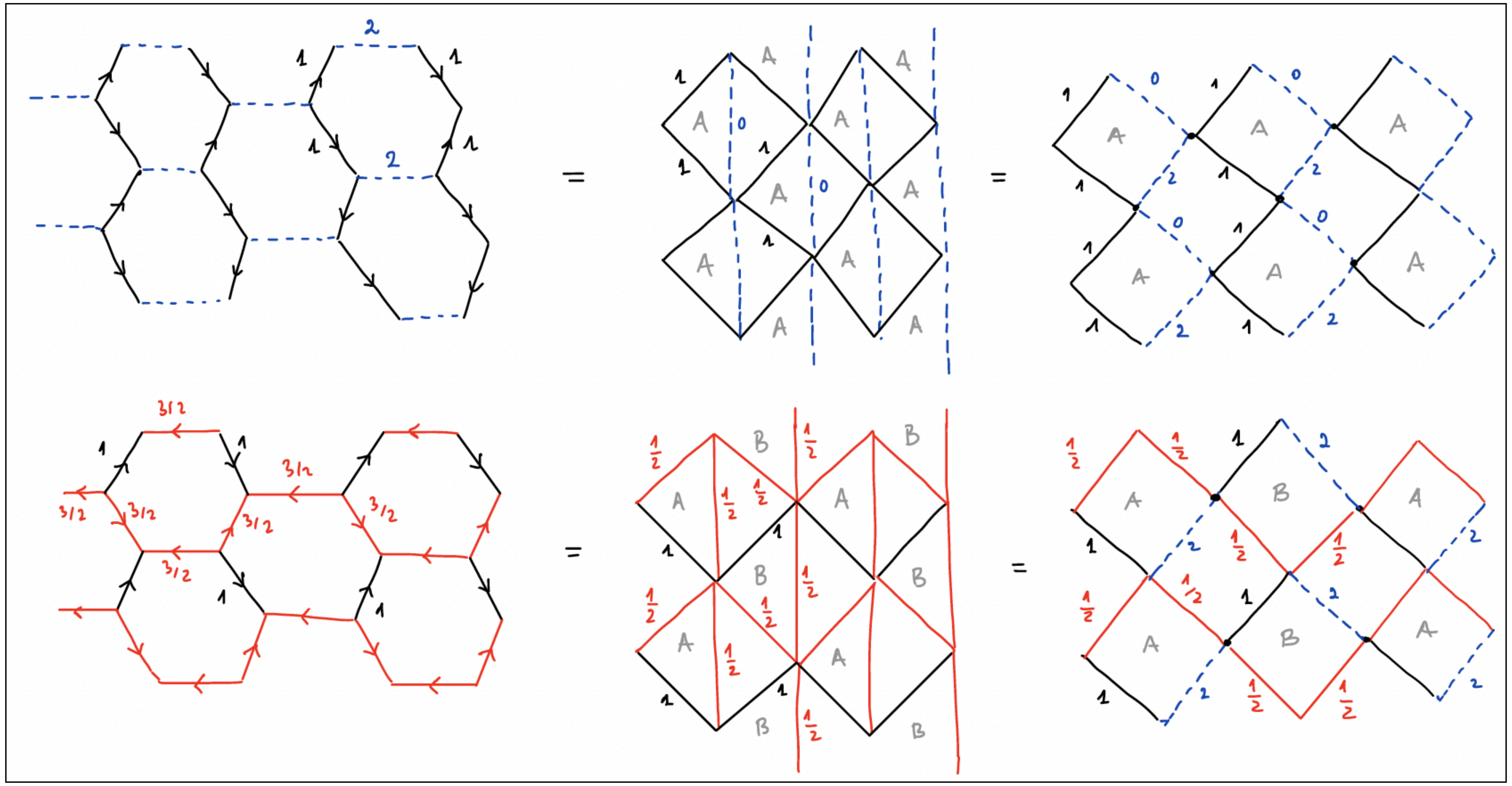




Dynamical Fishnet: case study

- *One can claim, to a good level of generality, that the theory spectrum can be still computed by Bethe-Salpeter method + OPE, but the graph-building kernel is not the "usual" Heisenberg magnet transfer-matrix.*

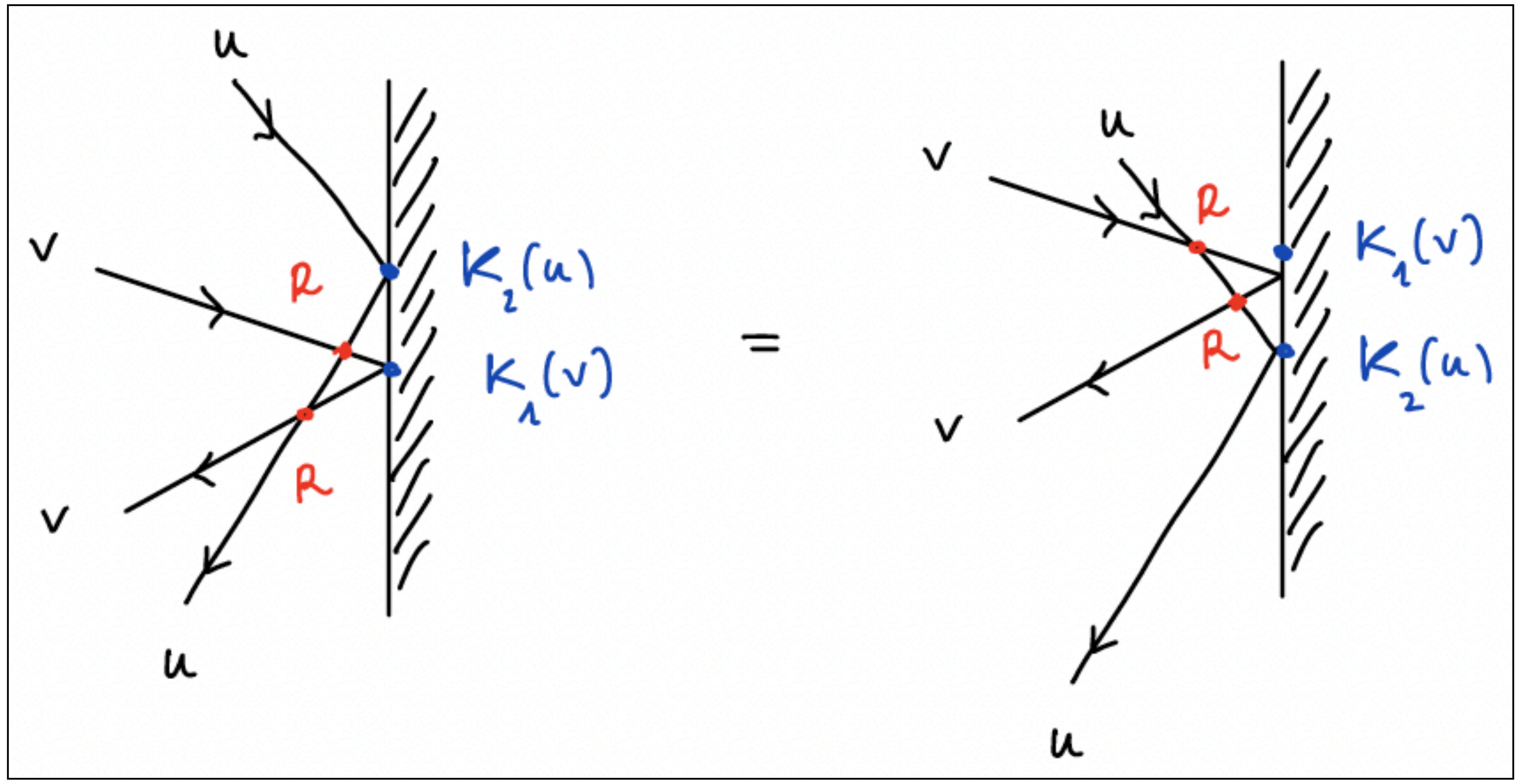
- *The bulk lattice structure of the Dynamical Fishnet theory can be analyzed by a few case studies, which show the emergence of a staggered square lattice, in opposition to the standard one.*



