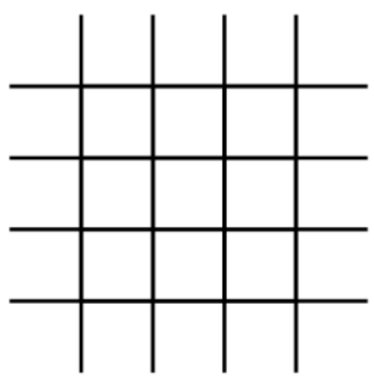
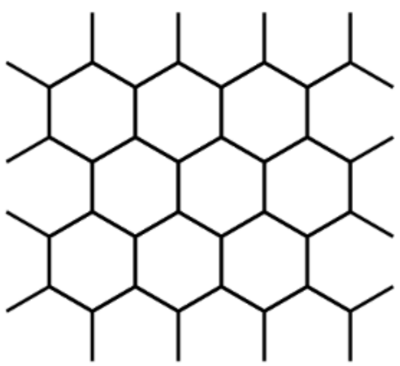
Fishnet Integrals in Two Dimensions



Christoph Nega



Joint work with:
Claude Duhr, Albrecht Klemm, Florian Löbbert and Franziska Porkert

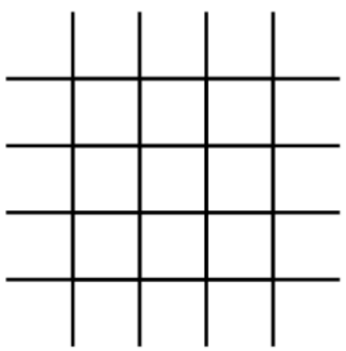
"Geometry from integrability: multi-leg fishnet integrals in two dimensions" [1]
"The Basso-Dixon formula and Calabi-Yau geometry" [2],
"Yangian-Invariant Fishnet Integrals in Two Dimensions as Volumes of Calabi-Yau Varieties" [3]

Bethe Fishnet Workshop 2024
Bonn
September 3, 2024

Plan of the Talk

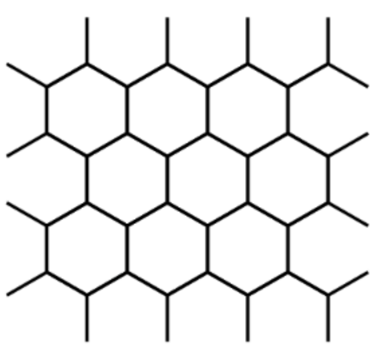
- Main **theme** of the talk:

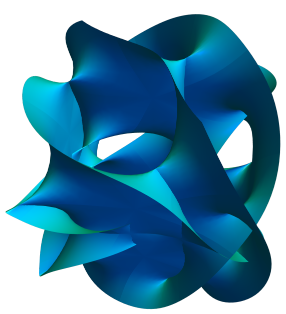
Interplay between Integrability (Symmetries) and Geometry (Calabi-Yau and Picard varieties)



square fishnets [3]

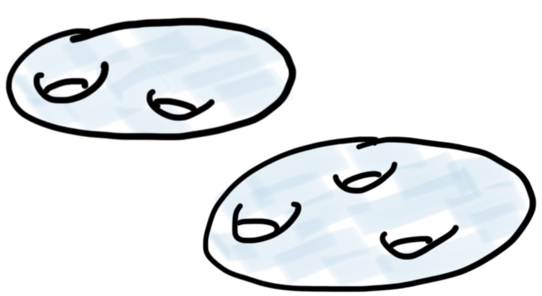
hexagonal fishnets [1]





Calabi-Yau varieties

Picard varieties



- In particular, we want to discuss **Fishnet integrals** in **two dimensions** and how we can compute them using their **symmetries** and associated **geometries**.

Table of Content

From SYM to the Fishnet Theory

- Let us start with **superconformal Yang-Mills** theory with SU(N) gauge symmetry:

$$\mathcal{L}_{\mathcal{N}=4} = \mathrm{tr}\left\{FF + D\Phi D\Phi + \bar{\Psi} D\Psi - g^2[\Phi,\Phi]^2 - g\Psi[\Phi,\Psi] - g\bar{\Psi}[\Phi,\bar{\Psi}]\right\}$$

field strength with gauge field A

six scalar fields

four spinors

- This theory has the following **symmetries**:
  - Conformal symmetry at the quantum level (beta-function vanishes)
  - The Lie algebra symmetry $\mathfrak{psu}(2,2|4)$
  - In the planar limit ($N \to \infty$) we have conformal and dual conformal symmetry

$\longrightarrow$ **Yangian symmetry (Integrability)**

[Dolan, Nappi, Witten; Drummond, Henn, Plefka;...]

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[Dolan, Nappi, Witten; Drummond, Henn, Plefka;...]

- From a γ-deformation of SYM we can construct the biscalar **fishnet theory** as a specific limit:

$$\mathcal{L}_{\mathcal{N}=4} \longrightarrow \mathcal{L}_{\gamma} \longrightarrow \mathcal{L}_{\text{fishnet}}$$

[Kazakov, Gürdogan; Kazakov, Olivucci]

$$\mathcal{L}_{\text{fishnet}} = N\mathrm{tr}\left\{-X(-\partial_\mu\partial^\mu)^\omega \bar{X} - Z(-\partial_\mu\partial^\mu)^{\frac{D}{2}-\omega}\bar{Z} + \xi^2 XZ\bar{X}\bar{Z}\right\}$$

for generic D and propagator powers ω

Fishnet Theory and Generalizations

- **Properties** of **fishnet theory**:
  - Yangian invariant (integrability)
  - Chiral structure of vertex allows only for a small number of Feynman diagrams.
  - Generalization to D spacetime dimensions with appropriately generalized propagator powers known, e.g. $D = 2, \omega = 1/2$.

[Chicherin, Kazakov, Löbbert, Müller, Zhang; Kazakov, Levkovich-Maslyuk, Mishnyakov]

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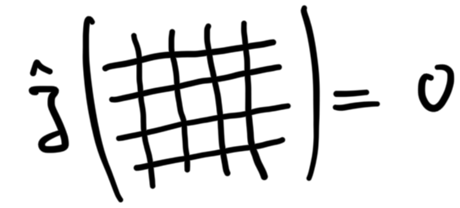

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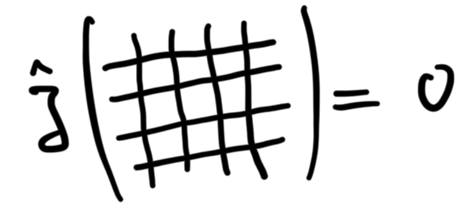

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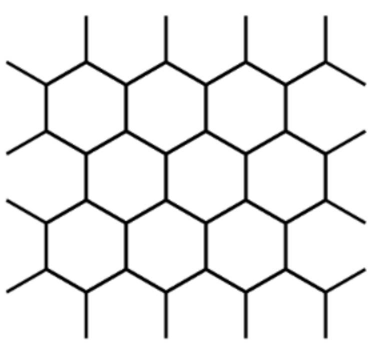
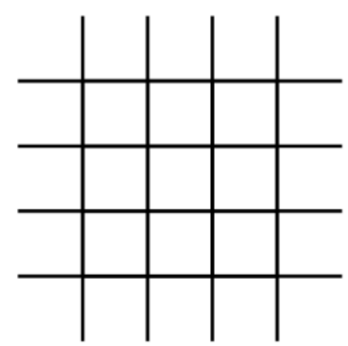
$V = 2D/(D-2)$



$D = 6$
$V = 3$



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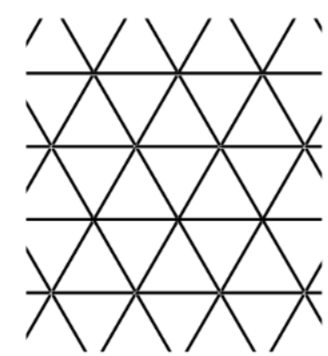

