# Conformal field theories from line defects and holography

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Based on work with L. Bianchi, D. Bonomi, G. Bliard, L. Griguolo, G. Peveri, D. Seminara

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*Fishnets: Conformal Field Theories and Feynman graphs*

# Motivation

Why CFT$_1$s are interesting?

- A simpler but still constraining setup to test ideas about higher-$d$ CFTs
- Non trivial CFT$_1$s naturally live on line defects, crucial for a deeper understanding of QFT dynamics.

# Motivation: Wilson loops as 1-dimensional defects

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In a CFT, for instance $\mathcal{N} = 4$ SYM in $d = 4$ or ABJM in $d = 3$, a Wilson line can be viewed as a conformal defect.

[Giombi Roiban Tseytlin 17] [Giombi Beccaria Tseytlin 18]
[Bianchi, Bliard, Forini, Griguolo, Seminara 20]

$$\langle W \rangle = P \exp\left( -i \int_{t_1}^{t_2} dt\, \mathcal{L}(t) \right)$$

A straight line breaks the original conformal symmetry to

a) dilatations, translations and special conformal transformations along the line

b) rotations around the line

+ part of the R-symmetry + part of the supersymmetry
(depending on the specific form of the loop)

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b) rotations around the line

Thus the Wilson loop implicitly defines a defect CFT$_1$.

Can we study this “simpler” CFT?

# A defect CFT$_1$

The set of correlators of operator insertions along the line

$$\langle \mathcal{O}(t_1)\mathcal{O}(t_2)\ldots\mathcal{O}(t_n)\rangle_W = \frac{\langle \mathrm{Tr}\mathcal{O}_1(t_1)W\mathcal{O}_2(t_2)\ldots\mathcal{O}_{n-1}(t_{n-1})W\mathcal{O}_n(t_n)\rangle}{\langle W\rangle}$$

where

$$\langle W\rangle = P\exp\big(-i\int_{t_1}^{t_2} dt\,\mathcal{L}(t)\big)$$

can be interpreted as characterizing a defect CFT$_1$.

It should be fully determined by its spectrum of dimensions and OPE coefficients.

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Also: operator insertions are equivalent to deformations of the Wilson line

[Drukker, Kawamoto 2006]

Complete knowledge of these correlators would, in principle, allow to compute the expectation value of general Wilson loops which are deformations of the line or circle.

# A defect CFT$_1$: the 1/2 BPS Wilson line in ABJM theory

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can be interpreted as characterizing a defect CFT$_1$.

It should be fully determined by its spectrum of dimensions and OPE coefficients.

Consider the $\mathcal{N}=6$ superconformal Chern-Simons-matter theory in $d=3$ (ABJM).

Its original symmetry, $OSp(6|4)$, is broken by the 1/2 BPS Wilson line to $SU(1,1|3)$, the $\mathcal{N}=6$ superconformal group in $d=1$.

Its bosonic subgroup is $SO(2,1)\times U(1)_M\times SU(3)_R$.

Operator insertions along the Wilson line are labelled by $[\Delta; m; j_1, j_2]$.

# The displacement supermultiplet

Among the possible operator insertions (defect operators), a special role is played by a set of “elementary excitations” with protected scaling dimension.
They fall into a short representation of the $SU(1,1|3)$ subalgebra
It is a chiral multiplet, the displacement supermultiplet

$$
\begin{array}{llll}
[\frac{1}{2},\frac{3}{2},0,0] & & & \mathbb{F},\ \bar{\mathbb{F}} \\
\searrow & & & \\
& [1,2,1,0] & & \mathbb{O}^a,\ \bar{\mathbb{O}}_a \quad a=1,2,3 \\
& \searrow & & \\
& & [\frac{3}{2},\frac{5}{2},0,1] & \mathbb{\Lambda}^a,\ \bar{\mathbb{\Lambda}}_a \quad a=1,2,3 \\
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& & \quad [2,3,0,0] & \mathbb{D},\ \bar{\mathbb{D}}
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$$

$$\partial_\mu T^{\mu m} = \delta^2(x^\perp)\,\mathbb{D}^m(t)$$

Translational invariance is broken, the stress tensor is no longer conserved and the usual conservation law needs to be modified by some additional terms localized on the defect.

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\end{array}
$$

**8F+8B**
like the DOF of
transverse
string fluctuations

All operators in the supermultiplet can be related to broken symmetry generators.

# The displacement supermultiplet

Their 2-point functions are particularly simple, e.g.

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\rangle = \frac{C_{\mathbb{D}}}{t_{12}^4}$$

where the normalization constant $C_{\mathbb{D}} = 12\, B_{1/2}(\lambda)$ has a physical meaning: [Bianchi Lemos Meineri 18] [Bianchi Lemos 19]

it coincides with the Bremsstrahlung function, one of the few unprotected observables known to each order in AdS/CFT. [Correa Henn Maldacena Sever 12] [Bianchi Griguolo Preti Seminara17] [Bianchi Preti Vescovi 18]

Their 3-point functions vanish by symmetry.

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Their 4-point functions, on the other hand, have a less costrained form

$$\langle \mathcal{O}_\Delta(t_1)\mathcal{O}_\Delta(t_2)\mathcal{O}_\Delta(t_3)\mathcal{O}_\Delta(t_4)\rangle = \frac{1}{(t_{12}t_{34})^{2\Delta}} G(\chi) \, .$$

G has non-trivial dependence on the coupling and conformal cross ratio $\chi = \frac{t_{12}t_{34}}{t_{13}t_{24}}$

They encode in particular scaling dimensions and structure constants of unprotected operators appearing in the OPE.

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superspace analysis

analytic bootstrap

direct (Witten) diagrammatics at strong coupling via AdS/CFT

# Chiral correlators in superspace

The supermultiplet accomodating the displacement operator is the chiral one.
We consider the chiral superfield ($y = x - \theta_a \bar{\theta}^a$)

$$\Phi(y,\theta) = \mathbb{F}(y) + \theta_a \mathbb{O}^a(y) + \theta_a \theta_b \, \epsilon^{abc} \, \mathbb{\Lambda}_c(y) + \theta_a \theta_b \theta_c \, \epsilon^{abc} \, \mathbb{D}(y) \, ,$$

The two-point function reads

$$\langle \Phi(y_1, \theta_1) \bar{\Phi}(y_2, \bar{\theta}_2) \rangle = \frac{C_\Phi}{\langle 1\bar{2} \rangle^{2\Delta}} \, ,$$

in terms of the chiral distance $\langle i\bar{j} \rangle = y_i - y_j - 2\theta_i^a \bar{\theta}_{a\,j}$.

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The most general form for the 4-point function is

$$\langle \Phi(y_1,\theta_1)\bar{\Phi}(y_2,\bar{\theta}_2)\Phi(y_3,\theta_3)\bar{\Phi}(y_4,\bar{\theta}_4)\rangle = \frac{1}{(\langle 1\bar{2}\rangle\, \langle 3\bar{4}\rangle)^{2\Delta}}\, f(\mathcal{X})\,,$$

since the only superconformal invariant is

$$\mathcal{X} = \frac{\langle 1\bar{2}\rangle\, \langle 3\bar{4}\rangle}{\langle 1\bar{4}\rangle\, \langle 3\bar{2}\rangle}\,.$$

The corresponding bosonic cross-ratio $\quad \chi = \dfrac{x_{12}x_{34}}{x_{13}x_{24}}$

# Four-point functions for the defect operators

Expanding in Graßmann variables we get a set of correlators for the elementary fields

$$z = \frac{\chi}{\chi - 1}$$

$$\langle \mathbb{F}(t_1)\bar{\mathbb{F}}(t_2)\mathbb{F}(t_3)\bar{\mathbb{F}}(t_4)\rangle = \frac{f(z)}{t_{12}t_{34}}$$

$$\langle \mathbb{O}^{a_1}(t_1)\bar{\mathbb{O}}_{a_2}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = \frac{4}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\left(f(z) + zf'(z) + z^2 f''(z)\right) + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\left(z^2 f'(z) - z^3 f''(z)\right)\Big]$$

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{D}(t_3)\bar{\mathbb{D}}(t_4)\rangle = \frac{64}{t_{12}^4 t_{34}^4}\Big[z^6(1-z)^3 f^{(6)}(z) - 3\, f^{(5)}(z) z^5 (1-z)^2 (7z+1) + 3\, f^{(4)}(z) z^4(-46z^3 + 63z^2 - 18z + 1) + 6\, f^{(3)}(z) z^3 (55z^3 - 39z^2 + 3z + 1) + 18\, f''(z)\,(-14z^5 + 3z^4 + z^2) - 36\, f'(z)\, z(1 - z^3) + 36\, f(z)\Big]$$

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = -\frac{16\,\delta^{a_3}_{a_4}}{t_{12}^4 t_{34}^4}\Big[(1-z)\, z^4 f^{(4)}(z) + (3z+1)\, z^3 f^{(3)}(z) + 3z^2\, f''(z) + 6z f'(z) + 6 f(z)\Big]$$

The correlation function $f(z)$ of the superconformal primary completely determines that of its superdescendants.

