Recursive Structure of Four-Point Fishnet Integrals

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Based on 2408.15331 with Florian Loebbert

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bctp Bethe Center for Theoretical Physics

Relations between $D = 2$ and $D > 2$?

CFT in $D = 2$ simpler than $D > 2$

Even global!

Basic reason: (anti-)holomorphic variables

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2$$

$\longrightarrow$ Factorization! $\quad \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{R}) \oplus \overline{\mathfrak{sl}(2,\mathbb{R})}$

$\longrightarrow$ Factorization of conformal blocks etc.

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$\longrightarrow$ Factorization of conformal blocks etc.

Can this carry over to $D > 2$?

Starting point: four-point functions

Conformal four-point kinematics

$$\frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2} = z\bar{z}, \quad \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{23}^2} = (1-z)(1-\bar{z})$$

$$x_1, x_2, x_3, x_4 \longrightarrow z, \bar{z}$$

Same in $D = 2$ and $D > 2$!

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Same in $D = 2$ and $D > 2$!

c.f. five-point kinematics

$$x_1, x_2, x_3, x_4, x_5 \longrightarrow z_1, \bar{z}_1, \quad z_2, \bar{z}_2, \quad w$$

Planes Angle

[Buric, Lacroix, Mann, Quintavalle, Schomerus '21]

In $D = 2$: $w = 0$ or $w = 1$!

Fishnet CFT

$$\mathcal{N}=4\ \text{SYM} \xrightarrow{\gamma\text{-deformation}} \gamma\text{-deformed }\mathcal{N}=4\ \text{SYM} \xrightarrow{\text{double-scaling limit}} \text{Fishnet CFT}$$

$$\mathscr{L} = N_c \,\mathrm{tr}\left[\phi_1^\dagger(-\partial^2)\phi_1 + \phi_2^\dagger(-\partial^2)\phi_2 + (4\pi)^2\xi^2\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2\right]$$

[Gürdogan, Kazakov '15]

Deformation & Generalization

$$\mathscr{L}_{D,\gamma} = N_c \,\mathrm{tr}\left[\phi_1^\dagger(-\partial^2)^\gamma\phi_1 + \phi_2^\dagger(-\partial^2)^{D/2-\gamma}\phi_2 + (4\pi)^2\xi^2\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2\right]$$

[Kazakov, Olivucci '18]

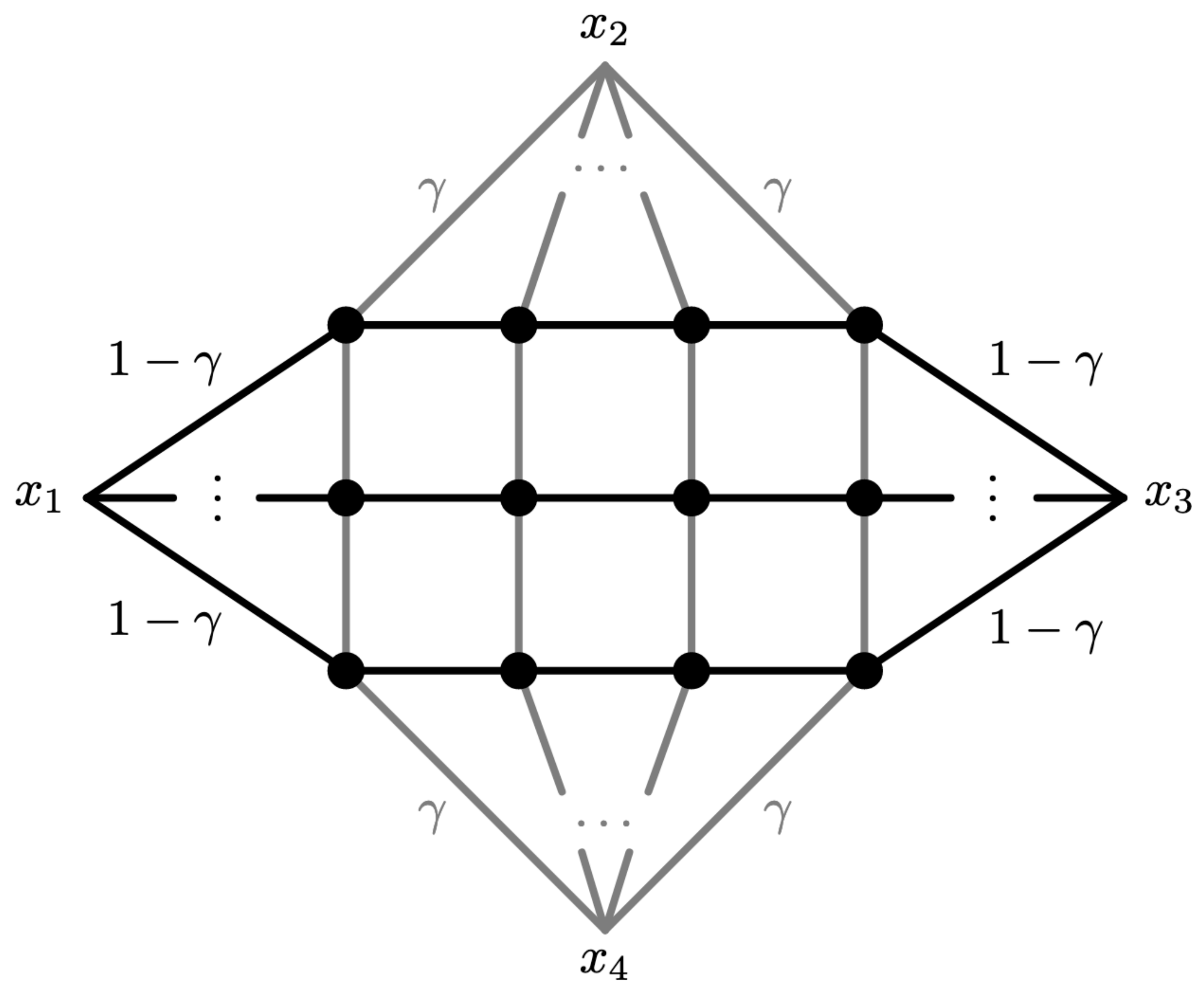
$\gamma \in (0, D/2)$

Integrability/Yangian symmetry $\longrightarrow$ Simplicity

Correlation functions given by fishnet integral $\longrightarrow$ Direct implications for Feynman integrals!

Impressive Results at Four points

In 2D deformed theory: $\left\langle \left(\phi_2(x_1)^\dagger\right)^M \left(\phi_1(x_2)^\dagger\right)^N \left(\phi_2(x_3)\right)^M \left(\phi_1(x_4)\right)^N \right\rangle$



$$= \det_{1\le i,j\le M}\left(\theta^{i-1}\bar{\theta}^{j-1}\phi_{2;\gamma}^{(M+N-1)}(z,\bar{z})\right)$$

$z\partial_z$

[Derkachov, Kazakov, Olivucci '18]

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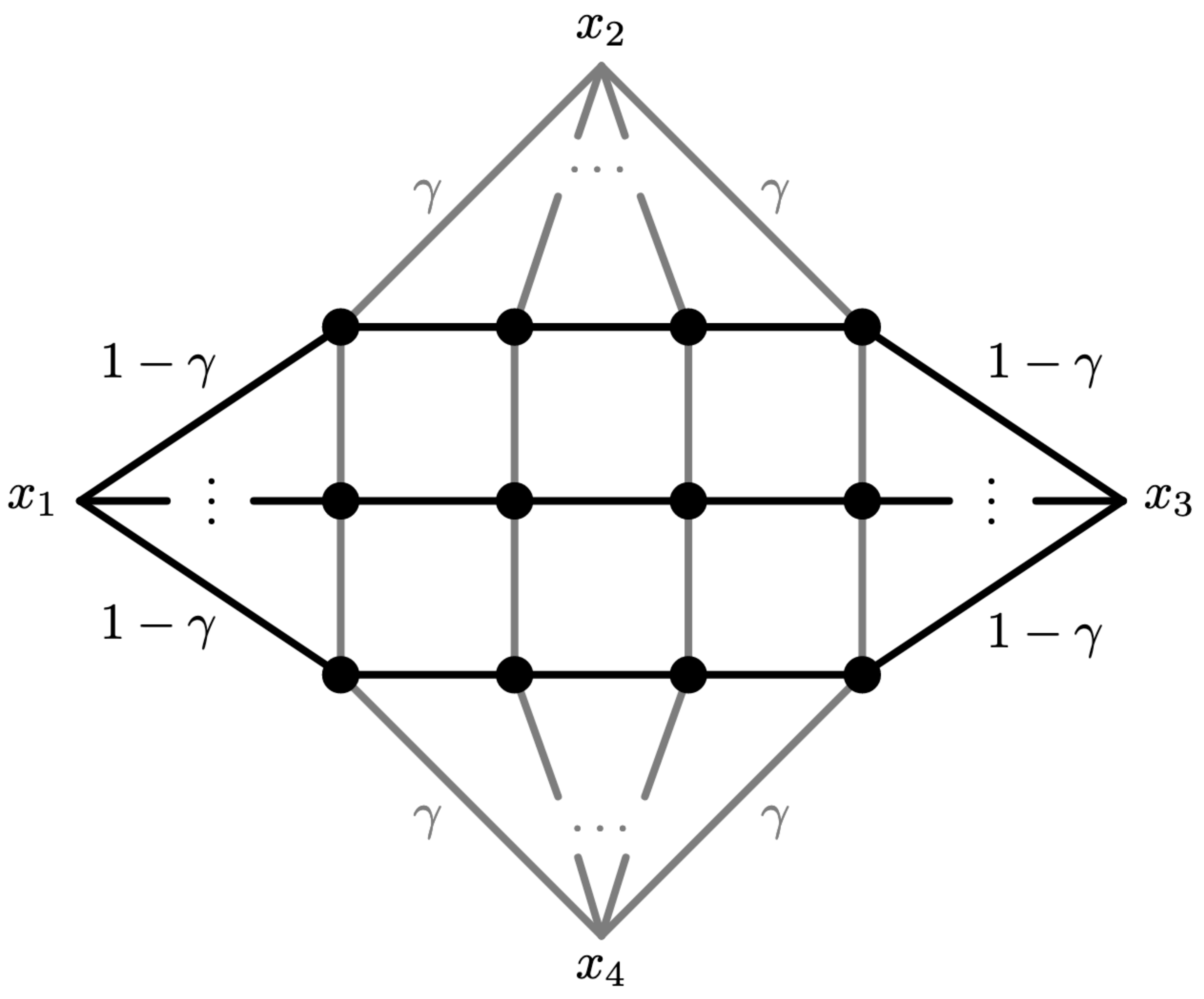
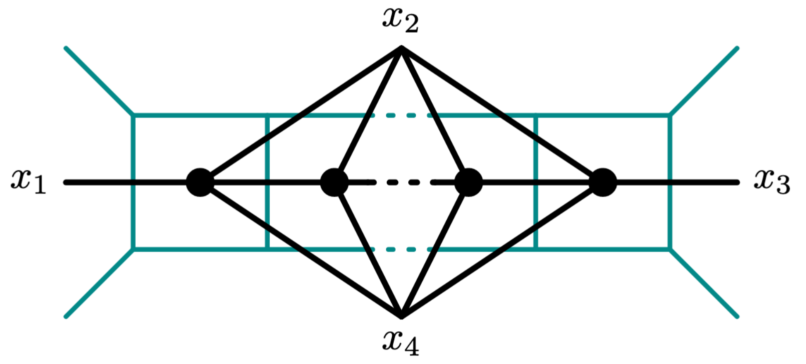


Ladder integrals



$z\partial_z$

$$= \det_{1\le i,j\le M}\left(\theta^{i-1}\bar{\theta}^{j-1}\phi_{2;\gamma}^{(M+N-1)}(z,\bar{z})\right)$$

