Feynman Integrals and Hypergeometric Functions

Recent Results and the Mathematica Implementations

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Overview

Gauss hypergeometric function ${}_2F_1$

- **Pochhammer symbol**

$$
\begin{aligned}
(x)_n &= \frac{\Gamma(x+n)}{\Gamma(x)}, \\
&= x(x+1)\ldots(x+n-1), \qquad x \in \mathbb{C} \setminus \mathbb{Z}_0^-, n \in \mathbb{Z}_0^+ \\
(1)_n &= n!
\end{aligned}
$$

The Gauss ${}_2F_1(a,b;c;x)$:

$$
\begin{aligned}
{}_2F_1(a,b;c;x) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n \qquad , \ |x| < 1 \\
&= 1 + \frac{abx}{c} + \frac{a(a+1)b(b+1)x^2}{2c(c+1)} + O\left(x^3\right)
\end{aligned}
$$

- The $w = {}_2F_1$ hypergeometric function satisfies the ordinary differential equation (ODE)

$$
x(1-x)\frac{d^2w}{dx^2} + [c-(a+b+1)x]\frac{dw}{dx} - abw = 0
$$

- 3 singular points : $0, 1$ and ∞.

Method to find analytic continuations

The Gauss ${}_2F_1$

- Integral representation

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a} dt $$

- Analytic continuations of ${}_2F_1$: finding relations between solutions of the ODE around each singular point.
- Connecting the solutions around $x = 0$ and $x = 1$.
- Use the following relation

$$ {}_2F_1(a,b;c;x) = A\,{}_2F_1(a,b;a+b+1-c;1-x) + B(1-x)^{c-a-b}{}_2F_1(c-a,c-b;1+c-a-b;1-x). $$

- Find A and B by substituting $x = 0$ and $x = 1$
-

$$ {}_2F_1(a,b;c;x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b+1-c;1-x) + (1-x)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}{}_2F_1(c-a,c-b;1+c-a-b;1-x). $$

Linear transformations

The Gauss ${}_2F_1$

- Similarly, analytic continuation around $x = \infty$:

$$
\begin{aligned}
{}_2F_1(a, b, c; x) = (-x)^{-a} \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} {}_2F_1\left(a, a-c+1, a-b+1; \frac{1}{x}\right) \\
+ (-x)^{-b} \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} {}_2F_1\left(b, b-c+1, b-a+1; \frac{1}{x}\right)
\end{aligned}
$$

- Pfaff-Euler transformations :

$$
\begin{aligned}
{}_2F_1(a, b; c; x) &= (1-x)^{-a} {}_2F_1\left(a, c-b; c; \frac{x}{x-1}\right) \\
{}_2F_1(a, b; c; x) &= (1-x)^{-b} {}_2F_1\left(b, c-a; c; \frac{x}{x-1}\right) \\
{}_2F_1(a, b; c; x) &= (1-x)^{c-a-b} {}_2F_1(c-a, c-b; c; x)
\end{aligned}
$$

Definitions

- Appell F_2 and F_4

$$F_2(a, b_1, b_2; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n} \frac{x^m y^n}{m!n!}$$

valid for $|x| + |y| < 1$

$$F_4(a, b; c_1, c_2; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n} \frac{x^m y^n}{m!n!}$$

valid for $\sqrt{|x|} + \sqrt{|y|} < 1$

- Lauricella $F_C^{(3)}$

$$F_C^{(3)} = \sum_{n_1,n_2,n_3=0}^{\infty} \frac{(a_1)_{n_1+n_2+n_3}(a_2)_{n_1+n_2+n_3}}{(c_1)_{n_1}(c_2)_{n_3}(c_3)_{n_2}} \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{n_1!n_2!n_3!}$$

with domain of convergence : $\sqrt{|z_1|} + \sqrt{|z_2|} + \sqrt{|z_3|} < 1$

The momentum representation

- Typically involve tensor and colour structures in numerator - do tensor reduction, colour decomposition
- Calculate the scalar integrals
- Momentum representation:

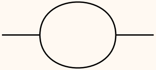
$$I_\Gamma(\nu, D) = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^{n}(-q_j^2 + m_j^2)^{\nu_j}}$$

l: number of loops
D: the space-time dimension
$\nu = (\nu_1, ..., \nu_n)$: propagator powers
k_r-s and q_j-s are the loop-momenta and internal-momenta for the Feynman graph Γ
q_j-s are combinations of external momentum and loop momentum.

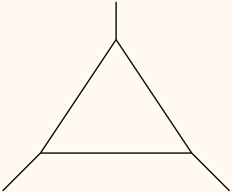
Feynman graphs/diagrams

- Tadpole :

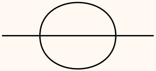
- bubble :



- 1-loop triangle :



- sunset :

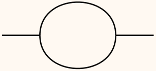


detailed study of sunset integral can be found in **BA** S. Friot, S. Ghosh '19 [1]

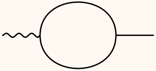
Satisfies differential equations

- These Feynman integrals satisfy differential equation
- For example

$$\frac{d}{dk^2}$$



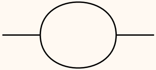
$$+\frac{1}{2}\left[\frac{1}{k^2}-\frac{(D-3)}{k^2+4m^2}\right]$$



$$+\frac{(D-2)}{4m^2}\left[\frac{1}{k^2}-\frac{1}{k^2+4m^2}\right]$$

$$=0$$

- with proper boudary condition, the solution $(x=k^2/(4m^2))$



$$=-\frac{(D-2)}{2m^2}\ .$$

$$\cdot\ {}_2F_1\left(1,2-\frac{D}{2};\frac{3}{2};-x\right)$$

- Tadpole can be expressed in terms of gamma functions

Relation to Feynman Integrals

- The dimension $d = 4 - 2\epsilon$
- One loop two-point function (B_0 function) : Anastasiou et. al. '00 [2]

$$F_4(1, \epsilon; 2-\epsilon, \epsilon; x, y), \quad F_4(\epsilon, 2\epsilon - 1; \epsilon, \epsilon; x, y), \ldots$$

  with $x = m_1^2/p^2$, $y = m_2^2/p^2$
- One loop three-point function :

$$F_2(\epsilon + 1, 1, 1; \epsilon + 1, 2 - \epsilon; x, y),$$
$$F_2(1, 1-\epsilon, 1; 1-\epsilon, 2-\epsilon; x, y), \ldots$$

  with $x = m_1^2/m_2^2$ and $y = q_1^2/m_2^2$
- The sunset integral with unequal masses : Berends et. al. '94 [3]

$$F_C^{(3)}(1, 2-\epsilon; 2-\epsilon, 2-\epsilon, 2-\epsilon; z_1, z_2, z_3),$$
$$F_C^{(3)}(1, \epsilon; 2-\epsilon, \epsilon, 2-\epsilon; z_1, z_2, z_3), \ldots$$

  with $z_1 = m_1^2/m_3^2$, $z_2 = m_2^2/m_3^2$ and $z_3 = p^2/m_3^2$

Domain of Convergences

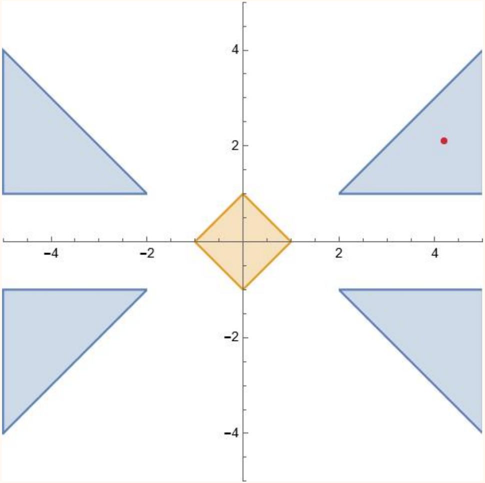


Figure: The defining domain of convergence of Appell F_2 (in orange), and of a analytic continuation of the same function that contains the red point (in blue) are plotted in real x-y plane