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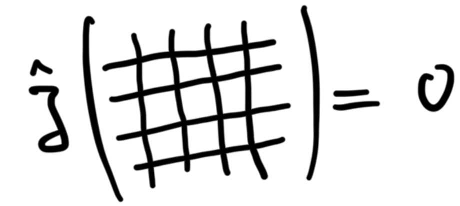

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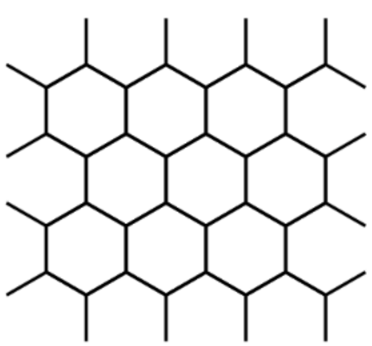

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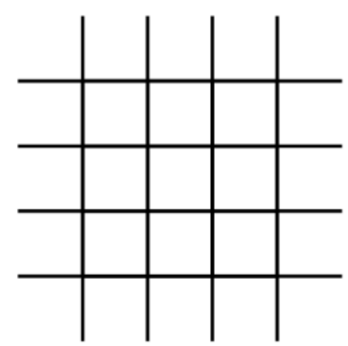

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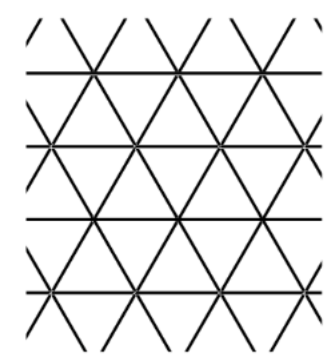

$D = 3$
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- For the hexagonal tiling there exists the **honeycomb fishnet theory** (generic D and propagator powers ω_i):

$$\mathcal{L}_{\text{honey}} = N \text{tr} \left\{ -X(-\partial_\mu \partial^\mu)^{\omega_1} \bar{X} - Y(-\partial_\mu \partial^\mu)^{\omega_2} \bar{Y} - Z(-\partial_\mu \partial^\mu)^{\omega_3} \bar{Z} + \xi_1^2 \bar{X} Y Z + \xi_2^2 X \bar{Y} \bar{Z} \right\}$$

[Kazakov, Olivucci]

Fishnet Integrals

- We can build fishnet integrals from the following **Feynman rules**:
  - Take a cut from a tiling:

or

  - Vertices: ξ_i
  - External points: α_i

  (considered in $\mathbb{R}^D$)

  - Propagators: $\frac{1}{[(\xi_i - \xi_j)^2]^{D/V}}$ or $\frac{1}{[(\xi_i - \alpha_j)^2]^{D/V}}$
  - Integrate over internal vertices.

$$\longrightarrow \quad I_G^{(D)}(\underline{\alpha}) = \int \Big[\prod_i \mathrm{d}^d \xi_i\Big]\Big[\prod_{i,j} \frac{1}{[(\xi_i - \xi_j)^2]^{D/V}}\Big]\Big[\prod_{i,j} \frac{1}{[(\xi_i - \alpha_j)^2]^{D/V}}\Big]$$

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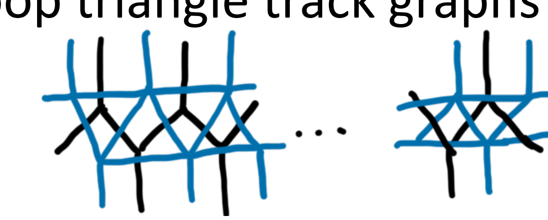
- We are particularly interested in the following **two families** of fishnet graphs and integrals:

ℓ-loop train track graphs $G_{1,\ell}$

ℓ-loop triangle track graphs Z_ℓ



Star-Triangle Relation

- **Conformal symmetry** relates a triple vertex integration to three propagators (**Star-Triangle Relation**):

$$\int \frac{\mathrm{d}^D\xi}{(\alpha_1-\xi)^{2\alpha}(\alpha_2-\xi)^{2\beta}(\alpha_3-\xi)^{2\gamma}} = \frac{X_{\alpha\beta\gamma}}{(\alpha_1-\alpha_2)^{2\gamma'}(\alpha_2-\alpha_3)^{2\alpha'}(\alpha_3-\alpha_1)^{2\beta'}}$$

gamma factors
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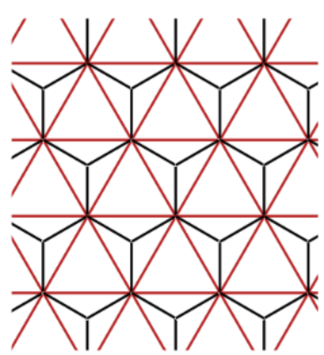
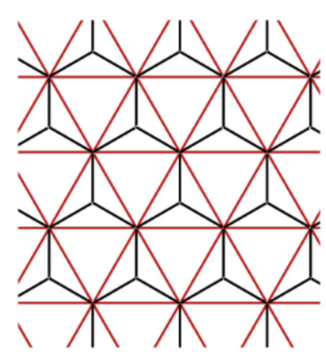
- With this identity we can map the **triangular tiling** to the **hexagonal one**:





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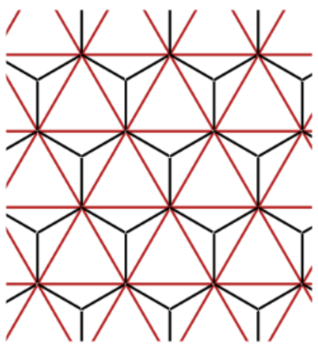
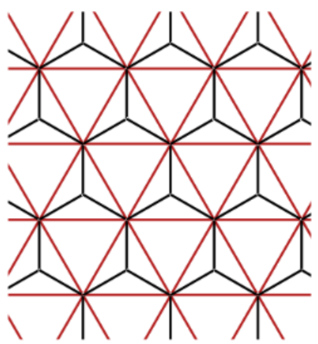

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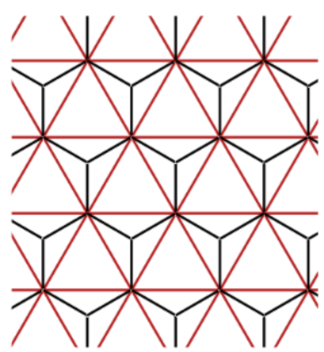

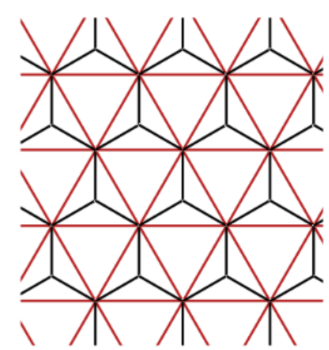

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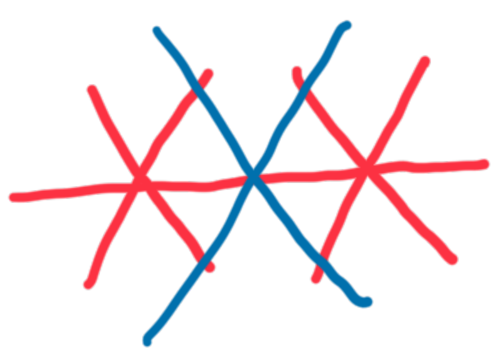

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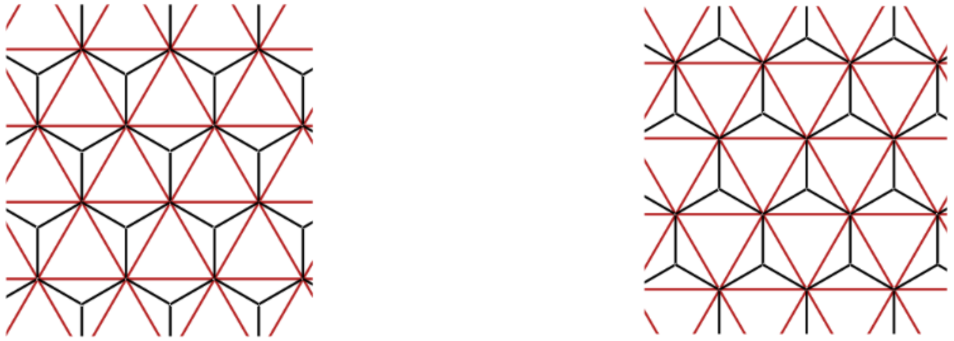

