Double scaling limit of rectangular fishnets

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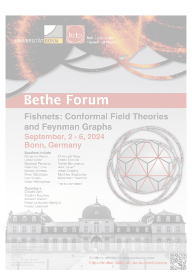


Fishnets: Conformal Field Theories and Feynman Graphs

Bethe Center for Theoretical Physics

Motivation:

Is there a dual description of open fishnets in terms of string world surface?

If so, it is expected to depend on

— the intrinsic geometry (the form of the fishnet)

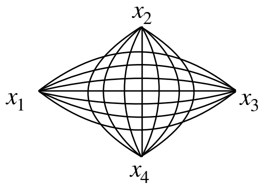
— the embedding of the boundary in the Minkowski space.

Explicit solutions to the thermodynamical (large size) limit of open fishnets could shade light on that.

The simplest case of a 4-point $m \times n$ rectangular fishnet have a simple matrix-model-like representation [Basso and Dixon, 2017].

— intrinsic geometry = aspect ratio n/m

— embedding of the boundary = coordinates of the four operators



[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021] observed that:

— For generic kinematics, the thermodynamical limit depend only on the intrinsic geometry

— The embedding shows up only in a **scaling limit** with a pair of the points getting light-like.

This talk: the solution in the most general **double scaling limit** where two pairs of points become nearly light-like.

1. Rectangular fishnet graphs

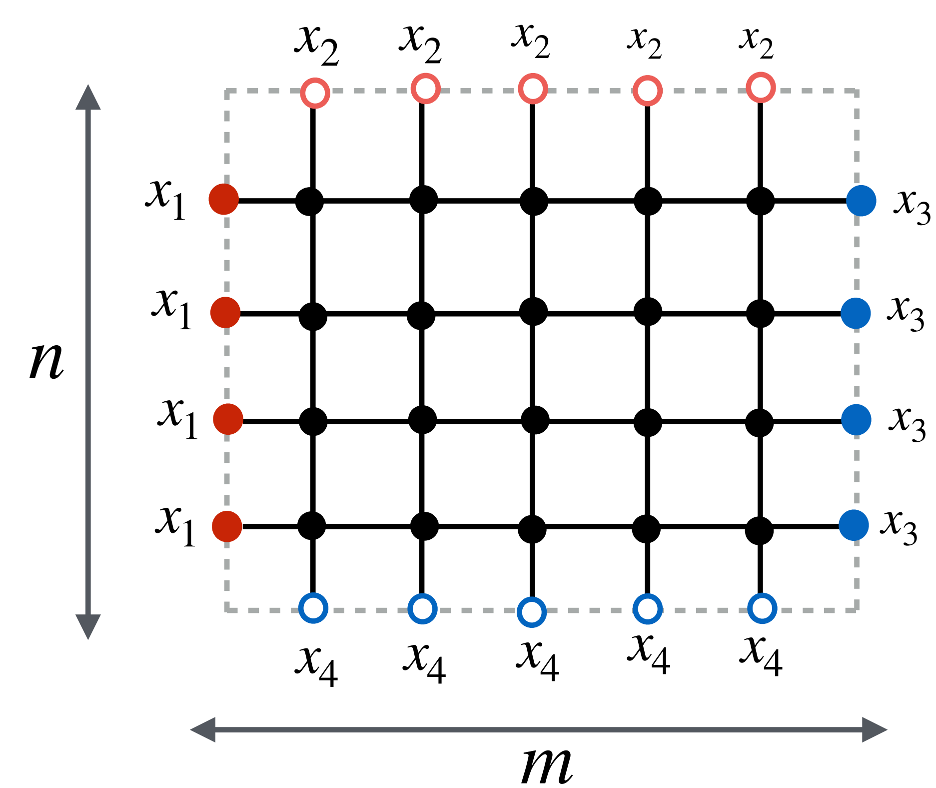
Open fishnets: single-trace correlators in the fishnet theory. In most cases described by a single planar graph (but not always!).