Staggered Two-Row transfer matrix

- *The two-row (two-columns) alternate transposition of the lattice cells suggest to look for integrability in terms of Sklyanin* ***Reflection Equations****.*

$$K_2(u)\,R_{12}(u-v)\,K_1(v)\,R_{12}(u+v) = R_{12}(u-v)\,K_1(v)\,R_{12}(u+v)\,K_2(u)$$



- *To this task, it is convenient to transform the Feynman Integrals by star-triangle identity and to focus on the "mirror-channel" expansion of the correlator.*

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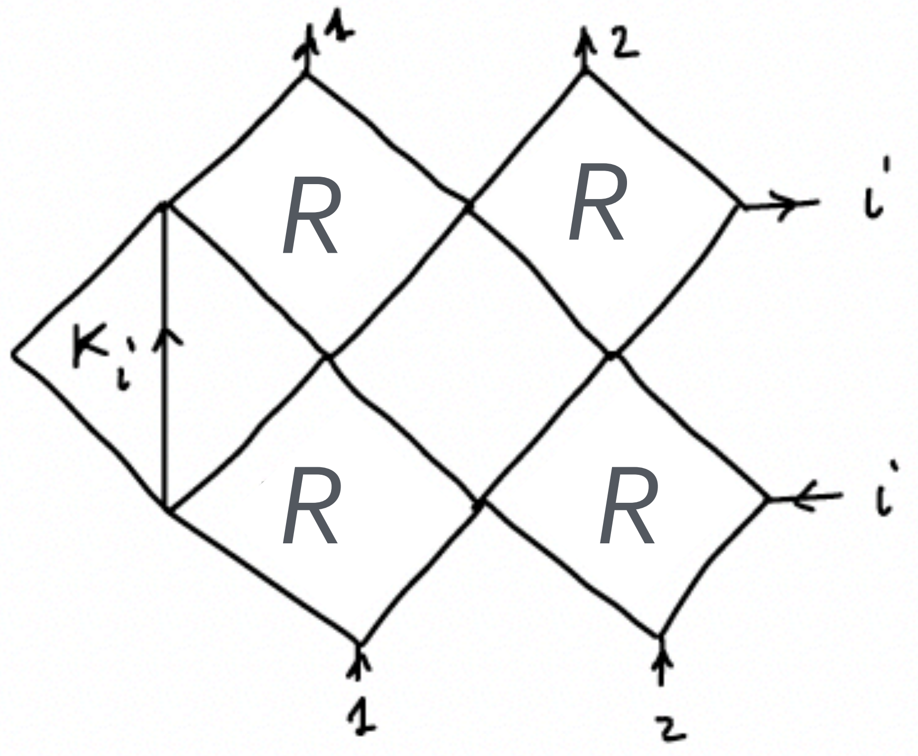
$$T_1(-v)^{-1}T_2(-u)^{-1}K_2(u)\,R_{12}(u-v)\,K_1(v)\,R_{12}(u+v)T_2(u)T_1(v) =$$
$$= T_1(-v)^{-1}T_2(-u)^{-1}R_{12}(u-v)\,K_1(v)\,R_{12}(u+v)\,K_2(u)T_2(u)T_1(v)$$

$$M_i^-(u) = T_i^{-1}(-u)\,K_i(u)\,T_i(u) =$$



$$M_2(u)\,R_{12}(u-v)\,M_1(v)\,R_{12}(u+v) = R_{12}(u-v)\,M_1(v)\,R_{12}(u+v)\,M_2(u)$$