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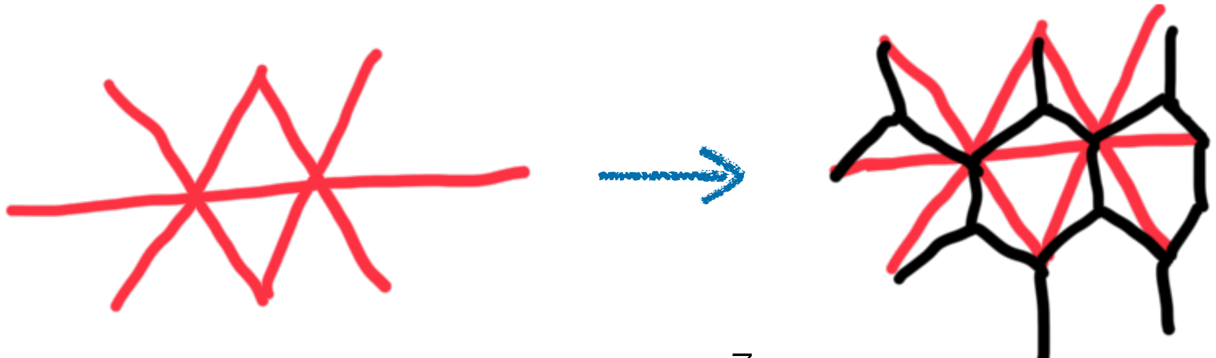

Yangian Symmetry of Fishnet Integrals

- Construction of the **Yangian** algebra $Y(\mathfrak{g})$:

Level 0:

$$\mathrm{J}^a = \sum_{j=1}^{n} \mathrm{J}_j^a$$

Lie algebra $\mathfrak{g}$ with generators J_j^a

Level 1:

$$\widehat{\mathrm{J}}^a = \frac{1}{2} f^a{}_{bc} \sum_{j<k} \mathrm{J}_j^c \mathrm{J}_k^b + \sum_{j=1}^{n} s_j \mathrm{J}_j^a$$

Commutation relations:

$$[\mathrm{J}^a, \mathrm{J}^b] = f^{ab}{}_c \mathrm{J}^c$$

$$\left[\mathrm{J}^a, \widehat{\mathrm{J}}^b\right] = f^{ab}{}_c \widehat{\mathrm{J}}^c$$

- For fishnets we consider the **conformal algebra** $\mathfrak{so}(1, D+1)$:

$$\mathrm{P}_\mu = -i\partial_\mu \qquad L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

$$D = -i(x_\mu \partial^\mu + \Delta) \qquad K_\mu = i(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2\Delta x_\mu)$$

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- Additionally, there are **two-side level one operators**:

$$\widehat{\mathrm{J}}_{jk}^a = \frac{1}{2} f^a{}_{bc} \mathrm{J}_j^c \mathrm{J}_k^b + \tilde{s}_j \mathrm{J}_j^a + \tilde{s}_k \mathrm{J}_k^a$$

with $\widehat{\mathrm{J}}_{jk}^a \dfrac{1}{x_{j0}^{2\nu_j} x_{k0}^{2\nu_k}} = 0$

Permutation Symmetry

- Consider the group of **permutations** of the external points leaving the **integral invariant**:

$$I_G^{(D)}(\sigma \cdot \underline{\alpha}) = I_G^{(D)}(\underline{\alpha})\,,\ \text{for all } \sigma \in \mathrm{Perm}_G$$

- Every **automorphism** of the **graph** acts as a permutation of the external points, i.e.

$$\mathrm{Aut}(G) \subset \mathrm{Perm}_G$$

- But there are **hidden relations** due to the star-triangle relation:

$$G \xrightarrow[\text{relation}]{\text{star-triangle}} G' \qquad \text{such that} \qquad \mathrm{Aut}(G) \subset \mathrm{Aut}(G')$$

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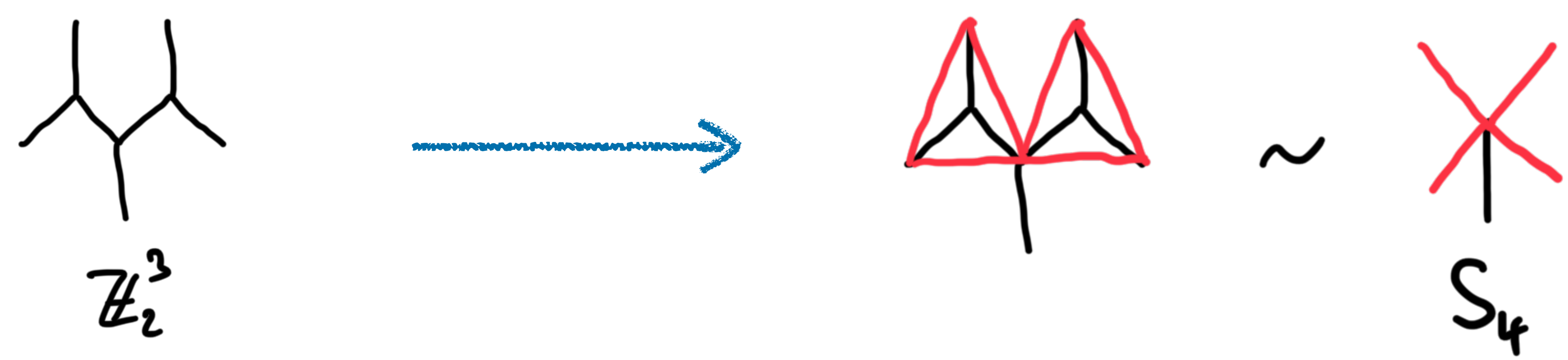
Example:



$$\mathrm{Aut}(G) = \mathbb{Z}_2^3 \subset \mathrm{Perm}_G = S_4$$

Yangian and Permutation Symmetry

- We can **combine Yangian** and **permutation symmetry**:

$$\mathrm{J}^a_\sigma := \sigma \mathrm{J}^a \sigma^{-1} \qquad \text{and} \qquad \widehat{\mathrm{J}}^a_\sigma := \sigma \widehat{\mathrm{J}}^a \sigma^{-1}$$

different representations due to ordering

- Obviously, on the level zero generators this has no effect but on the **level one generators**:

$$\widehat{\mathrm{P}}^\mu_\sigma I^{(D)}_G = 0$$

new differential equations

- In total, **fishnet integrals** have the **symmetry group**:

$$\mathrm{Perm}_G \ltimes \mathcal{Y}_G$$

permutations

Yangian algrebra including two-side generators

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$\longrightarrow$ **In D=2 we conjecture: YES IT DOES.**

Specialities in D=2

- Most importantly, we can use **complex variables** in two dimensions: $\mathbb{R}^2 \simeq \mathbb{C}$

$$a_j := \alpha_j^1 + i\alpha_j^2 \qquad \text{and} \qquad x_j := \xi_j^1 + i\xi_j^2$$

such that the fishnet integral becomes:

$$I_G(\underline{a}) = \int \left(\prod_{j=1}^{\ell} \frac{\mathrm{d}x_j \wedge \mathrm{d}\overline{x}_j}{-2i} \right) \frac{1}{|P_G(\underline{x},\underline{a})|^{4/V}} \qquad \text{with} \qquad P_G(\underline{x},\underline{a}) = \Big[\prod_{i,j} (x_i - x_j) \Big] \Big[\prod_{i,j} (x_i - a_j) \Big]$$