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$$\langle \mathbb{O}^{a_1}(t_1)\bar{\mathbb{O}}_{a_2}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = \frac{4}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\left(f(z)+zf'(z)+z^2f''(z)\right) + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\left(z^2f'(z)-z^3f''(z)\right)\Big]$$

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$$\begin{aligned}\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = -\frac{16\,\delta^{a_3}_{a_4}}{t_{12}^4 t_{34}^4}\Big[& (1-z)\, z^4 f^{(4)}(z) + (3z+1)\, z^3 f^{(3)}(z) \\ &+ 3z^2\, f''(z) + 6zf'(z) + 6f(z)\Big]\end{aligned}$$

The correlation function $f(z)$ of the superconformal primary completely determines that of its superdescendants.

non-trivial function of the coupling!

We can evaluate $f(z)$, and thus ALL the correlators, at strong coupling using string worldsheet worldsheet perturbation theory: via Witten diagrams.

# String dual - AdS$_2$ minimal surface

In AdS/CFT dictionary, the Wilson loop operator is dual to a minimal string surface ending on the contour defining the operator at the boundary.
For ABJM, the dual is a fundamental Type IIA string in a $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^3$ background.
The bosonic part of the Nambu-Goto string action reads

$$S_B = T \int d^2\sigma \sqrt{\det \frac{1}{z^2} \left(\partial_\mu x^r \partial_\nu x^r + \partial_\mu z \partial_\nu z\right) + 4\Big( \frac{\partial_\mu \bar{w}_a \partial_\nu w^a}{1+|w|^2} - \frac{\partial_\mu \bar{w}_a w^a \bar{w}_b \partial_\nu w^b}{(1+|w|^2)^2} \Big)}$$

where $T$ is the effective string tension

$$T = \frac{R^2}{2\pi\alpha'} = 2\sqrt{2\lambda}\,, \qquad \lambda = \frac{N}{k}\,.$$

The minimal surface dual to the 1/2-BPS Wilson line is given by

$$z = s\,, \qquad x^0 = t\,, \qquad x^i = 0\,, \qquad w^a = 0$$

The induced metric is just that of $AdS_2$

$$ds^2 = \frac{1}{s^2}(dt^2 + ds^2)$$

# String dual - AdS$_2$ minimal surface

This setup preserves same superconformal symmetry $SU(1,1|3)$ of our defect CFT$_1$
In particular, the isometry of AdS$_2$ is the conformal group in $d=1$,

Fluctuation modes over the minimal surface are scalar fields over AdS$_2$
and their dynamics is governed by the fluctuation Lagrangian

$$S_B \equiv T \int d^2\sigma \sqrt{g}\; L_B \,, \qquad L_B = \; L_2 + L_{4X} + L_{2X,2w} + L_{4w} + \dots \,,$$

$$L_2 = g^{\mu\nu}\partial_\mu X \partial_\nu \bar{X} + 2|X|^2 + g^{\mu\nu}\partial_\mu w^a \partial_\nu \bar{w}_a \;,$$

$$L_{4X} = 2|X|^4 + |X|^2 \left(g^{\mu\nu}\partial_\mu X \partial_\nu \bar{X}\right) - \tfrac{1}{2}\left(g^{\mu\nu}\partial_\mu X \partial_\nu X\right)\left(g^{\rho\kappa}\partial_\rho \bar{X} \partial_\kappa \bar{X}\right),$$

$$L_{2X,2w} = \left(g^{\mu\nu}\partial_\mu X \partial_\nu \bar{X}\right)\left(g^{\rho\kappa}\partial_\rho w^a \partial_\kappa \bar{w}_a\right) - 2\left(g^{\mu\nu}\partial_\mu X \partial_\nu w^a\right)\left(g^{\rho\kappa}\partial_\rho \bar{X} \partial_\kappa \bar{w}_a\right) \;,$$

$$L_{4w} = \; -\tfrac{1}{2}(w^a \bar{w}_a)(g^{\mu\nu}\partial_\mu w^b \partial_\nu \bar{w}_b) - \tfrac{1}{2}(w^a \bar{w}_b)(g^{\mu\nu}\partial_\mu w^b \partial_\nu \bar{w}_a) + \tfrac{1}{2}\left(g^{\mu\nu}\partial_\mu w^a \partial_\nu \bar{w}_a\right)^2$$
$$- \tfrac{1}{2}(g^{\mu\nu}\partial_\mu w^a \partial_\nu \bar{w}_b)\left(g^{\rho\kappa}\partial_\rho \bar{w}_a \partial_\kappa w^b\right) - \tfrac{1}{2}(g^{\mu\nu}\partial_\mu w^a \partial_\nu w^b)\left(g^{\rho\kappa}\partial_\rho \bar{w}_a \partial_\kappa \bar{w}_b\right) \;.$$

Effective 2d field theory of **1+3 complex scalars** in AdS$_2$ geometry

# String dual - $AdS_2$ minimal surface

This setup preserves same superconformal symmetry $SU(1,1|3)$ of our defect $CFT_1$!
In particular, the isometry of $AdS_2$ is the conformal group in $d=1$,

Fluctuation modes over the minimal surface are scalar fields over $AdS_2$

Then $AdS_2/CFT_1$ states that they should be dual to operators
inserted at the $d=1$ boundary with dimensions

$$\Delta(\Delta-1)=m^2 \qquad \text{bosons} \qquad \Delta=\frac{1}{2}+|m| \qquad \text{spinors}$$

Hence, we recover the eight bosonic operators in the super-displacement multiplet

| | | | | |
|---|---|---|---|---|
| $\Delta=\frac{1}{2}$ | $\mathbb{F},\ \bar{\mathbb{F}}$ | $\Longleftrightarrow$ | $\psi\,,\bar{\psi}$ | $m^2=0$ |
| $\Delta=1$ | $\mathbb{O}^a,\ \bar{\mathbb{O}}_a \quad a=1,2,3$ | $\Longleftrightarrow$ | $w^a\,,\bar{w}_a$ | $m^2=0$ |
| $\Delta=\frac{3}{2}$ | $\mathbb{\Lambda}^a,\ \bar{\mathbb{\Lambda}}_a \quad a=1,2,3$ | $\Longleftrightarrow$ | $\psi^a\,,\bar{\psi}_a$ | $m_F=\pm 1$ |
| $\Delta=2$ | $\mathbb{D},\ \bar{\mathbb{D}}$ | $\Longleftrightarrow$ | $X\,,\bar{X}$ | $m^2=2$ |

# Witten diagrams in $AdS_2$

The four-point functions of the dual operators at strong coupling can then be obtained from familiar AdS/CFT techniques by computing Witten diagrams in $AdS_2$.