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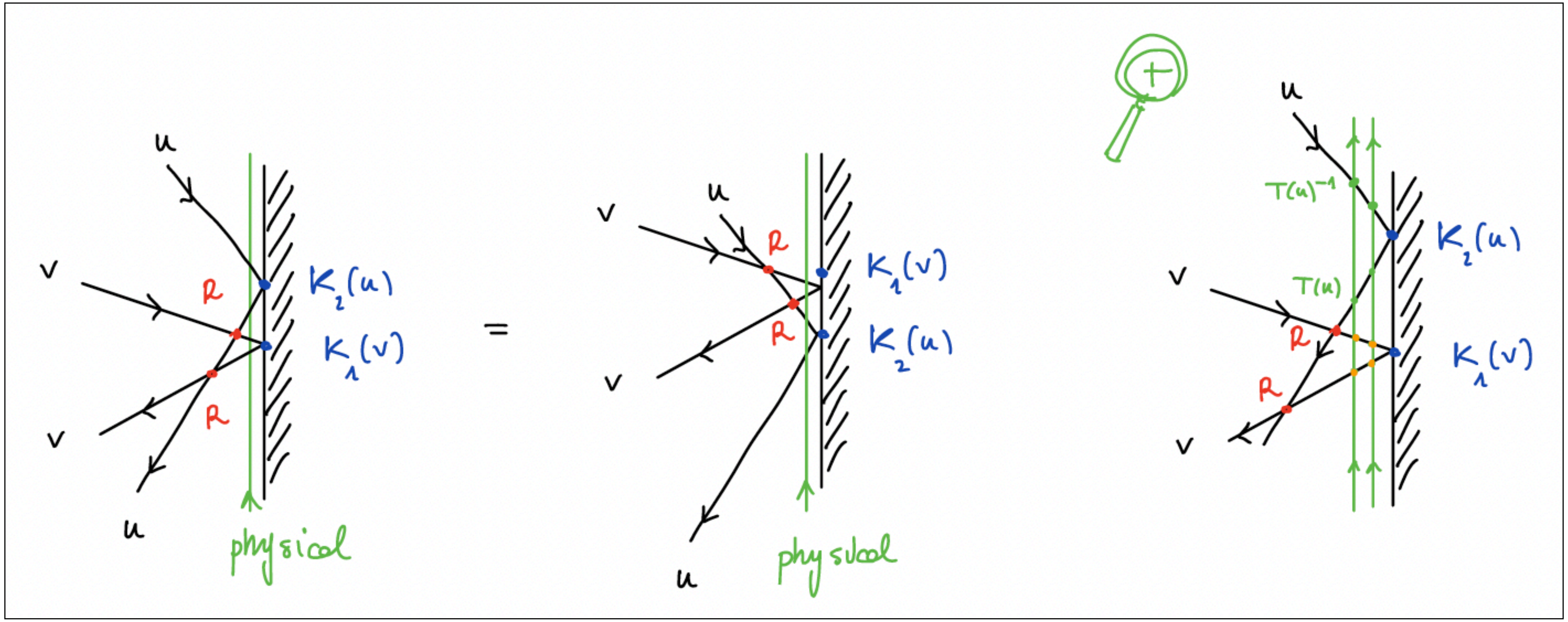
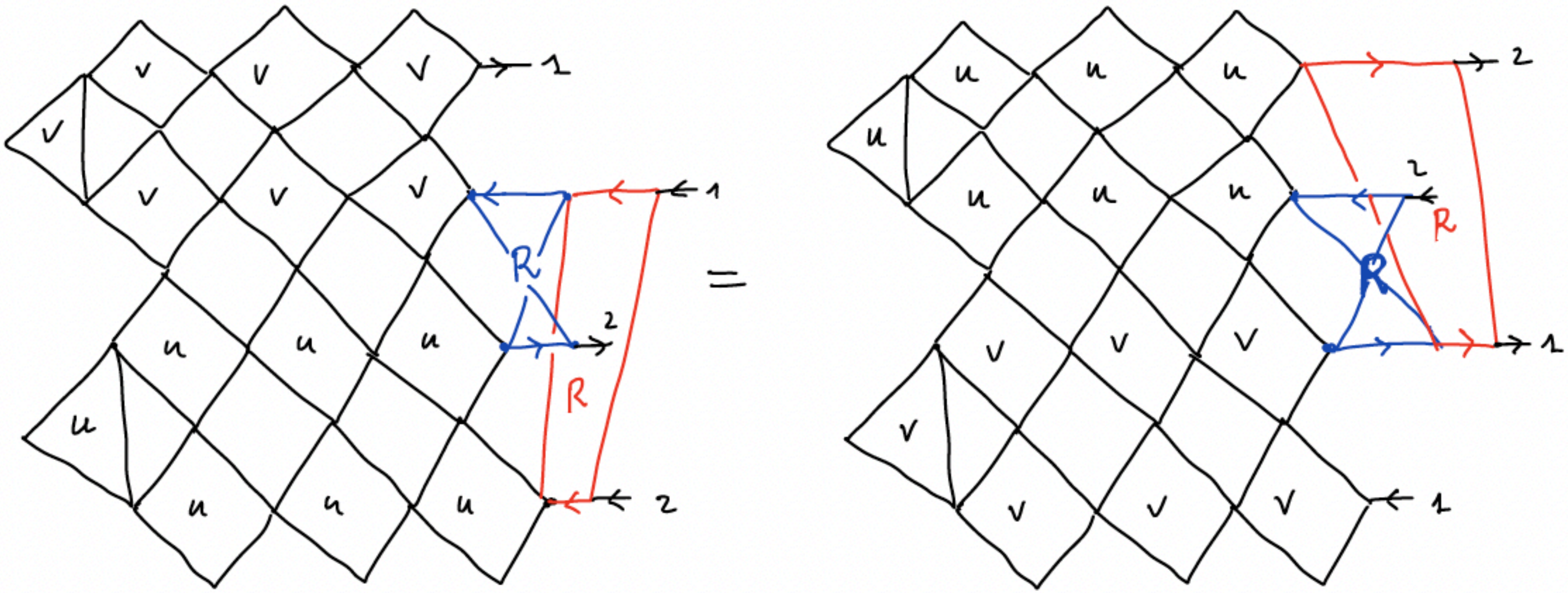