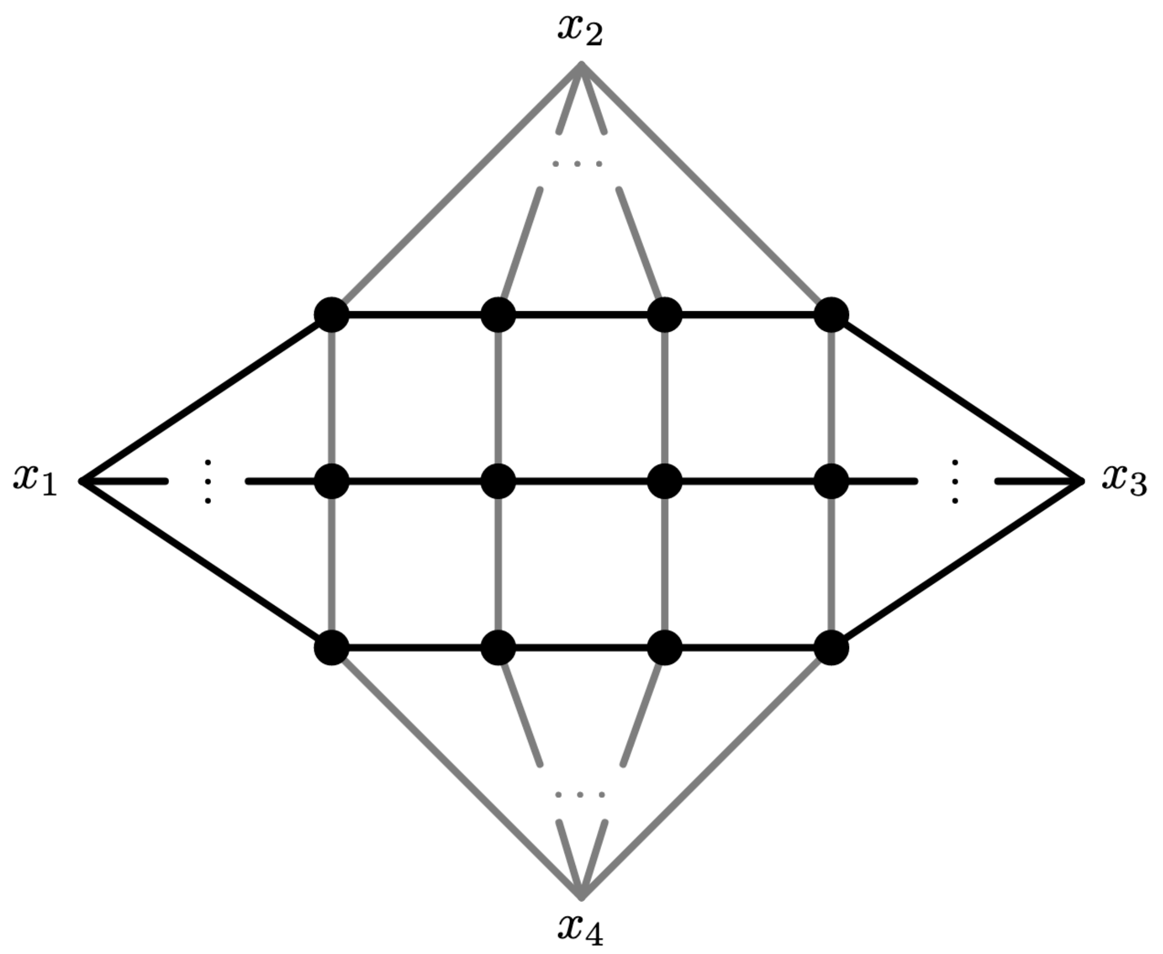
[Derkachov, Kazakov, Olivucci ’18]

$\sim |{}_{L+2}F_{L+1}|^2$

$\longrightarrow$ Factorization!

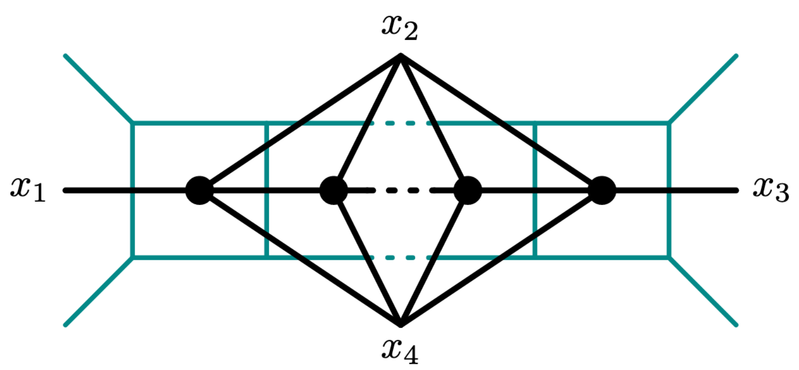
Impressive Results at Four points

In 4D undeformed theory: $\left\langle \left(\phi_2(x_1)^\dagger\right)^M \left(\phi_1(x_2)^\dagger\right)^N \left(\phi_2(x_3)\right)^M \left(\phi_1(x_4)\right)^N \right\rangle$



[Basso, Dixon '17]
[Basso, Dixon, Kosower, Krajenbrink, Zhong '21]

$$= \frac{1}{(z-\bar{z})^M} \det_{1\le i,j\le M} \left(f_{N-M+i+j-1}(z,\bar{z}) \right)$$



$$f_L(z,\bar{z}) = \sum_{n=0}^{L} \frac{(-1)^n (2L-n)!}{L!(L-n)!n!} \log(z\bar{z})^n \left[\mathrm{Li}_{2L-n}(z) - \mathrm{Li}_{2L-n}(\bar{z})\right]$$

[Usyukina, Davydychev '93]

Impressive Results at Four points

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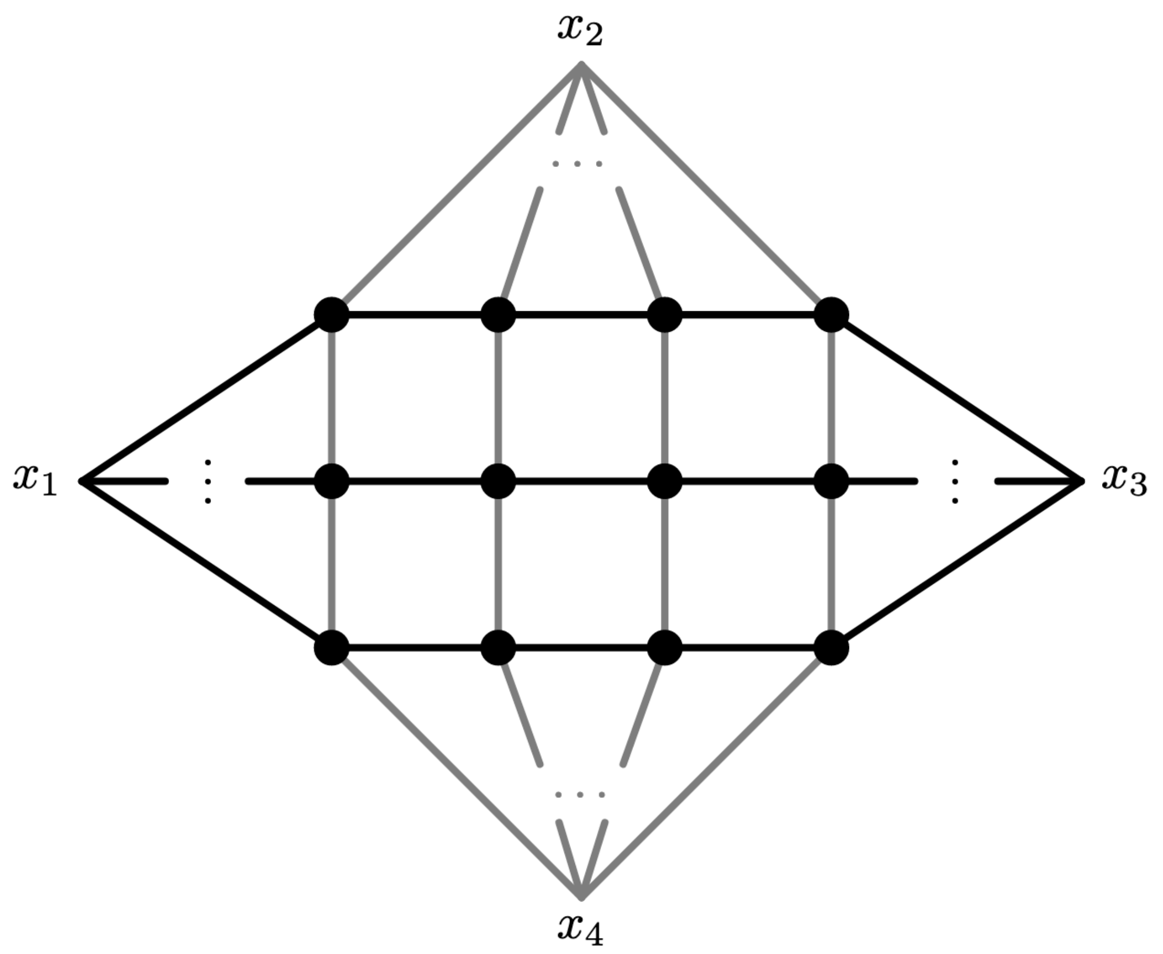

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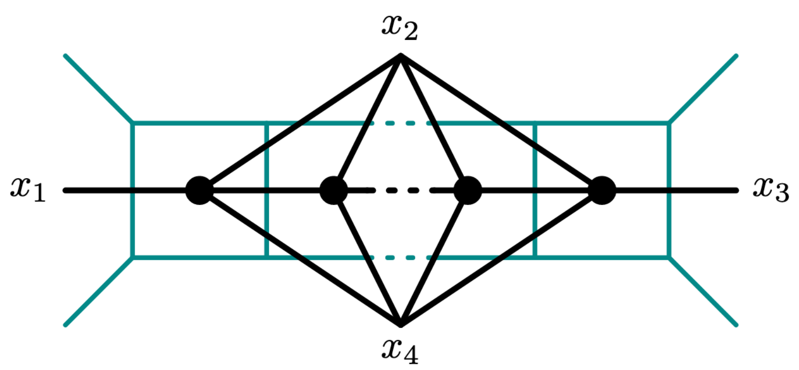

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[Usyukina, Davydychev '93]

$\longrightarrow$ Also factorized! Relation to $D=2$ factorization?

Conformal Schwinger Parametrization

Example: Star integral

Embedding space

$-X_i \cdot Y = (x_i - y)^2$

[Symanzik '72]
[Paulos, Spradlin, Volovich '12]

$$I_n = \int \mathrm{d}Y \prod_{i=1}^{n} \frac{1}{(-X_i \cdot Y)^{a_i}}$$

Schwinger trick

$$= \frac{1}{\prod_{i=1}^{n} \Gamma(a_i)} \prod_{i=1}^{n} \left(\int_0^\infty \mathrm{d}t_i \, t_i^{a_i - 1} \right) \left(\sum_{i=1}^{n} t_i \right)^{-D/2} \exp\left[\frac{(\sum_{i=1}^{n} t_i X_i)^2}{2 \sum_{i=1}^{n} t_i} \right]$$

Some more tricks

$$= \frac{2}{\prod_{i=1}^{n} \Gamma(a_i)} \left(\int_0^\infty \mathrm{d}t_i \, t_i^{a_i - 1} \right) \left(\sum_{i=1}^{n} t_i \right)^{\sum_{i=1}^{d} a_i - D} \exp\left[\frac{1}{2} \left(\sum_{i=1}^{n} t_i X_i \right)^2 \right]$$

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Simplifies for $\sum_{i=1}^{n} a_i = D$ $\longrightarrow$ Conformal Symmetry!

