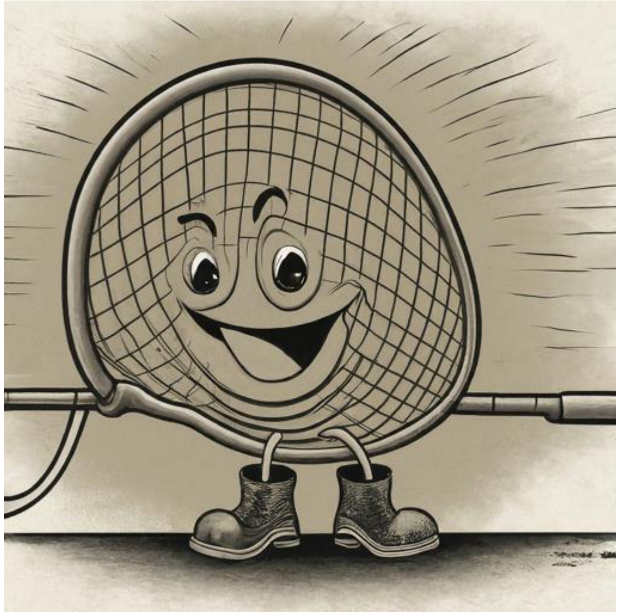
Fishnets & Bootstraps & Antipodal Self-Symmetry



Lance Dixon (SLAC)

Fishnets: Conformal Field Theories & Feynman Graphs

Bethe Center for Theoretical Physics

Bonn, September 4, 2024

SLAC NATIONAL ACCELERATOR LABORATORY

Outline & Summary

- There is a bizarre “Antipodal (Self) Duality” of (some) form factors / amplitudes in Planar N=4 SYM
- Involves writing the symbol backward, together with some kinematic map
- Can A(S)D extend to individual integrals?
- How about to “integrals that are also amplitudes”, e.g. four-point fishnet integrals?
- We’ll see that the answer to the latter question is yes, at least as a map on symbol letters

- Many words same backwards and forwards

ABBA

- Some also need a “letter map” (B↔R, C ↔ D):

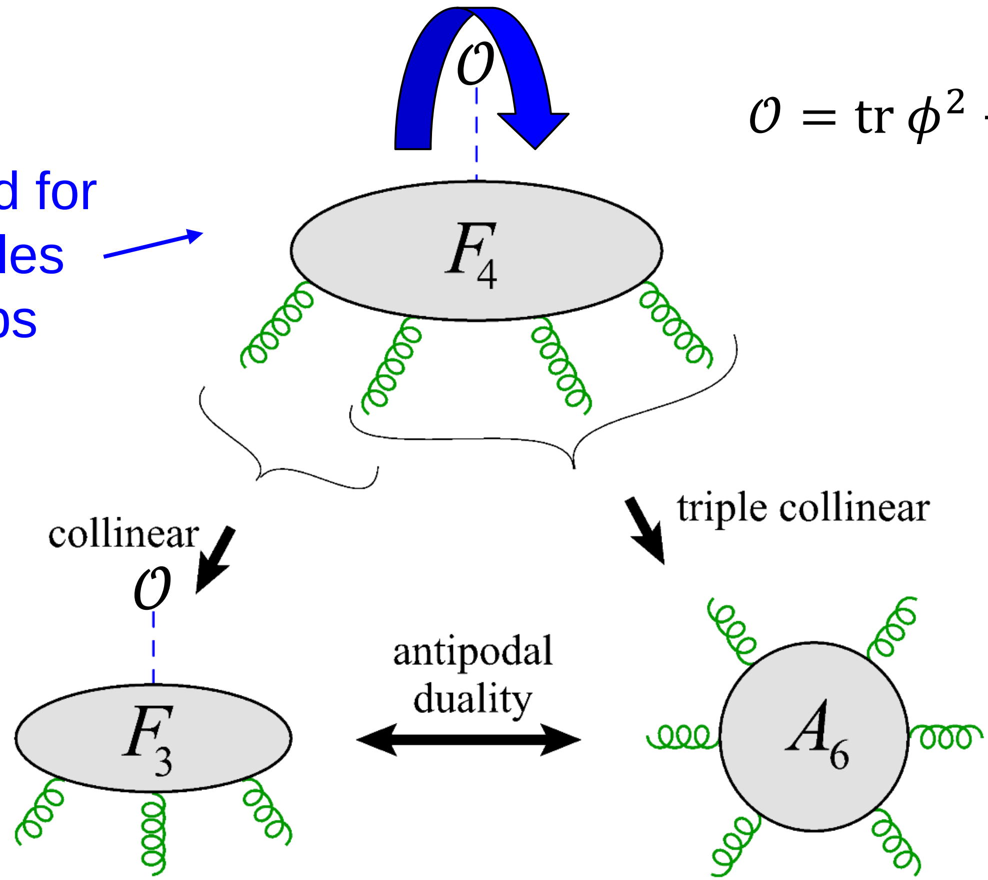
ABRACADABRA

Antipodal Self Duality for a 4-point form factor in planar N=4 SYM

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 2212.02410

$$\mathcal{O} = \mathrm{tr}\,\phi^2 + \cdots + \mathrm{tr}\, G_{\mu\nu}^{\mathrm{SD}} G^{\mu\nu\,\mathrm{SD}}$$

validated for 4 variables to 3 loops



antipode map reverses order of letters in symbol

also requires a kinematic map

validated for 2 variables to 8 loops!