Staggered Two-Row transfer matrix

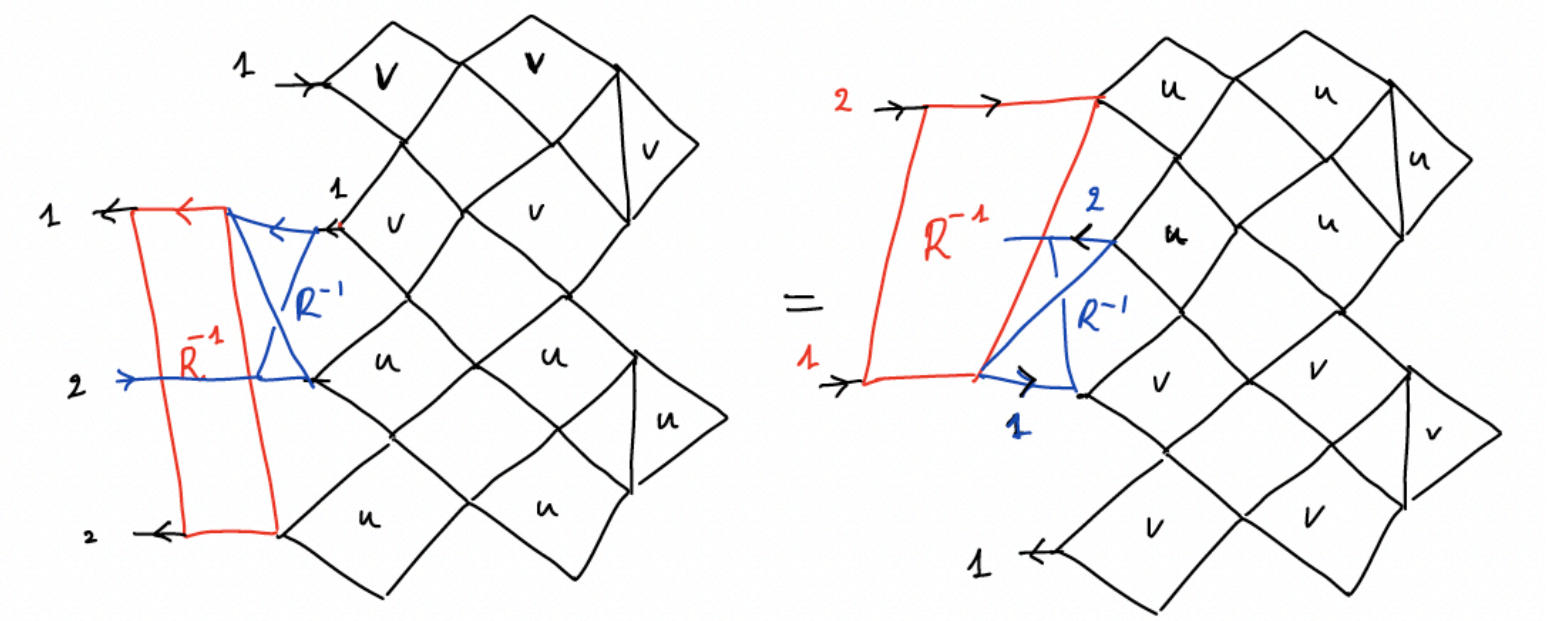
$$M_1^{(-)}(v)\,R_{12}(u-v)\,M_2^{(-)}(u)\,R_{12}(u+v) = R_{12}(u-v)\,M_2^{(-)}(u)\,R_{12}(u+v)\,M_1^{(-)}(v)$$



Staggered Two-Row transfer matrix

$$M_1^{(+)}(v)\, R_{12}(u-v)\, M_2^{(+)}(u)\, R_{12}(u+v) = R_{12}(-u-v-2)\, M_2^{(+)}(u)\, R_{12}(-u-v-2)\, M_1^{(+)}(v)$$

$$M^{(+)} = \left(\bar{\Sigma} \cdot M^{(-)}(-u-1) \cdot \Sigma\right)^t$$



Integrable Two-Row transfer matrix

$$M_1^{(-)}(v)\,R_{12}(u-v)\,M_2^{(-)}(u)\,R_{12}(u+v) = R_{12}(u-v)\,M_2^{(-)}(u)\,R_{12}(u+v)\,M_1^{(-)}(v)$$

$$M_1^{(+)}(v)\,R_{12}(u-v)\,M_2^{(+)}(u)\,R_{12}(u+v) = R_{12}(-u-v-2)\,M_2^{(+)}(u)\,R_{12}(-u-v-2)\,M_1^{(+)}(v)$$

$$M^{(+)} = \left(\,\bar{\Sigma}\cdot M^{(-)}(-u-1)\cdot\Sigma\right)^t$$

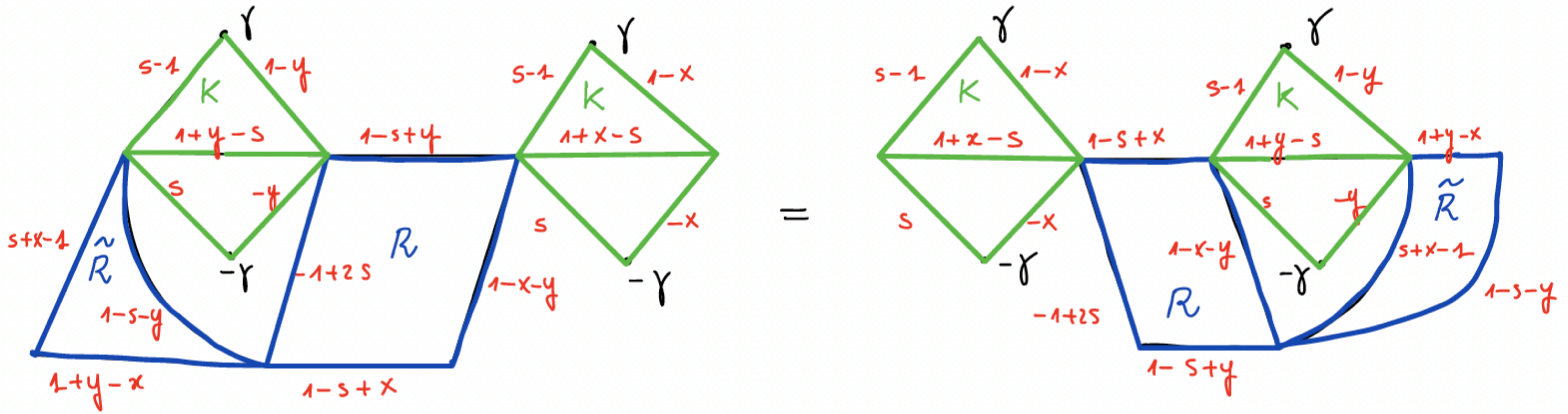
$$R_{12}(u-v)R_{12}(v-u) = 1$$

$$R_{12}(u)^{t_1} = f_{12}(u)\,(\Sigma_1\otimes 1)\,R_{12}(-u-1)\,(\bar{\Sigma}_1\otimes 1) \;= R_{12}(u)^{t_2} = f_{12}(u)\,(1\otimes\Sigma_2)\,R_{12}(-u-1)\,(1\otimes\bar{\Sigma}_2)$$

$$\hat{t}_0(u) = \mathrm{Tr}_0\left[M_0^{(+)}(u)M_0^{(-)}(u)\right] \quad\Longrightarrow\quad [\hat{t}_0\,(u)\,\hat{t}_{0'}\,(v)] = 0$$