Conformal Schwinger Parametrization

Generally:

$$I = \prod_i \left(\int_0^\infty \frac{\mathrm{d}t_i t_i^{a_i - 1}}{\Gamma(a_i)} \right) \exp\left(- \sum_{i,j} \alpha_{ij} x_{ij}^2 \right)$$

Sums of Schwinger parameters

$\updownarrow$

Graph Topology

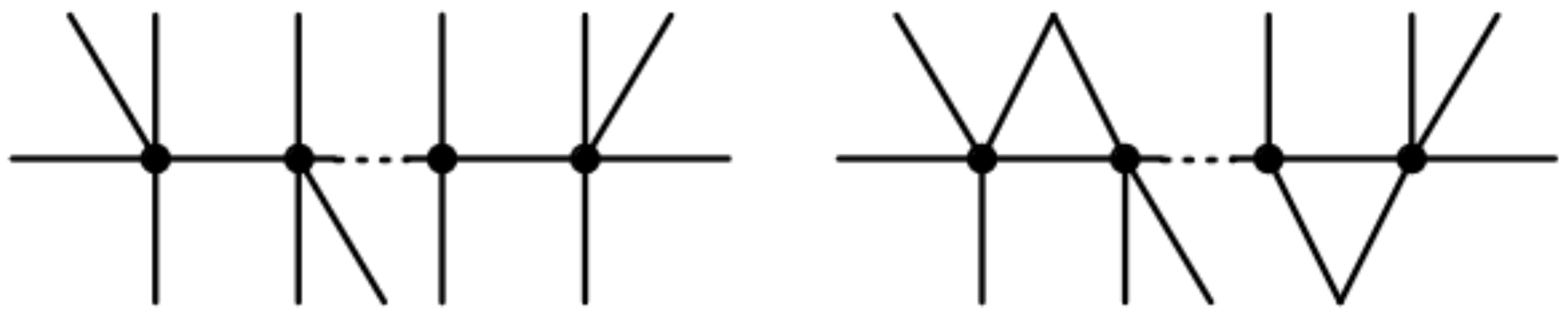
Schwinger parameters

$\updownarrow$

Propagators

Important case: track-like integrals

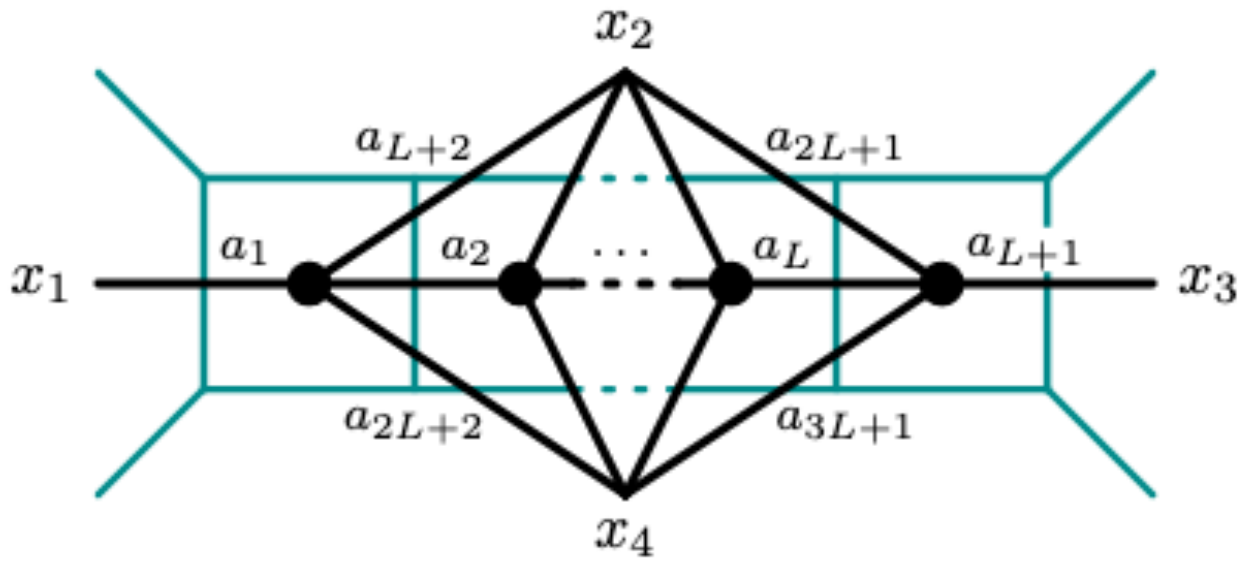
$\longrightarrow$ Dimensional recursion



Dimensional Recursion for Ladder Integrals

$$I_L = \prod_i \left(\int_0^\infty \frac{\mathrm{d}t_i t_i^{a_i - 1}}{\Gamma(a_i)} \right) \exp\left(-t_1 \ldots t_{L+1} x_{13}^2 + \ldots\right)$$

Independent of x_{13}^2



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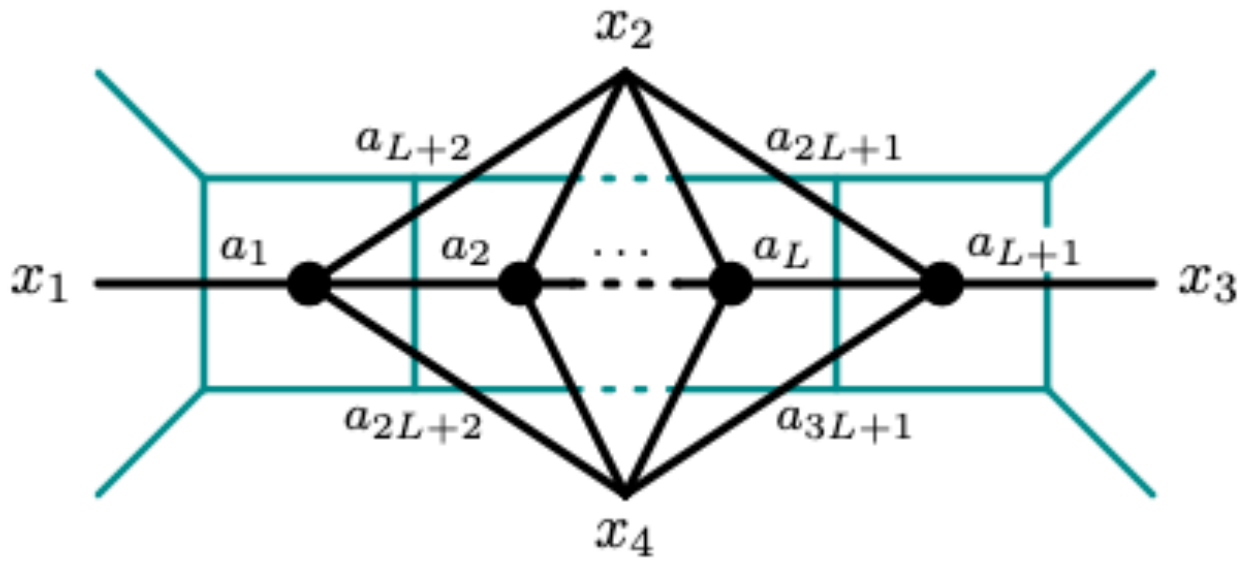

$$-\frac{\partial}{\partial x_{13}^2} I_L = \prod_i \left(\int_0^\infty \frac{\mathrm{d}t_i t_i^{a_i - 1}}{\Gamma(a_i)} \right) t_1 \ldots t_{L+1} \exp\left(-t_1 \ldots t_{L+1} x_{13}^2 + \ldots\right)$$

Raises $a_1, \ldots, a_{L+1}$ by 1

Conformal constraints

Raises D by 2!

Dimensional Recursion for Ladder Integrals

Conformally invariant statement

$$\phi^{(L)}_{D;\mathbf{a}}(z,\bar{z}) = \prod_{i=1}^{L+1} \frac{\Gamma(a_i - D/2+1)}{\Gamma(a_i)} \left[\frac{z\partial_z - \bar{z}\partial_{\bar{z}}}{z-\bar{z}}\right]^{D/2-1} \phi^{(L)}_{2;\mathbf{a}-(D/2-1)\mathbf{e}_{1,2,\ldots,L+1}}(z,\bar{z})$$

$D=2$

Shift of propagator powers

Dimensional Recursion for Ladder Integrals

Conformally invariant statement

$$\phi^{(L)}_{D;\mathbf{a}}(z,\bar{z}) = \prod_{i=1}^{L+1} \frac{\Gamma(a_i - D/2 + 1)}{\Gamma(a_i)} \left[\frac{z\partial_z - \bar{z}\partial_{\bar{z}}}{z - \bar{z}}\right]^{D/2-1} \phi^{(L)}_{2;\mathbf{a}-(D/2-1)\mathbf{e}_{1,2,\ldots,L+1}}(z,\bar{z})$$

$D = 2$

Shift of propagator powers

Known $D = 2$ result ⟶ Conformal ladders in all even D

[Duhr, Porkert '23]

[Loebbert, SFS '24]

$$\phi^{(L)}_{D;\mathbf{a}}(z,\bar{z}) = \prod_{i=1}^{L+1} \frac{\Gamma(a_i - D/2 + 1)}{\Gamma(a_i)} \left[\frac{z\partial_z - \bar{z}\partial_{\bar{z}}}{z - \bar{z}}\right]^{D/2-1} {}_{L+1}\mathscr{F}^{\mathrm{sv}}_{L}(\alpha_{\mathbf{a}}, \beta_{\mathbf{a}}, z)$$

Single-valued hypergeometric function

$\Pi_L(z)^\dagger \Sigma_L \Pi_L(z)$

Corollary: Calabi-Yau in Higher Dimensions

[Duhr, Klemm, Loebbert, Nega, Porkert ’22, ’23, ’24]

[See talk by Cristoph]

Ladder integrals in $D = 2$ and $a_i = 1/2$

$$\phi_{D=2}(z, \bar{z}) = \Pi(z)^{\dagger}\Sigma\Pi(z)$$

Bilinear in Calabi-Yau periods

Calabi-Yau periods

Intersection form

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[Duhr, Klemm, Loebbert, Nega, Porkert '22, '23, '24]

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$$\phi_{D=2}(z,\bar{z}) = \Pi(z)^{\dagger}\Sigma\Pi(z)$$

Bilinear in Calabi-Yau periods

Calabi-Yau periods

Intersection form

Dimensional Recursion

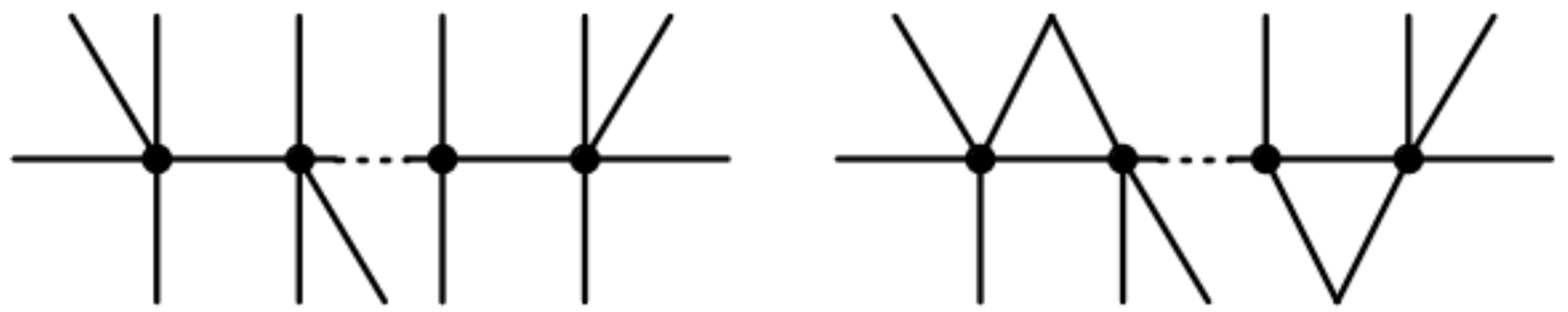
$\longrightarrow$ Ladder integrals in even D with $a_{\text{vert}} = 1/2,\ a_{\text{hor}} = (D-1)/2$

$$\phi_D(z,\bar{z}) = \left[\frac{z\partial_z - \bar{z}\partial_{\bar{z}}}{z-\bar{z}}\right]^{D/2-1} \Pi(z)^{\dagger}\Sigma\Pi(z)$$

Bilinear in derivatives of Calabi-Yau periods!