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

Multiple polylogarithms (MPLs)

- Characterize all form factors & amplitudes playing a role here
- At L loops, all results are weight $n = 2L$ MPLs, defined recursively as iterated integrals by

$$G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t)$$

and $G(\vec{0}_n, x) = \frac{(\ln x)^n}{n!}$

MPL Hopf algebra

Goncharov; Brown; Goncharov, Spradlin, Vergu, Volovich; Duhr, Gangl, Rhodes

- Differential definition: $dF = \sum_{s_k \in \mathcal{L}} F^{s_k} \, d \ln s_k$
- Hopf algebra "co-acts" on space of MPLs,
  $\Delta: F \to F \otimes F$
- Derivative dF is one piece of Δ: $\Delta_{n-1,1} F = \sum_{s_k \in \mathcal{L}} F^{s_k} \otimes \ln s_k$
- So we refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F
- $\boldsymbol{s_k}$ **are letters in the symbol alphabet** $\mathcal{L}$

Iterate to get **symbol**

- Apply $\{n\text{-}1,1\}$ coaction iterately:
- Define $\{n\text{-}2,1,1\}$ double coproducts, F^{s_k,s_j},

via derivatives of $\{n\text{-}1,1\}$ single coproducts F^{s_j}:

$$dF^{s_j} \equiv \sum_{s_k\in\mathcal{L}} F^{s_k,s_j}\, d\ln s_k$$

- And so on for $\{n\text{-}m,1,\ldots,1\}$ m^{th} coproducts of F.
- Maximal iteration, n times for weight n function, is

the **symbol,** ["ln" is implicit in s_{i_k}]

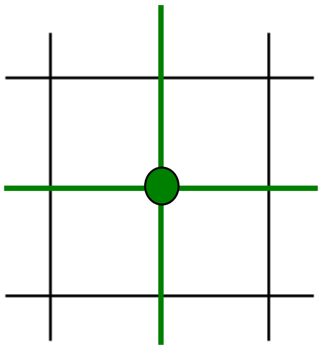
$$\mathcal{S}[F] \equiv \sum_{s_{i_1},\ldots,s_{i_n}\in\mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}}\, s_{i_1}\otimes\ldots\otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers (often integers!)

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Example

The (1,1) 4-point fishnet, i.e. the 1 loop box integral:



$$I_{1,1} = 2[\mathrm{L}i_2(z) - \mathrm{L}i_2(\bar{z})] + \ln(z\bar{z})\ln(\frac{1-z}{1-\bar{z}})$$

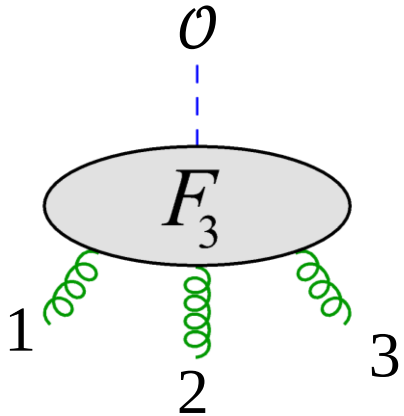
Its symbol is

$$\mathcal{S}[I_{1,1}] = \ z\bar{z} \otimes \frac{1-z}{1-\bar{z}} - (1-z)(1-\bar{z}) \otimes \frac{z}{\bar{z}}$$

Alphabet (same for all D=4 4-point fishnets):

$$\mathcal{L} = \{z, 1-z, \bar{z}, 1-\bar{z}\}$$

3-gluon form factor depends on 2 dimensionless variables u, v

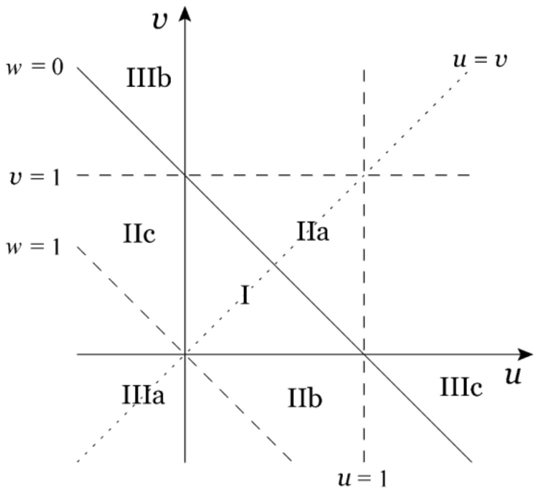


$k_1 + k_2 + k_3 = -k_{\mathcal{O}}$

$k_i^2 = 0 \quad s_{ij} = (k_i + k_j)^2$

$s_{123} = s_{12} + s_{23} + s_{31} = q^2$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}} = 1 - u - v$$



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

$D_3 \equiv S_3$ dihedral symmetry generated by:

a. cycle: $i \to i + 1 \pmod 3$, or

$u \to v \to w \to u$

b. flip: $u \leftrightarrow v$

N=4 amplitude is S_3 invariant

F_3 symbol alphabet has 6 letters

$$\mathcal{L} = \{\, a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \}$$

- Symbols of form factor $F_3^{(L)}$ at $L = 1, 2$ loops are just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[F_3^{(1)}\right] = (-1)\, b \otimes d + \text{dihedral}$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 4\, b \otimes d \otimes d \otimes d + 2\, b \otimes b \otimes b \otimes d + \text{dihedral}$$

Brandhuber, Travaglini, Yang, 1201.4170

known to 8 loops!