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- The conformal algebra splits into a **holomorphic** and **anti-holomorphic part** likewise the Yangian:

$$Y(\mathfrak{so}(1,3)) = Y(\mathfrak{sl}(2,\mathbb{R})) \oplus \overline{Y(\mathfrak{sl}(2,\mathbb{R}))}$$

- Thus, the whole **symmetry algebra** of the fishnet integrals splits:

$$\mathrm{Perm}_G \ltimes \mathcal{Y}_G = (\mathrm{Perm}_G \ltimes Y_G) \oplus (\mathrm{Perm}_G \ltimes \overline{Y}_G)$$

Yangian differential ideal
YDI(G)

set holomorphic differential operators
annihilating the Fishnet integral

Specialities in D=2

- The whole **integral** can also be split into **holomorphic** and **anti-holomorphic** parts:

$$I_G(\underline{a}) = \left|F_G(\underline{a})\right|^2 \phi_G(\underline{z}) = (-1)^{\frac{\ell(\ell-1)}{2}} (-2i)^{-\ell} \left|F_G(\underline{a})\right|^2 \int \overline{\Omega} \wedge \Omega$$

holomorphic rational function

vector of holomorphic cross ratios

with the holomorphic and conformal $(\ell, 0)$-form:

$$\Omega = \frac{1}{F_G(\underline{a})} \frac{\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_\ell}{P_G(\underline{x}, \underline{a})^{2/V}}$$

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- In the following, we will argue that this form gives rise to a **monodromy invariant bilinear** form:

$$\phi_G(\underline{z}) = (-i)^\ell \, \underline{\Pi}_G(\underline{z})^\dagger \Sigma_G \underline{\Pi}_G(\underline{z})$$

with the **period vector**:

$$\underline{\Pi}_G(\underline{z}) = \left(\int_{\Gamma_0} \Omega, \ldots, \int_{\Gamma_{b_\ell - 1}} \Omega \right)^T$$

associated to a **Calabi-Yau variety** or **Picard variety** for square and hexagonal fishnets, respectively.

Calabi-Yau Manifolds

Definition:

A **Calabi-Yau** (CY) n-fold X is a complex n-dimensional Kähler manifold equipped with a Kähler $(1,1)$-form ω. There are the (equivalent) additional properties:

- the first Chern class vanishes: $c_1(T_X) = 0$
- there exists a Ricci flat metric g: $R_{i\bar{j}}(g) = 0$
- there exists a no-where vanishing holomorphic $(n,0)$-form Ω
- the holonomy group of X is $\mathrm{SU}(N)$
- on X there exist two covariant constant spinors.

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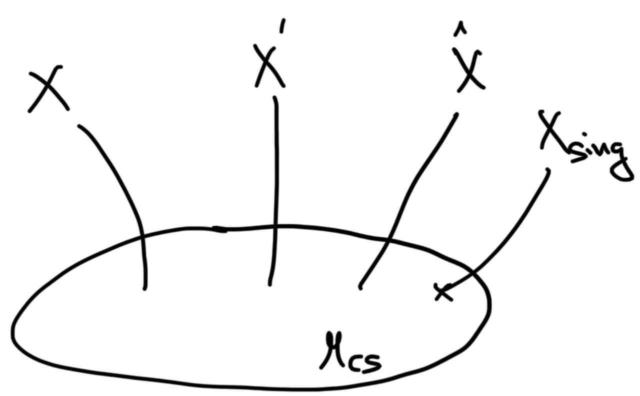
- Forms Ω and ω are both **characteristic** for a CY X $\longrightarrow$ (X, Ω, ω) cf. $(\mathcal{E},\ \mathrm{d}x/y,\ \mathrm{d}x \wedge \mathrm{d}y)$
- The **tangent space** of the **complex structure deformation space** of a CY $\mathcal{M}_{\mathrm{cs}}$ is given by $H^{n-1,1}(X)$.
- It is natural to consider **families** of CYs:



Constructions of CYs

How can we construct CYs?

- CYs can be defined via **polynomial constraints**:

> "Vanishing of the first Chern class $c_1(T_X)$ gives relation between ambient space and degree of the constraints."

- Single polynomial constraint: **Hypersurface CY**

Cubic one-fold: $\{Y^2Z - 4X^3 + g_2(t)XZ^2 + g_3(t)Z^3 = 0\} \subset \mathbb{P}^2$

Quintic three-fold: $\{X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - \psi X_0X_1X_2X_3X_4 = 0\} \subset \mathbb{P}^4$

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- For the fishnets we find:

Square tiling: $\{W = y^2 - P_G([\underline{x} : \underline{u}]; \underline{a}) = 0\} \subset (\mathbb{P}^1)^\ell$

Hexagonal tiling: $\{W = y^3 - P_G([\underline{x} : \underline{u}]; \underline{a}) = 0\} \subset (\mathbb{P}^1)^\ell$

Triangular tiling: no direct CY construction possible only via star-triangle relation

$P_G([\underline{x} : \underline{u}]; \underline{a})$ homogenized version of the fishnet graph polynomial

Periods of a CY

Definition:

Periods define a pairing between the homology $H_n(X)$ and the cohomology $H^n_{\mathrm{dR}}(X)$ of the CY X:

$$\begin{aligned} \Pi: \quad H_n(X) \times H^n_{\mathrm{dR}}(X) &\longrightarrow \mathbb{C} \\ (\Gamma, \alpha) \qquad\qquad &\longmapsto \int_\Gamma \alpha \end{aligned}$$

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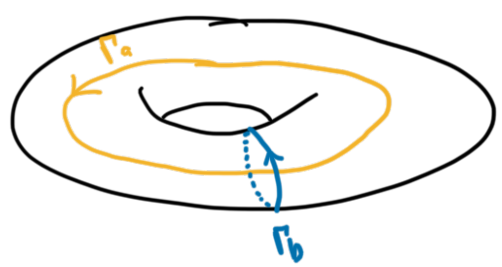
$$P_3 = Y^2 - X(X-1)(X-\lambda)$$



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Elliptic integrals
$K(\lambda), K(1-\lambda)$

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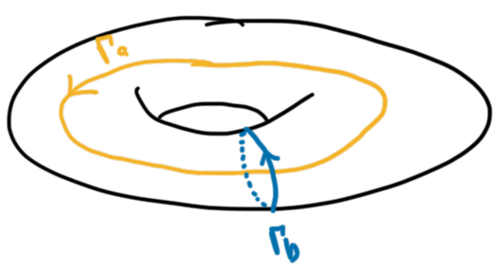

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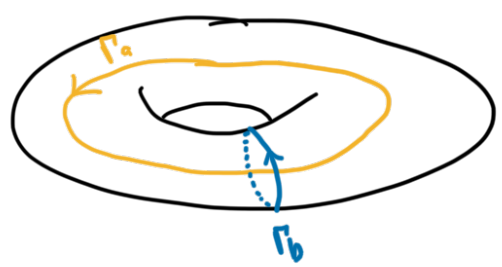

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"Periods describe the shape of a CY."

- Particularly interesting are the periods over Ω, which can be defined through the defining constraints:

$$\Omega = \int_{S^1} \frac{\mu}{P} \longrightarrow \Pi_i = \int_{\Gamma_i} \Omega$$

cf. $\Omega = \int_{S^1} \frac{\mathrm{d}X \wedge \mathrm{d}Y}{P_3} \sim \frac{\mathrm{d}X}{Y}$

- For generic CYs it is not even simple to explicitly define all cycles $\Gamma_i \in H_n(X, \mathbb{Z})$.

Computing Periods

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  - **Integration by Parts** identities
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$$\oint_{T^1} \mathrm{d}x \frac{1}{\sqrt{x(1-x)(x-z)}} = \oint_{T^1} \frac{\mathrm{d}x}{x} \frac{1}{\sqrt{(1-x)(1-z/x)}}$$

$$= \oint_{T^1} \frac{\mathrm{d}x}{x} \sum_{i,j} \binom{2i}{i}\binom{2j}{j} \frac{z^j}{4^{i+j}} x^{i-j} = 2\pi i \sum_{i=0}^{\infty} \binom{2i}{i}^2 \left(\frac{z}{4^2}\right)^i$$

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In a similar way, we have computed the **Picard-Fuchs differential ideal** for our fishnet integrals.