Dimensional recursion beyond ladders

Extends to any track-like integral, can make external propagators massive



Beyond 4 points: 2D kinematics $\neq$ generic kinematics

Recursion only reaches integral on 2D subspace

E.g. $n = 5$: $z_1, \bar{z}_1, \quad z_2, \bar{z}_2, \quad w \longrightarrow w = 0$ or $w = 1$ in $D = 2$

Two planes Angle

[Buric, Lacroix, Mann, Quintavalle, Schomerus '21]

Natural Representation For Dimensional Recursion

$z\partial_z - \bar{z}\partial_{\bar{z}}$ $\longrightarrow$ Shifts dimension

$z\partial_z + \bar{z}\partial_{\bar{z}}$ $\longrightarrow$ Shifts loop order!

[Petkou '21]
[Karydas, Li, Petkou, Vilatte '23]

(For $D = 4, \gamma = 1$)

$\longrightarrow$ Expand in eigenfunctions

Natural Representation For Dimensional Recursion

$$z = re^{i\varphi}$$
$$\bar{z} = re^{-i\varphi}$$

$$\frac{\partial}{\partial\varphi} \sim z\partial_z - \bar{z}\partial_{\bar{z}} \longrightarrow$$ Shifts dimension

$$r\frac{\partial}{\partial r} \sim z\partial_z + \bar{z}\partial_{\bar{z}} \longrightarrow$$ Shifts loop order!

(For $D = 4, \gamma = 1$)

[Petkou '21]
[Karydas, Li, Petkou, Vilatte '23]

$\longrightarrow$ Expand in eigenfunctions

$$\phi(z,\bar{z}) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \phi(n,\nu) \underbrace{(z\bar{z})^{i\nu}}_{e^{i\nu \log r}} \underbrace{\left(\frac{z}{\bar{z}}\right)^{n/2}}_{e^{in\varphi}}$$

Fourier-Mellin representation

$\updownarrow$

SoV representation

[Alfimov, Aprile, Basso, Cavaglia, Derkachov, Dixon, Ferrando, Gromov, Kazakov, Korchemsky, Kosower, Kozlowski, Krajenbrink, Levkovich-Maslyuk, Manashov, Olivucci, Sklyanin, Zhong,...]

Natural Representation For Dimensional Recursion

[Derkachov, Kazakov, Olivucci '19]

$$\phi^{(L)}_{2;\gamma}(z,\bar{z}) \sim \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \left((-1)^n \frac{\Gamma\left(\frac{1-\gamma}{2}+\frac{n}{2}-i\nu\right)\Gamma\left(\frac{1-\gamma}{2}-\frac{n}{2}+i\nu\right)}{\Gamma\left(\frac{\gamma+1}{2}+\frac{n}{2}+i\nu\right)\Gamma\left(\frac{\gamma+1}{2}-\frac{n}{2}-i\nu\right)} \right)^{L+1} (z\bar{z})^{i\nu} \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}$$

$D = 2$

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[Derkachov, Kazakov, Olivucci '19]

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$D = 2$

$z\partial_z - \bar{z}\partial_{\bar{z}}$

Operator action

[Fleury, Komatsu '16]
[Derkachov, Olivucci '19, '20]

$$\phi^{(L)}_{4;\gamma}(z,\bar{z}) \sim \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} n \left((-1)^n \frac{\Gamma\left(\frac{1-\gamma}{2}+\frac{n}{2}-i\nu\right)\Gamma\left(\frac{1-\gamma}{2}-\frac{n}{2}+i\nu\right)}{\Gamma\left(\frac{\gamma+1}{2}+\frac{n}{2}+i\nu\right)\Gamma\left(\frac{\gamma+1}{2}-\frac{n}{2}-i\nu\right)} \right)^{L+1} (z\bar{z})^{i\nu} \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}$$

$D = 4$

Loop Recursion

$$\left(z\partial_z + \bar{z}\partial_{\bar{z}}\right) L_L(z,\bar{z}) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{2i\nu n}{\left(\nu^2 + \frac{n^2}{4}\right)^{L+1}} (z\bar{z})^{i\nu} \left(\frac{z}{\bar{z}}\right)^{n/2}$$

$$\phi_L(z,\bar{z}) = \frac{1}{z-\bar{z}} L_L(z,\bar{z})$$

$$= -\frac{1}{L} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} n \frac{\partial}{\partial\nu} \left(\frac{i}{\left(\nu^2 + \frac{n^2}{4}\right)^{L}} \right) (z\bar{z})^{i\nu} \left(\frac{z}{\bar{z}}\right)^{n/2}$$

$$= -\frac{1}{L} \log(z\bar{z}) \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{n}{\left(\nu^2 + \frac{n^2}{4}\right)^{L}} (z\bar{z})^{i\nu} \left(\frac{z}{\bar{z}}\right)^{n/2}$$

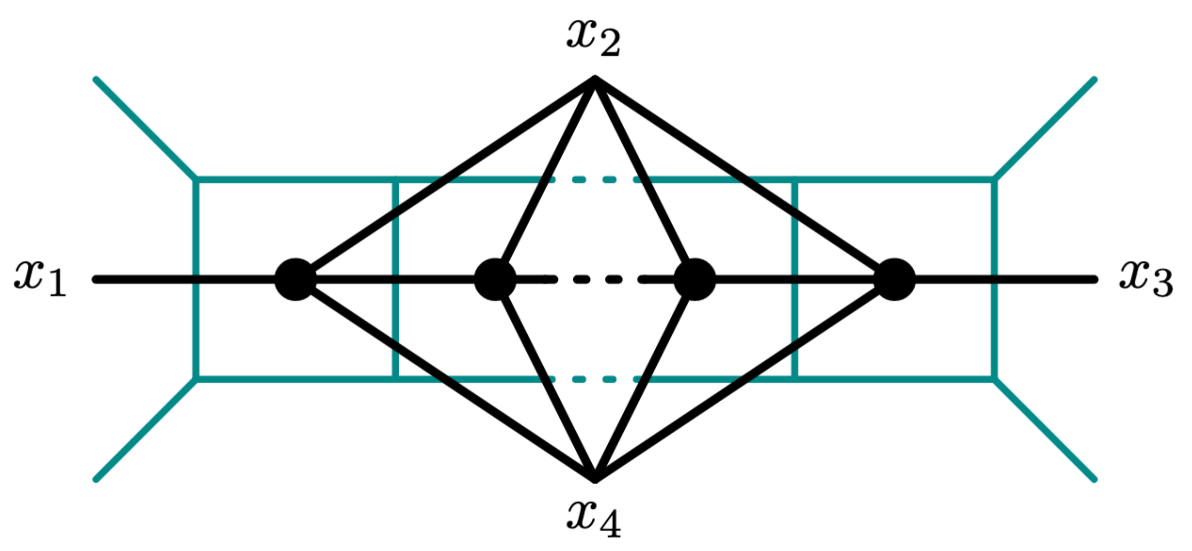
$$= -\frac{1}{L} \log(z\bar{z}) L_{L-1}(z,\bar{z})$$

Reduces loop order!

[Petkou ’21]
[Karydas, Li, Petkou, Vilatte ’23]

Loop Recursion

Ladder integrals: $$-\frac{1}{\log(z\bar{z})}\left(z\partial_z + \bar{z}\partial_{\bar{z}}\right) L_L(z,\bar{z}) = -\frac{1}{L} L_{L-1}(z,\bar{z})$$



1. vs 2. order

$$-z\bar{z}\partial_z\partial_{\bar{z}} L_L = L_{L-1}$$

Laplacian

[Drummond, Henn, Smirnov, Sokatchev '06]
[Drummond, Henn, Trnka '10]

Loop Recursion

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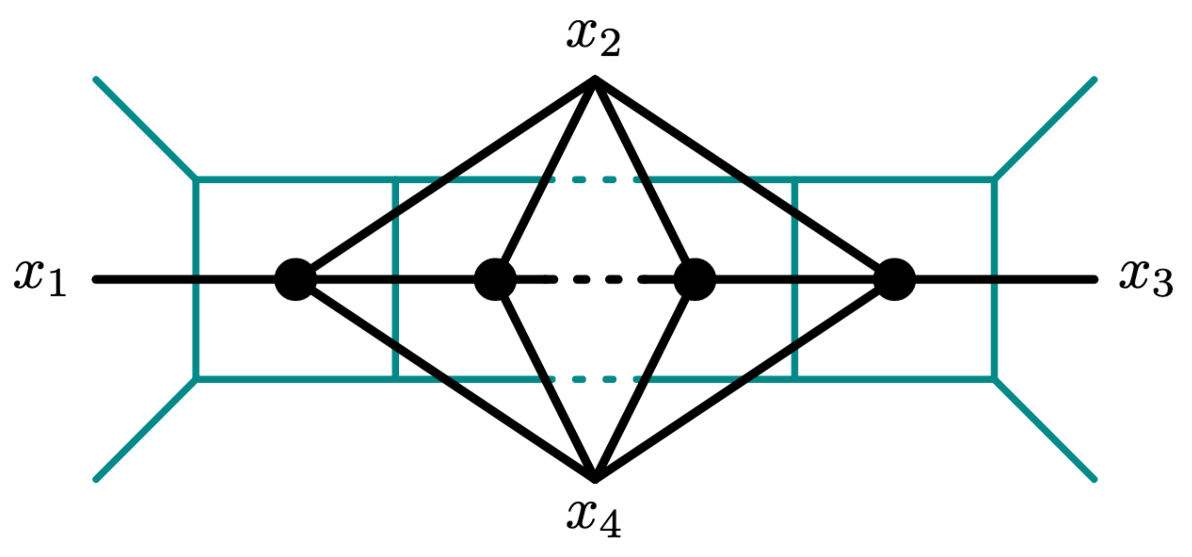


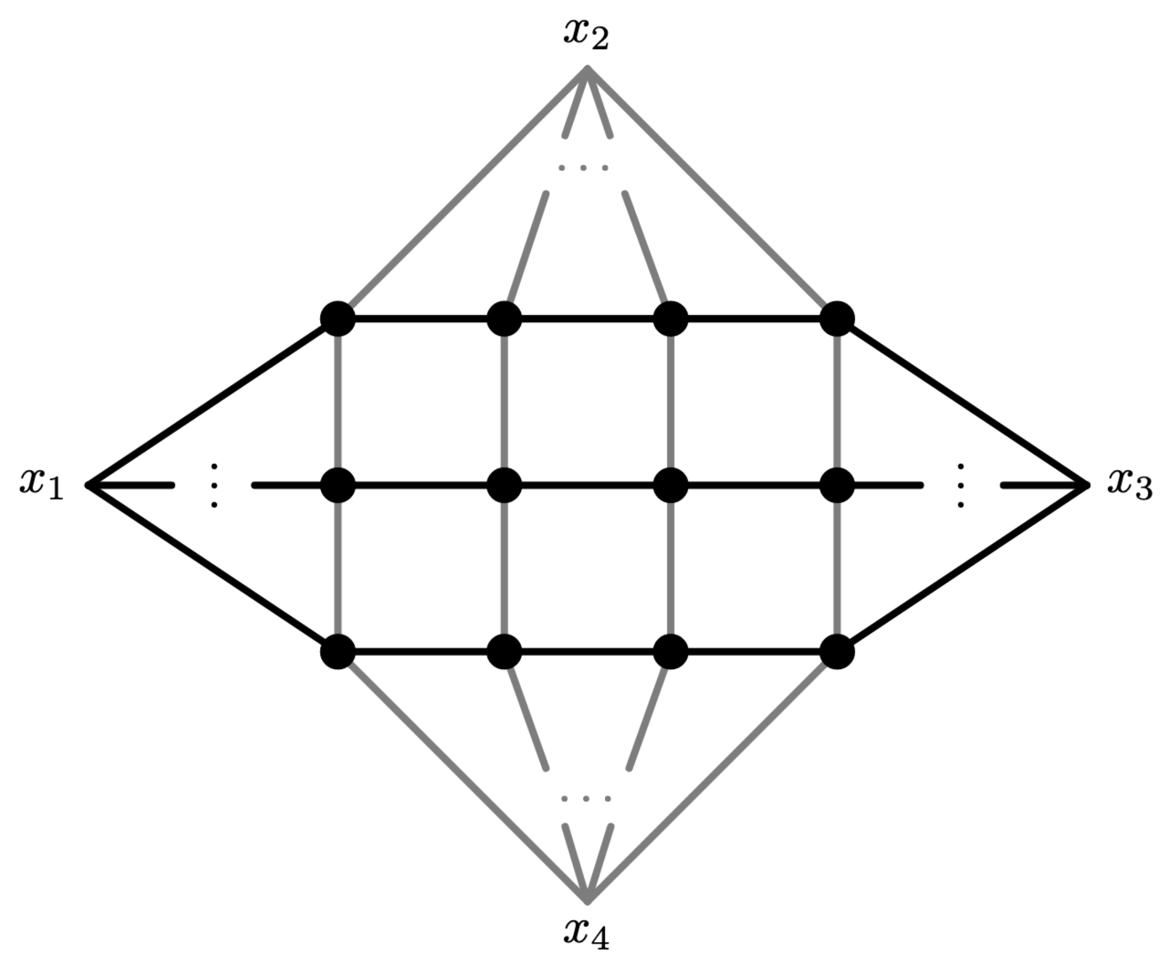
1. vs 2. order

$$-z\bar{z}\partial_z\partial_{\bar{z}} L_L = L_{L-1}$$ Laplacian

[Drummond, Henn, Smirnov, Sokatchev ’06]
[Drummond, Henn, Trnka ’10]

Also natural action on more general four-point fishnet integrals?