[Aprile, Basso, Caetano, Chicherin, Derkachov, Dixon, Duhr, Ferrando, Fleury, Gromov, Kazakov, Klemm, Korchemsky, Loebbert, Müller, Münkler, Nega, Negro, Olivucci, Porkert, Preti, Sever, Sizov, Staudacher, Stawinski, Zhong, …]

— Interesting mathematical objects: SoV, Y-B, Yangian, Calabi-Yau, …

The simplest **4-point correlators in the fishnet CFT** [Basso-Dixon]

$$G_{m,n}(x_1,x_2,x_3,x_4) = \left\langle \mathrm{Tr}\{\phi_2(x_1)^n \phi_1(x_2)^m \phi_2^\dagger(x_3)^n \phi_1^\dagger(x_4)^m\} \right\rangle$$



Can be looked at as a lattice model defined on a rectangle with four different Dirichlet b.c. on the edges

$$G_{m,n}(x_1,x_2,x_3,x_4) = \int_{\mathbb{R}^4} \prod_{r \in \text{bulk}} d^4x(r) \prod_{r \bullet\!\!-\!\!\bullet r'} \frac{1}{|x(r)-x(r')|^2}$$

- Fluctuation variable $x \in \mathbb{R}^4$,
- nearest-neighbour interaction $|x-y|^{-2}$ •——•

— Exactly solvable open spin chain with $SO(1,5)$ symmetry [Derkachov-Olivucci, 2020], using the SOV techniques in [Derkachov-Korchemsky-Manashov,2001].

— Continuum limit, if exists, is different from that for cylindrical fishnets [Basso-Zhong, Gromov-Sever]

Conformal symmetry

$G_{m,n}(x_1, x_2, x_3, x_4)$ is a correlation function of spinless fields with dimensions $\Delta_2 = \Delta_4 = m,\ \Delta_1 = \Delta_3 = n$

By the conformal invariance, the correlator depends, up to a standard factor, on the positions x_1, x_2, x_3, x_4 through the two conformal invariant cross ratios

$$U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = \frac{z\bar{z}}{(1-z)(1-\bar{z})}, \qquad V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = \frac{1}{(1-z)(1-\bar{z})}$$

By conformal transformation $x_1 = (0,0),\ x_2 = (z, \bar{z}),\ x_3 = (\infty, \infty),\ x_4 = (1,1)$

Parametrisation by hyperbolic angles: $z = -e^{-\sigma-\varphi}, \quad \bar{z} = -e^{-\sigma+\varphi}$ (in Minkowski kinematics $\sigma, \varphi \in \mathbb{R}$)

$$G_{m,n}(x_1, x_2, x_3, x_4) = \frac{g^{2mn}}{(x_{13}^2)^n (x_{24}^2)^m} \times I_{m,n}^{\mathrm{BD}}(z, \bar{z})$$

Basso-Dixon integral

2. Basso-Dixon integral

Matrix-model like integral conjectured by Basso and Dixon (2017) using the AdS/CFT integrability and proved by Derkachov and Olivucci (2019-2020).

$$I^{\mathrm{BD}}_{m,n}(z,\bar{z}) = (2\cosh\sigma + 2\cosh\varphi)^m$$

$$\times \sum_{a_1,\ldots,a_m=1}^{\infty} \prod_{j=1}^{m} \frac{\sinh(a_i\varphi)}{\sinh\varphi}\, a_j\, (-1)^{a_j-1} \int \prod_{j=1}^{m} \frac{du_j}{2\pi}\, \exp(2i\sigma u_j)$$

$$\times \prod_{i=1}^{m} \left(u_j^2 + \frac{a_j^2}{4}\right)^{-m-n} \prod_{i<j} \left[(u_i-u_j)^2 + \frac{(a_i+a_j)^2}{4}\right]\left[(u_i-u_j)^2 + \frac{(a_i-a_j)^2}{4}\right]$$

"Dual integral representation"[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