Integrable Two-Row transfer matrix

- *The problem of finding a solution* ***K(u)*** *of the RE compatible with the solution of Yang-Baxter equation* ***R(u)*** *is complicated by the presence of Fermionic loops in the graph, ie. Pauli matrices in the numerators.*
- *To start with, we follow the strategy used for Basso-Dixon diagrams in the works [Derkachov, Kazakov, E.O.] and [Derkachov, E.O.], where we considered first the solution of the algebraic problem for 2D Fishnets, then generalized to 4D by including fermions.*
- *A recent paper by Sergey Derkachov et al. [2406.19864] provides us with a great starting point.*



SL(2,**C**) Reflection Equation

- *A recent paper by Sergey Derkachov et al. [2406.19864] provides us with a great starting point.*
- *First, they consider the finite-dimensional reflection algebra:*

$$K(u) = \begin{pmatrix} \mathrm{i}\alpha & u - \frac{1}{2} \\ \gamma^2\left(u - \frac{1}{2}\right) & \mathrm{i}\alpha \end{pmatrix}$$

$$\boxed{\mathbb{R}(u-v)\,K(u)\,\mathbb{R}(u+v-1)\,K(v) = K(v)\,\mathbb{R}(u+v-1)\,K(u)\,\mathbb{R}(u-v)}$$

- *Then, they lift the representation to (infinite-dimensional) principal series:*

$$\mathcal{K}(\boldsymbol{s},\boldsymbol{x}) = [z+\gamma]^{g-s}\,[z-\gamma]^{1-s-g}\,[\partial_z]^{x-s}\,[z+\gamma]^{x-g}\,[z-\gamma]^{x+g-1}$$

- *Finally, they infer an equation of infinite-dimensional operators:*

$$L(u) = \begin{pmatrix} u+S & S_- \\ S_+ & u-S \end{pmatrix} = \begin{pmatrix} u_1 + 1 + z\partial_z & -\partial_z \\ z^2\partial_z + (u_1 - u_2 + 1)z & u_2 - z\partial_z \end{pmatrix}$$

$$\boxed{\begin{aligned} &\mathcal{K}(\boldsymbol{s},\boldsymbol{x})\,L(u+x-1,u-s)\,K(u)\,L(u+s-1,u-x) \\ &= L(u+s-1,u-x)\,K(u)\,L(u+x-1,u-s)\,\mathcal{K}(\boldsymbol{s},\boldsymbol{x}) \end{aligned}}$$

SL(2,**C**) Reflection Equation

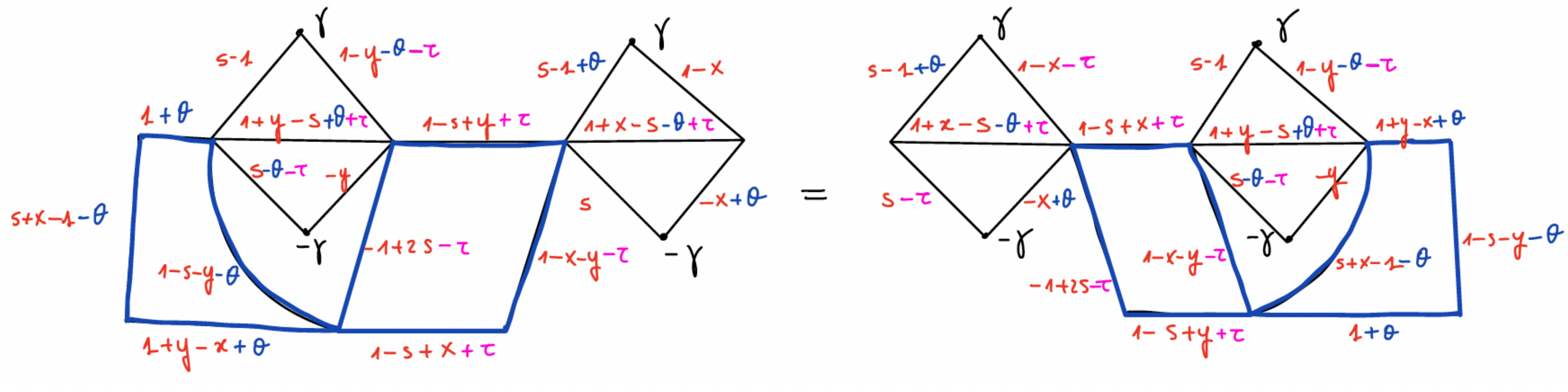
- *Finally, they consider both representations to be infinite-dimensional (principal series):*

$$\mathcal{K}(\boldsymbol{s},\boldsymbol{x}) = [z+\gamma]^{g-s}\,[z-\gamma]^{1-s-g}\,[\partial_z]^{x-s}\,[z+\gamma]^{x-g}\,[z-\gamma]^{x+g-1}$$