Mathematica packages

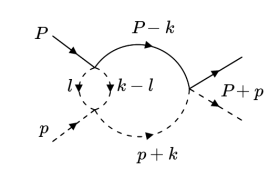
GKZ approch :: Mellin-Barnes method :: Method of Olsson :: multivariate hypergeometric functions

- `FeynGKZ.wl` : To express Feynman integrals in terms of multivariate hypergeometric functions **BA**, S. Banik, S.Bera, S. Datta, '23 [4]
- `MBConicHulls.wl` : Study of N-fold Mellin-Barnes integrals **BA**, S. Banik, S. Friot, S. Ghosh, '21 [5]
- `Olsson.wl` : Automated package to find ACs of MHFs **BA**, S. Bera, S. Friot, T. Pathak, '21 [6]
- `AppellF2.wl, AppellF1.wl, AppellF3.wl` : Study of Appell F_2 **BA**, S. Bera, S. Friot, O. Marichev, T. Pathak, '21 [7]
  S. Bera, T. Pathak, '24 [8]
- `LauricellaFD.wl, LauricellaSaranFS.wl` :
  Numerical evaluation of triple variable Lauricella Saran $F_D^{(3)}$, $F_S^{(3)}$ functions
  S. Bera, T. Pathak, '24 [8]
- `MultiHypExp.wl` : Series expansion of MHFs about their parameters S. Bera, '22, '23 [9, 10]

Mathematica packages

Algebraic relation among Feynman integrals :: Method of regions :: Chisholm approximation

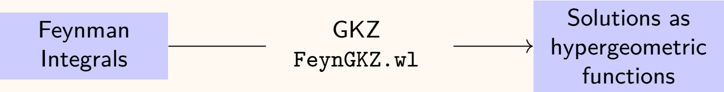
- **AlgRel.wl** : To find algebraic relations among Feynman integrals **BA**, S. Bera, T. Pathak, '23 [11]
- **ASPIRE** : New approach to MoR **BA**, A. Pal, S. Ramanan, R. Sarkar, '18 [12]
  In the context of $\pi - K$ scattering at the threshold two loop fish diagram is considered



- **Chisholm D.wl** : To find rational approximant for bi-variate series S. Bera, T. Pathak, '23 [13]

Part I

based on **BA**, S. Banik, S.Bera, S. Datta, '23 [4]


Feynman Integrals
GKZ
FeynGKZ.wl
Solutions as hypergeometric functions

Work flow

- Feynman integral in Lee-Pomeransky (LP) representation can be thought of as solution of a set of partial differential equations
- These set of PDEs are know as Gel'fand-Kapranov-Zelevinsky (GKZ) systems
- Using the GKZ approach, hypergeometric series solution of these integrals can be obtained
- There are two different approaches
- **Algebraic** : GD method $\approx$ 'generalized Frobenius method'
- **Geometrically**: the triangulation method : the triangulation of the polytope associated with the LP polynomial is considered.
- **triangulation** (in 2D) : breaking a polygon into triangles.
  Example: A rectangle could be broken in exactly two ways

The Lee-Pomeransky representation

- We saw the momentum representation of Feynman integrals
- An alternate form R. Lee, A. Pomeransky '13 [14] :

$$I_\Gamma(\nu, D) = \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2}-\omega)}\left(\prod_{i=1}^{n}\int_{\alpha_i=0}^{\infty}\frac{d\alpha_i\,\alpha_i^{\nu_i-1}}{\Gamma(\nu_i)}\right)G(\alpha)^{-\frac{D}{2}}$$
$$= \frac{\Gamma(\frac{D}{2})}{\Gamma(\frac{D}{2}-\omega)\Gamma(\nu)}\int_{\mathbb{R}_+^n} d\alpha\,\alpha^{\nu-1}G(\alpha)^{-\frac{D}{2}}$$

- Lee-Pomeransky polynomial: $G(\alpha) = U(\alpha) + F(\alpha)$.

The Lee-Pomeransky representation (contd.)

- Generalized G-polynomial:

$$G_z(\alpha) = \sum_{a_j \in A} z_j \alpha^{a_j} = \sum_{j=1}^{N} z_j \prod_{i=1}^{n} \alpha_i^{a_{ij}}$$

$z_j \to$ generic/indeterminate

- Generalized Feynman integral:

$$I_{G_z}(\nu, \nu_0) = \Gamma(\nu_0) \int_{\mathbb{R}_+^n} d\alpha\, \alpha^{\nu-1} G_z(\alpha)^{-\nu_0}$$

where, $\nu_0 = \frac{D}{2}$

Solving the GKZ system (contd.)

- Triangulate Δ_{G_z}!
- Triangulation structure: $T = \{\sigma_1, ..., \sigma_r\}$.
- $\sigma_i \subset \{1, ..., N\}$ is some index set.
- Can always obtain a regular triangulation! I.M. Gelfand, M. M. Kapranov, and A. Zelevinsky [15]
- Can always obtain a unimodular regular triangulation ($\text{vol}_0(\sigma_i) = 1$)! W. Bruns and J. Gubeladze '09 [16] Finn F. Knudsen '73 [17]

The associated GKZ system (contd.)

- Start with Feynman integral

$$I_{G_z}(\nu, \nu_0)$$

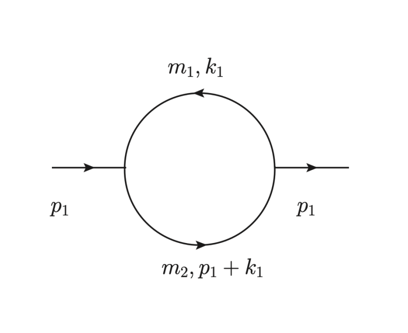
- Find its associated PDEs : $H_{\mathcal{A}}(\nu, \nu_0)$

$$\Rightarrow H_{\mathcal{A}}(\nu, \nu_0) I_{G_z}(\nu, \nu_0) = 0$$

- $I_{G_z}(\nu, \nu_0) \to$ *GKZ hypergeometric function*! L. de la Cruz '19 [18] Klausen '19 [19]
- **Algebraically**: the SST algorithm Saito, Sturmfels and Takayama [20] $\longrightarrow$ the Gröbner deformation (GD) method
- **Geometrically**: the triangulation method
- Both are equivalent! (Triangulations are in one-to-one correspondence with square free initial ideals which gives the series representations, i.e., the structure of each series is determined by square free initial ideals or the triangulations.)
- GD method $\approx$ 'generalized Frobenius method'

$$\phi_v := \sum_{u \in L} \frac{[v]_{u_-}}{[u+v]_{u_+}} z^{u+v}$$

Bubble diagram with two unequal masses



The corresponding integral in momentum-representation:

$$I_\Gamma(\nu_1, \nu_2, D; p_1^2) = \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \frac{1}{(-k_1^2 + m_1^2)^{\nu_1}(-(p_1 + k_1)^2 + m_2^2)^{\nu_2}}$$

with two unequal masses m_1 and m_2, and external momentum p_1.