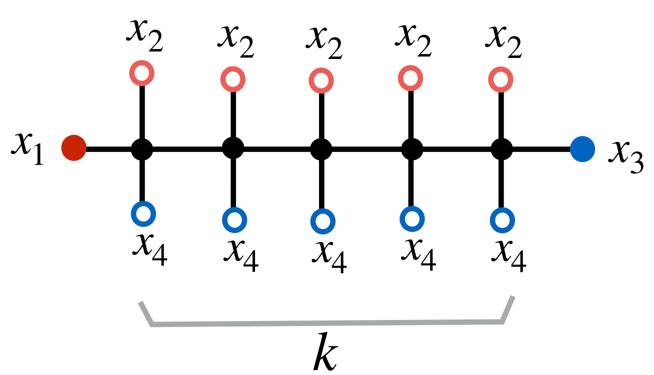
By Fourier transformation $u \to i\partial/\partial t,\ \partial/\partial u \to -it$, the discrete sum can be done explicitly

$$I^{\mathrm{BD}}_{m,n}(z,\bar{z}) = \frac{1}{\mathcal{N}}\frac{1}{m!}\int_{|\sigma|}^{\infty} \prod_{j=1}^{m} dt_j\, (t_j^2-\sigma^2)^{(n-m)}\, \frac{\cosh\sigma + \cosh\varphi}{\cosh t_j + \cosh\varphi} \prod_{j,k=1}^{m}(t_j+t_k)\prod_{j<k}(t_j-t_k)^2$$

"Gluing ladders into fishnet"

B-D integral generalises the integral for the ladder diagrams ($m = 1$)

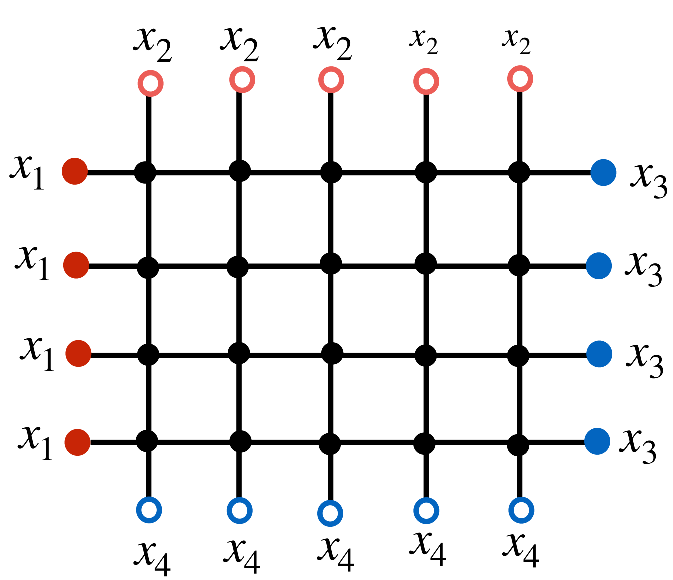
k-ladder diagram:



$$f_k(z, \bar{z}) = \int_{|\sigma|}^{\infty} \frac{\cosh \sigma + \cosh \varphi}{\cosh t + \cosh \varphi} (t^2 - \sigma^2)^{k-1}\ 2t dt$$

Broadhurst-Davydychev, 2010

$m \times n$ fishnet ($n = m + \ell$):



$$I^{\mathrm{BD}}_{m,m+\ell} = \frac{1}{\mathcal{N}} \det \left(\left[f_{j+k+\ell-1} \right]_{j,k=1,\ldots,m} \right)$$

B. Basso, L. Dixon 1705.03545,

Effective matrix model:

— The B-D integral takes the form of the partition function of the $O(n)$ matrix model with $n = -2$ and unusual confining potential:

$$I_{m,n}^{\mathrm{BD}} = \mathcal{Z}_m(\ell, \sigma, \varphi), \qquad \ell \equiv n - m \quad \text{- "bridge"}$$

$$\mathcal{Z}_m(\ell, \sigma, \varphi) = \frac{1}{\mathcal{N}} \frac{1}{m!} \int_{|\sigma|}^{\infty} \prod_{j=1}^{m} dt_j \, e^{-V(t_j)} \prod_{j,k=1}^{m} (t_j + t_k) \prod_{j<k}^{m} (t_j - t_k)^2$$

$$V(t) = \log \frac{\cosh t + \cosh \varphi}{\cosh \sigma + \cosh \varphi} - \ell \log(t^2 - \sigma^2)$$

+ infinite wall at $t = |\sigma|$

• unusual confining potential:

— grows slowly (linearly) at $t \to \pm\infty$;

— $V'(t)$ has an infinite array of simple poles on the imaginary axis.