For the 4-point function of fields e.g. in AdS

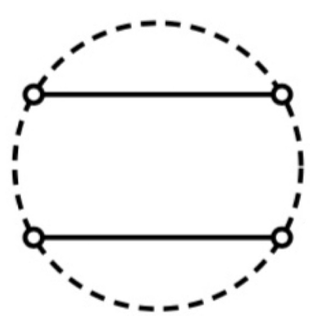
$$\langle X(t_1)\,\bar{X}(t_2)\,X(t_3)\,\bar{X}(t_4)\rangle = \frac{1}{t_{12}^2 t_{34}^2} G(z)\ ,$$

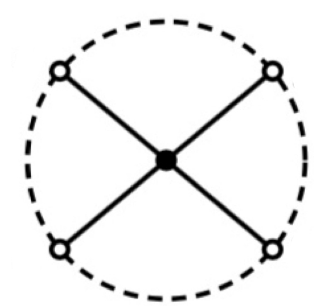
where $G(z)$ has the strong coupling expansion

$$G(z) = G^{(0)}(z) + \frac{1}{T} G^{(1)}(z) + \dots$$

**disconnected contribution (diagrams with 2 ''boundary-to-boundary" propagators)**

**tree-level contact diagrams (4-vertices with 4 bulk-to-boundary propagators attached)**





# Summary of 4-point function results

The correlators of string worldsheet excitations read

$$\langle X(t_1)\,\bar{X}(t_2)\,X(t_3)\,\bar{X}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^4}\Big[1+z^4+\tfrac{1}{T}\big[-8z^4-(3-8z)z^4(\ln z-\ln(1-z)) - z^3-\tfrac{7}{6}z^2-z-(8-3z)\tfrac{\ln(1-z)}{z}-8\big]\Big]$$

$$\langle w^{a_1}(t_1)\,\bar{w}_{a_2}(t_2)\,w^{a_3}(t_3)\,\bar{w}_{a_4}(t_4)\rangle = \frac{1}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\big[1+\tfrac{1}{2\,T}\big(z^2\ln z-\big(z^2-\tfrac{4}{z}+3\big)\ln(1-z)-z+4\big)\big] +\delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\big[z^2+\tfrac{1}{2\,T}\big((3-4z)z^2\ln z+(4z^3-3z^2-1)\ln(1-z)+(4z-1)z\big)\big]\Big]$$

$$\langle X(t_1)\,\bar{X}(t_2)\,w^{a_3}(t_3)\,\bar{w}_{a_4}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^2}\,\delta^{a_3}_{a_4}\big[1+\tfrac{1}{T}\big(2(z-2)\tfrac{\ln(1-z)}{z}-4\big)\big]$$

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The superspace analysis of correlators for defect operators gives

$$\langle X(t_1)\,\bar{X}(t_2)\,X(t_3)\,\bar{X}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^4}\Big[1+z^4+\tfrac{1}{T}\big[-8z^4-(3-8z)z^4(\ln z-\ln(1-z)) - z^3-\tfrac{7}{6}z^2-z-(8-3z)\tfrac{\ln(1-z)}{z}-8\big]\Big]$$

$$\Updownarrow$$

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{D}(t_3)\bar{\mathbb{D}}(t_4)\rangle = \frac{64}{t_{12}^4 t_{34}^4}\Big[z^6(1-z)^3 f^{(6)}(z) - 3\,f^{(5)}(z) z^5 (1-z)^2(7z+1) + 3\,f^{(4)}(z)z^4(-46z^3+63z^2-18z+1) + 6\,f^{(3)}(z)z^3(55z^3-39z^2+3z+1) + 18\,f''(z)\,(-14z^5+3z^4+z^2) - 36\,f'(z)\,z(1-z^3)+36\,f(z)\Big]$$

$$\langle w^{a_1}(t_1)\,\bar{w}_{a_2}(t_2)\,w^{a_3}(t_3)\,\bar{w}_{a_4}(t_4)\rangle = \frac{1}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\big[1+\tfrac{1}{2\,T}\big(z^2\ln z-\big(z^2-\tfrac{4}{z}+3\big)\ln(1-z)-z+4\big)\big] + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\big[z^2+\tfrac{1}{2\,T}\big((3-4z)z^2\ln z+(4z^3-3z^2-1)\ln(1-z)+(4z-1)z\big)\big]\Big]$$

$$\Updownarrow$$

$$\langle \mathbb{O}^{a_1}(t_1)\bar{\mathbb{O}}_{a_2}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = \frac{4}{t_{12}^2 t_{34}^2}\Big[\delta^{a_1}_{a_2}\delta^{a_3}_{a_4}\,\big(f(z)+zf'(z)+z^2f''(z)\big) + \delta^{a_1}_{a_4}\delta^{a_3}_{a_2}\,\big(z^2f'(z)-z^3f''(z)\big)\Big]$$

$$\langle X(t_1)\,\bar{X}(t_2)\,w^{a_3}(t_3)\,\bar{w}_{a_4}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^2}\,\delta^{a_3}_{a_4}\big[1+\tfrac{1}{T}\big(2(z-2)\tfrac{\ln(1-z)}{z}-4\big)\big]$$

$$\Updownarrow$$

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{O}^{a_3}(t_3)\bar{\mathbb{O}}_{a_4}(t_4)\rangle = -\frac{16\,\delta^{a_3}_{a_4}}{t_{12}^4 t_{34}^4}\Big[(1-z)\,z^4 f^{(4)}(z)+(3z+1)\,z^3 f^{(3)}(z)+3z^2\,f''(z)+6zf'(z)+6f(z)\Big]$$

# Summary of 4-point function results

The correlators of string worldsheet excitations read

The superspace analysis of correlators for defect operators gives

$$\langle X(t_1)\,\bar{X}(t_2)\,X(t_3)\,\bar{X}(t_4)\rangle = \frac{1}{t_{12}^4 t_{34}^4}\Big[1+z^4+\tfrac{1}{T}\big[-8z^4-(3-8z)z^4(\ln z-\ln(1-z)) - z^3-\tfrac{7}{6}z^2-z-(8-3z)\tfrac{\ln(1-z)}{z}-8\big]\Big]$$

$$\Updownarrow$$

$$\langle \mathbb{D}(t_1)\bar{\mathbb{D}}(t_2)\mathbb{D}(t_3)\bar{\mathbb{D}}(t_4)\rangle = \frac{64}{t_{12}^4 t_{34}^4}\Big[z^6(1-z)^3 f^{(6)}(z) - 3\,f^{(5)}(z)z^5(1-z)^2(7z+1) + 3\,f^{(4)}(z)z^4(-46z^3+63z^2-18z+1) + 6\,f^{(3)}(z)z^3(55z^3-39z^2+3z+1) + 18\,f''(z)\,(-14z^5+3z^4+z^2) - 36\,f'(z)\,z(1-z^3)+36\,f(z)\Big]$$

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Is there a single f(z) solving simultaneously these non-trivial ODEs?

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&\qquad -z^3-\tfrac{7}{6}z^2-z-(8-3z)\tfrac{\ln(1-z)}{z}-8\big]\Big]
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These differential equations are all solved by the simple function

$$\boxed{f(z)=1-z+\frac{1}{T}\Big(1-z-(3-z)z\ln z+\frac{(1-z)^3}{z}\ln(1-z)\Big)+\mathcal{O}\Big(\frac{1}{T^2}\Big)},$$

This is the strong coupling expansion of the function governing all correlation functions of operators in the displacement supermultiplet.

Also derived using analytic bootstrap

# CFT data at strong coupling

The four-point function has an OPE expansion in superblocks

$$\langle \mathbb{F}(t_1)\bar{\mathbb{F}}(t_2)\mathbb{F}(t_3)\bar{\mathbb{F}}(t_4)\rangle = \frac{1}{t_{12}t_{34}} f(z) = \frac{1}{t_{12}t_{34}} \sum_h c_h (-z)^h \, {}_2F_1(h,h,2h+3,z)\,.$$

eigenfunctions of the super-Casimir
of *N*=6 algebra in d=1

[Dolan, Osborn 2011]

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eigenfunctions of the super-Casimir
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At this order, operators exchanged in the OPE are just the identity and the tower of operators $\mathcal{O}\partial^n\mathcal{O}$, built out of the elementary excitations. Therefore,

$$h = 1 + n + \frac{1}{T}\gamma_n^{(1)} \qquad c_n = c_n^{(0)} + \frac{1}{T} c_n^{(1)}$$

$$f(z) = 1 - z + \frac{1}{T}\Big(1 - z - (3-z)z\ln z + \frac{(1-z)^3}{z}\ln(1-z)\Big) + \mathcal{O}\Big(\frac{1}{T^2}\Big)\,,$$

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$$h = 1 + n + \frac{1}{T}\gamma_n^{(1)} \qquad\qquad c_n = c_n^{(0)} + \frac{1}{T}\, c_n^{(1)}$$

$$c_n^{(1)} = \partial_n \left(c_n^{(0)}\, \gamma_n^{(1)}\right)$$

$$\gamma_n^{(1)} = 3 + 4n + n^2$$

$$c_n = \sqrt{\pi}2^{-2n-3}(n+3)\frac{\Gamma(n+1)}{\Gamma\left(n+\frac{5}{2}\right)}\Big[(n+2) + \frac{1}{T}\big[4n^2 - 2\left(n^3+6n^2+11n+6\right)\ln 2 + 15n$$
$$+(n+1)(n+2)(n+3)\psi^{(0)}(n+1) - (n+1)(n+2)(n+3)\psi^{(0)}\left(n+\tfrac{5}{2}\right) + 13\big]\Big]$$

"inverting" for the coefficients in the sum, namely using orthogonality relations for the hypergeometric functions.