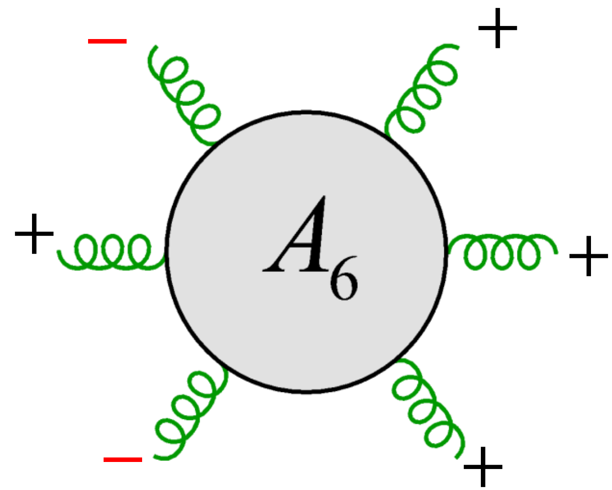
LD, Gürdoğan, A. McLeod, M. Wilhelm, 2204.11901

dihedral cycle: $a \to b \to c \to a$, $\quad d \to e \to f \to d$

dihedral flip: $a \leftrightarrow b$, $\quad d \leftrightarrow$ e

6-gluon MHV amplitude

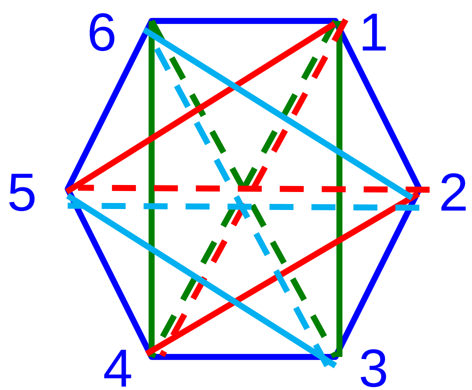
- Dual to Wilson hexagon, invariant under dual conformal transformations; it only depends on 3 dual conformal cross ratios, $\hat{u}, \hat{v}, \hat{w}$:



$$\widehat{u} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

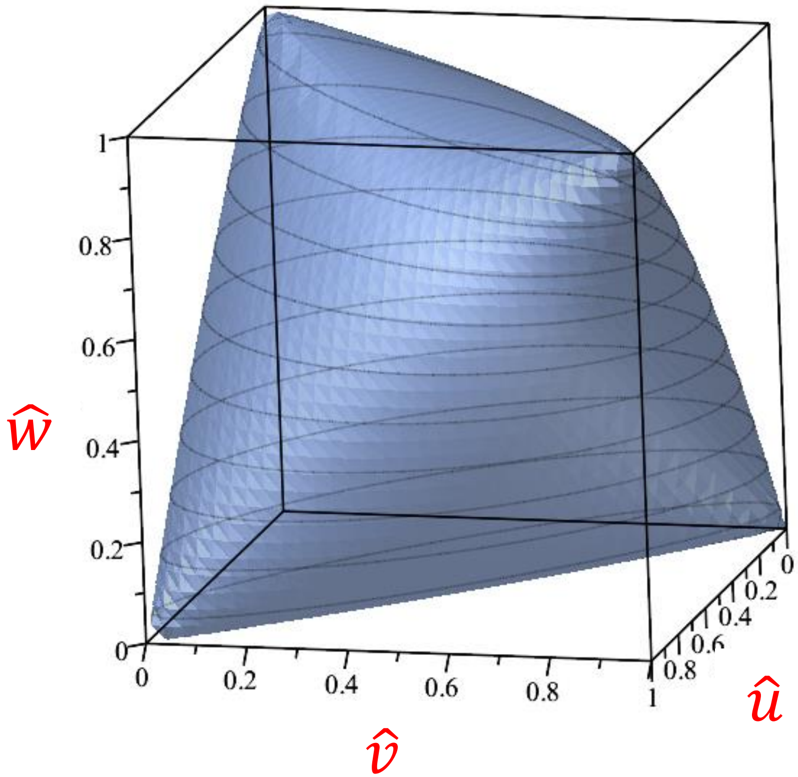
$$\widehat{v} = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

$$\widehat{w} = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$



D_6 dihedral symmetry: cycle $(\mathrm{mod}\ 6)$ and flip, but it acts on $\hat{u}, \hat{v}, \hat{w}$ as $D_3 = S_3$

Parity-preserving surface



$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

kinematics lies in a 3d subspace of 4d spacetime
→ parity invariant

6-gluon symbol alphabet

- $\mathcal{L}_6 = \{ \hat{u}, \hat{v}, \hat{w}, 1-\hat{u}, 1-\hat{v}, 1-\hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w \}$ → $1 \text{ for } \Delta = 0$

→ $\mathcal{L}'_6 = \{ \hat{a} = \frac{\hat{u}}{\hat{v}\hat{w}},\ \hat{b} = \frac{\hat{v}}{\hat{w}u},\ \hat{c} = \frac{\hat{w}}{\hat{u}\hat{v}},\ \hat{d} = \frac{1-\hat{u}}{\hat{u}},\ \hat{e} = \frac{1-\hat{v}}{\hat{v}},\ \hat{f} = \frac{1-\hat{w}}{\hat{w}} \}$

- Symbols of amplitude $A_6^{(L)}$ on $\Delta = 0$ at $L = 1, 2$ loops are just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[A_6^{(1)}\right] = (-\frac{1}{2})\hat{b}\otimes\hat{d} + \text{dihedral}$$

$$\mathcal{S}\left[A_6^{(2)}\right] = \hat{b}\otimes\hat{d}\otimes\hat{d}\otimes\hat{d} + \frac{1}{2}\hat{b}\otimes\hat{b}\otimes\hat{b}\otimes\hat{d} + \text{dihedral}$$

$$\ldots$$

was known to 7 loops

Goncharov, Spradlin, Vergu, Volovich, 1006.5703, …,
Caron-Huot, LD, Dulat, McLeod, von Hippel, 1903.10890

Antipodal duality (AD)

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

$$F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)$$

Antipode map S, at symbol level, reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

Kinematic map in terms of underlying variables is:

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u + v + w = 1$ to parity-preserving surface

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

Kinematic map on letters

$\sqrt{\hat{a}} = d, \quad \hat{d} = a,$ *plus cyclic relations*

Works through 8 loops (even beyond symbol)!