3. Thermodynamical limit $(m, n \to \infty)$

Effective action: $\mathcal{S} \equiv \sum_{j=1}^{m} V(t_j) - \sum_{k\neq j}^{m} \log(t_k^2 - t_j^2) - \sum_{j=1}^{m} \log(2t_j)$

$$-V'(t_j) + \sum_{k\neq j}^{m} \frac{2}{t_j - t_k} + \sum_{k=1}^{m} \frac{2}{t_j + t_k} = 0 \qquad (j = 1,...,m)$$

$\Rightarrow$ Riemann-Hilbert problem for the meromorphic function

$$H(t) \equiv -\frac{1}{2}V'(t) + G(t) - G(-t),$$

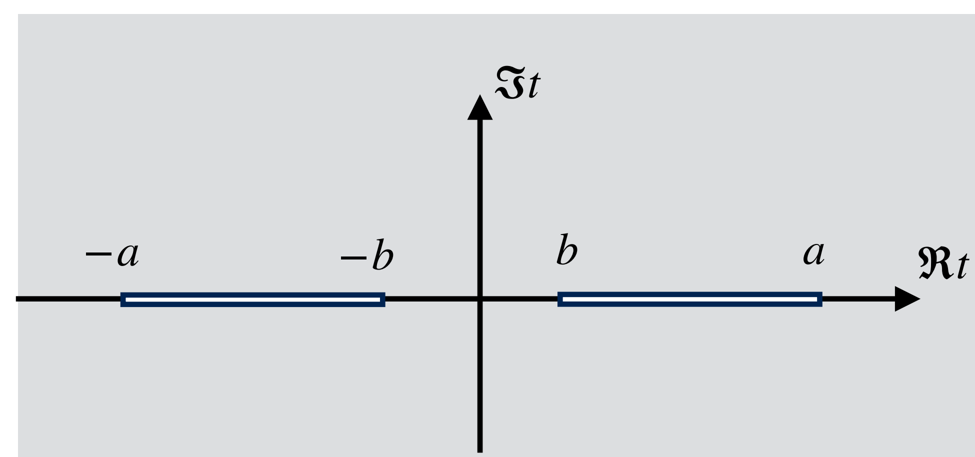
$$G(t) = \sum_{k=1}^{m} \frac{1}{t - t_k} = \int_b^a \frac{dt'\rho(t')}{t - t'}$$

R-H problem:

1) $H(t)$ is analytic in the t-plane with two symmetric cuts $[b, a]$ and $[-a, -b]$ and a puncture at $t = \infty$.

2) $H(t)$ satisfies for t on the real axis

$$H(t + i0) + H(t - i0) = 0 \text{ on the cuts,}$$
$$H(t + i0) - H(t - i0) = 0 \text{ outside the cuts}$$



3) Asymptotics at infinity: $H(t)) = -\frac{1}{2}V'(t) + \frac{2m}{t} + O(t^{-3})$

Explicit solution of the R-H problem [M. Gaudin, unpublished notes, 1988]

The solution for any potential $V(t)$ analytic around the real axis takes form of elliptic integral

$$H(t) = -2\int_b^a \frac{dt_1}{2\pi}\,\frac{y(t)}{y(t_1)}\,\frac{tV'(t) - t_1 V'(t_1)}{t^2 - t_1^2}, \qquad y = \frac{1}{a}\sqrt{(a^2 - t^2)(t^2 - b^2)}$$

$$\int_b^a \frac{dt}{y(t)} V'(t) = 0, \qquad \frac{1}{a}\int_b^a \frac{dt}{y(t)}\, t^2\, V'(t) = 2\pi m \quad \Rightarrow \quad a, b$$

Our potential is given by different analytic expressions in different kinematical domains ⇒ different scaling regimes in the thermodynamical limit

$$V(t) = \log \frac{\cosh t + \cosh \varphi}{\cosh \sigma + \cosh \varphi} - \ell \log(t^2 - \sigma^2)$$