# Intermediate conclusions

- We have considered a class of four-point correlators in the $\mathrm{CFT}_1$ defined on the 1/2-BPS Wilson line in the 3d superconformal ABJM theory.
- Superconformal symmetry determines four-point correlators of the displacement supermultiplet in terms of a single function, that we evaluate at strong coupling using holography and Witten diagrams and the analytic boostrap. We can extract CFT data.
- Further progress on the ABJM Wilson line: topological sector (kinematical defect) [Gorini, Griguolo, Guerrini, Penati, Seminara, Soresina 22], integrability for the cusp-deformed WL [Correa, Giraldo-Rivera, Lagares 23] three-loop (in AdS) correlators via analytic bootstrap

# Intermediate conclusions and questions

- We have considered a class of four-point correlators in the $\mathrm{CFT}_1$ defined on the 1/2-BPS Wilson line in the 3d superconformal ABJM theory.
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- What happens beyond tree-level? Witten diagrams with loops in AdS should be well-defined, since the 2d theory is supposed to be UV finite.
  However, issues of regularization appear.

  Is there a representation ("momentum space") in which these computations simplify and the scattering nature of the correlator becomes transparent?

# Conformal correlators and Mellin space

Higher dimensions

[Mack 2009] [Penedones 2010]

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{13}^{2\Delta}x_{24}^{2\Delta}}\,F(u,v)\,, \qquad u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,, \qquad v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$

$$F(u,v) = \int_{\mathcal{C}} d\gamma_{12}d\gamma_{14}\,M(\gamma_{12},\gamma_{14})\,\Gamma^2(\gamma_{12})\,\Gamma^2(\gamma_{14})\Gamma^2(\Delta-\gamma_{12}-\gamma_{14})\,u^{-\gamma_{12}}v^{-\gamma_{14}}$$

$M(\gamma_{12},\gamma_{14})$ has the properties of a scattering amplitude:

- Crossing symmetry
- Poles corresponding to operators exchanged in the OPE
- Asymptotic behavior compatible with the Regge limit
- Simple expression for Witten diagrams

# Conformal correlators and Mellin space in $d=1$

In $d=1$, just one independent cross ratio and thus one independent Mellin variable
Reduce the higher-dimensional case is subtle.
Then, inherently one-dimensional formulation inspired by same guiding principles.

Notice that, in fact, a **family** of Mellin amplitudes can be defined $t = \frac{x_{12}\, x_{34}}{x_{14} x_{23}} > 0$

$$\mathcal{M}_a(s) = \int_0^\infty dt\, f(t) \left(\frac{t}{1+t}\right)^a t^{-1-s} \qquad \mathcal{M}_a^{-1}[\mathcal{M}_a(s)] = \int_{\mathcal{C}} \frac{ds}{2\pi i}\, f(t)\, t^s \left(\frac{t}{1+t}\right)^{-a} \mathcal{M}_a(s)$$

$a=0$ $\rightarrow$ Mellin transform of $f(t)$ in $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{(x_{12}\, x_{34})^{2\Delta_\phi}}\, f(t)$

$a = -2\Delta_\phi$ $\rightarrow$ Mellin transform of the crossing-symmetric $g(t) = \left(\frac{t}{1+t}\right)^{2\Delta_\phi} f(t)$

$a = -2\Delta_\phi + 1$ leads to simple results in a perturbative expansion around GFF

# Definition and properties

$$M(s) = \frac{1}{\Gamma(s)\Gamma(2\Delta - s)} \int_0^\infty dt\, t^{-1-s}\, f(t) \qquad t = \frac{x_{12}\, x_{34}}{x_{14} x_{23}} > 0$$

with inverse

$$f(t) = \int_{\mathcal{C}} \frac{ds}{2\pi i}\, \Gamma(s)\, \Gamma(2\Delta - s)\, M(s)\, t^s$$

where $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}}\, f(t)\,.$

Crossing $f(t) = t^{2\Delta_\phi} f(1/t)$ translates to $M(s) = M(2\Delta - s)$

reminiscent of the crossing $S(\mathrm{s}) = S(4m^2 - \mathrm{s})$ in two (flat) dimensions.

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where $\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} f(t)\,.$

To obtain a definition on the whole s-complex plane an analytic continuation of the region of convergence ( $2\Delta_\phi - \Delta_0 < \mathrm{Re}(s) < \Delta_0$ ) is required.

($\Delta_0$ : dimension of the lightest operator exchanged)

Subtraction procedure [Costa, Penedones, Zhiboedov 2021]

$$f_0(t) = f(t) - \left(\frac{t}{1+t}\right)^{-2\Delta} \sum_{\Delta+k=\Delta_0}^{\Delta_\phi} c_\Delta C_{\Delta,k} t^{\Delta+k}$$

$$\psi_0(s) = \int_0^1 dt t^{-1-s} f_0(t) + \sum_{\Delta+k=\Delta_0}^{\Delta_\phi} c_\Delta C_{\Delta,k} \frac{1}{s - \Delta - k}$$

$$M(s) = \frac{\psi_0(s) + \psi_\infty(s)}{\Gamma(s)\Gamma(2\Delta_\phi - s)}$$

# Nonperturbative Mellin amplitude in $d = 1$

Adding more and more poles we can further extend the area of analyticity obtaining a representation valid in the whole complex plane

$$M(s) = \frac{\psi_0(s) + \psi_\infty(s)}{\Gamma(s)\Gamma(2\Delta_\phi - s)}$$

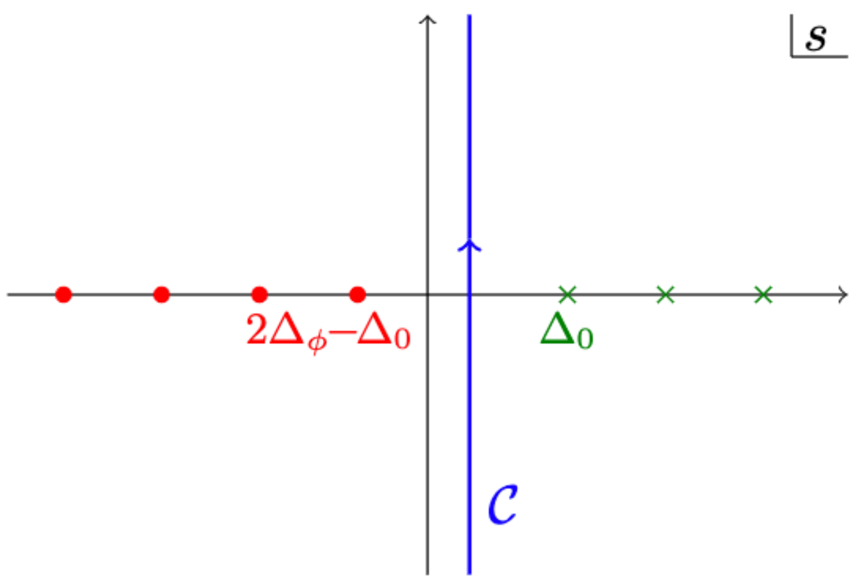
$$\psi_0(s) = \sum_\Delta \sum_{k=0}^{\infty} c_\Delta \frac{(-1)^{k+1}\Gamma(\Delta+k)^2\Gamma(2\Delta)}{k!\Gamma(\Delta)^2\Gamma(2\Delta+k)} \frac{1}{s-\Delta-k}\,,$$