Bubble diagram with two unequal masses (contd.)

After successfully loading the package and installing its dependencies, specify the integral in its momentum representation as:

```
In[3]:=   MomentumRep = {{k1, m1, a1}, {p1 + k1, m2, a2}};
          LoopMomenta = {k1};
          InvariantList = {p1^2 → −s};
          Dim = 4 − 2ϵ;
          Prefactor = 1;
```

Bubble diagram with two unequal masses (contd.)

Now derive the $\mathcal{A}$-matrix:

```
In[4]:=   FindAMatrixOut = FindAMatrix[{MomentumRep, LoopMomenta,
           InvariantList, Dim, Prefactor}, UseMB → False];
```

Prints ⇒ The Symanzik polynomials $\to \mathrm{U} = \mathrm{x}_1 + \mathrm{x}_2$

$,\mathrm{F} = \mathrm{m}_1^2\mathrm{x}_1^2 + \mathrm{s}\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_1^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_2^2$

The Lee-Pomeransky polynomial $\to \mathrm{G} =$

$\mathrm{x}_1 + \mathrm{m}_1^2\mathrm{x}_1^2 + \mathrm{x}_2 + \mathrm{s}\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_1^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_2^2$

The associated $\mathcal{A}$−matrix $\to \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{pmatrix}$, which has codim = 2.

Normalized Volume of the associated Newton Polytope → 3

Time Taken 1.50005 seconds

Bubble diagram with two unequal masses (contd.)

Compute the unimodular regular triangulations J. Rambau [21]

```
In[5]:=   Triangulations = FindTriangulations[FindAMatrixOut];
```

```
Prints ⇒  Finding all regular triangulations ...
          Found 5 Regular Triangulations, out of which 3 are Unimodular
          The 3 Unimodular Regular Triangulations →
          1 ::  {{1,2,3},{2,3,4},{3,4,5}}
          2 ::  {{1,2,3},{2,4,5},{2,3,5}}
          3 ::  {{2,4,5},{1,3,5},{1,2,5}}
          Time Taken 0.126965 seconds
```

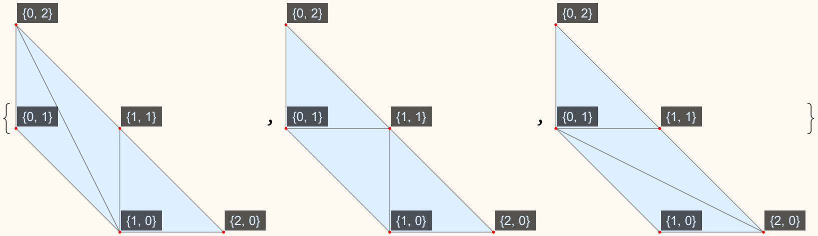


Figure: Visualization of 3 **unimodular regular triangulations**

Bubble diagram with two unequal masses (contd.)

Calculate the Γ-series:

In[7]:= SeriesSolution = SeriesRepresentation[Triangulations,2];

Prints ⇒ Unimodular Triangulation → 2

Number of summation variables → 2

Non-generic limit → $\{z_1 \to m_1^2,\ z_2 \to s + m_1^2 + m_2^2,\ z_3 \to 1,\ z_4 \to m_2^2,\ z_5 \to 1\}$

The series solution is the sum of following 3 terms.

Term 1 ::

$$\left(\left((-1)^{-n_1-n_2}\,\mathrm{Gamma}[-2+\epsilon+a_1-n_1-n_2]\,\mathrm{Gamma}[4-2\epsilon-a_1-a_2+n_2]\,\mathrm{Gamma}[a_2+2n_1+n_2]\,(m_1^2)^{2-\epsilon-a_1}\left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}\right)^{n_1}\left(\frac{m_1^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{-a_2}\right)\Big/\left(\mathrm{Gamma}[a_1]\,\mathrm{Gamma}[4-2\epsilon-a_1-a_2]\,\mathrm{Gamma}[a_2]\,\mathrm{Gamma}[1+n_1]\,\mathrm{Gamma}[1+n_2]\right)\right)$$

Term 2 ::

$$\left(\left((-1)^{-n_1-n_2}\,\mathrm{Gamma}[-2+\epsilon+a_2-n_1-n_2]\,\mathrm{Gamma}[4-2\epsilon-a_1-a_2+n_2]\,\mathrm{Gamma}[a_1+2n_1+n_2]\,(m_2^2)^{2-\epsilon-a_2}\left(\frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}\right)^{n_1}\left(\frac{m_2^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{-a_1}\right)\Big/\left(\mathrm{Gamma}[a_1]\,\mathrm{Gamma}[4-2\epsilon-a_1-a_2]\,\mathrm{Gamma}[a_2]\,\mathrm{Gamma}[1+n_1]\,\mathrm{Gamma}[1+n_2]\right)\right)$$

Term 3 ::

$$\left(\left((-1)^{-n_1-n_2}\mathrm{Gamma}[2-\epsilon-a_2+n_1-n_2]\,\mathrm{Gamma}[2-\epsilon-a_1-n_1+n_2]\,\mathrm{Gamma}[-2+\epsilon+a_1+a_2+n_1+n_2]\left(\frac{m_1^2}{s+m_1^2+m_2^2}\right)^{n_1}\left(\frac{m_2^2}{s+m_1^2+m_2^2}\right)^{n_2}(s+m_1^2+m_2^2)^{2-\epsilon-a_1-a_2}\right)\Big/\left(\mathrm{Gamma}[a_1]\,\mathrm{Gamma}[4-2\epsilon-a_1-a_2]\,\mathrm{Gamma}[a_2]\,\mathrm{Gamma}[1+n_1]\,\mathrm{Gamma}[1+n_2]\right)\right)$$

Time Taken 0.066558 seconds

Bubble diagram with two unequal masses (contd.)

Check for an expression in terms of known hypergeometric functions using `Olsson.wl` :

```
In[8]:=   GetClosedForm[SeriesSolution];
```

Prints ⇒ Closed form found with Olsson!