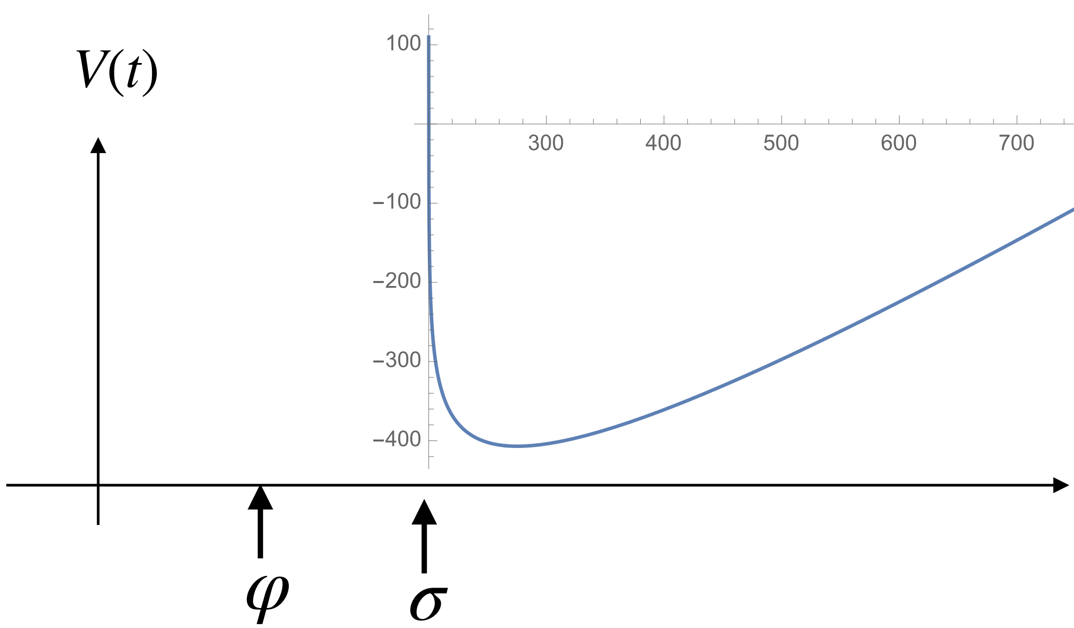
— Infinite potential wall at $t = \sigma$, grows linearly at $t \to \infty$

$\Rightarrow$ the "eigenvalues" are confined to the interval $t > \sigma$ and spread at distance $\sim m$

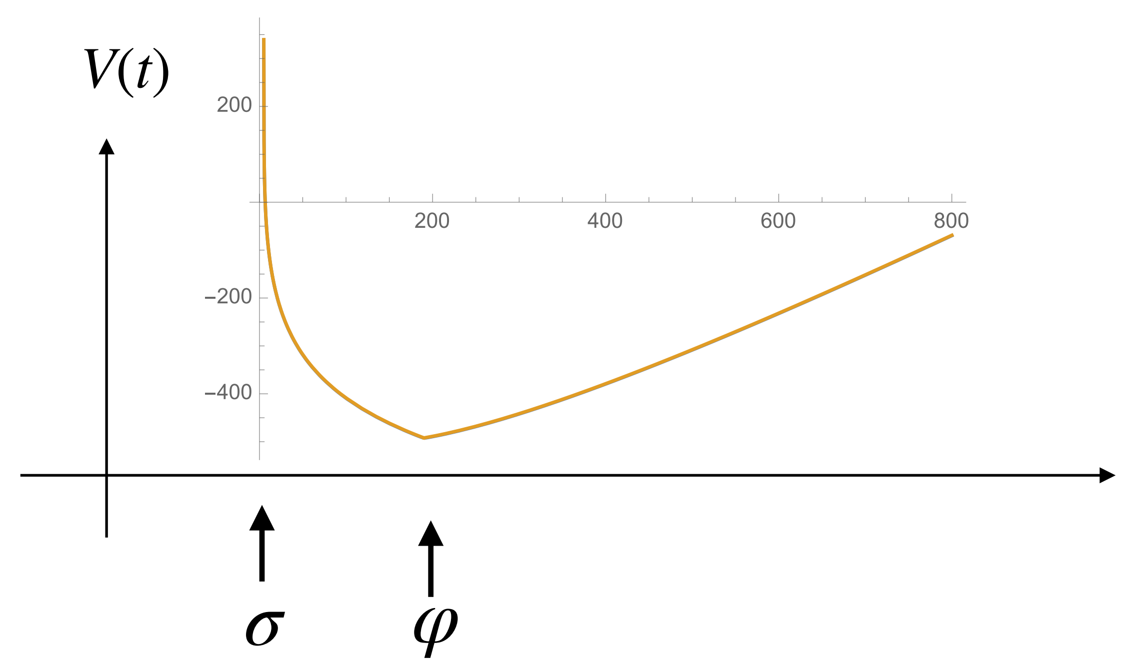
— Infinite array of logarithmic poles which lead to a cusp at $t = \varphi$ if $\varphi \sim m$