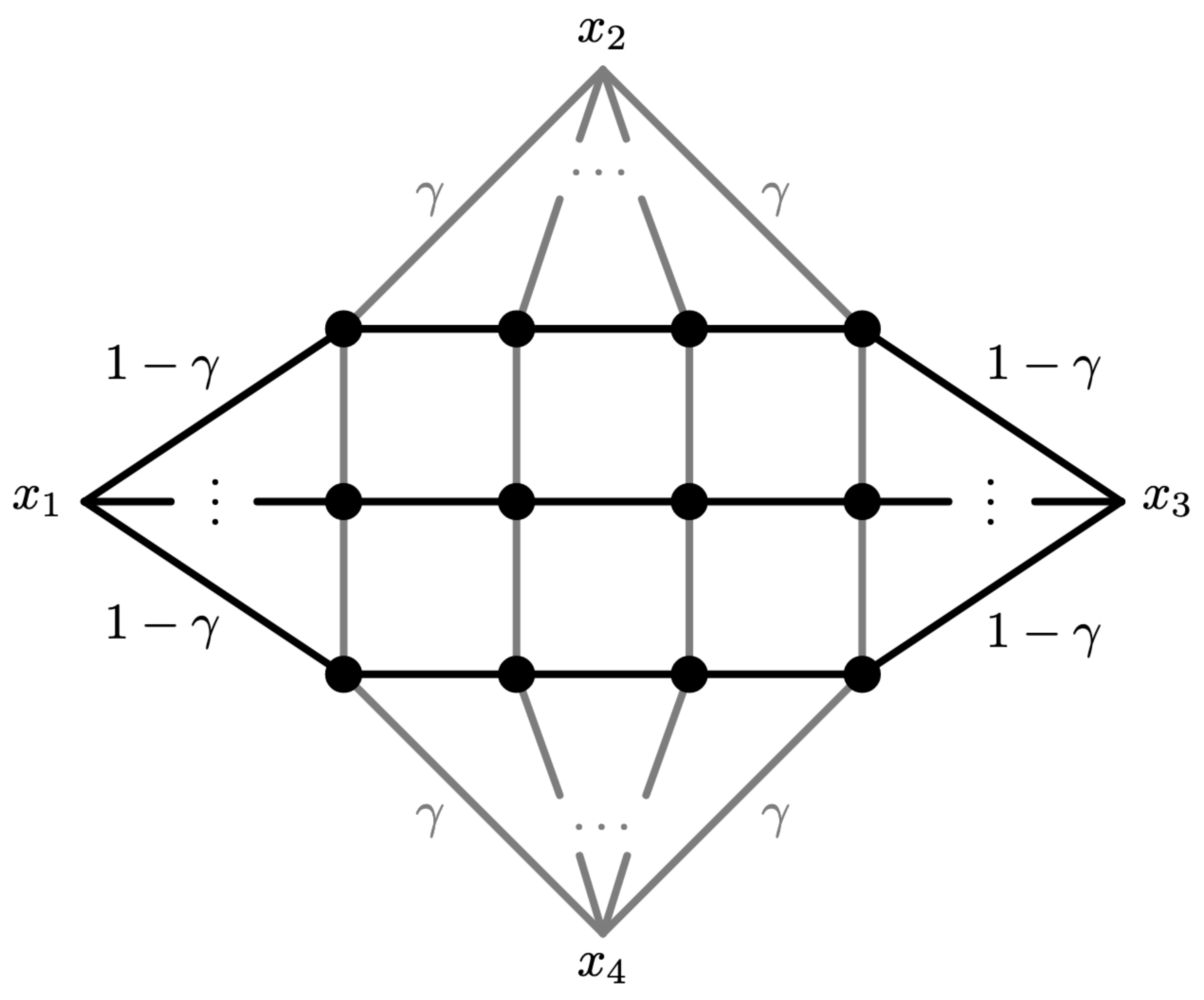
$$-\frac{1}{\log(z\bar{z})}\left(z\partial_z + \bar{z}\partial_{\bar{z}}\right)$$



$= ?$

Useful warm-up: deformed 2D fishnets

Recall: 2D Basso-Dixon formula



Ladder integrals

[Derkachov, Kazakov, Olivucci '18]

$z\partial_z$

$$= \det_{1\le i,j\le M}\left(\theta^{i-1}\bar{\theta}^{j-1}\phi_{2;\gamma}^{(M+N-1)}(z,\bar{z})\right) \equiv \Phi_{M,N}$$

Bi-directional Wronskian

Known to satisfy recursive equation: 2D Toda molecule equation [Ma '11]

Useful warm-up: deformed 2D fishnets

Illustration:

$$\theta \begin{vmatrix} \phi^{(3)} & \bar{\theta}\phi^{(3)} \\ \theta\phi^{(3)} & \theta\bar{\theta}\phi^{(3)} \end{vmatrix} = \phi^{(3)}\theta^2\bar{\theta}\phi^{(3)} - \bar{\theta}\phi^{(3)}\theta^2\phi^{(3)} = \begin{vmatrix} \phi^{(3)} & \bar{\theta}\phi^{(3)} \\ \theta^2\phi^{(3)} & \theta^2\bar{\theta}\phi^{(3)} \end{vmatrix}$$

Useful warm-up: deformed 2D fishnets

Illustration:

$$\theta \begin{vmatrix} \phi^{(3)} & \bar{\theta}\phi^{(3)} \\ \theta\phi^{(3)} & \theta\bar{\theta}\phi^{(3)} \end{vmatrix} = \phi^{(3)}\theta^2\bar{\theta}\phi^{(3)} - \bar{\theta}\phi^{(3)}\theta^2\phi^{(3)} = \begin{vmatrix} \phi^{(3)} & \bar{\theta}\phi^{(3)} \\ \theta^2\phi^{(3)} & \theta^2\bar{\theta}\phi^{(3)} \end{vmatrix}$$

$\longrightarrow$ $\theta, \bar{\theta}$ map minors of Φ_K to minors of Φ_K

$$\Phi_K = \begin{vmatrix} \phi^{(K)} & \bar{\theta}\phi^{(K)} & \bar{\theta}^2\phi^{(K)} & \dots \\ \theta\phi^{(K)} & \theta\bar{\theta}\phi^{(K)} & \theta\bar{\theta}^2\phi^{(K)} & \dots \\ \theta^2\phi^{(K)} & \theta^2\bar{\theta}\phi^{(K)} & \theta^2\bar{\theta}^2\phi^{(K)} & \dots \\ \vdots & \vdots & \vdots & \end{vmatrix}$$

Basso-Dixon Formula as Toda Equation

$$A\begin{bmatrix} i \\ i \end{bmatrix} A\begin{bmatrix} j \\ j \end{bmatrix} - A\begin{bmatrix} i \\ j \end{bmatrix} A\begin{bmatrix} j \\ i \end{bmatrix} - A\begin{bmatrix} i & j \\ i & j \end{bmatrix} A\begin{bmatrix} \end{bmatrix} = 0$$

Identity between minors of matrix

Basso-Dixon Formula as Toda Equation

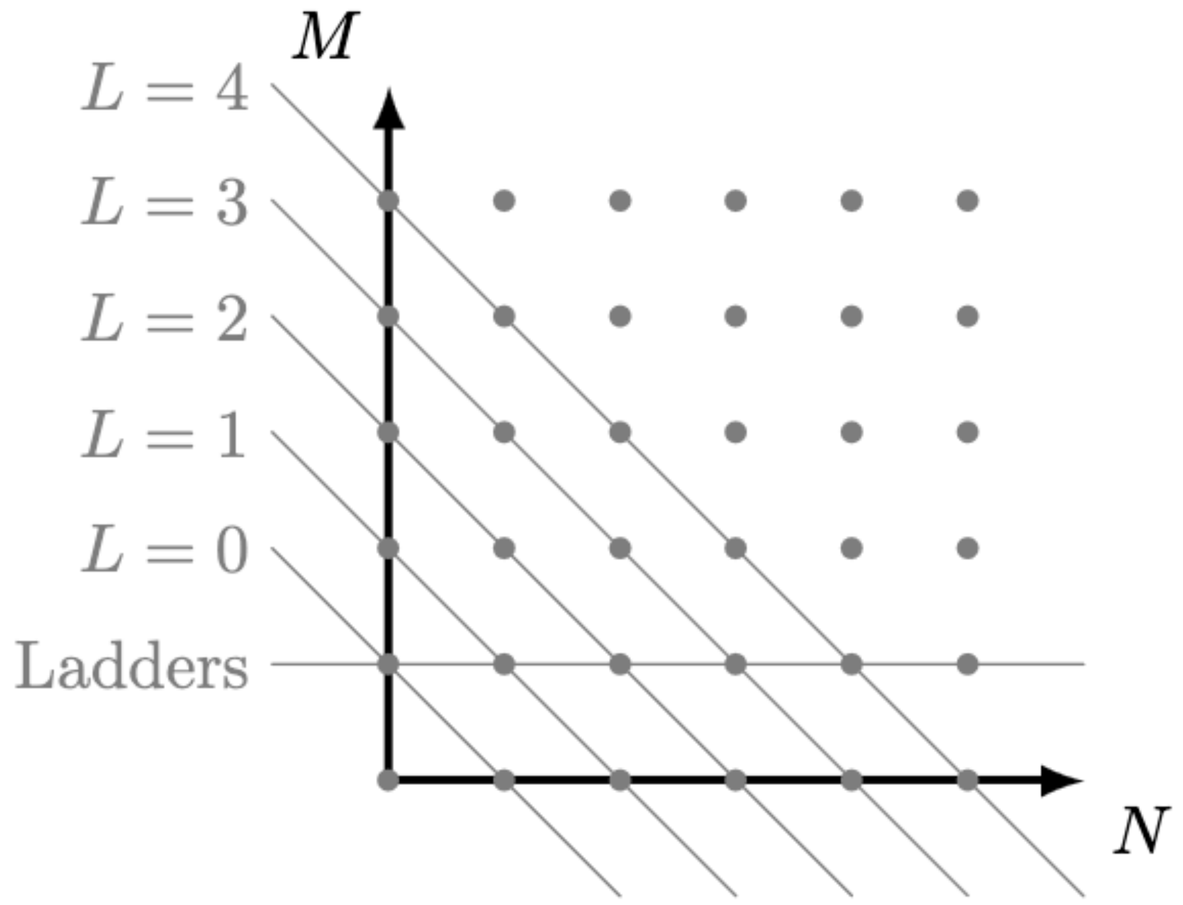
$$A\begin{bmatrix} i \\ i \end{bmatrix} A\begin{bmatrix} j \\ j \end{bmatrix} - A\begin{bmatrix} i \\ j \end{bmatrix} A\begin{bmatrix} j \\ i \end{bmatrix} - A\begin{bmatrix} i & j \\ i & j \end{bmatrix} A\begin{bmatrix} \; \end{bmatrix} = 0$$