Computing Periods

- A **basis of the solution space** $\{\varpi_i\}$ to these differential equations can be obtained by standard techniques, e.g. **Frobenius Method**.

- This is particularly simple if a **MUM point** (= total degeneration of indicials) exists:

logarithmic structure reflects
the cohomology of the CY

$$
\begin{aligned}
\varpi_0 &= \text{power series in z} \\
\varpi_1 &= \varpi_0 \log(z) + \Sigma_1 \\
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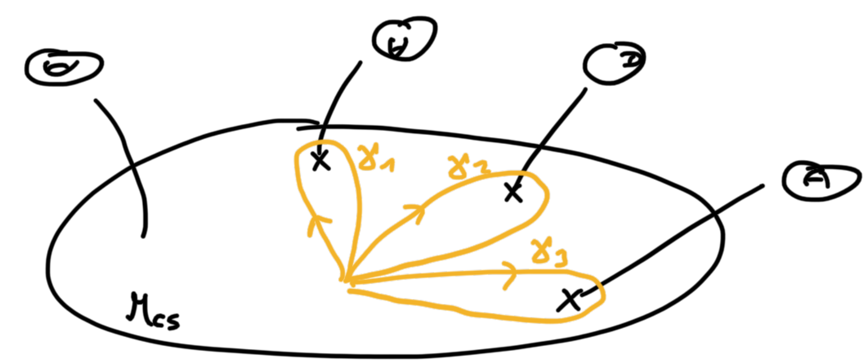
- Finally, a **basis change** from $\{\varpi_i\}$ to $\{\Pi_i\}$ (basis over $\mathbb{Z}$ or $\mathbb{Z}[\alpha]$) has to be determined. This change of basis can be found from **monodromy considerations**:



  - There exist special points in $\mathcal{M}_{\text{cs}}$ where the CY gets singular.
  - Analytic continuation around these points corresponds to a monodromy: $\Pi \longmapsto M_{\gamma_i}\Pi$
  - All monodromies have to respect the intersection pairing Σ between the periods.

    $\Longrightarrow$ In a good basis $\{\Pi_i\}$ all monodromies M_{γ_i} have to be "integral", i.e. $M_{\gamma_i} \in \mathcal{O}(\Sigma, \mathbb{Z})$
  - The deformation method produces for hypergeometric CYs directly a **rational monodromy basis**. [Kerr]

- If all monodromies are known, one can also determine Σ by requiring: $M^T \Sigma M = \Sigma$

Conjeture

The Picard-Fuchs Ideal for Calabi-Yau varieties of square and hexagonal Fishnet integrals is equal to the Yangian Differential Operator Ideal. Therefore, these fishnet integrals are completely fixed by their symmetry algebra.

$$\mathrm{Perm}_G \ltimes \mathcal{Y}_G$$

Griffiths Transversality

- Is there a **better/faster way** on a CY to determine Σ than computing all monodromies?

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- Is there a **better/faster way** on a CY to determine Σ than computing all monodromies?

- On a CY there exists the phenomenon of **Griffiths transversality**:

$$\begin{aligned} \Omega &\in H^{n,0}(X) \\ \partial_z \Omega &\in H^{n,0}(X) \oplus H^{n-1,1}(X) \\ \partial_z^2 \Omega &\in H^{n,0}(X) \oplus H^{n-1,1}(X) \oplus H^{n-2,2}(X) \\ &\vdots \\ \partial_z^n \Omega &\in H^{n,0}(X) \oplus \ldots \oplus H^{0,n}(X) \end{aligned}$$

- Consideration of type forbids many integrals:

$$\boxed{\int_X \Omega \wedge \partial_z^k \Omega = \Pi^T \, \Sigma \, \partial_z^k \Pi = \begin{cases} 0, & k < n \\ C_n, & k = n \end{cases}}$$

The rational function C_n is called the Yukawa Coupling.

- We can use these relations to easily determine Σ.

Monodromy Invariant Bilinear Form

- On a CY there exists a natural real, positiv and **monodromy invariant** object namely the exponential of the **Kähler potential**:

$$\boxed{i^{n^2}\int_X \Omega\wedge\bar{\Omega} = i^{n^2}\Pi^\dagger\Sigma\Pi = e^{-K(z,\bar{z})}}$$

Monodromy invariance follows from:

$$\Pi^\dagger\Sigma\Pi \longrightarrow (M_{\gamma_i}\Pi)^\dagger\,\Sigma\,M_{\gamma_i}\Pi = \Pi^\dagger M_{\gamma_i}^\dagger\Sigma M_{\gamma_i}\Pi = \Pi^\dagger\Sigma\Pi$$

if $M_{\gamma_i}^\dagger = M_{\gamma_i}^T$

- This is particularly satisfied for our **basis of solutions** determined by the **deformation method**. [Kerr]

- The Fishnet integral is now just given by this monodromy invariant bilinear form:

$$I_G(\underline{a}) = (-i)^\ell|F_G(\underline{a})|^2\underline{\Pi}_G(\underline{z})^\dagger\Sigma_G\underline{\Pi}_G(\underline{z})$$

Picard Varieties

- Another useful geometry for fishnet integrals are so-called **Picard curves**:

Tripple covering of $\mathbb{P}^1$:

$$y^3 = \tilde{P}(x, \underline{a})$$

with $\deg(\tilde{P}(x, \underline{a})) > 3$

double covering of $\mathbb{P}^1$:

$$y^2 = \hat{P}(x, \underline{b})$$

for $\deg(\hat{P}(x, \underline{b})) = 3, 4$ we get an elliptic curve

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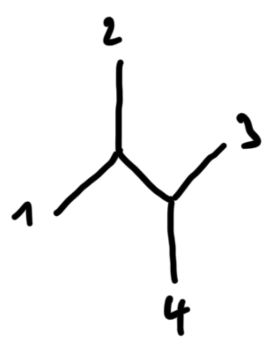
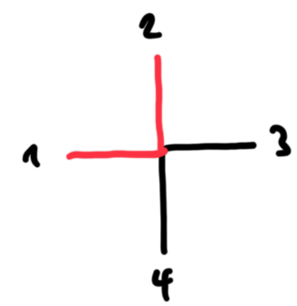


$\xrightarrow{\text{star-triangle relation}}$



$$y^3 = P = (x_1 - a_1)(x_1 - a_2)(x_1 - x_2)(x_2 - a_3)(x_2 - a_4)$$

singular **K3** variety

$$y^3 = \tilde{P} = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

singular **genus two** Picard curve

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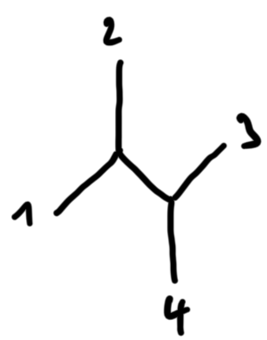
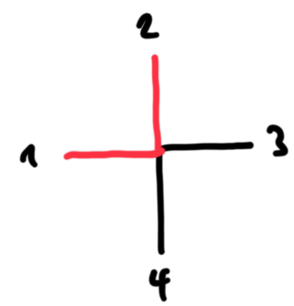
- For the **hexagonal Fishnet integrals** we find using the **start-triangle relation**:



$\xrightarrow[\text{relation}]{\text{star-triangle}}$



$$y^3 = P = (x_1 - a_1)(x_1 - a_2)$$
$$(x_1 - x_2)(x_2 - a_3)(x_2 - a_4)$$

singular **K3** variety

$$y^3 = \tilde{P} = (x - a_1)(x - a_2)(x - a_3)^2(x - a_4)^2$$

singular **genus two** Picard curve

- We can generalize Picard curves also to **Picard varieties**:

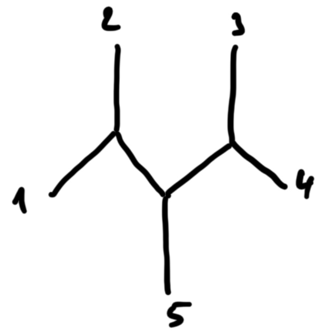
Tripple covering of $(\mathbb{P}^1)^r$:

$$y^3 = \tilde{P}(\underline{x}, \underline{a})$$

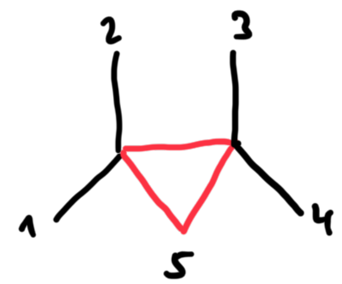
From the star-triangle relation we find in this way usually **singular Picard varieties**.