Term 1 ::

$$\frac{1}{\text{Gamma}[a_1]}\,\text{Gamma}[-2+\epsilon+a_1]\;\text{H3}\left[a_2, 4-2\epsilon-a_1-a_2, 3-\epsilon-a_1, \frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}, \frac{m_1^2}{s+m_1^2+m_2^2}\right]\; m_1^4\,(m_1^2)^{-\epsilon-a_1}(s+m_1^2+m_2^2)^{-a_2}$$

Term 2 ::

$$\frac{1}{\text{Gamma}[a_2]}\,\text{Gamma}[-2+\epsilon+a_2]\;\text{H3}\left[a_1, 4-2\epsilon-a_1-a_2, 3-\epsilon-a_2, \frac{m_1^2 m_2^2}{(s+m_1^2+m_2^2)^2}, \frac{m_2^2}{s+m_1^2+m_2^2}\right]\; m_2^4\,(m_2^2)^{-\epsilon-a_2}(s+m_1^2+m_2^2)^{-a_1}$$

Term 3 ::

$$\left(\left(\text{G1}\left[-2+\epsilon+a_1+a_2, 2-\epsilon-a_1, 2-\epsilon-a_2, -\frac{m_2^2}{s+m_1^2+m_2^2}, -\frac{m_1^2}{s+m_1^2+m_2^2}\right]\,\text{Gamma}[2-\epsilon-a_1]\,\text{Gamma}[2-\epsilon-a_2]\;\text{Gamma}[-2+\epsilon+a_1+a_2]\,(s+m_1^2+m_2^2)^{2-\epsilon-a_1-a_2}\right)\Big/\left(\text{Gamma}[a_1]\;\text{Gamma}[4-2\epsilon-a_1-a_2]\,\text{Gamma}[a_2]\right)\right)$$

Time Taken 0.05827 seconds

Bubble diagram with two unequal masses (contd.)

Evaluate the sum of the Γ-series terms numerically:

In[9] :=	SumLim = 30; ParameterSub = $\{\epsilon \to 0.001, a_1 \to 1, a_2 \to 1, s \to 10, m_1 \to 0.4, m_2 \to 0.3\}$; NumericalSum[SeriesSolution, ParameterSub, SumLim];
Prints ⇒	Numerical result = 997.382 Time Taken 0.222572 seconds

Summary

- Feynman integrals are solutions of GKZ hypergeometric system
- Feynman integrals can be expressed in terms of multivariate hypergeometric functions (MHFs)
- Two equivalent approaches : Gröbner deformation method and triangulation approach
- We have also studied their interconnection in [4]
- The power of propagators and the dimensional parameter appear as Pochhammer parameters
- The ratio of scales appear as variable of MHFs
- One then goes on to find ACs or series expansion of MHFs about the dimensional parameter

Part II

based on **BA**, S. Friot, S. Bera, T. Pathak, '21 [6]

The Olsson.wl package

Olsson - ROC2

(B. Ananthanarayan, S. Friot, S. Bera, T. Pathak) [6]

- ROC2.wl : an independent package that finds the region of convergence (ROC) of a double hypergeometric series, is a part of Olsson.wl
- The command Olsson takes the arguments as

```
In[1]:= Olsson[q,summation_index_List,expression,options]
```

- summation_index_List is the list of summation indices and q is an integer that can take value from 1 to Length[summation_index_List]
- The available options of Olsson.wl are

sum,one,inf,PET1,PET2,PET3,sim,roc

Commands and options of `Olsson.wl`

`sum,one,inf,PET1,PET2,PET3,sim,roc`

- The option `sum` takes the summation of the `expression` wrt `q`-th entry of the `summation_index_List`
- The option `one` performs the AC of ${}_2F_1(\ldots, z)$ around $z = 1$

$$
{}_2F_1(a, b, c; z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b, a+b-c+1; 1-z) + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b} {}_2F_1(c-a, c-b, c-a-b+1; 1-z)
$$

- The option `inf` performs the AC of ${}_2F_1(\ldots, z)$ around $z = \infty$

$$
\begin{aligned}
{}_2F_1(a, b, c; z) = & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} {}_2F_1\left(a, a-c+1, a-b+1; \frac{1}{z}\right) \\
& + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} {}_2F_1\left(b, b-c+1, b-a+1; \frac{1}{z}\right)
\end{aligned}
$$

and similar AC from ${}_pF_{p-1}(\ldots, z)$

Commands and options of `Olsson.wl`

- The options `PET1,PET2,PET3` does the Pfaff-Euler transformations

$$
{}_2F_1(a, b; c; z) = (1 - z)^{-a} {}_2F_1\left(a, c - b; c; \frac{z}{z-1}\right)
$$
$$
{}_2F_1(a, b; c; z) = (1 - z)^{-b} {}_2F_1\left(b, c - a; c; \frac{z}{z-1}\right)
$$
$$
{}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z)
$$

- `sim` option simplifies the gamma functions, Pochhammer symbols assuming that each of the summation index belongs to $\mathbb{N}_0$

- The `roc` option find the region of convergence (ROC) of the final expression, provided they are double hypergeometric functions

- This option calls the `ROC2.wl` package to find the ROC

Demonstration of `Olsson.wl`

Options `sum`, `inf`

- Storing the summand in the variable `F2`

In[2]:=
```
F2 = (Pochhammer[a,m+ n]Pochhammer[b1,m]Pochhammer[b2,n])/(Pochhammer[c1,m]Pochhammer[c2,n]) x^m/m! y^n/n!;
```

- The `sum` can be used as

In[3]:=
```
Olsson[1,{m,n}, F2 ,sum->True]
```

Out[3]=
```
(y^n HypergeometricPFQ[{b1, a + n},{c1},x] Pochhammer[a,n]Pochhammer[b2,n])/(n! Pochhammer[c2,n])
```

- The option `inf` can be used as

In[4]:=
```
Olsson[1,{m,n}, F2 ,inf->True]
```

Out[4]=
```
((-x)^-b1 y^n Gamma[c1]Gamma[a-b1+n] HypergeometricPFQ[...,1/x] ... )/(n!
Gamma[-b1+c1]Gamma[a+n]Pochhammer[c2,n])
+((-x)^-a-n y^n Gamma[c1]Gamma[-a+b1-n] HypergeometricPFQ[...,1/x] ...)/(n!
Gamma[b1]Gamma[-a+c1-n]Pochhammer[c2,n])
```

Demonstration of Olsson.wl

Options **sim**, **roc**

- The output can simplified using **sim** option

In[5]:= `Olsson[1,{m,n}, F2 ,inf→True, sim→True]`

Out[5]=

$$\frac{(-x)^{-a-n} x^{-m} y^{n} \text{Gamma}[-a+b1]\text{Gamma}[c1]\text{Pochhammer}[a,m+n]\text{Pochhammer}[b2,n]\text{Pochhammer}[1+a-c1,m+n]}{m!n!\text{Gamma}[b1]\text{Gamma}[-a+c1]\text{Pochhammer}[1+a-b1,m+n]\text{Pochhammer}[c2,n]} + \frac{(-1)^{n}\ (-x)^{-b1} x^{-m}\ y^{n} \text{Gamma}[a-b1]\text{Gamma}[c1]\text{Pochhammer}[b1,m]\text{Pochhammer}[b2,n]\ \text{Pochhammer}[1+b1-c1,m]}{m!n!\text{Gamma}[a]\text{Gamma}[-b1+c1]\text{Pochhammer}[1-a+b1,m-n]\text{Pochhammer}[c2,n]}$$