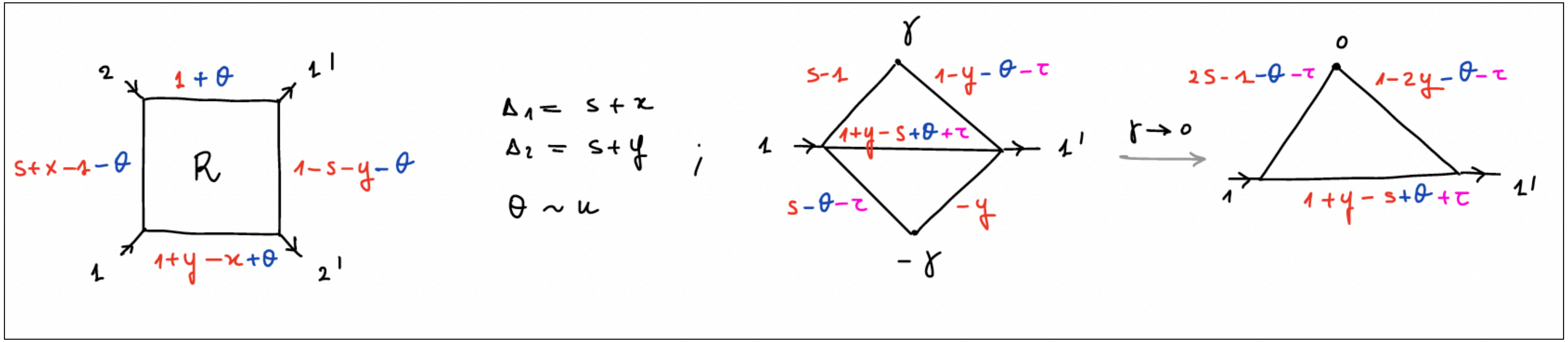
$$\boxed{\mathcal{K}_1(\boldsymbol{x})\,\widetilde{\mathbb{R}}_{12}(\boldsymbol{x},\boldsymbol{y})\,\mathcal{K}_2(\boldsymbol{y})\,\mathbb{R}_{12}(\boldsymbol{x},\boldsymbol{y}) = \mathbb{R}_{12}(\boldsymbol{x},\boldsymbol{y})\,\mathcal{K}_2(\boldsymbol{y})\,\widetilde{\mathbb{R}}_{12}(\boldsymbol{x},\boldsymbol{y})\,\mathcal{K}_1(\boldsymbol{x})}$$

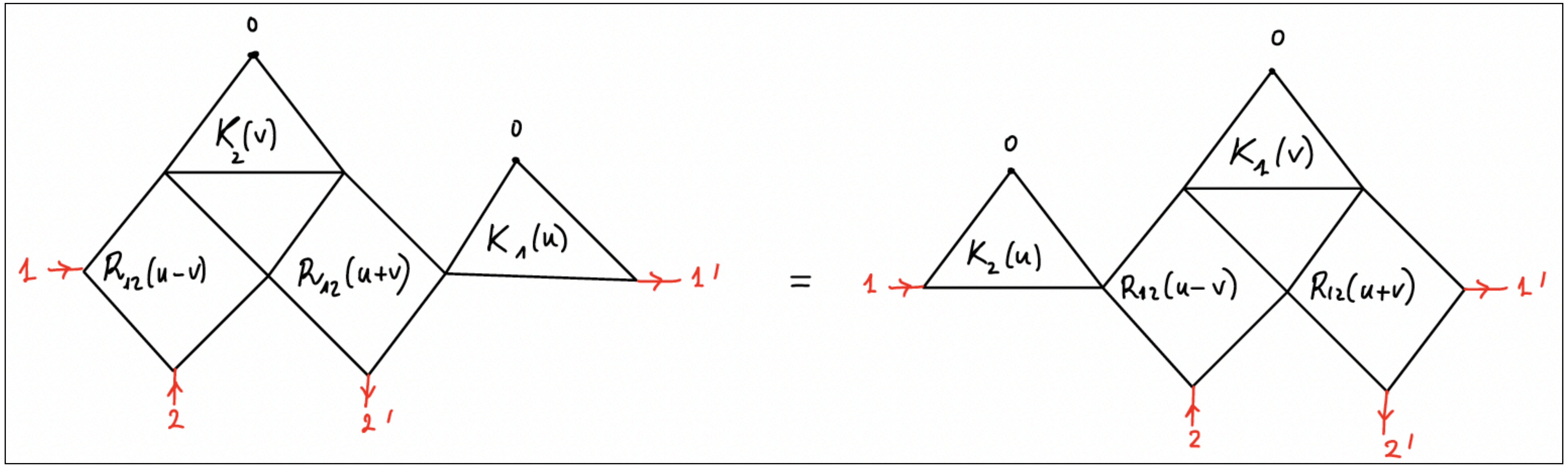
SL(2,**C**) Reflection Equation

- *In [2406.19864] they actually do not get the Reflection Equation in the picture, but some reduction of it with no spectral parameter...*
- *Starting from their equation, we can repeat its proof by star-triangle identity, and "inject" more parameters (here called* θ, τ*) thus "restoring" two spectral-parameter-dependent R-matrices in the principal series of SL(2,*C*).*