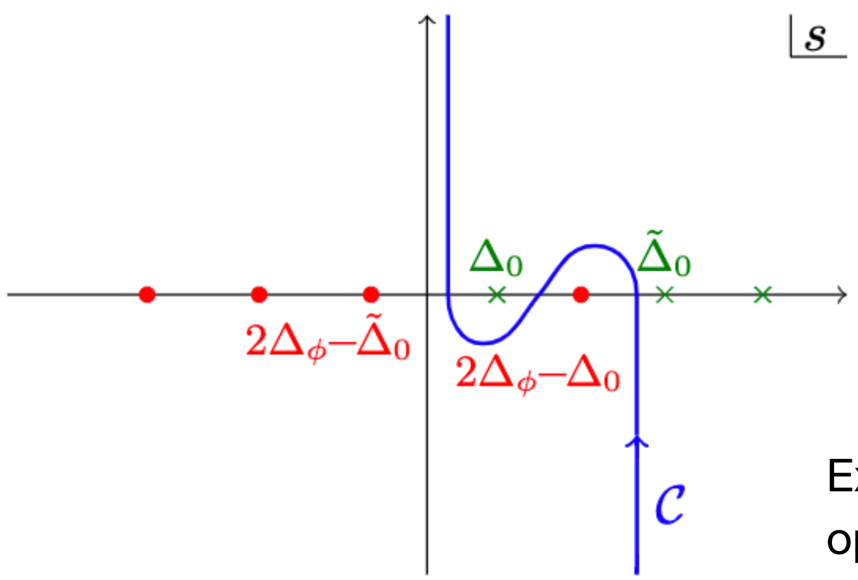
Right poles $s_R = \Delta + k\,, \qquad k = 0, 1, 2, \ldots$

$$\psi_\infty(s) = \sum_\Delta \sum_{k=0}^{\infty} c_\Delta \frac{(-1)^{k}\Gamma(\Delta+k)^2\Gamma(2\Delta)}{k!\Gamma(\Delta)^2\Gamma(2\Delta+k)} \frac{1}{s-2\Delta_\phi+\Delta+k}$$

Left poles $s_L = 2\Delta_\phi - \Delta - k\,, \qquad k = 0, 1, 2, \ldots$

and the contour $\mathcal{C}$ is chosen so to leave *right* poles on its right and *left* poles on its left.




Ex. of one light operator exchanged

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Adding more and more poles we can further extend the area of analyticity obtaining a representation valid in the whole complex plane

$$M(s) = \frac{1}{\Gamma(s)\,\Gamma(2\Delta_\phi - s)} \sum_{\Delta,k} a_\Delta\, C_{\Delta,k} \left[\frac{1}{s-k-\Delta} + \frac{1}{2\Delta_\phi - s - k - \Delta}\right]$$

Summing over k gives the Mellin counterpart of the conformal block expansion

$$M(s) = \sum_{\Delta} \frac{G_\Delta(s) + G_\Delta(2\Delta_\phi - s)}{\Gamma(s)\,\Gamma(2\Delta_\phi - s)} \qquad G_\Delta(s) = \frac{{}_3F_2(\Delta, \Delta, \Delta - s; 2\Delta, 1 + \Delta - s; 1)}{\Delta - s}$$

- $M(s)$ is crossing-invariant
- Asymptotic behavior: $M(s) \sim \frac{1}{s^a}$ , $a > 1$

  (controlled by the Regge limit of the correlator and ensured by the prefactor)

- $M(s)$ has poles for physical exchanged operators
- $M(s)$ has zeros (generically) at $s = 2\Delta_\phi + k$ , $k = 0, 1, 2, ..$

  (canceling unwanted OPE contributions)

From this bounded, meromorphic function and its properties some nonperturbative sum rules can be derived. However the most efficient use of this Mellin formalism happens at **perturbative level.**

# Perturbation theory: quartic interactions with derivatives in $AdS_2$

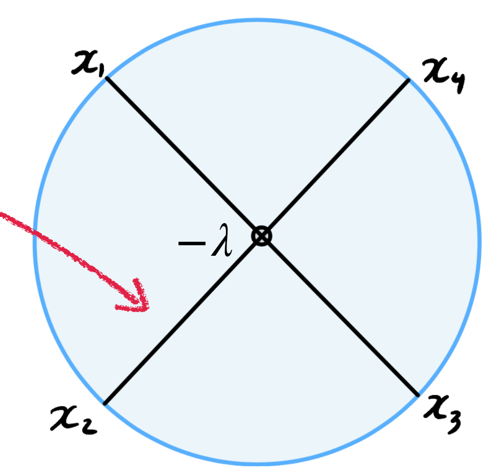
$$S = \int dxdz\,\sqrt{g}\,\left[\,g^{\mu\nu}\,\partial_\mu\Phi\,\partial_\nu\Phi + m^2_{\Delta_\phi}\Phi^2 + g_L\,(\partial^L\Phi)^4\,\right], \qquad L = 0, 1, \ldots$$

where $ds^2 = \frac{1}{z^2}(dx^2 + dz^2)$. Here $(\partial^L\Phi)^4$ denotes a complete and independent set of quartic vertices with four fields and up to $4L$ derivatives.

For $L = 0$, this is $\phi^4$ theory: correlators are $\bar{D}$-functions.

$K_\Delta(y, x; x_i)$

$$< \phi_\Delta(x_1)\phi_\Delta(x_2)\phi_\Delta(x_2)\phi_\Delta(x_2) > = -\lambda\int\frac{dydx}{y^2}\prod_{i=1}^{4}\left(\frac{y}{y^2 + (x - x_i)^2}\right)^\Delta$$
$$= \frac{C_\Delta}{(x_{12}x_{34})^{2\Delta}}\bar{D}_\Delta(z)$$




No closed from expression is known, in cross ratio space, for general $\Delta$.

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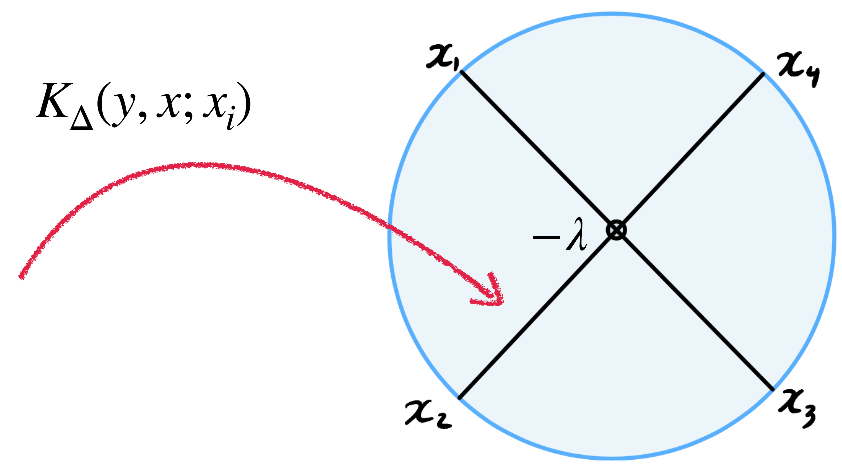


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In Mellin space their explicit expressions are simpler

$$\bar{D}_{1,1,1,1} = -\frac{2\log(1-\chi)}{\chi} - \frac{2\log(\chi)}{1-\chi} \longrightarrow M_{1111}(s) = 2\,\Gamma(s-1)\Gamma(-s)$$

$$\bar{D}_{2,2,2,2} = -\frac{2\left(\chi^2-\chi+1\right)}{15(1-\chi)^2\chi^2} + \frac{\left(2\chi^2-5\chi+5\right)\log(\chi)}{15(\chi-1)^3} - \frac{\left(2\chi^2+\chi+2\right)\log(1-\chi)}{15\chi^3} \longrightarrow M_{2222}(s) = 2(2-s+s^2)\,\Gamma(s-3)\Gamma(-2-s)$$

# Perturbation theory: quartic interactions with derivatives in $AdS_2$

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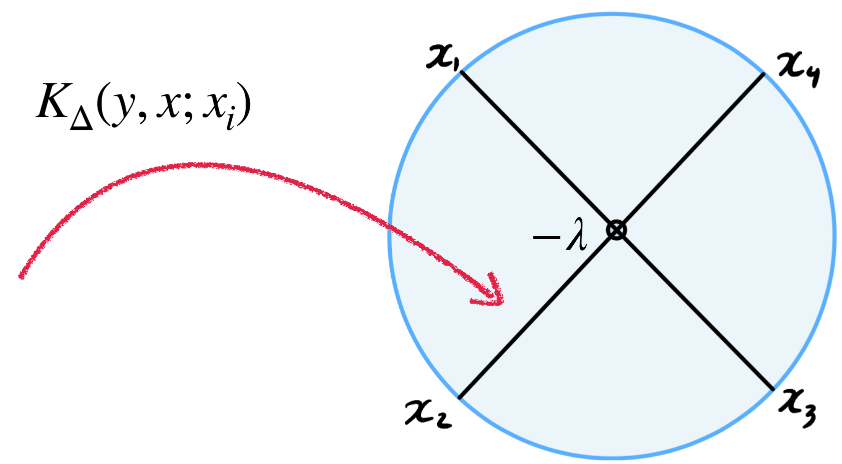


No closed from expression is known, in cross ratio space, for general $\Delta$.