LD, Liu, 2308.08199

L	number of terms in symbol
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	92,954,568
8	1,671,656,292

(number of nonzero integers that match)

But why?!

Antipodal Duality Evidence by Counting: Dimensions of $\{n, 1, 1, \ldots, 1\}$ Coproducts

$F_3^{(L)}$

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	3	1														
$L=2$	1	3	6	3	1												
$L=3$	1	3	9	12	6	3	1										
$L=4$	1	3	9	21	24	12	6	3	1								
$L=5$	1	3	9	21	46	45	24	12	6	3	1						
$L=6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L=7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L=8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

$A_6^{(L)}$

$\sim$ on $\Delta = 0$

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	3	1														
$L=2$	1	3	6	4	1												
$L=3$	1	3	6	12	12	3	1										
$L=4$	1	3	6	12	24	29	12	3	1								
$L=5$	1	3	6	12	24	45	62	30	12	3	1						
$L=6$	1	3	6	12	24	45	85	134	78	31	12	3	1				
$L=7$	1	3	6	12	24	45	85	155	257	190	84	31	12	3	1		
$L=8$	1	3	6	12	24	45	85	155	279	466	437	199	84	31	12	3	1

This count is really of the parity-even functions

Antipodal Self Duality Evidence by Counting

- 4-point form factor $F_4^{(L)}$ has palindromic coproducts

weight n	0	1	2	3	4	5	6
$L=2$	1	8	32	8	1	–	–
$L=3$	1	8	56	253	56	8	1

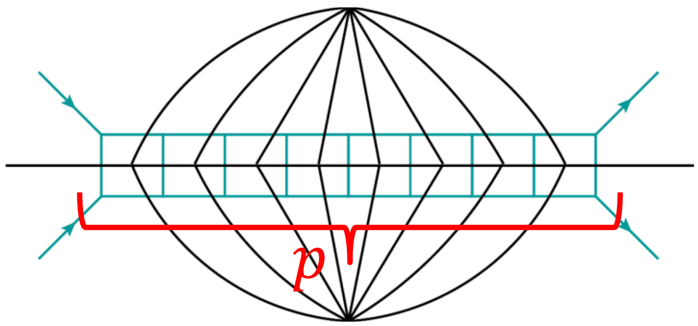
- Only palindromic on a parity-preserving surface
- Only in “remainder function normalization”
- 93 letter alphabet → limited to 3 loops so far

How about for integrals?

Arkani-Hamed, Yuan, 1712.09991

- One loop n-gon integral
- $$I[Q] = \int \frac{d^n x \, \delta(1 - \sum_i x_i)}{\sum_{i,j} Q_{ij} x_i x_j}$$
- Q_{ij} depends on kinematics
- Symbol antipode takes $Q \to Q^{-1}$
- However, if Q is for massless kinematics, Q^{-1} typically is not

How about ladder integrals?



palindromic!

Usyukina, Davydychev, PLB305, 136 (1993)

$$L_p = \sum_{j=p}^{2p} \frac{j!\,[-\ln(z\bar{z})]^{2p-j}}{p!(j-p)!(2p-j)!}[\mathrm{Li}_j(z) - \mathrm{Li}_j(\bar{z})]$$

$$u = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} \equiv \frac{z\bar{z}}{(1-z)(1-\bar{z})}\,, \quad v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \equiv \frac{u}{z\bar{z}}$$

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$(1,1)$	1	2	1														
$(1,2)$	1	2	3	2	1												
$(1,3)$	1	2	3	4	3	2	1										
$(1,4)$	1	2	3	4	5	4	3	2	1								
$(1,5)$	1	2	3	4	5	6	5	4	3	2	1						
$(1,6)$	1	2	3	4	5	6	7	6	5	4	3	2	1				
$(1,7)$	1	2	3	4	5	6	7	8	7	6	5	4	3	2	1		
$(1,8)$	1	2	3	4	5	6	7	8	9	8	7	6	5	4	3	2	1

Palindromic not sufficient for ladders!

- First entries: $\{\, z\bar{z},\ (1-z)(1-\bar{z})\, \}$
- Last entries (for $p > 1$): $\{\, z,\ \bar{z}\, \}$
- First pair of entries: 2 of 3 are products
- Last pair of entries (for $p > 2$): All 3 of 3 are products:

  $$\{\, z \otimes z, \qquad z \otimes \bar{z} + \bar{z} \otimes z, \qquad \bar{z} \otimes \bar{z}\, \}$$
- Since the dimensions of first-pair space and last-pair space differ after projecting out products, there can be no antipodal map

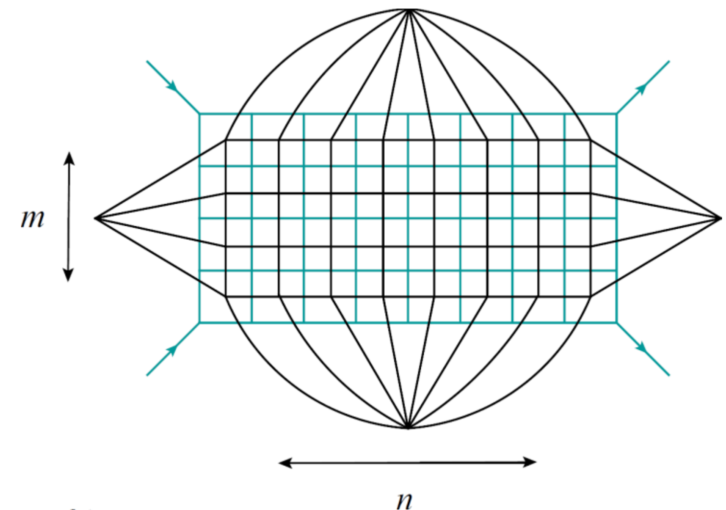
Strongly deformed planar N=4 SYM

Gürdoğan, Kazakov, 1512.06704; Caetano, Gürdoğan, Kazakov, 1612.05895

- Simplest limit has only 2 scalars
- Motivates considering (among other things!) 4-point fishnet integrals,

$$< \phi_2^n(x_1)\phi_2^{\dagger n}(x_2)\phi_1^m(x_3)\phi_1^{\dagger m}(x_4) > =$$