- We have obtained the AC of F_2 around $(\infty, 0)$

- The associated ROC can be found using **roc** option

In[6]:= `Olsson[1,{m,n}, F2 ,inf→True, sim→True, roc→True]`

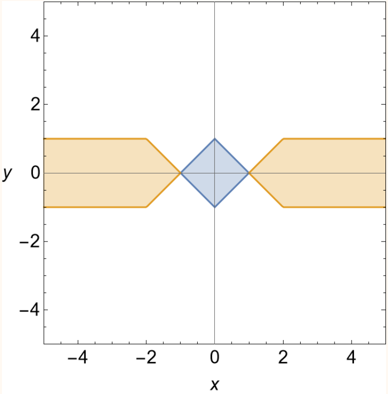
Out[6]=

$$\{\frac{1}{\text{Abs}[x]}<1\&\&\text{Abs}[\frac{y}{x}]<1\&\&\text{Abs}[\frac{y}{x}]<1-\frac{1}{\text{Abs}[x]}$$
$$\&\&\frac{1}{\text{Abs}[x]}<1\&\&\text{Abs}[y]<1\&\&\text{Abs}[y]<-1+\text{Abs}[x],\ldots\}$$

Demonstration of `Olsson.wl`

ROC

- The ROCs plotted in real $x - y$ plane :



- The other options `one, PET1, PET2, PET3` work similarly.
- repetitive use of these options can be made to find new ACs.

- the resulting series can be recognized using the serrecog or serrecog2var command

```
In[7]:= Plus@@(serrecog2var[{m,n},#]&/@(List@@Last[%6]))
Out[7]=
```

$$\frac{(-x)^{-b1}\ \text{Gamma}[a-b1]\ \text{Gamma}[c1]}{\text{Gamma}[a]\text{Gamma}[-b1+c1]}\ \text{FTilde}[\{\ldots\},\{\frac{1}{x},-y\}] + \frac{(-x)^{-a}\ \text{Gamma}[-a+b1]\ \text{Gamma}[c1]}{\text{Gamma}[b1]\text{Gamma}[-a+c1]}\text{KdF}[\{\ldots\},\{\frac{1}{x},-\frac{y}{x}\}]$$

- we recover the well-known analytic continuation of Appell F_2.

- The serrecog2var command can recognize all 14 Appell-Horn series in two variables.

- The serrecog command can recognize bi-variate KdF, mirror-KdF (FTilde), Lauricella functions in any number of variables.

Physics Applications

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The one-loop pentagon to higher orders in ϵ

Vittorio Del Duca,[a] Claude Duhr,[b] E. W. Nigel Glover[c] and Vladimir A. Smirnov[d]

Physics Applications

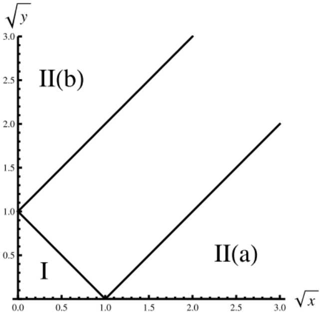


Figure 1. The three regions contributing to the scalar massless pentagon in Euclidean kinematics.

Physics Applications

$$
\begin{aligned}
&\mathcal{I}_{\mathrm{ND}}^{(IIa)}(s,s_1,s_2,t_1,t_2)\\
&\quad= -\frac{1}{\epsilon^3}\,y_2^{-\epsilon}\,\Gamma(1-2\epsilon)\,\Gamma(1+\epsilon)^2\,F_4\Big(1-2\epsilon,1-\epsilon,1-\epsilon,1-\epsilon;-y_1,y_2\Big)\\
&\qquad+\frac{1}{\epsilon^3}\,\Gamma(1+\epsilon)\,\Gamma(1-\epsilon)\,F_4\Big(1,1-\epsilon,1-\epsilon,1+\epsilon;-y_1,y_2\Big)\\
&\qquad-\frac{1}{\epsilon^2}\,y_1^{\epsilon}\,y_2^{-\epsilon}\Bigg\{\big[\ln y_1+\psi(1-\epsilon)-\psi(-\epsilon)\big]\,F_4\big(1,1-\epsilon,1+\epsilon,1-\epsilon;-y_1,y_2\big)\\
&\qquad+\frac{\partial}{\partial\delta}\,F_{0,2}^{2,1}\left(\begin{matrix}1+\delta & 1+\delta-\epsilon\\ - & -\end{matrix}\middle|\begin{matrix}1 & - & - & -\\ 1+\delta & 1-\epsilon & 1+\epsilon+\delta & -\end{matrix}\middle|-y_1,y_2\right)_{|\delta=0}\Bigg\}\\
&\qquad+\frac{1}{\epsilon^2}\,y_1^{\epsilon}\Bigg\{\big[\ln y_1+\psi(1+\epsilon)-\psi(-\epsilon)\big]\,F_4\big(1,1+\epsilon,1+\epsilon,1+\epsilon;-y_1,y_2\big)\\
&\qquad+\frac{\partial}{\partial\delta}\,F_{0,2}^{2,1}\left(\begin{matrix}1+\delta & 1+\delta+\epsilon\\ - & -\end{matrix}\middle|\begin{matrix}1 & - & - & -\\ 1+\delta & 1+\epsilon & 1+\epsilon+\delta & -\end{matrix}\middle|-y_1,y_2\right)_{|\delta=0}\Bigg\}.
\end{aligned}
\tag{5.14}
$$

Physical applications

- We find analytic continuations of Appell F_4 and Kampé de Fériet functions that covers the positive quadrant in (x, y) plane
- The ananlytic continuations suggest that, the region I should be devided in to two parts $I(a)$ and $I(b)$
- The ACs of multivariate hypergeometric functions can be derived using `Olsson.wl`
- The ROC of double hypergeometric functions only can be found
- The expressions obtained in this way are error-free.
- The ACs are derived in no time

Part III

based on **BA**, S. Bera, S. Friot, O. Marichev and T. Pathak, '21 [7]

The AppellF2.wl package

Usage and demonstration

AppellF2

(B. Ananthanarayan, S. Bera, S. Friot, O. Marichev and T. Pathak [7])

- It can find the value of F_2 for **generic complex values of Pochhammer parameters** and arbitrary **real values of x, y except the points on the singular lines**
- Usage :

```
In[8]:= AppellF2[a,b1,b2,c1,c2,x,y,precision,terms,F2show→ True]
```

- For example,

```
In[9]:= AppellF2[2.2345,3.363,0.242,8.3452,0.657,-2.311,5.322,
        10,100,F2show→ True]

Out[9]=
        0.09333639793-0.06847416686 I
```

- Other commands

F2findall,F2expose,F2ROC,F2evaluate

Challenges in numerical evaluation

- We found a total of 44 ACs for Appell F_2
- All the ACs should obey the cut structures of F_2
- The cut of F_2 lies from 1 to ∞ along the real axis for each of the variables