Calabi-Yau Varieties vs. Picard Varieties

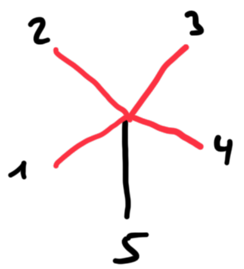
- Using the star-triangle relation we can produce **different geometries** associated to a given Fishnet integral:



CY three-variety



Picard two-variety



Genus three Riemann curve

→ Due to the star-triangle relation we can **not** associate a **unique geometry** to a Fishnet integral. Even the **dimensions** are **different**.

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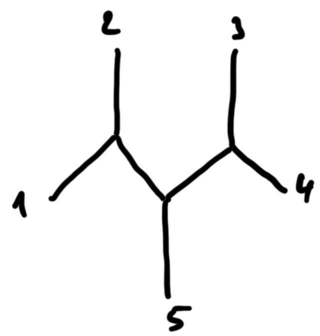

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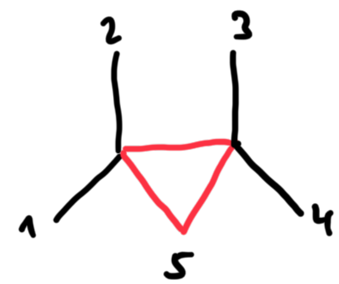

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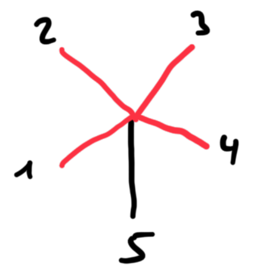
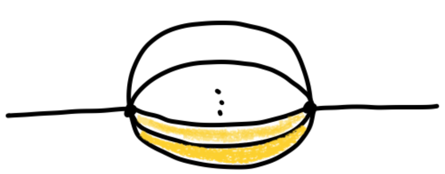


Genus three Riemann curve

Due to the star-triangle relation we can **not** associate a **unique geometry** to a Fishnet integral. Even the **dimensions** are **different**.

- **Similar observations** have been also made in the following cases:

  - Banana integrals:



$\mathcal{F} = 0$ hypersurface CY

$P_1 = P_2 = 0$ complete intersection CY

[Bönisch, Duhr, Fischbach, Klemm, CN]

  - Genus drop in Feynman integrals:

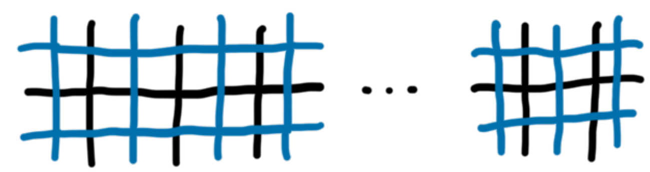
genus three

genus two

[Marzucca, McLeod, Page, Pögel, Weinzierl]

Examples: Train Track Graphs

- Our first examples are the so-called **train track graphs** $G_{1,\ell}$:

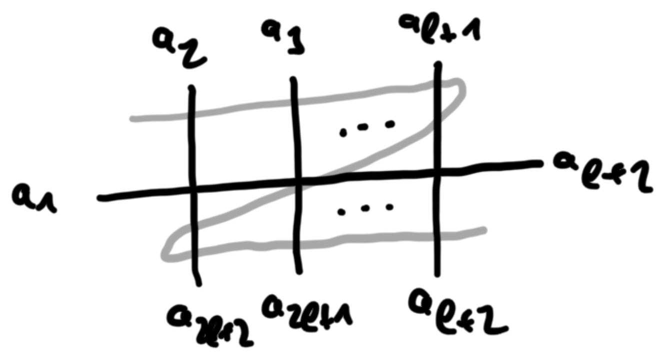


- These graphs have **no hidden symmetries** such that we find for the **permutation symmetry group**:

$$\mathrm{Perm}_{G_{1,\ell}} = \mathrm{Aut}(G_{1,\ell}) = \begin{cases} S_4\,, & \ell = 1 \\ S_3^2 \times \mathbb{Z}_2^{\ell-2} \times \mathbb{Z}_2\,, & \ell > 1\,, \end{cases}$$

- The following **cross ratios** give rise to a **MUM point**:

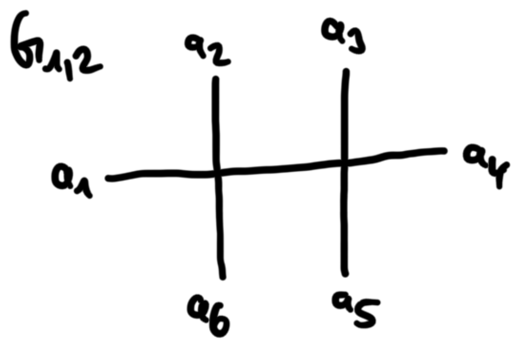
$$z_k = \frac{1}{4}\,\chi_{1,k+1,k+2,\ell+2}\,, \quad z_\ell = \frac{1}{4^{3-l}}\,\chi_{1,\ell+1,2\ell+2,\ell+2}\,, \quad z_{\ell+k} = \frac{1}{4}\,\chi_{1,2\ell+3-k,2\ell+2-k,\ell+2}$$



- For the **prefactor** we have chosen:

$$F^{(2)}_{G_{1,\ell}}(\underline{a}) = \frac{|a_1 - a_{\ell+2}|^{\ell-1}}{|a_{\ell+3} - a_1||a_{\ell+4} - a_1| \cdots |a_{2\ell+2} - a_1||a_2 - a_{\ell+2}||a_3 - a_{\ell+2}| \cdots |a_{\ell+1} - a_{\ell+2}|}$$

Examples: Two-Loop Train Track



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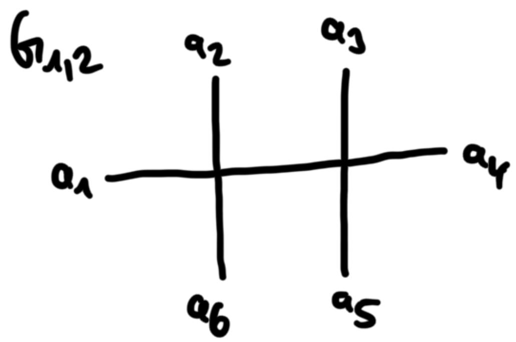