In Mellin space their explicit expressions are simpler and closed expressions can be found

$$\hat{M}_{\Delta_\phi}(s) = \pi \csc(\pi s) \left( \pi \cot(\pi s) P_{\Delta_\phi}(s) - \sum_{k=1}^{2\Delta_\phi - 1} \frac{P_{\Delta_\phi}(k)}{s - k} \right)$$

$$P_{\Delta_\phi}(s) = 2 \frac{\Gamma(\Delta_\phi)^4}{\Gamma(2\Delta_\phi)} {}_4F_3(\{\tfrac{1}{2}, s, 1 - \Delta_\phi, 2\Delta_\phi - s\}; \{1, 1, \Delta_\phi + \tfrac{1}{2}\}; 1)$$

# Perturbation theory: quartic interactions with derivatives in $AdS_2$

$$S = \int dx dz \sqrt{g} \left[ g^{\mu\nu} \partial_\mu \Phi \, \partial_\nu \Phi + m^2_{\Delta_\phi} \Phi^2 + g_L \, (\partial^L \Phi)^4 \right], \qquad L = 0, 1, \ldots$$

where $ds^2 = \frac{1}{z^2}(dx^2 + dz^2)$. Here $(\partial^L \Phi)^4$ denotes a complete and independent set of quartic vertices with four fields and up to $4L$ derivatives.

For quartic bulk interactions of the kind $(\partial^L \phi)^4$

$$\boxed{M_{(\partial^L \phi)^4} = \sum_{l=0}^{L} a_l \sum_{k=0}^{2l} c_{k,l} M_{\Delta+l}(s+k) \qquad 2c_{k,l} = \frac{\Gamma(l+1)}{\Gamma(k+1)\Gamma(l-k+1)} + \delta_{k,0} + \delta_{k,2l}}$$

With such **closed** formulas we can successfully extract **new** CFT data in closed form.

$$\hat{\gamma}^{(1)}_{L,n}(\Delta_\phi) = \hat{\mathcal{G}}_{L,n}(\Delta_\phi) \hat{\mathcal{P}}_{L,n}(\Delta_\phi)$$

$$\hat{\mathcal{G}}_{L,n}(\Delta_\phi) = \frac{\sqrt{\pi} 4^{-2\Delta-L+1} \Gamma(2\Delta)^2 \Gamma(L+\frac{1}{2}) \Gamma(L+\Delta)^4 \Gamma(L+2\Delta-\frac{1}{2}) \Gamma(n+\Delta+\frac{1}{2}) \Gamma(L-n+\Delta)}{\Gamma(L+1) \Gamma(L+\Delta+\frac{1}{2})^2 \Gamma(L+2\Delta) \Gamma(n+\Delta)^3 \Gamma(2n+2\Delta-\frac{1}{2}) \Gamma(L+n+\Delta+\frac{1}{2})}$$

$\hat{\mathcal{P}}_{L,n}(\Delta_\phi)$ is a polynomial in $n$ and in $\Delta_\phi$ of degree $6L$.

Verified in [Knop, Mazac 22]

Obtained comparing residues at poles of. $M_{(\partial^L \phi)^4}$ with those of Mellin block expansion

# From OPE inversion formula to dispersion relation

- The OPE expresses a four-point correlator as a discrete sum of conformal blocks

$$\mathcal{G}(z) = \sum_{\Delta} a_{\Delta} G_{\Delta}(z) \qquad\qquad z = \frac{x_{12}\, x_{34}}{x_{13}\, x_{24}}$$

- Another expansion - the conformal partial wave decomposition - is in terms of a complete basis of orthonormal functions (principal series, $\Delta \in \frac{1}{2} + i\mathbb{R}$ and discrete series).

$$\mathcal{G}(z) = \int_{\frac{1}{2}}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{I_{\Delta}}{n_{\Delta}} \Psi_{\Delta}(z) + \sum_{m=0}^{\infty} \frac{4m-1}{4\pi^2} \tilde{I}_{2m} \Psi_{2m}(z)$$

$$\Psi_{\Delta}(z) = \kappa_{1-\Delta} G_{\Delta}(z) + \kappa_{\Delta} G_{1-\Delta}(z)\,, \qquad \kappa_{\Delta} = \frac{\sqrt{\pi}\,\Gamma(\Delta - \frac{1}{2})\Gamma(\frac{1-\Delta}{2})^2}{\Gamma(1-\Delta)\Gamma(\frac{\Delta}{2})^2}$$

$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{I_{\Delta}}{2\kappa_{\Delta}} G_{\Delta}(z) + \sum_{m=0}^{\infty} \frac{\Gamma^2(2m+2)}{2\pi^2 \Gamma(4m+3)} \tilde{I}_{2m+2} G_{2m+2}(z)$$

From the poles of the coefficients one recovers the OPE expansion

$$a_{\Delta} = -\text{Res}\left[\frac{I_{\Delta'}}{2\kappa_{\Delta'}}\right]_{\Delta'=\Delta}$$

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$$= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{I_{\Delta}}{2\kappa_{\Delta}} G_{\Delta}(z) + \sum_{m=0}^{\infty} \frac{\Gamma^2(2m+2)}{2\pi^2 \Gamma(4m+3)} \tilde{I}_{2m+2} G_{2m+2}(z)$$

- Because of the orthonormality one can perform a (trivial) inversion

$$I_{\Delta} = \int_{-\infty}^{\infty} dz\, z^{-2}\, \Psi_{\Delta}(z)\, \mathcal{G}(z) \quad \text{for } \Delta \in \frac{1}{2} + i\mathbb{R}\,, \qquad \widetilde{I}_{\Delta} = \int_{-\infty}^{\infty} dz\, z^{-2}\, \Psi_{\Delta}(z)\, \mathcal{G}(z) \quad \text{for } \Delta \in 2\mathbb{N}$$

# From OPE inversion formula to dispersion relation

A more powerful inversion can be derived from a contour-deformation argument based on the analytic structure of the correlator and its (Regge) behavior at infinity

[Caron-Huot 17]

$$I_\Delta = 2\int_0^1 dw\, w^{-2} H_\Delta(w)\, \mathrm{dDisc}[\mathcal{G}(w)]$$

known explicitly for all integer (bos) and half-integers (ferm) dimensions $\Delta_\phi$ of the external operators.

[Simmons-Duffin, Stanford, Witten 2017] [Mazac 2018]

$$\tilde{I}_m = \frac{4\Gamma^2(m)}{\Gamma(2m)}\int_0^1 dw\, w^{-2} G_m(w)\, \mathrm{dDisc}[\mathcal{G}(w)]$$

sl(2,R) conformal block

makes use of the **double discontinuity** of the correlator

$$\mathrm{dDisc}[\mathcal{G}(z)] = \mathcal{G}(z) - \frac{\mathcal{G}^{\curvearrowleft}(z) + \mathcal{G}^{\smile}(z)}{2} \quad \text{for } z \in (0,1)$$

$\mathcal{G}^{\curvearrowleft}(z)$: value of *G*(z) moving counterclockwise around the branch cut at $z$=1, vv for $\mathcal{G}^{\smile}(z)$.

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$H_\Delta(w)$: known explicitly for all integer (bos) and half-integers (ferm) dimensions $\Delta_\phi$ of the external operators.

$G_m(w)$: sl(2,R) conformal block

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$\mathcal{G}^{\circlearrowleft}(z)$: value of *G*(z) moving counterclockwise around the branch cut at $z$=1, vv for $\mathcal{G}^{\circlearrowright}(z)$.

**It provides an analytic continuation of the coefficients (in higher d, this means we can think of spin as a expansion parameter).**

The dDisc of a correlator is **much simpler** than the correlator itself, in perturbation theory. Crucially **can be computed at any order from lower order data**!