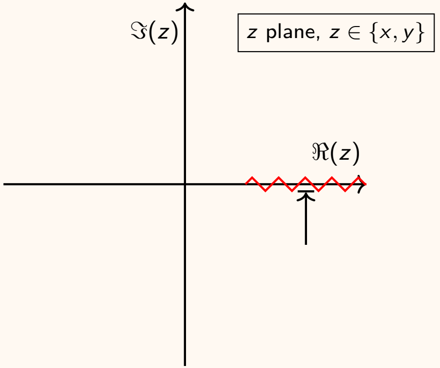


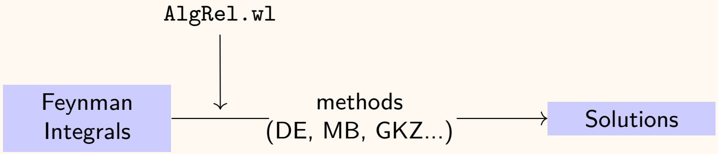
Figure: The red wiggly line denotes the cut of Appell F_2 and the arrow indicates the path of approach when the function is evaluated on the cut

- The value of F_2 on the cut is evaluated with '$-i\epsilon$ prescription'

For x-cut, $$F_2[\ldots, x, y] = \lim_{\epsilon \to 0^+} F_2[\ldots, x - i\epsilon, y]$$

Part IV

based on **BA**, S. Bera, T. Pathak, '23 [11]


AlgRel.wl
Feynman Integrals
methods
(DE, MB, GKZ...)
Solutions

Algebraic relations among Feynman integrals

- Motivated by the works of O. Tarasov O. Tarasov '22 and references within [22]
- Integral with general propagators :

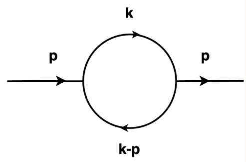
$$I_2((q_1 - q_2)^2, m_1, m_2) = \int \frac{d^d k}{d_1 d_2}$$

where

$$d_i = (k + q_i)^2 - m_i^2$$

- k : loop-momentum, q_i : combination of external momentum, m_i : mass of the propagator
- When $q_1 = 0$ and $q_2 = -p \longrightarrow$ one-loop bubble integral

$$I_2(p^2, m_1, m_2) = \int \frac{d^d k}{(k^2 - m_1^2)((k-p)^2 - m_2^2)}$$



Algebraic relations among Feynman integrals

- Partial fraction

$$\frac{1}{d_1 d_2} = \frac{x_1}{D_1 d_1} + \frac{x_2}{D_1 d_2}$$

where $D_i = (k + P_i)^2 - M_i^2$

-

$$D_1 = x_1 d_2 + x_2 d_1$$

- Comparing the coefficients of k^2, k and k^0

$$\begin{aligned} & x_1 + x_2 = 1 \\ & x_1 \mathbf{q_2} + x_2 \mathbf{q_1} = \mathbf{P_1} \\ & -M_1^2 + P_1^2 - (-m_2^2 + q_2^2)x_1 - (-m_1^2 + q_1^2)x_2 = 0 \end{aligned}$$

- Solve for the unknowns

$$x_1, x_2, \mathbf{P_1}$$

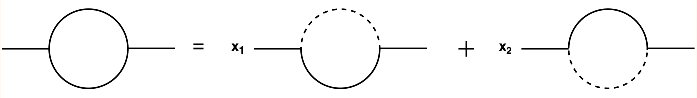
- The parameter M_1 is free to choose. We choose $M_1 = 0$

Algebraic relations among Feynman integrals

- for one-loop bubble integral

$$I_2(p^2, m_1, m_2) = x_1 I_2((P_1 + p)^2, 0, m_2) + x_2 I_2(P_1^2, m_1, 0)$$

- diagrammatically



- The general result for the massive bubble diagram can be written in terms of the Appell F_4 function I. Gonzalez and V. H. Moll [23]
-

$$I_2(p, m_1, m_2) = \frac{(m_2^2)^{\frac{d}{2}-2}\Gamma(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})}{\Gamma(\frac{d}{2})} F_4\left(2-\frac{d}{2}, 1; \frac{d}{2}, 2-\frac{d}{2}; \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}\right)$$
$$+ \frac{(m_1^2)^{\frac{d}{2}-1}\Gamma(1-\frac{d}{2})}{m_2^2} F_4\left(\frac{d}{2}, 1; \frac{d}{2}, \frac{d}{2}; \frac{p^2}{m_2^2}, \frac{m_1^2}{m_2^2}\right)$$

Algebraic relations among Feynman integrals

- Appell F_4

$$F_4(a,b,c,d,x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n} \frac{x^m y^n}{m!n!}$$

  valid for $\sqrt{|x|} + \sqrt{|y|} < 1$
- The analytic expression result for $I_2(p,m,0)$ is C.G. Bollini, J.J. Giambiagi. '72 [24] E. E. Boos, A. I. Davydychev '91 [25]

$$I_2^{(d)}(p^2;m^2,0) = -\Gamma(1-\frac{d}{2}) m_2^{d-4} \, {}_2F_1\left[\begin{array}{ll} 1, 2-\frac{d}{2}; & \frac{p^2}{m^2} \\ \frac{d}{2}; & \end{array}\right]$$

- We found a reduction formula $F_4 \to {}_2F_1$
- Problem of finding analytic continuations of $F_4 \longrightarrow$ Problem of finding analytic continuations of ${}_2F_1$
- Reduction in the ratios of the original Feynman integral
- Reduction of computational complexity as we have to evaluate integrals with less massive propagators

Reduction formula of hypergeometric functions

- Reduction formulas for multi-variable hypergeometric function

$$F_4(1,1;1,1;x,y) = \frac{1}{\sqrt{(x+y-1)^2-4xy}}$$

$$F_4\left(\frac{3}{2},1;\frac{1}{2},\frac{3}{2};x,y\right) = \frac{x-y+1}{x^2-2x(y+1)+(y-1)^2}$$

$$F_4\left(\frac{5}{2},1;-\frac{1}{2},\frac{5}{2};x,y\right) = \frac{(x-y+1)\left(x^2-2x(y+5)+(y-1)^2\right)}{\left(x^2-2x(y+1)+(y-1)^2\right)^2}$$

$$F_4\left(\frac{1}{2},1;\frac{3}{2},\frac{1}{2};x,y\right) = \frac{\tanh^{-1}\left(\frac{-\sqrt{-2(x+1)y+(x-1)^2+y^2}+x-y+1}{2\sqrt{x}}\right)}{\sqrt{x}}$$

Integrals with more propagators

- What about product such as $\frac{1}{d_1 d_2 d_3}$?
- $$\begin{aligned} \frac{1}{d_1 d_2 d_3} &= \frac{x_1}{D_1 d_1 d_3} + \frac{x_2}{D_1 d_2 d_3} \\ &= \frac{x_1 x_3}{D_1 D_2 d_1} + \frac{x_1 x_4}{D_1 D_2 d_3} + \frac{x_2 x_5}{D_1 D_3 d_2} + \frac{x_2 x_6}{D_1 D_3 d_3} \end{aligned}$$
- In a similar manner we can use this recursively for product of N-propagators depending only on one loop momenta.
- The final result is a sum of 2^{N-1} terms where N is the total number of denominators we started with