Identity between minors of matrix

$$\downarrow$$

Recursive equation

$$\Phi_{M,N}\theta\bar{\theta}\Phi_{M,N} - \theta\Phi_{M,N}\bar{\theta}\Phi_{M,N} - \Phi_{M+1,N-1}\Phi_{M-1,N+1} = 0$$



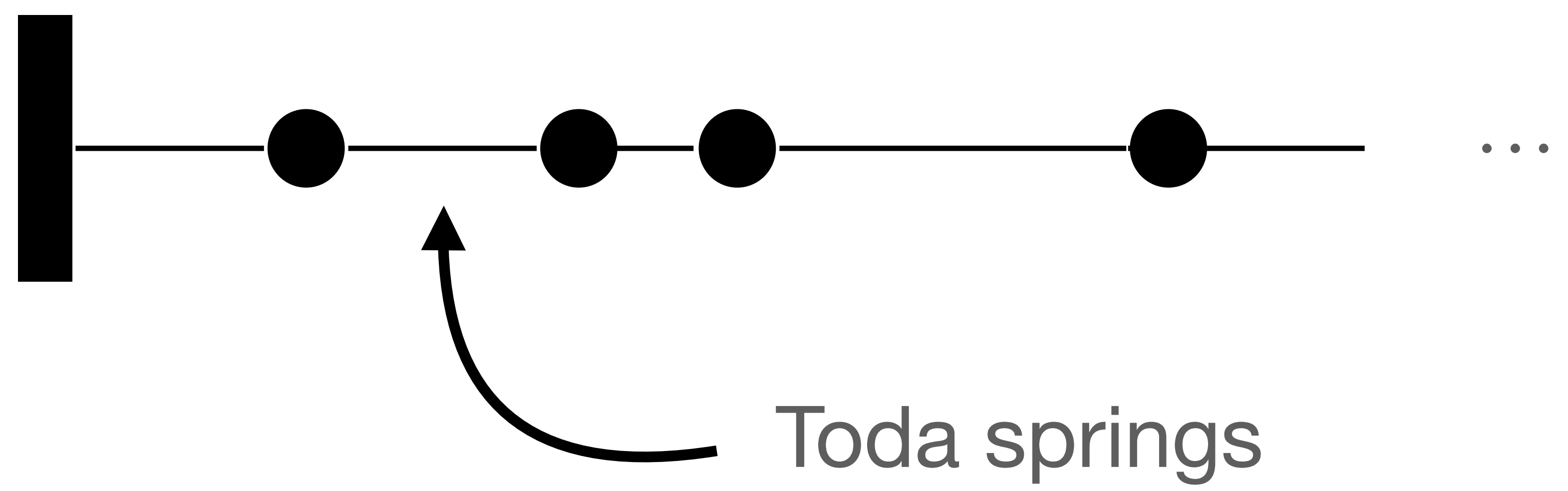
Together with boundary condition $\Phi_{0,N} = 1,\ \Phi_{1,L} = \phi^{(L)}$
fully equivalent to Basso-Dixon formula

Basso-Dixon Formula as Toda Equation

$$\Phi_{M,N}\theta\bar{\theta}\Phi_{M,N} - \theta\Phi_{M,N}\bar{\theta}\Phi_{M,N} - \Phi_{M+1,N-1}\Phi_{M-1,N+1} = 0$$

Known as semi-infinite 2D Toda molecule equation

[Leznov, Saviliev ’81]
[Popowicz ’83]
[Hirota ’88]



Toda springs

$V(r) = \frac{a}{b}\left(e^{-br} - 1\right) + ar$

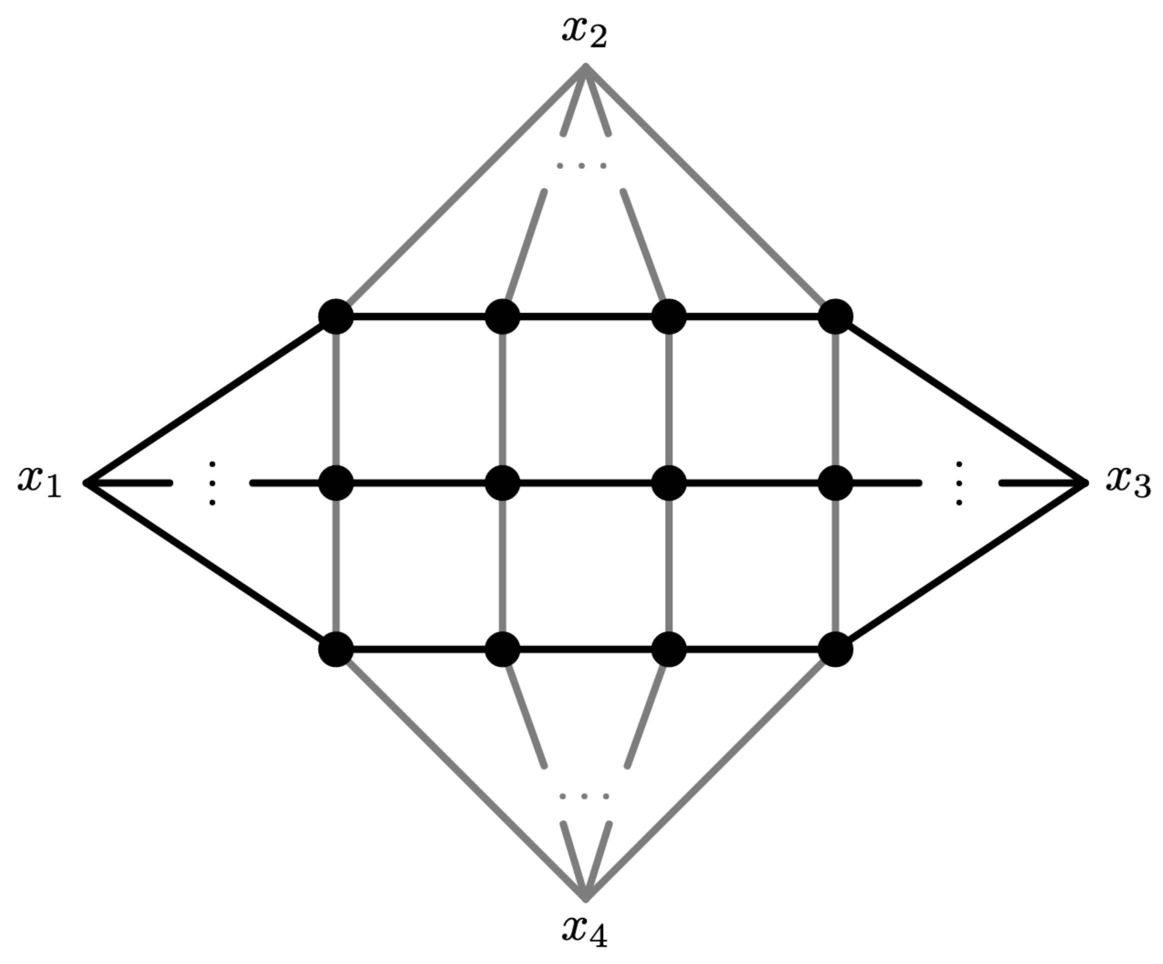
[Toda ’70]

→ Classically integrable system!

c.f. [Alexandrov, Bajnok, Beccaria, Belitsky, Boldis, Kanning, Kazakov, Korchemsky, Laurent, Olivucci, Staudacher, Tseytlin, Tsuboi, Vieira, Zabrodin,…]

Back to 4D fishnets

Recall 4D Basso-Dixon formula



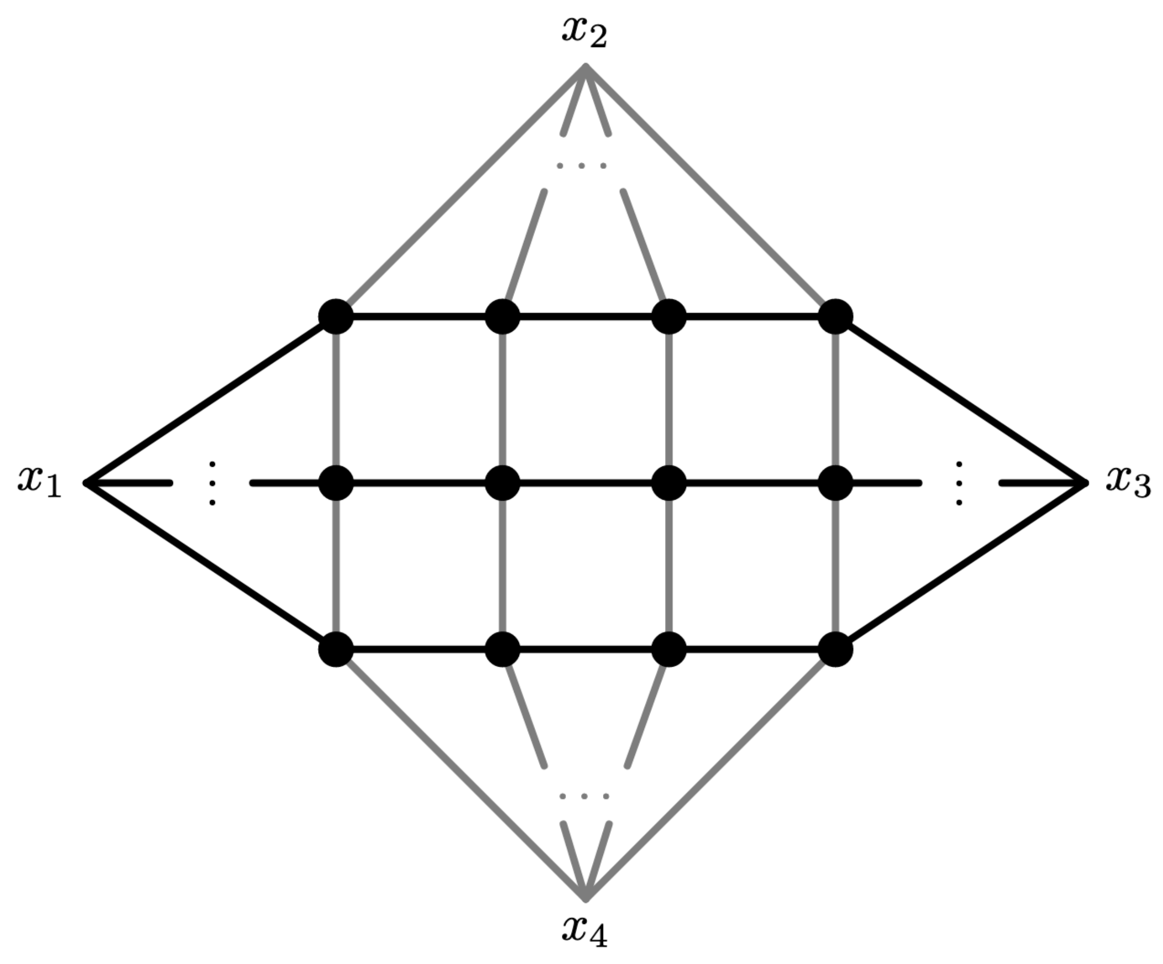
$$= \frac{1}{(z-\bar{z})^M} \det_{1\le i,j\le M} \left(f_{N-M+i+j-1}(z,\bar{z}) \right)$$

[Basso, Dixon '17]
[Basso, Dixon, Kosower, Krajenbrink, Zhong '21]

Back to 4D fishnets

Recall 4D Basso-Dixon formula

[Basso, Dixon '17]
[Basso, Dixon, Kosower, Krajenbrink, Zhong '21]



$$= \frac{1}{(z-\bar{z})^M} \det_{1\le i,j\le M} \left(f_{N-M+i+j-1}(z,\bar{z}) \right)$$

$$\sim \Psi_{M,N} = \det_{1\le i,j\le M} \left(c_{i+j} R_L^{i+j-2} f_{N+M-1}(z,\bar{z}) \right)$$

$$c_k = (M+N-k)!$$

$$R_L = -\frac{1}{\log(z\bar{z})} \left(z\partial_z + \bar{z}\partial_{\bar{z}} \right)$$

Wronskian matrix (with coefficients)! $\longrightarrow$ Is there a Toda(-like) equation?

Toda equation for 4D fishnets?