# Dispersion relation for CFT$_1$ correlators

The double discontinuity can then be taken as the starting point to reconstruct the full correlator

$$
\begin{aligned}
\mathcal{G}(z) &= \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{I_\Delta}{2\kappa_\Delta} G_\Delta(z) + \sum_{m=0}^{\infty} \frac{\Gamma^2(2m+2)}{2\pi^2\Gamma(4m+3)} \tilde{I}_{2m+2} G_{2m+2}(z) \\
&= \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{H_\Delta^{B/F}(w)}{\kappa_\Delta} G_\Delta(z) \\
&+ \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] \sum_{m=0}^{\infty} \frac{2\Gamma(2m+2)^4}{\pi^2\Gamma(4m+4)\Gamma(4m+3)} G_{2m+2}(w) G_{2m+2}(z) \\
&\equiv \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] K_{\Delta_\phi}(z,w)\,,
\end{aligned}
$$

$$
\begin{aligned}
K_{\Delta_\phi}(z,w) &= \frac{w\, z^2 (w-2)\log(1-w)}{\pi^2 (w-z)(w+z-wz)} - \frac{z\, w^2 (z-2)\log(1-z)}{\pi^2 (w-z)(w+z-wz)} \\
&\pm \frac{z^2}{\pi^2}\left[\log(1-w)\frac{(1-2w)w^{2-2\Delta_\phi}}{(w-1)wz^2+z-1} + \frac{\log(1-z)}{z}\frac{w^{2-2\Delta_\phi}}{wz-1} + \log(z)\frac{(1-2w)w^{2-2\Delta_\phi}}{(w-1)wz^2+z-1} + (w \to \tfrac{w}{w-1})\right] \\
&- w^{2-2\Delta_\phi} \sum_{n=0}^{2\Delta_\phi - 4} a_n^{\Delta_\phi}(w)\, \underset{\substack{\downarrow \\ \text{sl(2, R) Casimir}}}{\mathcal{C}^n} \left[\frac{2}{\pi^2}\left(\frac{z^2\log(z)}{1-z} + z\log(1-z)\right)\right]
\end{aligned}
$$

The kernel of the integral can be evaluated explicitly at each given integer and half-integer dimension $\Delta_\phi$ of the external identical operators.

# Dispersion relation for CFT1 correlators

The double discontinuity can then be taken as the starting point to reconstruct the full correlator

$$
\begin{aligned}
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&= \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{d\Delta}{2\pi i} \frac{H_\Delta^{B/F}(w)}{\kappa_\Delta} G_\Delta(z) \\
&+ \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] \sum_{m=0}^{\infty} \frac{2\Gamma(2m+2)^4}{\pi^2\Gamma(4m+4)\Gamma(4m+3)} G_{2m+2}(w) G_{2m+2}(z) \\
&\equiv \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] K_{\Delta_\phi}(z,w)\,,
\end{aligned}
$$

$$
\begin{aligned}
K_{\Delta_\phi}(z,w) &= \frac{w\,z^2(w-2)\log(1-w)}{\pi^2(w-z)(w+z-wz)} - \frac{z\,w^2(z-2)\log(1-z)}{\pi^2(w-z)(w+z-wz)} \\
&\pm \frac{z^2}{\pi^2}\left[\log(1-w)\tfrac{(1-2w)w^{2-2\Delta_\phi}}{(w-1)wz^2+z-1} + \tfrac{\log(1-z)}{z}\tfrac{w^{2-2\Delta_\phi}}{wz-1} + \log(z)\tfrac{(1-2w)w^{2-2\Delta_\phi}}{(w-1)wz^2+z-1} + (w \to \tfrac{w}{w-1})\right] \\
&- w^{2-2\Delta_\phi} \sum_{n=0}^{2\Delta_\phi-4} a_n^{\Delta_\phi}(w)\, \mathcal{C}^n \left[\tfrac{2}{\pi^2}\left(\tfrac{z^2\log(z)}{1-z} + z\log(1-z)\right)\right]
\end{aligned}
$$

The kernel, crossing symmetric (in z), Regge bounded, and definite positive, explicitly depends on the dimension $\Delta_\phi$ of the external operators ($\neq$ **from higher d**).

# Dispersion relation in perturbation theory

The double discontinuity can then be taken as the starting point to reconstruct the full correlator,

$$\mathcal{G}(z) = \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] K_{\Delta_\phi}(z,w)\,,$$

$$\mathrm{dDisc}_t[\mathcal{G}(z)] = \mathcal{G}(z) - \frac{\mathcal{G}^{\circlearrowleft}(z) + \mathcal{G}^{\circlearrowright}(z)}{2}$$

- Much simpler than correlator!

$$\mathrm{dDisc}[\log(1-z)] = 0\,,$$
$$\mathrm{dDisc}[\log^2(1-z)] = 4\pi^2$$

- On conformal blocks, dDisc acts as

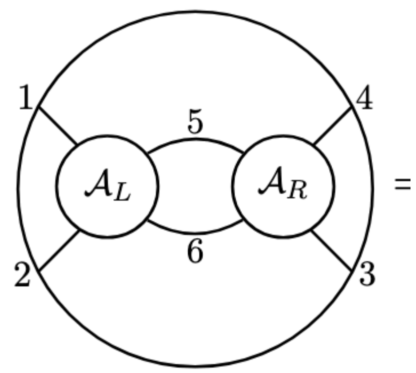
$$\mathrm{dDisc}\Big[\tfrac{z^{2\Delta_\phi}}{(1-z)^{2\Delta_\phi}} G_\Delta(1-z)\Big] = 2\sin^2 \tfrac{\pi}{2}(\Delta - 2\Delta_\phi) \tfrac{z^{2\Delta_\phi}}{(1-z)^{2\Delta_\phi}} G_\Delta(1-z)$$

If the correlator is evaluated in a perturbative expansion about generalised free theory, this implies that each given order dDisc is given in terms of lower order data

E.g. $$\mathrm{dDisc}[\mathcal{G}^{(2)}(z)] = \pi^2 \sum_n \frac{1}{2} a_n^{(0)} \left(\gamma_n^{(1)}\right)^2 \frac{z^{2\Delta_\phi}}{(1-z)^{2\Delta_\phi}} G_{2\Delta_\phi + 2n}(1-z)$$

# Double discontinuity in perturbation theory

Direct connection of Ddisc with “unitarity” cut operators in AdS, which act on bulk amplitudes putting virtual lines on shell [Alday, Caron-Huot 17] [Meltzer Perlmutter Sivaramakrishan 19]

$\mathcal{A} =$ 

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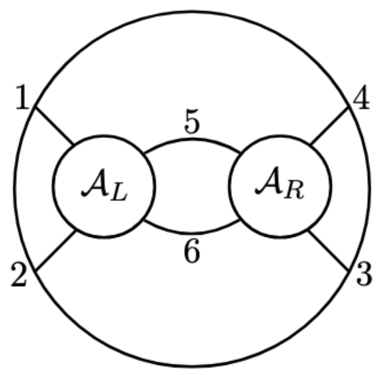
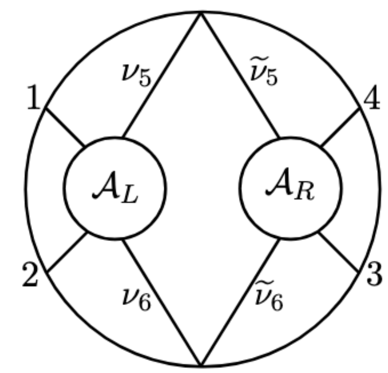
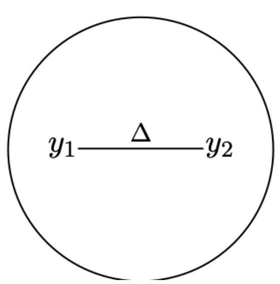


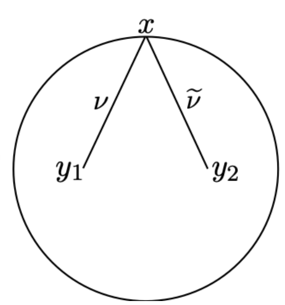

$= \int d\nu_{5,6} \int d^d x_{5,6} P(\nu_5, \Delta_5) P(\nu_6, \Delta_6)$




Split representation of bulk-to-bulk propagator in terms of two bulk-to-boundary propagators




$= \int d\nu d^d x P(\nu, \Delta)$




$$G_\Delta(y_1, y_2) = \int_{-\infty}^{\infty} d\nu \, P(\nu, \Delta) \int_{\partial \mathrm{AdS}} d^d x K_{\frac{d}{2}+i\nu}(x, y_1) K_{\frac{d}{2}-i\nu}(x, y_2),$$

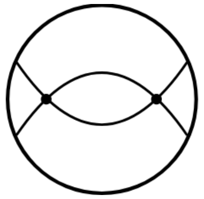
A propagator goes on-shell when localised onto a pole of $P(\nu, \Delta)$

A “Cut operator” can be defined, effect same as Ddisc (vanishes on contact diagrams, etc)

Effective to 4 point, 1-loop (no general unitarity), to be developed.