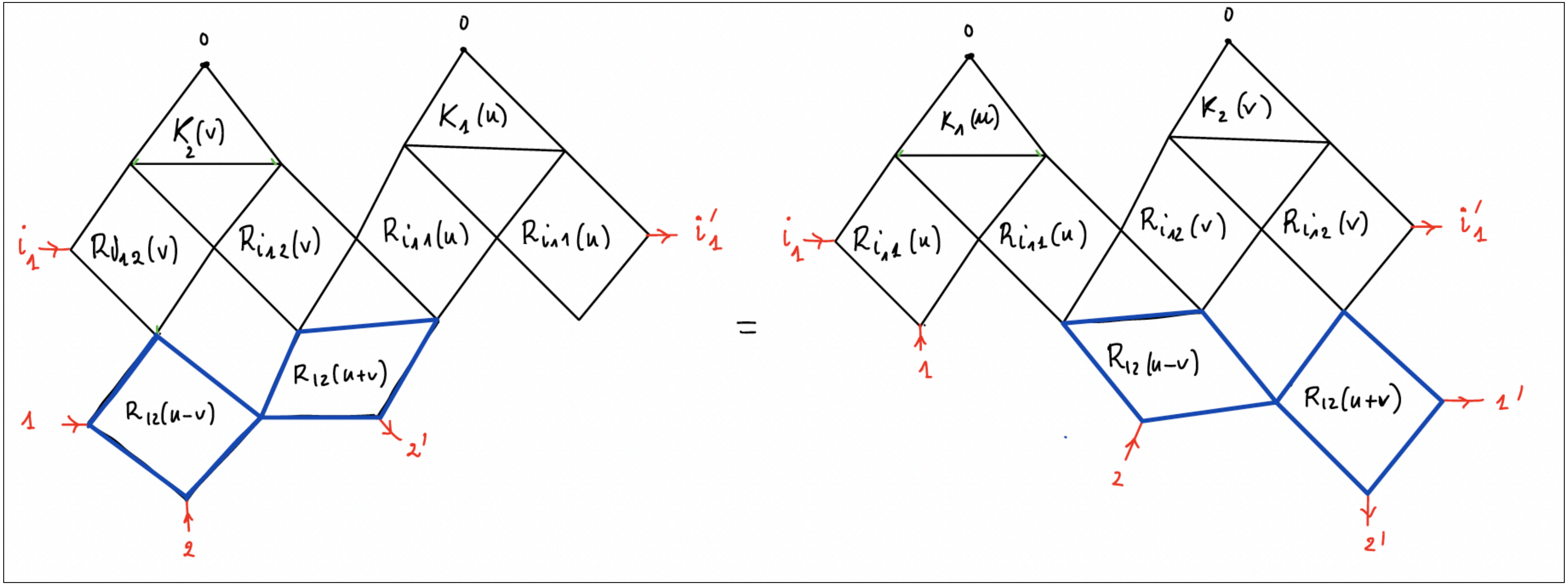
SL(2,**C**) Reflection Equation





$$K_2(u)\,R_{12}(u-v)\,K_1(v)\,R_{12}(u+v) = R_{12}(u-v)\,K_1(v)\,R_{12}(u+v)\,K_2(u)$$

SL(2,**C**) Reflection Equation

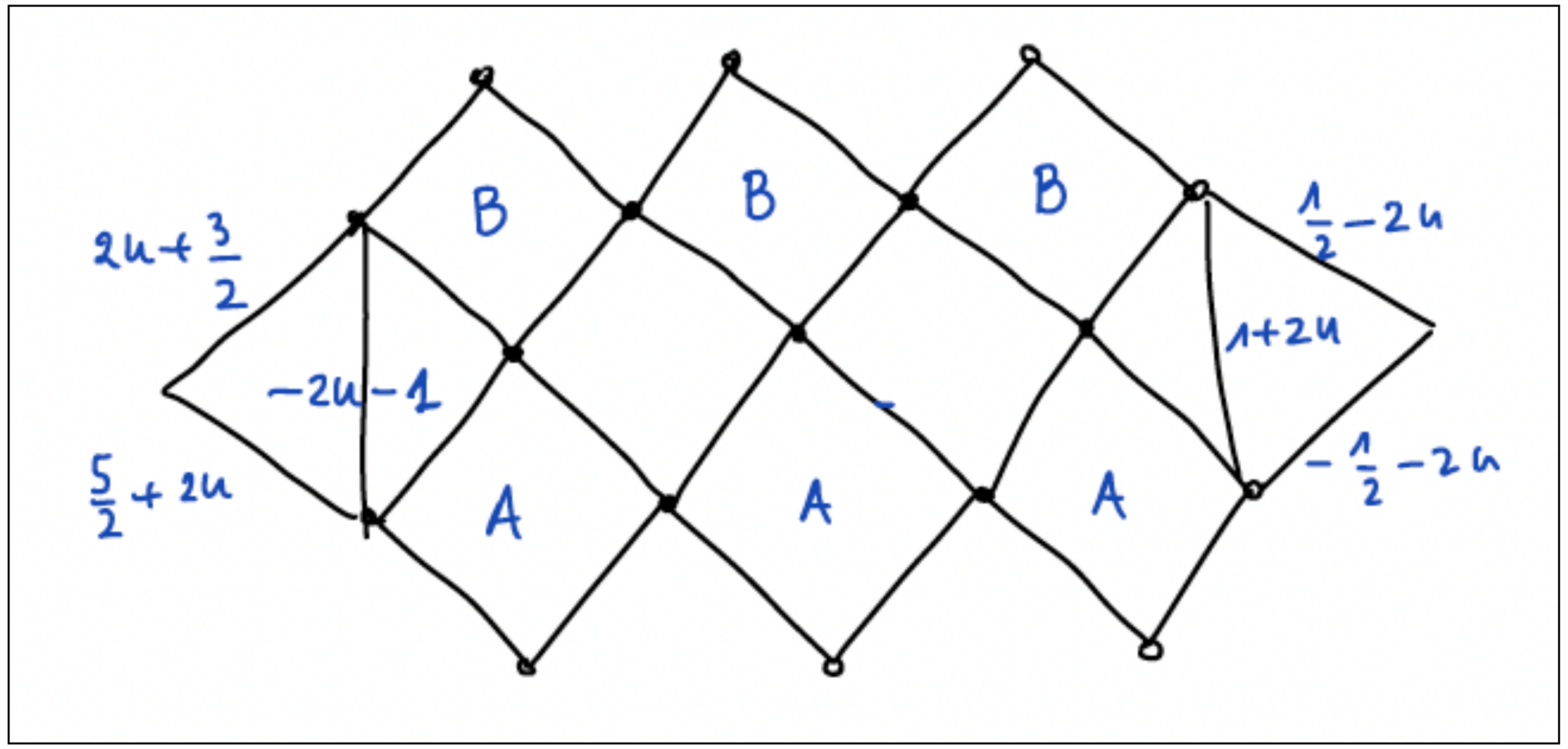


$$M_1^{(-)}(v)\, R_{12}(u-v)\, M_2^{(-)}(u)\, R_{12}(u+v) = R_{12}(u-v)\, M_2^{(-)}(u)\, R_{12}(u+v)\, M_1^{(-)}(v)$$

Staggered transfer matrix in p.s.