Again minors are mapped to minors of matrix

$$\begin{pmatrix} c_2 f_K & c_3 R_L f_K & c_4 R_L^2 f_K & \cdots \\ c_3 R_L f_K & c_4 R_L^2 f_K & c_5 R_L^3 f_K & \cdots \\ c_4 R_L^2 f_K & c_5 R_L^3 f_K & c_6 R_L^4 f_K & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

But additional complications due to coefficients! E.g. $\Psi_{2,4}$

Additional coefficients

Additional terms

$$R_L \begin{vmatrix} c_2 f_5 & c_3 R_L f_5 \\ c_3 R_L f_5 & c_4 R_L^2 f_5 \end{vmatrix} = 2 \begin{vmatrix} c_2 f_5 & c_4 R_L^2 f_5 \\ c_3 R_L f_5 & c_5 R_L^3 f_5 \end{vmatrix}$$

$$R_L^2 \begin{vmatrix} c_2 f_5 & c_3 R_L f_5 \\ c_3 R_L f_5 & c_4 R_L^2 f_5 \end{vmatrix} = \begin{vmatrix} c_2 f_5 & c_4 R_L^2 f_5 \\ c_4 R_L^2 f_5 & c_6 R_L^4 f_5 \end{vmatrix} + \mathscr{M}_{2,4}$$

$$1152 \begin{vmatrix} f_2 & f_3 \\ f_3 & f_4 \end{vmatrix}$$

Toda-like equation for 4D fishnets

Identity between minors of matrix

$$\frac{c_{2M+2}}{c_{2M}}\Psi_{M,N}R_L^2\Psi_{M,N} - \left(\frac{c_{2M+1}}{c_{2M}}\right)^2 \left(R_L\,\Psi_{M,N}\right)^2 - \Psi_{M-1,N+1}\Psi_{M+1,N-1} = \Psi_{M,N}\mathscr{M}_{M,N}$$

[Loebbert, SFS ’24]

Additional coefficients

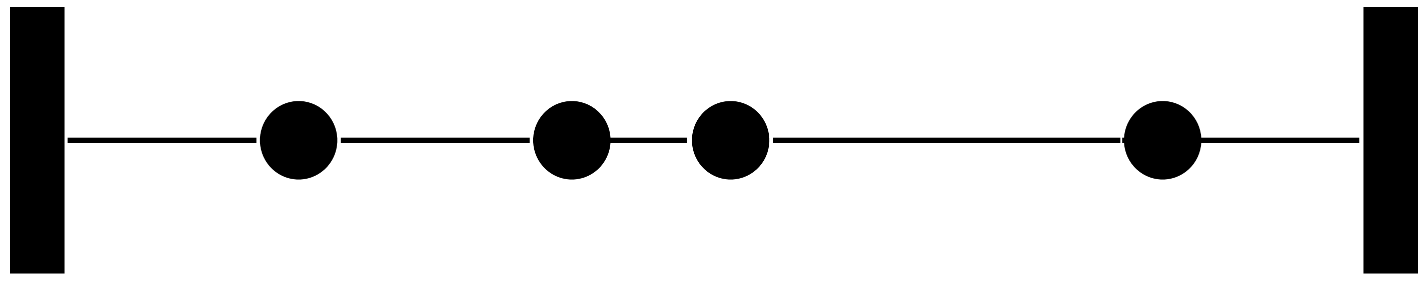
Additional term

Toda-like equation for 4D fishnets

Identity between minors of matrix

$$\tau_n \ddot{\tau}_n - (\dot{\tau}_n)^2 - \tau_{n-1}\tau_{n+1} = 0$$

(Almost) 1D Toda molecule equation



$$\frac{c_{2M+2}}{c_{2M}} \Psi_{M,N} R_L^2 \Psi_{M,N} - \left(\frac{c_{2M+1}}{c_{2M}}\right)^2 \left(R_L \Psi_{M,N}\right)^2 - \Psi_{M-1,N+1}\Psi_{M+1,N-1} = \Psi_{M,N}\mathscr{M}_{M,N}$$

[Loebbert, SFS ’24]

Additional coefficients

Additional term

Fishnet integrals as tau functions?

2D deformed fishnets

$$\Phi_{M,N}\theta\bar{\theta}\Phi_{M,N} - \theta\Phi_{M,N}\bar{\theta}\Phi_{M,N} - \Phi_{M+1,N-1}\Phi_{M-1,N+1} = 0$$

Admits Hirota form $\longrightarrow$ Solutions are tau functions!

4D undeformed fishnets

$$\frac{c_{2M+2}}{c_{2M}}\Psi_{M,N}R_L^2\Psi_{M,N} - \left(\frac{c_{2M+1}}{c_{2M}}\right)^2 \left(R_L\Psi_{M,N}\right)^2 - \Psi_{M-1,N+1}\Psi_{M+1,N-1} = \Psi_{M,N}\mathscr{M}_{M,N}$$

Close to Hirota form $\longrightarrow$?

Octagon as tau-function?

Octagon form factor

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_4(x_4)\rangle \sim \sum_{l=0}^{K/2} \mathbb{O}_l^2$$

$\mathrm{Tr}\left[(y_1\cdot\Phi_1(x_1))^K\right]$

Half-BPS Operator

$K \gg 1$

[Coronado '18]

→ Admits determinant representation

[Kostov, Petkova, Serban '19]
[Belitsky, Korchemsky '19,'20]

→ 4D Fishnet integrals in weak coupling expansion!

$$\mathbb{O}_l \sim \sum_{M=0}^{\infty} g^{2M(M+l)}\left[\Psi_{M,M+l} + \mathcal{O}(g^2)\right]$$

[Belitsky, Korchemsky '20]
[Olivucci, Vieira '21]

→ Satisfies Toda equations in limits! → Octagon as tau-function?

Web of Recursions

$(D = 4, \gamma = 1)$ $R_L = -\frac{z\partial_z + \bar{z}\partial_{\bar{z}}}{\log z\bar{z}}$ ⟵ Two operators ⟶ $R_D = \frac{z\partial_z - \bar{z}\partial_{\bar{z}}}{z - \bar{z}}$

$\downarrow$

[Petkou '21]
[Karydas, Li, Petkou, Vilatte '23] $R_L L_L \sim L_{L-1}$ Loop recursion

$\downarrow$

Toda (like) equation [Loebbert, SFS '24]

$$\frac{c_{2M+2}}{c_{2M}}\Psi_{M,N}R_L^2\Psi_{M,N} - \left(\frac{c_{2M+1}}{c_{2M}}\right)^2 \left(R_L\Psi_{M,N}\right)^2 - \Psi_{M-1,N+1}\Psi_{M+1,N-1} = \Psi_{M,N}\mathscr{M}_{M,N}$$

C.f. $D = 2, \gamma$ generic: 2D Toda molecule equation

$$\Phi_{M,N}\theta\bar{\theta}\Phi_{M,N} - \theta\Phi_{M,N}\bar{\theta}\Phi_{M,N} - \Phi_{M+1,N-1}\Phi_{M-1,N+1} = 0$$

c.f. [Ma '11]

Web of Recursions

$(D=4,\gamma=1)$ $R_L = -\frac{z\partial_z + \bar{z}\partial_{\bar{z}}}{\log z\bar{z}}$ ← Two operators → $R_D = \frac{z\partial_z - \bar{z}\partial_{\bar{z}}}{z-\bar{z}}$

[Petkou '21]
[Karydas, Li, Petkou, Vilatte '23]

$R_L L_L \sim L_{L-1}$ Loop recursion

Dimensional recursion

$$\frac{z\partial_z - \bar{z}\partial_{\bar{z}}}{z-\bar{z}}\phi_D^{(L)} \sim \phi_{D+2}^{(L)}$$

Toda (like) equation

[Loebbert, SFS '24]

$$\frac{c_{2M+2}}{c_{2M}}\Psi_{M,N}R_L^2\Psi_{M,N} - \left(\frac{c_{2M+1}}{c_{2M}}\right)^2\left(R_L\Psi_{M,N}\right)^2 - \Psi_{M-1,N+1}\Psi_{M+1,N-1} = \Psi_{M,N}\mathscr{M}_{M,N}$$

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c.f. [Ma '11]

Back to question from beginning!

General 2D Feynman Integrals

$\boxed{\text{Factorization}}$

External points $z_i = x_i^1 + i x_i^2$

Internal points $w_i = y_i^1 + i y_i^2$

$\longrightarrow$

Measure $\dfrac{\mathrm{d}^2 y_i}{\pi} = \dfrac{\mathrm{d}w_i \wedge \mathrm{d}\bar{w}_i}{2\pi i}$

Propagators $(x_i - y_j)^2 = |z_i - w_j|^2$

General 2D Feynman Integrals

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Propagators $(x_i - y_j)^2 = |z_i - w_j|^2$

$\longrightarrow$ Double Copy!

$$\phi(\mathbf{z}, \bar{\mathbf{z}}) = \Pi(\mathbf{z})^\dagger \Sigma \Pi(\mathbf{z})$$

[Duhr, Porkert '23]

$\longrightarrow$ Lift to higher dimensions!

Double Copy in Higher Dimension

Double copy in 2D

$$\phi^{(L)}_{D=2}(z,\bar{z}) = \Pi(z)^\dagger \Sigma \Pi(z)$$

Dimensional recursion

$$\phi^{(L)}_{D=4}(z,\bar{z}) \sim \frac{z\partial_z - \bar{z}\partial_{\bar{z}}}{z-\bar{z}} \Pi(z)^\dagger \Sigma \Pi(z) = \frac{1}{z-\bar{z}} \left[\Pi(z)^\dagger \Sigma (z\partial_z \Pi(z)) - (z\partial_z \Pi(z))^\dagger \Sigma \Pi(z) \right]$$

$$= \frac{1}{z-\bar{z}} \begin{pmatrix} \Pi(z) \\ z\partial_z \Pi(z) \end{pmatrix}^\dagger \begin{pmatrix} 0 & \Sigma \\ -\Sigma & 0 \end{pmatrix} \begin{pmatrix} \Pi(z) \\ z\partial_z \Pi(z) \end{pmatrix}$$

Again in double copy form!

Double Copy in Higher Dimension

Special propagator powers from limits

$$\phi^{(1)}_{D=4,\gamma=1}(z,\bar{z}) = \frac{1}{z-\bar{z}}\Pi(z)^\dagger \Sigma \Pi(z)$$

$\mathcal{O}\left((\gamma-1)^2\right)$ $\mathcal{O}\left(\frac{1}{\gamma-1}\right)$

4D unit propagator
power box

$$= \frac{1}{z-\bar{z}}\left[\pi_0(z)^\dagger \sigma_0 \pi_0(z) + \pi_{-1}(z)^\dagger \sigma_0 \pi_1(z) + \pi_1(z)^\dagger \sigma_0 \pi_{-1}(z)\right]$$

Different form, but still double copy

Double Copy in Higher Dimension

Special propagator powers from limits

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4D unit propagator
power box

$$= \frac{1}{z-\bar{z}}\left[\pi_0(z)^{\dagger}\sigma_0\pi_0(z) + \pi_{-1}(z)^{\dagger}\sigma_0\pi_1(z) + \pi_1(z)^{\dagger}\sigma_0\pi_{-1}(z)\right]$$

Different form, but still double copy

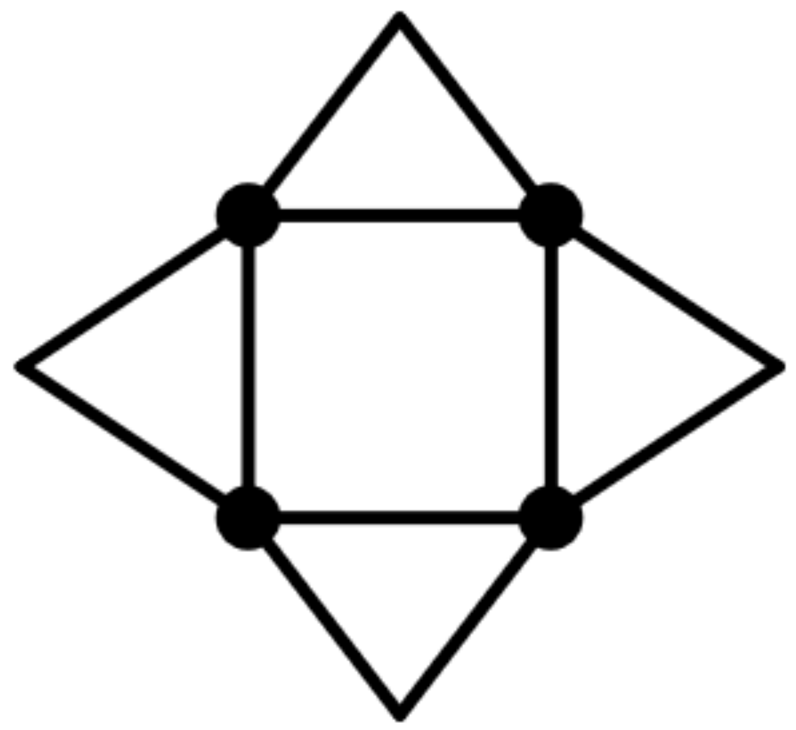
$$= \frac{1}{z-\bar{z}}\left[2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(\bar{z}) + \log(z\bar{z})\log\left(\frac{1-z}{1-\bar{z}}\right)\right]$$

Bloch Wigner!