- Using the previous MUM point variables the **Yangian Differential Ideal** is generated by:

$$\theta_i = z_i \partial_i$$

$$\begin{aligned}
\mathcal{D}_{G_{1,2},1} &= \theta_1^2 - 2z_1\left(\theta_1-\theta_2\right)\left(1+2\theta_1+2\theta_2\right) - 4z_1z_2\left(1+2\theta_2-2\theta_3\right)^2 \\
&\quad - 32z_1z_2z_3\left(1+2\theta_2-\theta_3\right)\left(1+2\theta_3\right)\ , \\
\mathcal{D}_{G_{1,2},2} &= \theta_1\theta_2 - \theta_3\left(\theta_2-\theta_3\right) + 2z_3\left(\theta_2-\theta_3\right)\left(1+2\theta_3\right) - 4z_1z_2\left(1+2\theta_1\right)\left(1+2\theta_2-2\theta_3\right) \\
&\quad - 4z_1z_2z_3\left(1+2\theta_1\right)\left(4+8\theta_3\right)\ , \\
\mathcal{D}_{G_{1,2},3} &= (\theta_1-\theta_2)\theta_3 + 4z_3\left(\theta_1-\theta_2\right)\left(\theta_2-\theta_3\right) \\
&\quad + 4z_2z_3\left(-4\theta_1\left(1+\theta_2\right)+\left(1+2\theta_2\right)^2 - 4\theta_2\theta_3+4\theta_3^2\right) + 32z_2z_3^2\left(\theta_2-\theta_3\right)\left(1+2\theta_3\right)
\end{aligned}$$

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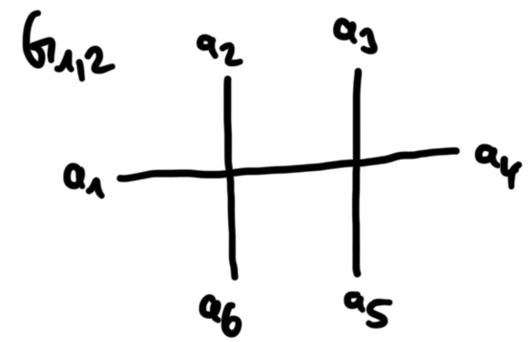

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&\quad - 32 z_1 z_2 z_3\left(1 + 2\theta_2 - \theta_3\right)\left(1 + 2\theta_3\right)\,, \\
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&\quad + 4z_2 z_3\left(-4\theta_1\left(1 + \theta_2\right) + \left(1 + 2\theta_2\right)^2 - 4\theta_2\theta_3 + 4\theta_3^2\right) + 32 z_2 z_3^2\left(\theta_2 - \theta_3\right)\left(1 + 2\theta_3\right)
\end{aligned}$$

- We find **5=1+3+1 solutions** as expected from a three-parameter K3 surface.

- These solutions can be constructed from the **deformation method**:

$$\begin{aligned}
\Phi_{G_{1,2},0}(\underline{z}) &= \varpi(\underline{z};0) \\
\Phi_{G_{1,2},1,i}(\underline{z}) &= \partial_{\rho_i}\varpi(\underline{z};\underline{\rho})|_{\underline{\rho}=0} \\
\Phi_{G_{1,2},2}(\underline{z}) &= \left[\partial_{\rho_2}^2 + 2\left(\partial_{\rho_1}\partial_{\rho_2}\right.\right. \\
&\qquad \left.\left. + \partial_{\rho_1}\partial_{\rho_3} + \partial_{\rho_2}\partial_{\rho_3}\right)\right]\varpi(\underline{z};\underline{\rho})|_{\underline{\rho}=0}
\end{aligned}$$

$$\varpi(\underline{z};\underline{\rho}) = \sum_{\underline{n}=0}^{\infty} c(\underline{n}+\underline{\rho})\underline{z}^{\underline{n}+\underline{\rho}}$$

$$c(\underline{n}) = (n_1)(n_3)(n_2 - n_1)(n_2 - n_3)(n_1 - n_2 + n_3)$$

Examples: Two-Loop Train Track

- We get a **rational monodromy basis** after normalizing the logarithms:

$$\underline{\Pi}_{G_{1,2}}(\underline{z}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2\pi i} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2\pi i} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\pi i} & 0 \\ -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{(2\pi i)^2} \end{pmatrix} \underline{\Phi}_{G_{1,2}}(\underline{z})$$

- The **two-loop train track integral** is then given by

$$\phi_{G_{1,2}}(\underline{z}) = -\underline{\Pi}_{G_{1,2}}(\underline{z})^\dagger \Sigma_{G_{1,2}} \underline{\Pi}_{G_{1,2}}(\underline{z})$$

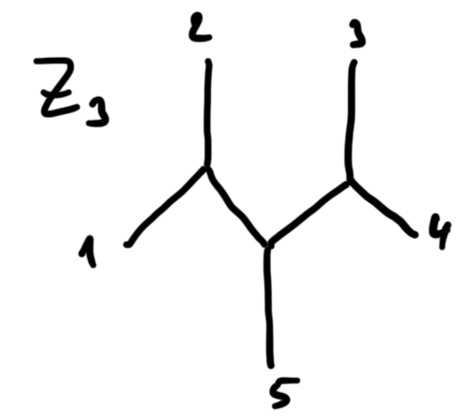
with **intersection form**:

$$\Sigma_{G_{1,2}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Examples: Triangle Tracks

- Let us particularly discuss the **three-loop triangle track integral**. Its permutation symmetry is given by:

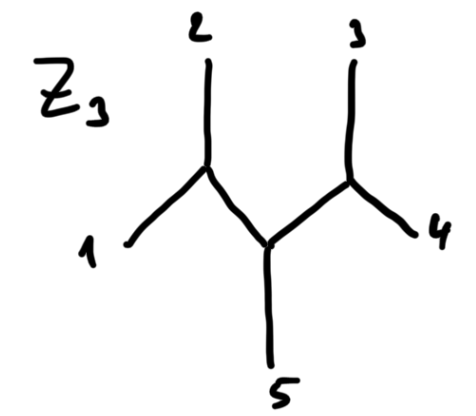
$$\mathrm{Perm}_{Z_3} = S_4$$



- Convenient **variables** are given by the following two cross ratios:

$$z_1 = \frac{1}{3^3}\chi_{1,5,3,4}\,, \qquad z_2 = \chi_{1,3,2,4}$$

Examples: Triangle Tracks



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$$\begin{aligned}
\mathcal{D}_{Z_3,1} &= \theta_1\theta_2 - 9z_1\left(1+3\theta_1-3\theta_2\right)\theta_2 - 3z_1z_2\left[2+9\theta_2\left(1+\theta_2\right)\right]\,,\\
\mathcal{D}_{Z_3,2} &= \theta_2\left(-1+3\theta_2\right) + z_2\left[-3\theta_1^2+\theta_1\left(1+3\theta_2\right)-\theta_2\left(1+3\theta_2\right)\right]\\
&\quad + 27z_1z_2\left[6\theta_1^2+\theta_1\left(2-6\theta_2\right)-3\theta_2\left(1+\theta_2\right)\right] - 9z_1z_2\left[27z_1\left(2+3\theta_1\right)\left(1+3\theta_1-3\theta_2\right)\right.\\
&\quad \left.-z_2\left(2+9\theta_2\left(1+\theta_2\right)\right)\right]
\end{aligned}$$

which is the set of differential equations of an **Appell hypergeometric system**:

$$\begin{aligned}
\Phi_{Z_3,0}(\underline{z}) &= F_1(2/3,1/3,1/3,1;3^3z_1z_2,3^3z_1)\\
&= 1+6z_1+\left(90z_1^2+6z_1z_2\right)+\left(1680z_1^3+45z_1^2z_2\right)+\mathcal{O}(z_i^4)\,,\\
\Phi_{Z_3,1}(\underline{z}) &= \Phi_0(\underline{z})\log(z_1)+\left(15z_1-\frac{z_2}{2}\right)+\left(\frac{513z_1^2}{2}+3z_1z_2-\frac{z_2^2}{5}\right)+\mathcal{O}(z_i^3)\,,\\
\varphi_{Z_3,0}(\underline{z}) &= z_2^{1/3}\left[1+\frac{\lambda z_2}{6}+\lambda^2\left(9z_1z_2+\frac{5z_2^2}{63}\right)+\lambda^3\left(\frac{15}{7}z_1z_2^2+\frac{4z_2^3}{81}\right)+\mathcal{O}(z_i^4)\right]
\end{aligned}$$