# Correlators from dispersion in perturbation theory

- Checked on the one-loop correlator of four the $\lambda\phi^4$ theory in AdS2



- dCFT1 defined by 1/2 BPS Wilson line in N=4 sYM: state of the art is 4th order in strong coupling (=3 loops in AdS) obtained with perturbative Ansatz

rational functions

[Ferrero, Meneghelli 21,23]

$$G(z) = \sum_{\ell=0}^{\infty} G^{(\ell)}(z) \qquad \text{where} \qquad G^{(\ell)}(z) = \sum_{i=1}^{N(\ell)} r_i(z)\, \mathcal{T}_i(z)$$

$$\mathcal{T}_i(z) \in \{\text{HPLs of transcendentality } \mathtt{t} \leq \mathtt{t}_{\mathtt{max}}(\ell)\} \qquad N(\ell) = \sum_{\mathtt{t}=0}^{\mathtt{t}_{\mathtt{max}}(\ell)} 2^{\mathtt{t}} = 2^{1+\mathtt{t}_{\mathtt{max}}(\ell)} - 1$$

$\mathtt{t}_{\mathtt{max}}(\ell) = \ell$

Unknowns are some coefficients in an educated guess for the rational functions $r_i(z)$
Ansatz constrained by:
a) AdS unitarity (highest logarithmic singularities fixed in terms of lower order ones)
b) Crossing symmetry, Braiding symmetry, Regge bound and supersymmetric localization
fix the remaining terms ~ 1, log(z), log(1-z).

# Correlators from dispersion in perturbation theory

The dispersion relation bypasses the need of an Ansatz incorporating all constraints!

$$\mathcal{G}(z) = \int_0^1 dw\, w^{-2} \mathrm{dDisc}[\mathcal{G}(w)] K_{\Delta_\phi}(z,w)$$

with a **caveat:** the **regularization** procedure necessary order by order in perturbation theory (where the Regge behaviour is worse than in the full nonperturbative correlator) implies subtractions which depend on a few **unknown** OPE data (i.e. data at same pert. order).

@ 1 loop: $a_0^{(2)},\ \gamma_0^{(2)}$

@ 2 loops: $a_0^{(3)}, a_1^{(3)}\ \ \gamma_0^{(3)}\ \gamma_1^{(3)}$

@ 3 loops: $a_0^{(4)}, a_1^{(4)}\ \ \gamma_0^{(4)}\ \gamma_1^{(4)}$

These **can be fixed**, **in the N=4 SYM case**, using inputs from supersymmetric localization or constraints from integrated correlators. [Cavaglia'Gromov Julius Preti 22] [Drukker, Kong, Sakkas 22]

This kind of leftover ambiguity is not surprising in this context,
e.g. in higher d there is a low spin ambiguity.

# Correlators from dispersion in perturbation theory: STRATEGY

1) Compute $\mathrm{dDisc}[\mathcal{G}^{(\ell)}(z)]$ from lower order CFT data.

$$\mathrm{dDisc}[\mathcal{G}^{(\ell)}(z)] = \sum_{k=0}^{\ell} G^{(\ell)}_{\log^k}(z)\,\log^k z \qquad z\to 0 \text{ OPE limit}$$

$\equiv$ compute terms proportional to $\log^k z$ with $k>1$ from lower order data

eg. $$\mathrm{dDisc}[\mathcal{G}^{(2)}] = \pi^2 \sum_n \frac{1}{4}\, a_n^{(0)} (\gamma_n^{(1)})^2 \frac{z^{2\Delta_\phi}}{(1-z)^{2\Delta_\phi}}\, G_{2\Delta_\phi+2n}(1-z)$$

MIXING: because of degeneracy between operators in the free theory this is rather an average

$$\langle a_n^{(0)} (\gamma_n^{(1)})^2 \rangle := \sum_{\mathcal{O}} (\mu_{\mathcal{O}}^{(0)})^2 (\gamma_{\mathcal{O}}^{(1)})^2$$

eigenstates of dil. operator     OPE coeff with external operators

However, at previous order, from one correlator it is only possible to extract

$$\langle a_n^{(0)} \gamma_n^{(1)} \rangle := \sum_{\mathcal{O}} (\mu_{\mathcal{O}}^{(0)})^2 \gamma_{\mathcal{O}}^{(1)}$$

For a solution of the mixing problem in this setup, Ferrero Meneghelli 2023

# Correlators from dispersion in perturbation theory: STRATEGY

Usually solved considering more correlators (since $\mu_0$ depend on external operators, the result of the average depends on the four-point function one is considering
⟹ enough correlators, enough inequivalent averages $\langle a_n^{(0)} \gamma_n^{(1)} \rangle$ to calculate the actual $\gamma_n^{(1)}$.

2) Regge boundedness of correlator in Regge limit $(\frac{1}{2}+it)^{-2\Delta\phi}\, G(\frac{1}{2}+it) < \infty$ for $t \to \infty$ is broken perturbatively. Assume mild $\gamma_n^{(\ell)} \sim n^{\ell+1}$ ⟹ $G(\frac{1}{2}+it) \sim t^{\ell}$

⟹ Reflected in the inversion formula $H_\Delta^{\text{unb.}}(\omega)$

⟹ In the dispersion relation $K^{\text{unbounded}}$ (with extra poles that may spoil convergence of the integral defining the correlator)

⟹ Subtraction at level of correlator $G^{reg}(z) = G^{(\ell)} -$ subtractions

⟹ 'dDisc $[G^{reg}]$ enters the dispersion relation multiplying $K^{\text{unbounded}}$ and one then just demands that the integral of the dispersion relation converges

3) the subtractions then depend on specific unknown at the given order

⟹ to fix these unknown, use constraints from localization/integrated correlators of Cavaglià Gromov Julius Preti

# Conclusions

- We have considered a class of four-point correlators in the $\mathrm{CFT}_1$ defined on the 1/2-BPS Wilson line in the 3d superconformal ABJM theory.

- Superconformal symmetry determines four-point correlators of the displacement supermultiplet in terms of a single function, that we evaluate at strong coupling using holography and Witten diagrams and the analytic boostrap. We can extract CFT data.

- We defined a Mellin amplitude for $\mathrm{CFT}_1$ four-point functions; bounded, meromorphic function of a single complex variable, whose analytical properties are inferred from physical requirements on the correlator.
  Closed-form expressions for Mellin transform of tree-level contact interactions with an arbitrary number of derivatives in a bulk AdS2 field theory, and for first correction to the scaling dimension of "two-particle" operators exchanged.

- Derived from the inversion formula a dispersion relation for $\mathrm{CFT}_1$ four-point functions, an integral over the double discontinuity of the correlator.

# Outlook

- Higher-order analysis, multi-point correlators, non-identical in the same setup

- Organising principles/hidden symmetries?

  Recent observation of integrable structure underlying contact Witten diagrams [Rigatos, Zhou 22]
  If generalizes to other classes of diagrams this would open a playground of applications of integrability in AdS spaces.

- Despite/with the help of these analytic bootstrap tools, a motivation to develop technology for Witten diagrams remains, thanks to the general observation that (for a class of boundary correlators related to inflationary correlators)

  perturbation theory in rigid de Sitter -> Witten diagrams in **E**AdS

  [Sleight, Taronna 20,21] [Di Pietro, Gorbenko, Komatsu 21]

  It would be great to develop loop-technology for $AdS_2$ models with derivative interactions

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Thank you.