- *Gluing the RE of type (+) with the RE of type (-) it follows a genuine commutation of transfer matrices acting only on physical spaces for generic spectral parameters u,v.*

$$\hat{t}(u) = \mathrm{Tr}_0 \left[M_0^{(+)}(u) M_0^{(-)}(u) \right]$$

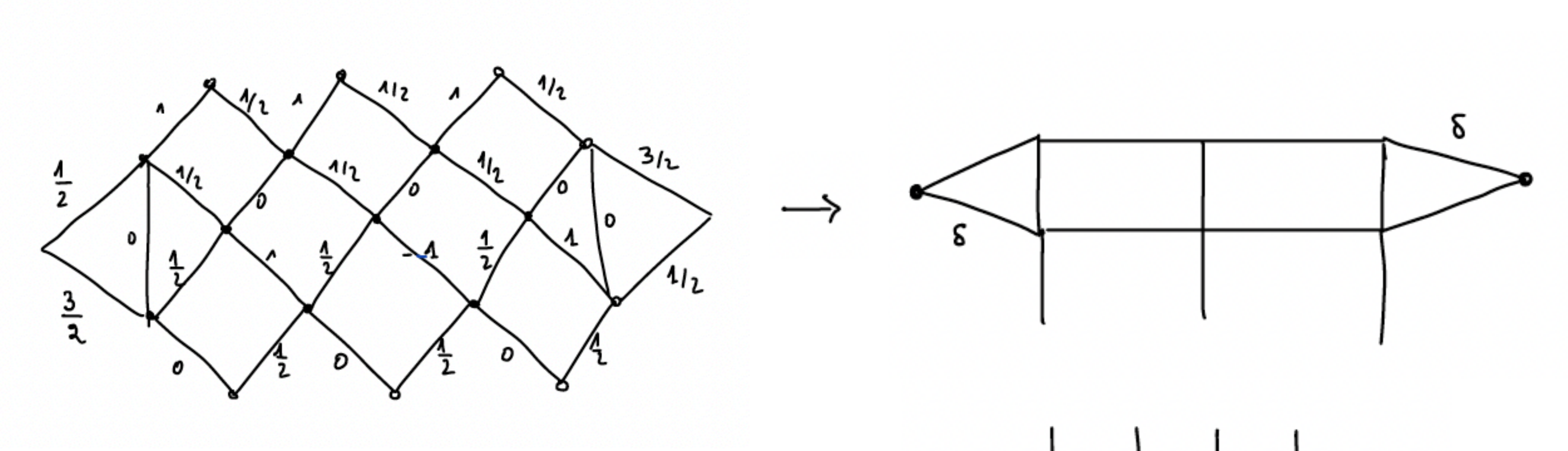


- *Such objects define an Integrable lattice model with staggering, in the principal series representation of SL(2,C). It should serve as an archetype for studying similar setups in Fishnet theory.*

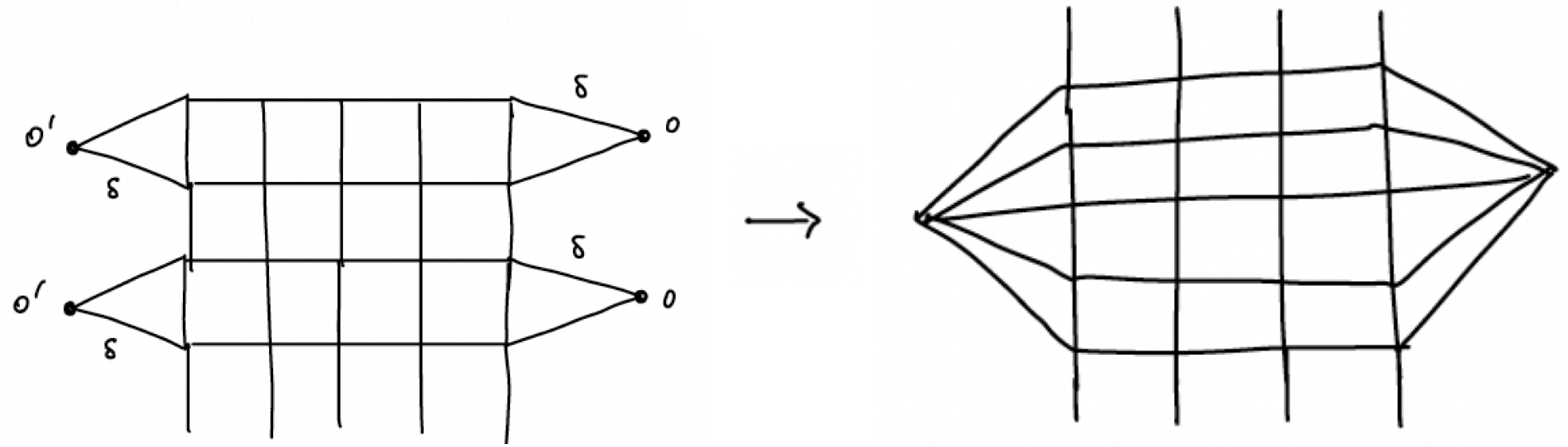
Reduction to square lattice

- *A fun example relates the staggered lattice to the regular square-lattice Fishnet in two dimension, but with different boundary conditions ("domain wall").*

$$\Delta_1 = \Delta_2 = 1/2\,,\; u \to -\frac{1}{2}\;:$$



- *Convolution of two "two-rows":*



Open Question: generators of the Yangian?

- *Yangian symmetry has been worked out starting with Volodya, Florian et al. papers, when a Fishnet diagram has disk topology and completely split boundary (ie. non-integrated) points.*

- *Yangian symmetry of such diagrams follows from RLL = LLR equation satisfied by inf-dim R with the Lax-operatorof the chain. There is an analogue **K**′LKL = LKL**K**′ identity [2406.19864]:*

$$\boxed{\mathscr{K}(u)L_1(u)K(u)L_1(u) = L_1(u)K(u)L_1(u)\mathscr{K}(u)}$$

$$T(u) = L_N(u)\cdots L_1(u)K(u)L_1(u)\cdots L_n(u)$$

- *Could one infer twisted-Yangian symmetry, eg. by adapting "lasso techniques"?*
  [hint: The staggered models describe, as a particular case, a class of square-lattice Fishnets that have split-points on two sides and identified points on the other two.]

Summary:

- *Fishnet Theory integrability is described by Integrable spin-magnets that differs from choice of boundary conditions (eg. direct vs mirror channels) but also with different bulks, eg. staggered magnets.*

- *We need to study staggered models (and generally lattice structures with higher periodicity) and the corresponding integrability equations (YBE —> RE).*

- *Formulate spectral equations (boundary TQ equations).*

- *Formulate analogue of Yangian symmetry.*


Vielen Dank!