AlgRel.wl

In[10]:=

```
<<AlgRel.wl

AlgRel.wl v1.0
Authors : B. Ananthanarayan, Souvik Bera, Tanay Pathak
```

In[11]:=

```
AlgRel[{Propagator's number},{k,q,m},{P,M},x,Substitutions]
```

Out[11]=

```
{{Algebraic relation},{Values}}
```

Consider the example of Bubble integral. To obtain the result for it we can use the following command

In[12]:=

```
AlgRel[{1, 2},{k,q,m},{P, M}, x,{q[1]-> 0,q[2]->-p,M[1]->0}]
```

Out[12]=

$$\{\{\frac{x[1]}{((k+P[1])^2)(-m[1]^2+(k)^2)}+\frac{x[2]}{((k+P[1])^2)(-m[2]^2+(k-p)^2)}\},$$

$$\{x[1]\text{->}\frac{p^2+m[1]^2-m[2]^2+\sqrt{(p^2+m[1]^2-m[2]^2)^2-4p^2(m[1]^2)}}{p^2},\ldots\}\}$$

Summary

- Reduction in complexity of the original integral by reducing it to a sum of simpler integrals
- We can always convert a general N-point, 1-loop massive integral, into a sum of integrals with just 1 massive propagator.
- We also developed a suitably modified recursive algorithm for implementation in MATHEMATICA : `AlgRel.wl`
- Obtaining non-trivial and elusive reduction formulas for the multi-variable hypergeometric functions.

Future directions

- Dispersion relation and Feynman integrals
- Hodge structure of Feynman integrals
- Relation with algebraic geometry, number theory, combinatorics
- Iterated Chen Integrals
- Theory of differential equations
- Theory of chords
- ...

References I

[1] B. Ananthanarayan, Samuel Friot, and Shayan Ghosh. *Eur. Phys. J. C* 80.7 (2020), p. 606. arXiv: 1911.10096 [hep-ph].

[2] C. Anastasiou, E. W. Nigel Glover, and C. Oleari. *Nucl. Phys. B* 572 (2000), pp. 307–360. arXiv: hep-ph/9907494.

[3] Frits A. Berends et al. *Z. Phys. C* 63 (1994), pp. 227–234.

[4] B. Ananthanarayan et al. *Comput. Phys. Commun.* 287 (2023), p. 108699. arXiv: 2211.01285 [hep-th].

[5] B. Ananthanarayan et al. *Phys. Rev. Lett.* 127.15 (2021), p. 151601. arXiv: 2012.15108 [hep-th].

[6] B. Ananthanarayan et al. *accepted for publication in CPC* (Dec. 2021). arXiv: 2201.01189 [cs.MS].

[7] B. Ananthanarayan et al. *Comput. Phys. Commun.* 284 (2023), p. 108589. arXiv: 2111.05798 [math.CA].

[8] Souvik Bera and Tanay Pathak. (Mar. 2024). arXiv: 2403.02237 [hep-ph].

[9] Souvik Bera. *Nucl. Phys. B* 989 (2023), p. 116145. arXiv: 2208.01000 [math-ph].

References II

[10] Souvik Bera. *Comput. Phys. Commun.* 297 (2024), p. 109060. arXiv: 2306.11718 [hep-th].

[11] B. Ananthanarayan, Souvik Bera, and Tanay Pathak. *Nucl. Phys. B* 995 (2023), p. 116345. arXiv: 2307.04852 [hep-ph].

[12] B. Ananthanarayan et al. *Eur. Phys. J. C* 79.1 (2019), p. 57. arXiv: 1810.06270 [hep-ph].

[13] Souvik Bera and Tanay Pathak. (Sept. 2023). arXiv: 2309.07687 [cs.MS].

[14] Roman N. Lee and Andrei A. Pomeransky. *JHEP* 11 (2013), p. 165. arXiv: 1308.6676 [hep-ph].

[15] Israel M. Gelfand, Mikhail M. Kapranov, and Andrei Zelevinsky. “Hypergeometric functions, Toric varieties and Newton polyhehra”. 1991.

[16] W. Bruns and J. Gubeladze. *Polytopes, Rings, and K-Theory*. Springer Monographs in Mathematics. Springer New York, 2009. ISBN: 9780387763569. URL: https://books.google.co.in/books?id=pbgg1pFxW8YC.

References III

[17] Finn F. Knudsen. "Construction of nice polyhedral subdivisions". *Toroidal Embeddings I*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 109–164. ISBN: 978-3-540-37755-9. URL: https://doi.org/10.1007/BFb0070321.

[18] Leonardo de la Cruz. *JHEP* 12 (2019), p. 123. arXiv: 1907.00507 [math-ph].

[19] René Pascal Klausen. *JHEP* 04 (2020), p. 121. arXiv: 1910.08651 [hep-th].

[20] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama. *Gröbner deformations of hypergeometric differential equations*. Vol. 6. Springer Science & Business Media, 2013.

[21] Jörg Rambau. "TOPCOM: Triangulations of Point Configurations and Oriented Matroids". *Proceedings of the International Congress of Mathematical Software*. 2002. URL: http://www.zib.de/PaperWeb/abstracts/ZR-02-17.

[22] O. V. Tarasov. *JHEP* 06 (2022), p. 155. arXiv: 2203.00143 [hep-ph].

References IV

[23] Ivan Gonzalez and Victor H. Moll. *Advances in Applied Mathematics* 45.1 (2010), pp. 50–73. ISSN: 0196-8858. URL: https://www.sciencedirect.com/science/article/pii/S0196885809001225.

[24] CG Bollini and JJ Giambiagi. *Physics Letters B* 40.5 (1972), pp. 566–568.

[25] E. E. Boos and Andrei I. Davydychev. *Theor. Math. Phys.* 89 (1991), pp. 1052–1063.

Back up slides

Solving the GKZ system (contd.)

Triangulation method

- We saw:

$$\mathcal{A} = \begin{pmatrix} 1 \\ A \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_N \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{(n+1)\times N}$$

- A defines an assembly of N points (a point configuration) in $\mathbb{Z}^n$

$$\mathrm{Conv}(A) := \left\{ \sum_{j=1}^{N} k_j a_j \middle| k \in \mathbb{R}_{\geq 0}^{N}, \sum_{j=1}^{N} k_j = 1 \right\}$$

- Newton polytope of $G_z(\alpha)$:

$$\Delta_{G_z} := \mathrm{Conv}(A)$$

Solving the GKZ system (contd.)

- Regular triangulations can be used to construct a basis for the finite-dimensional solution space of $H_{\mathcal{A}}(\underline{\nu})$
- Each element: Γ-series
- Whole solution: linear combination of the Γ-series elements
- Unimodularity: one $\sigma_i \rightarrow$ one Γ-series
- Might as well use just the unimodular regular triangulations to construct a basis!