Examples: Triangle Tracks

- In this case, we have to compute the **intersection form** computing **all monodromies** and requiring that:

$$M^T \Sigma_{Z_3} M = \Sigma_{Z_3}$$

From this we then find:

$$\Sigma_{Z_3} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -i\sqrt{3} \end{pmatrix}$$

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- With the intersection form we find similarly as before for the **three-loop triangle track integral**:

$$I_{Z_3}(\underline{a}) = i \frac{|a_{14}|^{2/3}}{|a_{12}|^{4/3}|a_{13}|^{2/3}|a_{45}|^{4/3}|a_{34}|^{2/3}} \underline{\Pi}_{Z_3}(\underline{z})^\dagger \Sigma_{Z_3} \underline{\Pi}_{Z_3}(\underline{z})$$

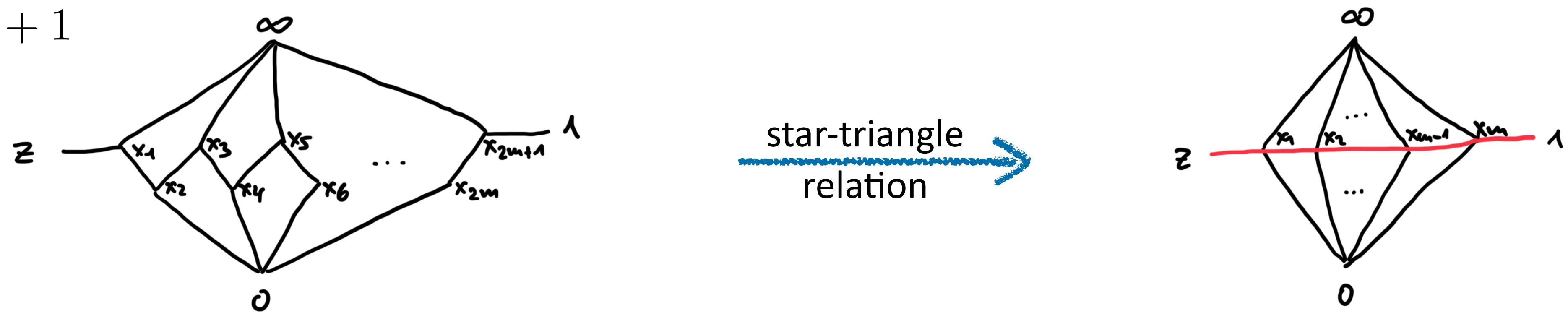
with

$$\underline{\Pi}_{Z_3}(\underline{z}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2\pi i} & 0 \\ 0 & 0 & \frac{2i\pi}{\Gamma\left(\frac{1}{3}\right)^3} \end{pmatrix} \underline{\Phi}_{Z_3}(\underline{z})$$

Examples: 4-pt Limit of Triangle Tracks

- We can also consider the **4-pt limit** of the **triangle track integrals** (similarly as the ladder graphs):

$\ell = 2m + 1$



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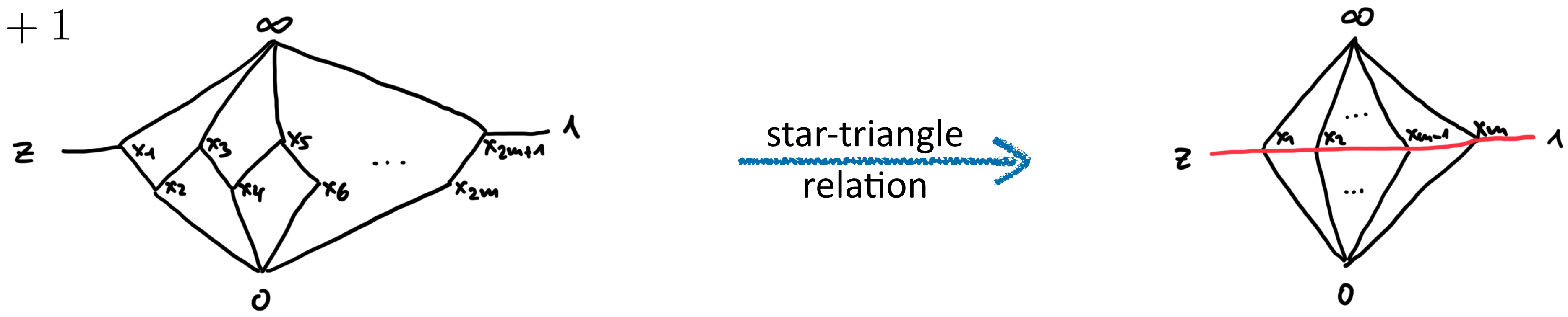

- These integrals are related to interesting **hypergeometric period integrals**:

$$z = \frac{1}{(3\sqrt{3})^{m+1}} \chi_{1,4,2,3}$$

$$\mathcal{L}_m^{\text{Odd}} = \theta^{m+1} - (\sqrt{3})^{m+1} z(1+3\theta)^{m+1}\,,$$

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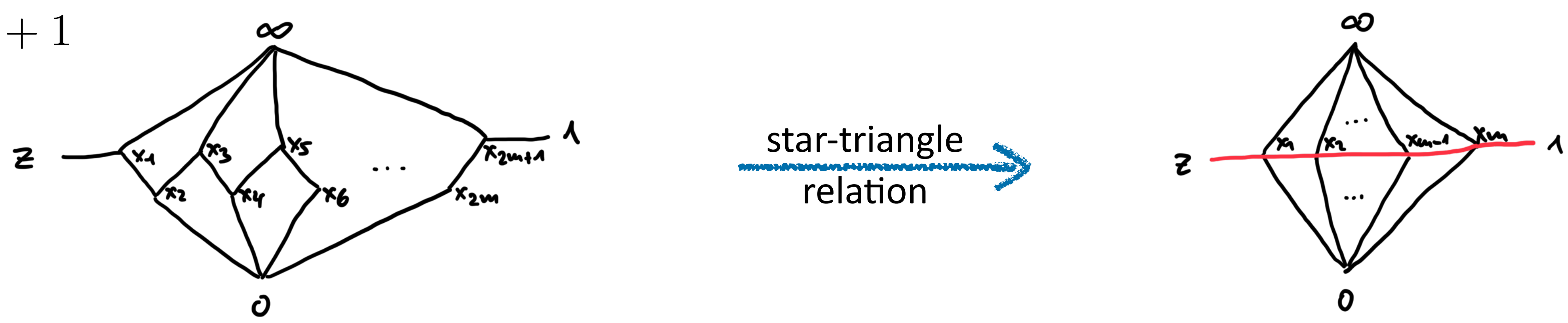

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- These hypergeometric systems give rise to $\mathbb{Z}[\alpha]$-**integral monodromies** ($\alpha = e^{i\pi/3}$):

$$M_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad M_{\frac{1}{3^6}} = \begin{pmatrix} 1 & 2-\alpha & -(1+\alpha) & 3(1-\alpha) \\ 0 & 1-\alpha & -1+\alpha & -(1+\alpha) \\ 0 & 1-\alpha & 0 & 1-2\alpha \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} -2+3\alpha & 7-11\alpha & -2(1-2\alpha) & -3\alpha \\ \alpha & -(4+\alpha) & 2 & -2+\alpha \\ 2\alpha & -2(1+3\alpha) & 1+2\alpha & -1-\alpha \\ 0 & 2 & -1 & 1 \end{pmatrix}$$

- To construct the $\mathbb{Z}[\alpha]$-integral monodromy basis we need **interesting transcendental numbers**:

$\pi, \sqrt{3}, \zeta(n)$

$\zeta(n, 1/3)$ Hurwitz ζ-function

Conclusion

- We have analyzed **Fishnet integrals** in **two spacetime dimensions** with special emphasis on the interplay between their symmetries and geometries.

- We have seen that in **two dimensions** the Fishnet integrals are **fully determined** by their symmetries, i.e. **Yangian** and **permutation symmetry**.

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$\rightarrow$ Sven's talk

Thank you for your attention