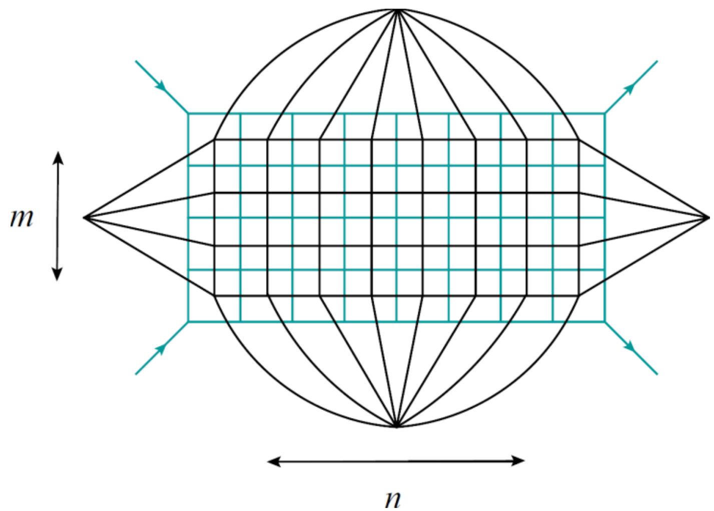
$$\propto I_{m,n}(u,v) = I_{m,n}(z,\bar{z})$$



$$u = \frac{x_{14}^2 x_{23}^2}{x_{12}^2 x_{34}^2} \equiv \frac{z\bar{z}}{(1-z)(1-\bar{z})}\,, \quad v = \frac{x_{13}^2 x_{24}^2}{x_{12}^2 x_{34}^2} \equiv \frac{u}{z\bar{z}}$$

Counting for 4-point fishnet integrals

Basso, LD, 1705.03545; Derkachov and Olivucci, 1912.07588, 2007.15049;
Basso et al., 2105.10514; talks by Basso, Stawinski



$$I_{m,n} = \frac{1}{\mathcal{N}} \det_{1 \le i,j \le m} (M_{i+j+n-m-1})$$

$$M_p \equiv p!\,(p-1)!\; L_p$$

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$(2,2)$	1	2	3	4	6	4	3	2	1										
$(2,3)$	1	2	3	4	6	8	10	10	10	7	4	2	1						
$(3,3)$	1	2	3	4	6	8	11	14	17	20	17	14	11	8	6	4	3	2	1

palindromic only for $n = m$ – **square**!

The (1,1) [box] symbol

$$\mathcal{S}[I_{1,1}] = \; z\bar{z} \otimes \frac{1-z}{1-\bar{z}} - (1-z)(1-\bar{z}) \otimes \frac{z}{\bar{z}}$$

- Clearly invariant under antipodal symmetry combined with the letter map:

$$\mathrm{M:} \quad \bar{z} \to \frac{1}{\bar{z}} \qquad\qquad 1-\bar{z} \to \frac{1}{1-\bar{z}}$$

- And overall (-1)
- Note that M is **not** a map of the underlying variables $(z, \bar{z})$
- Same true for $R_{6,7,8}^{(2)}$ symbol map Liu, 2207.11815

Square fishnet antipodal conjecture

$$S\{\mathcal{S}[I_{m,m}]\} = (-1)^m \mathrm{M}\{\mathcal{S}[I_{m,m}]\}$$

where S is the antipode and M is the letter map:

$$\bar{z} \to \frac{1}{\bar{z}} \qquad\qquad 1-\bar{z} \to \frac{1}{1-\bar{z}}$$

- Checked so far for $m = 1,\ 2,\ \ 3$ where the number of symbol terms is

$$8, \qquad 2048, \qquad 72{,}351{,}744$$

One more check*

$m = 4$ is palindromic:

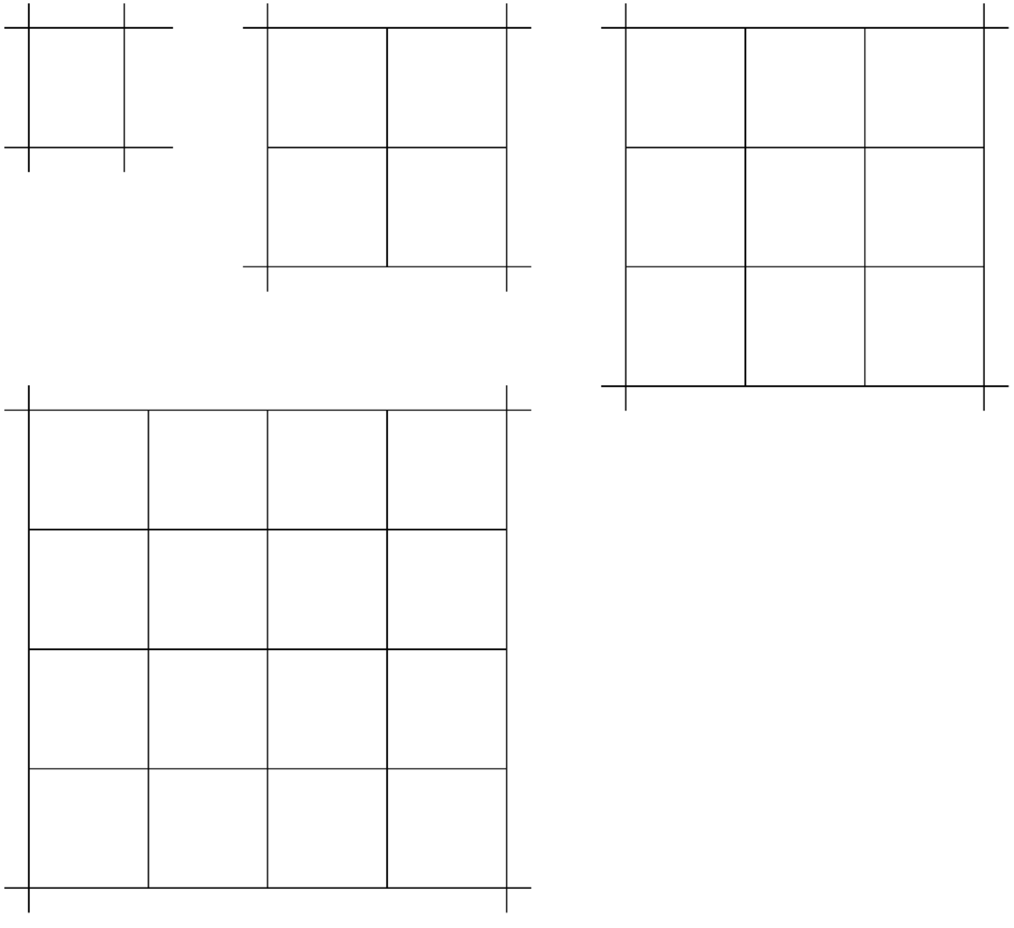
1 2 3 4 6 8 11 14 18 23 29 36 45 53 62 70 78 70 62 53 45 36 29 23 18 14 11 8 6 4 3 2 1

*at weight 32

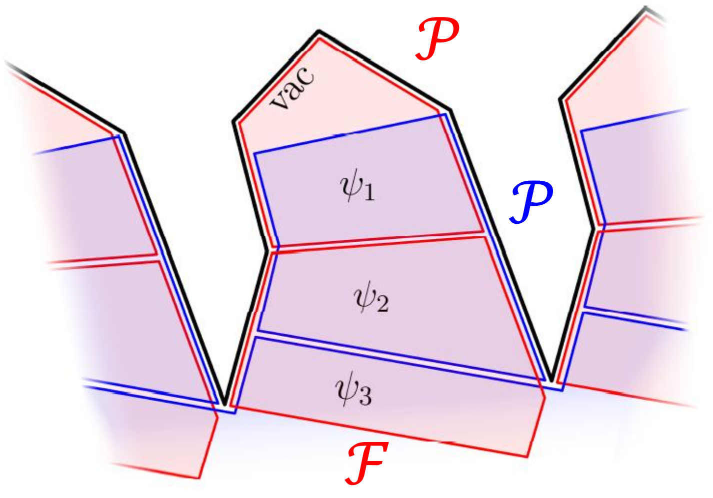
Summary & Open Questions

- 6-gluon amplitude ←→ 3-gluon form factor in planar N=4 SYM by strange new antipodal duality, swaps role of branch cuts and derivatives
- Embedded in 4-gluon form factor antipodal self-duality
- **Who ordered that?**
- Can now find at least antipodal self-"symmetry" in square 4d fishnet integrals
- 3-dimensional kinematics seems to play a crucial role in all cases (parity preserving surfaces, or only 3 momenta). Why?
- Where else might it hold? 2d fishnet integrals?
- Can we show it's true for $I_{m,m}$ for any m?
- How much more can we exploit it to learn more about amplitudes, form factors, and integrals?

Extra Slides



Form Factor OPE



- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$.
- For $\mathrm{tr}\phi^2$: **Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569**

FFOPE kinematical variables for F_4

$$u_1 = \frac{T^2 T_2^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_2 = \{1 + T^2 + \frac{S^2[(1+F_2^2)S_2 T_2 + F_2(1+S_2^2+T^2+T_2^2)]}{F_2 S_2^2}\}^{-1}$$

$$u_3 = \frac{S^2}{(T^2+1)(S^2+T^2+T_2^2+1)}$$

$$u_4 = \frac{S^2 T^2}{S_2^2} u_2$$

$$v_1 = \frac{T_2^2+1}{S^2+T^2+T_2^2+1}$$

- OPE limit takes $T, T_2 \to 0$, interpolates between 2-collinear limit $T_2 \to 0$ and 3-collinear limit $T \to 0$,

AD explains many patterns in F_3

- Every term in the symbol starts with a, b, c; never d, e, f
- Physical reason related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- Empirically, 12 pairs of adjacent letters are forbidden:

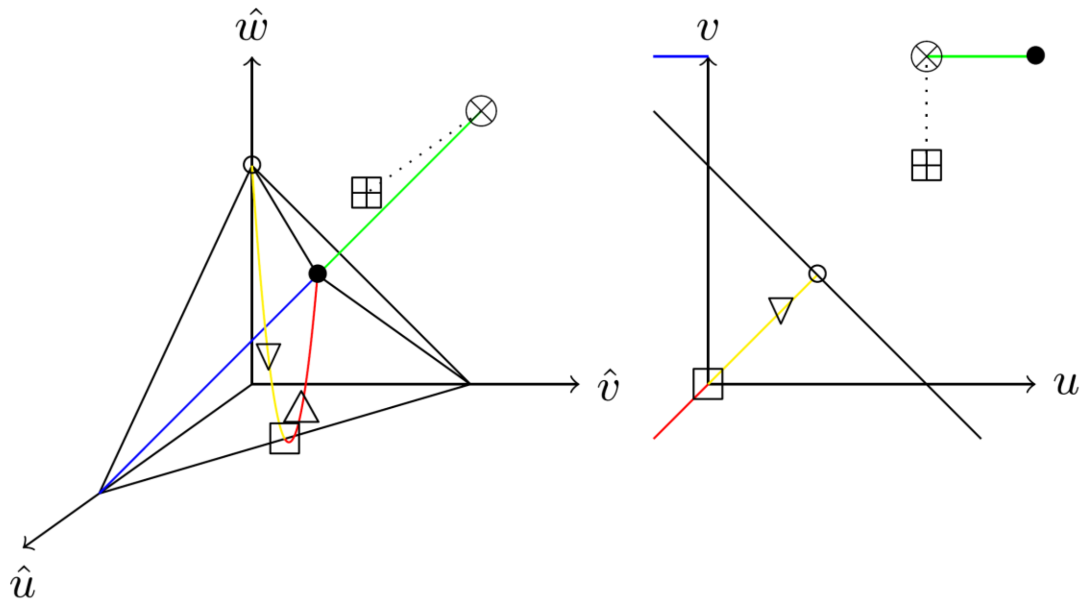
$$\ldots a \otimes d \ldots, \quad \ldots b \otimes e \ldots, \quad \ldots c \otimes f \ldots$$
$$\ldots d \otimes a \ldots, \quad \ldots e \otimes b \ldots, \quad \ldots f \otimes c \ldots$$
$$\ldots d \otimes e \ldots, \quad \ldots e \otimes f \ldots, \quad \ldots f \otimes d \ldots$$
$$\ldots e \otimes d \ldots, \quad \ldots f \otimes e \ldots, \quad \ldots d \otimes f \ldots$$

- Resemble constraints from causality: Steinmann relations Steinmann, Helv. Phys. Acta (1960)
- But not really, which mystified us for a while…
- However, the relations are antipodally dual to the (extended) Steinmann relations for A_6 !!

Many special dual points

There is an "f" alphabet at all these points: a way of writing multiple zeta values (MZV's) so that coaction is manifest.

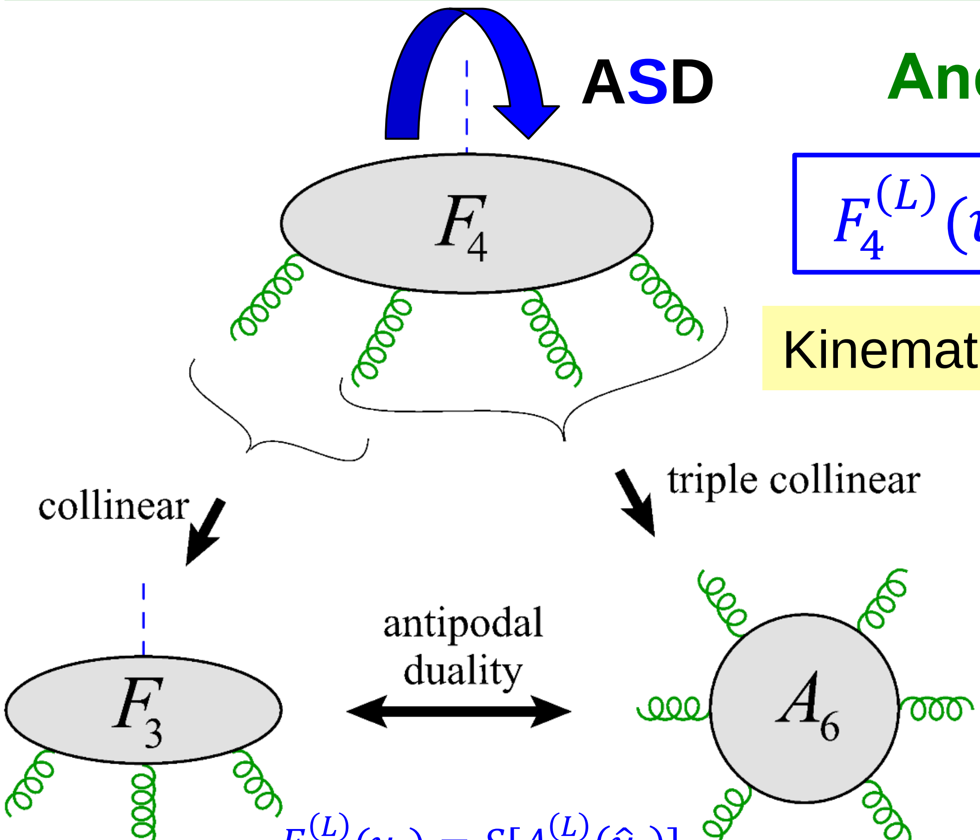
F. Brown, 1102.1310;
O. Schnetz, HyperlogProcedures



	$(\hat{u},\hat{v},\hat{w})$	(u,v,w)	functions
▽	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\sqrt[6]{1}$
□	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,1)$	$\mathrm{Li}_2(\frac{1}{2})$ + logs
●	$(1,1,1)$	$\lim_{u\to\infty}(u,u,1-2u)$	MZVs
○	$(0,0,1)$	$(\frac{1}{2},\frac{1}{2},0)$	MZVs + logs
△	$(\frac{3}{4},\frac{3}{4},\frac{1}{4})$	$(-1,-1,3)$	$\sqrt[6]{1}$
⊞	(∞,∞,∞)	$(1,1,-1)$	alternating sums
⊗	$\lim_{\hat{v}\to\infty}(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(1,v,-v)$	MZVs
green–blue line	$(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(u,v,1-u-v)$	HPL$\{0,1\}$
red–yellow line	$(\hat{u},\hat{u},(1-2\hat{u})^2)$	$(u,u,1-2u)$	HPL$\{-1,0,1\}$