Double Copy for 4D Basso-Dixon

Basso-Dixon formula for window

$$\phi^{(2,2)}(z,\bar{z}) \sim \frac{1}{(z-\bar{z})^2}(f_1 f_3 - f_2^2)$$



Double Copy for 4D Basso-Dixon

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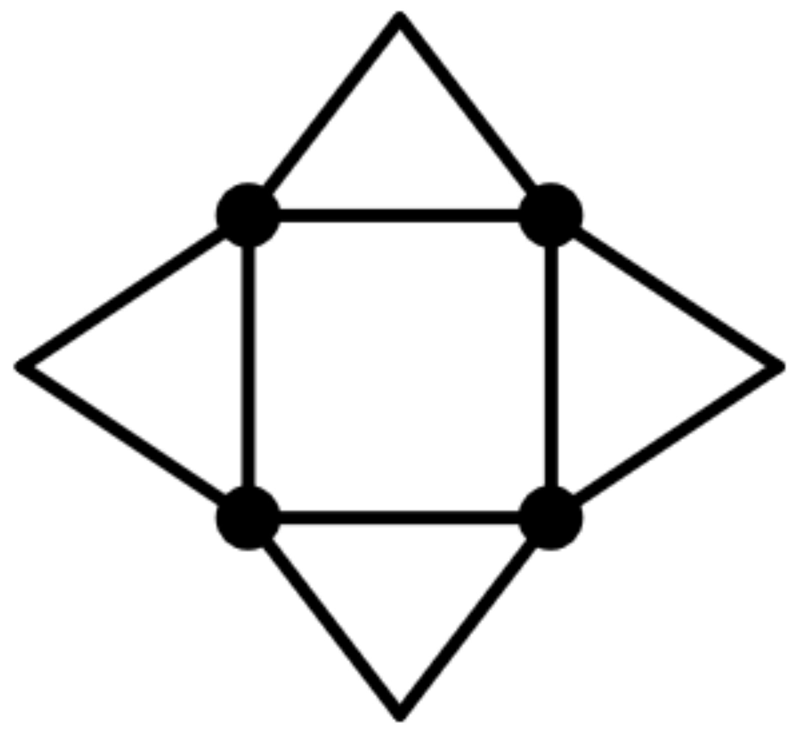


Each product: schematically

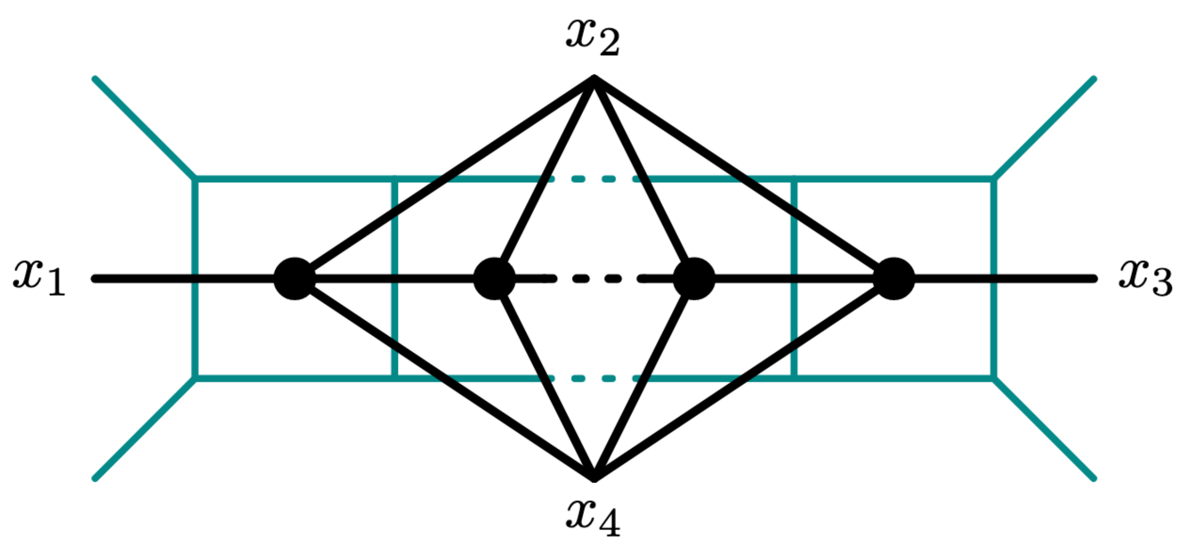
$$ff \sim \bar{\pi}(\bar{z})\Sigma\pi(z)\bar{\pi}(\bar{z})\Sigma\pi(z)$$

$$= \bar{\pi}(\bar{z})\bar{\pi}(\bar{z})\Sigma\Sigma\pi(z)\pi(z)$$

New periods: products of ladder periods

Double copy form!

Conclusion

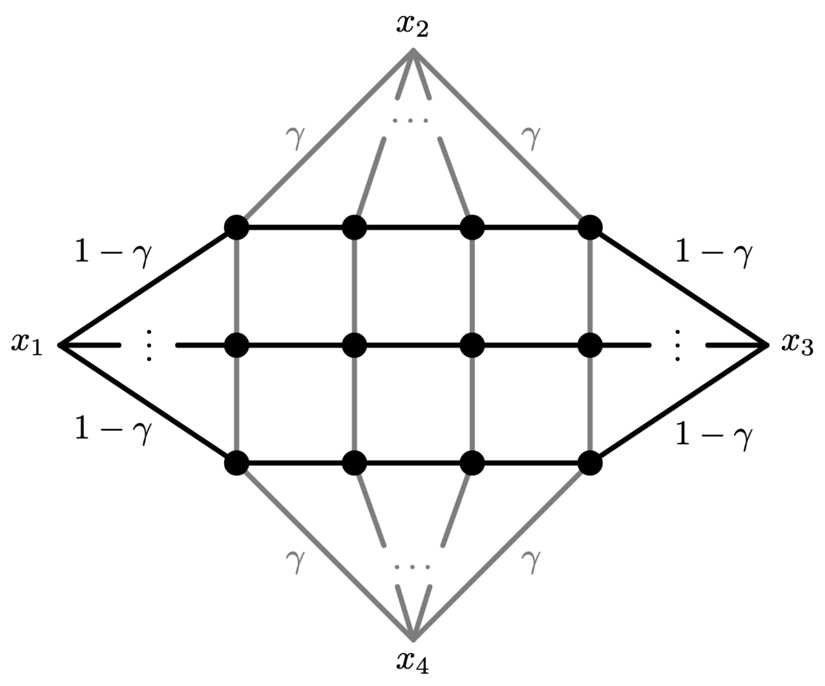


Dimensional recursion for track-like integrals

→ Ladder integrals for all even D

→ New Calabi-Yau integrals in $D > 2$

→ Double copy beyond $D = 2$



Loop recursion/Toda equations

→ 2D Toda molecule equation for 2D deformed Basso-Dixon integrals

→ 1D Toda molecule-like equation for 4D undeformed Basso-Dixon integrals

→ Connection to classical integrability, tau functions

Conformal field theory ⟷ Feynman integrals ⟷ Integrability

Dimensional recursion

New expressions?

Conformal blocks beyond 4 points?

Conformal field theory ⟷ Feynman integrals ⟷ Integrability

Dimensional recursion

New expressions?

Conformal blocks beyond 4 points?

Conformal partial wave expansion

New angle on
Intersection matrix?

[Mimachi, Yoshida '02]

Double copy

Conformal field theory ↔ Feynman integrals ↔ Integrability

Dimensional recursion

New expressions?

Conformal blocks beyond 4 points?

Determinant formulas
for other Feynman integrals?

Loop recursion

Basso-Dixon integrals
as tau functions?

Conformal partial wave expansion

New angle on
Intersection matrix?

[Mimachi, Yoshida ’02]

Double copy

Conformal field theory ⟷ Feynman integrals ⟷ Integrability

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for other Feynman integrals?

Loop recursion ⟷ Basso-Dixon integrals
as tau functions?

Conformal partial wave expansion

New angle on
Intersection matrix?

[Mimachi, Yoshida '02]

Double copy

Other QFT observables?
Octagon,…

[Alexandrov, Bajnok, Beccaria, Belitsky, Boldis, Kanning, Kazakov, Korchemsky, Laurent, Olivucci, Staudacher, Tseytlin, Tsuboi, Vieira, Zabrodin,…]

Backup Slides

Polylogarithmic Limit

$$\phi^{(L)}_{4;\gamma}(z,\bar{z}) = \left(\frac{\Gamma(1-\gamma}{\Gamma(2-\gamma)}\right)^{L+1}\left[\frac{z\partial_z - \bar{z}\partial_{\bar{z}}}{z-\bar{z}}\right]\phi^{(L)}_{2,\gamma}(z,\bar{z})$$

Divergent for $\gamma \to 1$!

Idea: Look at SoV representation [Karydas, Li, Petkou, Vilatte '23]

$$\phi^{(L)}_{2;\gamma}(z,\bar{z}) = \left(\frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}\right)^{L+1}(z\bar{z})^{\frac{\gamma-1}{2}}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}\left((-1)^n\frac{\Gamma\left(\frac{1-\gamma}{2}+\frac{n}{2}-i\nu\right)\Gamma\left(\frac{1-\gamma}{2}-\frac{n}{2}+i\nu\right)}{\Gamma\left(\frac{\gamma+1}{2}+\frac{n}{2}+i\nu\right)\Gamma\left(\frac{\gamma+1}{2}-\frac{n}{2}-i\nu\right)}\right)^{L+1}(z\bar{z})^{i\nu}\left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}}$$

Precisely cancels divergence!

$\gamma \to 1$

$$\phi^{\mathrm{reg},(L)}_{2;\gamma=1}$$

Polylogarithmic Limit

Explicitly

Residue Theorem

$$\phi^{\mathrm{reg},(L)}_{2;\gamma=1} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\mathrm{d}\nu}{2\pi} \frac{1}{\left(\nu^2 + \frac{n^2}{4}\right)^{L+1}} (z\bar{z})^{i\nu} \left(\frac{z}{\bar{z}}\right)^{n/2}$$

$$\phi^{\mathrm{reg},(L)}_{2;\gamma=1} = \sum_{n=0}^{L} \frac{(-1)^n (2L-n)!}{L!(L-n)!n!} \log(z\bar{z})^n \left(\mathrm{Li}_{2L+1-n}(z) - \mathrm{Li}_{2L+1-n}(\bar{z})\right) - \frac{(-1)^L}{2(2L+1)!} \log(z\bar{z})^{2L+1}$$

Yields familiar ladder formula through dimensional recursion

$$\phi^{(L)}_{4;\gamma=1} = \frac{1}{z-\bar{z}} \sum_{n=0}^{L} \frac{(-1)^n (2L-n)!}{L!(L-n)!n!} \log(z\bar{z})^n \left(\mathrm{Li}_{2L-n}(z) - \mathrm{Li}_{2L-n}(\bar{z})\right)$$

[Usyukina, Davydychev ’93]