Feynman Integral

- Momentum representation:

$$I_\Gamma(\nu, D) = \int \prod_{r=1}^{l} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \frac{1}{\prod_{j=1}^{n}(-q_j^2 + m_j^2)^{\nu_j}}$$

l: number of loops; D: the space-time dimension; $\nu = (\nu_1, ..., \nu_n)$: propagator powers
k_r-s and q_j-s are the loop-momenta and internal-momenta for the Feynman graph Γ.

- Lee-Pomeransky representation:

$$I_\Gamma(\nu, D) = \frac{\Gamma(\frac{D}{2})}{\Gamma\left(\frac{(l+1)D}{2} - \sum_i \nu_i\right)} \left(\prod_{i=1}^{n} \int_{\alpha_i=0}^{\infty} \frac{d\alpha_i\, \alpha_i^{\nu_i - 1}}{\Gamma(\nu_i)}\right) G(\alpha)^{-\frac{D}{2}}$$

- Lee-Pomeransky polynomial: $G(\alpha) = U(\alpha) + F(\alpha)$.

The Lee-Pomeransky representation (contd.)

- Generalized Feynman integral:

$$I_{G_z}(\nu, \nu_0) = \int_{\mathbb{R}^n_+} d\alpha\, \alpha^{\nu-1} G_z(\alpha)^{-\nu_0}$$

  where, $\nu_0 = \frac{D}{2}$

- Generalized G-polynomial:

$$G_z(\alpha) = \sum_{a_j \in A} z_j \alpha^{a_j}$$

  $z_j \rightarrow$ generic/indeterminate

- Construct

$$\mathcal{A} = \begin{pmatrix} 1 \\ A \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_N \end{pmatrix}$$

The associated GKZ system (contd.)

- We describe the Gel'fand-Kapranov-Zelevinsky (GKZ) system as follows:

$$H_{\mathcal{A}}(\underline{\nu}) = I_{\mathcal{A}} \cup \langle \mathcal{A} \cdot \theta + \underline{\nu} \rangle$$

  where

$$\mathcal{A} = \{a_{ij}; i \in \{1, ..., n+1\}, j \in \{1, ..., N\}\} | a_{ij} = 1; i = 1\}$$
$$\underline{\nu} = (\nu_0, \nu_1, ..., \nu_n)^T$$

- $\mathcal{A} \to (n+1) \times N$ matrix; $n+1 \leq N$
- $\theta = (\theta_1, \ldots, \theta_N)^T$; $\theta_i = z_i \partial_i \to$ Euler operators
- **Assume**: (1,...,1) lies in $\mathbb{Q}$-row span of $\mathcal{A}$

In layman's terms

- Start with Feynman integral

$$I_{G_z}(\nu, \nu_0)$$

- Find its associated PDEs : $H_{\mathcal{A}}(\nu, \nu_0)$

$$\Rightarrow H_{\mathcal{A}}(\nu, \nu_0) I_{G_z}(\nu, \nu_0) = 0$$

- $H_{\mathcal{A}}(\nu, \nu_0)$ is called Gel'fand-Kapranov-Zelevinsky (GKZ) system or $\mathcal{A}$-hypergeometric system
- Solve the PDEs:
  - Algebraic way : Gröbner deformation method (GD) (Saito, Sturmfels and Takayama [20], de la Cruz [18])
  - Geometric way : Triangulation method (Klausen [19])
- Both are equivalent
- GD ≈ 'generalized Frobenius method'

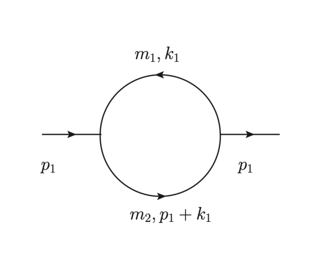
$$\phi_v := \sum_{u \in L} \frac{[v]_{u_-}}{[u+v]_{u_+}} z^{u+v}$$

Bubble diagram with two unequal masses (contd.)

FeynGKZ

(B. Ananthanarayan, S. Banik, S. Bera, S. Datta [4])

Load the package
and its dependencies




```
In[3]:=    MomentumRep = {{k1, m1, a1}, {p1 + k1, m2, a2}};
           LoopMomenta = {k1};
           InvariantList = {p1^2 -> -s};
           Dim = 4 - 2ϵ;
           Prefactor = 1;
```

Bubble diagram with two unequal masses (contd.)

Now derive the $\mathcal{A}$-matrix:

```
In[4]:=    FindAMatrixOut = FindAMatrix[{MomentumRep, LoopMomenta,
            InvariantList, Dim, Prefactor}, UseMB → False];
```

Prints ⇒ The Symanzik polynomials $\rightarrow \mathrm{U} = \mathrm{x}_1 + \mathrm{x}_2$

$, \mathrm{F} = \mathrm{m}_1^2\mathrm{x}_1^2 + \mathrm{s}\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_1^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_2^2$

The Lee-Pomeransky polynomial $\rightarrow \mathrm{G} =$

$\mathrm{x}_1 + \mathrm{m}_1^2\mathrm{x}_1^2 + \mathrm{x}_2 + \mathrm{s}\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_1^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_1\mathrm{x}_2 + \mathrm{m}_2^2\mathrm{x}_2^2$

The associated $\mathcal{A}$−matrix $\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 \end{pmatrix}$, which has codim $= 2$.

Normalized Volume of the associated Newton Polytope $\rightarrow 3$

Time Taken 1.50005 seconds

Bubble diagram with two unequal masses (contd.)

Compute the unimodular regular triangulations J. Rambau [21]

```
In[5]:=   Triangulations = FindTriangulations[FindAMatrixOut];
```

```
Prints ⇒  Finding all regular triangulations ...
          Found 5 Regular Triangulations, out of which 3 are Unimodular
          The 3 Unimodular Regular Triangulations →
          1 :: {{1,2,3},{2,3,4},{3,4,5}}
          2 :: {{1,2,3},{2,4,5},{2,3,5}}
          3 :: {{2,4,5},{1,3,5},{1,2,5}}
          Time Taken 0.126965 seconds
```

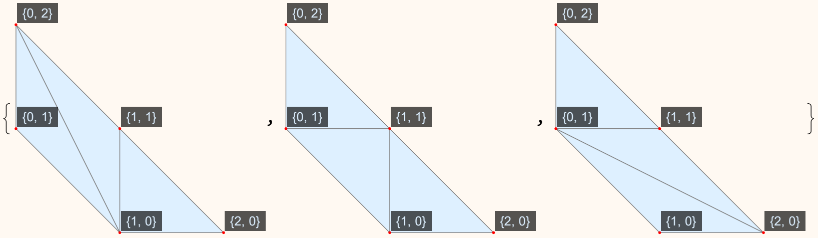


Figure: Visualization of 3 **unimodular regular triangulations**

Bubble diagram with two unequal masses (contd.)

Evaluate the sum of the Γ-series terms numerically:

```
In[9]:=    SumLim = 30;
           ParameterSub = {ϵ → 0.001, a1 → 1, a2 → 1, s → 10, m1 → 0.4, m2 → 0.3};
           NumericalSum[SeriesSolution, ParameterSub, SumLim];
------------------------------------------------------------------------
Prints ⇒   Numerical result = 997.382
           Time Taken 0.222572 seconds
```