Antipodal Self Duality

Given an antipodal duality relating 2-collinear and 3-collinear limits of F_4, it's natural to search for a self-duality of F_4 that holds for all parity-preserving bulk kinematics



$$F_3^{(L)}(u_i) = S[A_6^{(L)}(\hat{u}_i)]$$

And it's there!

$$F_4^{(L)}(u_i, v_i) = S[F_4^{(L)}(g(u_i), g(v_i))]$$

Kinematic map g simple in FFOPE variables:

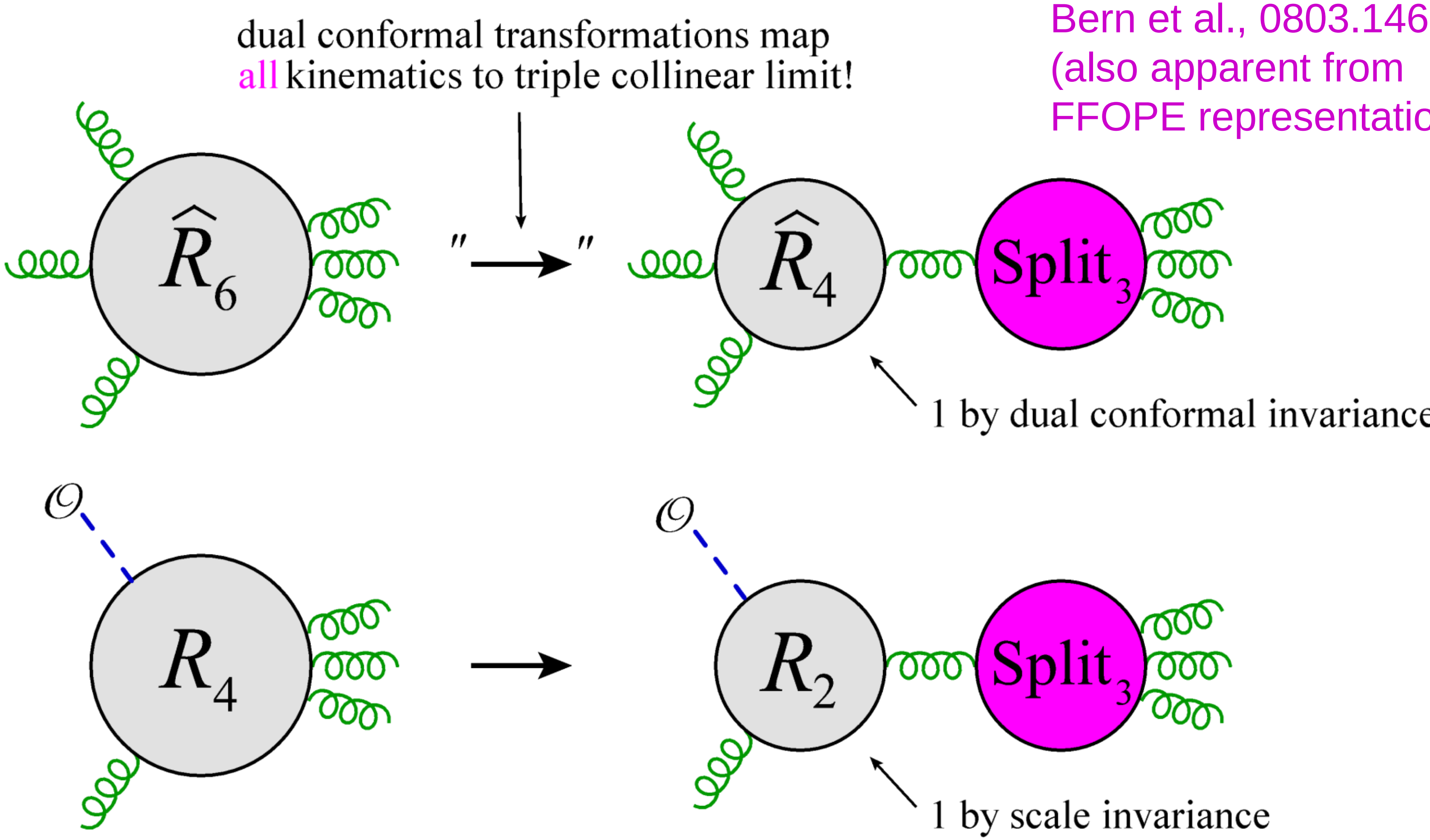
$$g: \quad T_2 \to \frac{T}{S}, \quad S_2 \to \frac{1}{TS}$$

$$T \to \sqrt{\frac{T_2}{S_2}}, \qquad S \to \sqrt{\frac{1}{T_2 S_2}}$$

$$F_2 = 1$$

Triple Collinear Limit of 4-point form factor → 6-gluon amplitude

Bern et al., 0803.1465;
(also apparent from
FFOPE representation)



ASD beyond 2 loops

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 23mm.nnnnn

- Bootstrapped symbol of F_4 at 3 loops, using same 113 letter (2-loop) alphabet.
- We again find a **unique result**, which obeys all the FFOPE predictions we could check.
- 2 loop symbol uses only 34 of the letters [3,784 terms]
- 3 loop symbol uses only 88 of the letters [3,621,202 terms]
- **ASD holds at 3 loops!**
- 4 loops in progress