If $\sigma, \varphi \sim m$, the solution depends on whether $|\varphi| < |\sigma|$ or $|\varphi| > |\sigma|$:

$$\frac{\partial V(t)}{\partial t} \underset{t\to\infty}{\to} \mathrm{sgn}(t)\,\theta\left(|t| - |\varphi|\right) - \frac{2t}{\sigma^2 - t^2}\ell, \qquad t \in \mathbb{R}$$



Regime I: $|\varphi| \leq |\sigma|$



Regime II: $|\varphi| > |\sigma|$

Bulk, scaling and double scaling limits

- In the **bulk thermodynamical limit**

$m \to \infty$ with $\hat{\ell} = \ell/m$, σ and φ finite,

$V(t) \to |t| - \ell \log(t^2) \Rightarrow$ the solution depends only on ℓ

Generic position of the four points

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

- In the **scaling limit**

$m, \sigma \to \infty$ with $\hat{\ell} = \ell/m$, $\hat{\sigma} \sim \sigma/m$ and φ finite,

$V(t) = \log|t| - \ell \log(t^2 - \sigma^2) \Rightarrow$ the solution depends on ℓ and σ

Euclidean short-distance limit (always in regime I)

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

- In the **double scaling limit**

$m, \sigma, \varphi \to \infty$ with $\hat{\ell} = \ell/m$, $\hat{\sigma} \sim \sigma/m$, $\hat{\varphi} \sim \varphi/m$ finite

Double light-cone limit (regime I and regime II)

[I.K., 2022]

$V(t) \to \max\left(|t|, |\varphi|\right) - \max\left(|\sigma|, \varphi|\right) - \ell \log(t^2 - \sigma^2) \Rightarrow$ the solution depends on ℓ, σ and φ

This is the most general limit containing the other two as particular cases.

3. Solution in the double scaling limit

We are looking for the spectral density $\rho(t)$ and the free-energy density in the double scaling limit

$$\ell, m, \sigma, \varphi, t \to \infty \qquad \text{with } \hat{\sigma} = \frac{\sigma}{m}, \ \hat{\varphi} = \frac{\varphi}{m}, \ \hat{\ell} = \frac{\ell}{m}, \ \hat{t} = \frac{t}{m} \ \text{ finite.}$$

However we will work with the original variables keeping in mind that they all scale as m.

The **"Free energy"** defined as $\mathcal{F}_m(\ell, \sigma, \varphi) \equiv \log \mathcal{Z}_m(\ell, \sigma, \varphi)$ is an extended quantity: it grows as "area" $mn = m(m + \ell)$

$$\hat{\mathcal{F}}(\hat{\sigma}, \hat{\varphi}, \hat{\ell}) = \lim_{m\to\infty} \frac{\mathcal{F}_m(\ell, \sigma, \varphi)}{m(m+\ell)} \qquad \text{— free energy per unit area (finite).}$$

Assume we have computed the function $H(t)$. Then $\Phi(t) = \Phi(-t) = \int_t^\infty H(t)dt$ gives the effective potential of a probe particle at the point $t \in \mathbb{C}$ in the collective field of the other particles. The effective potential is constant on the support of the spectral density: $\Phi(t) = \Phi_0, \quad b < |t| < a$. The constant $\Phi_0 = \Phi(a)$ is the energy needed to bring a new particle from $t = \infty$ to $t = a$, hence $\partial_m \mathcal{S} = \Phi_0$. The second derivative of the free energy is simply related to the positions of the branch points

$$\partial_m^2 \mathcal{F} = -\partial_m^2 \log \mathcal{N} - \partial_m \Phi(a) = 2 \log \frac{a^2 - b^2}{4(2m + \ell)^2}$$

The solution in **regime I** $(\ell, \sigma, \varphi \sim m, \ \ |\varphi| < |\sigma|)$

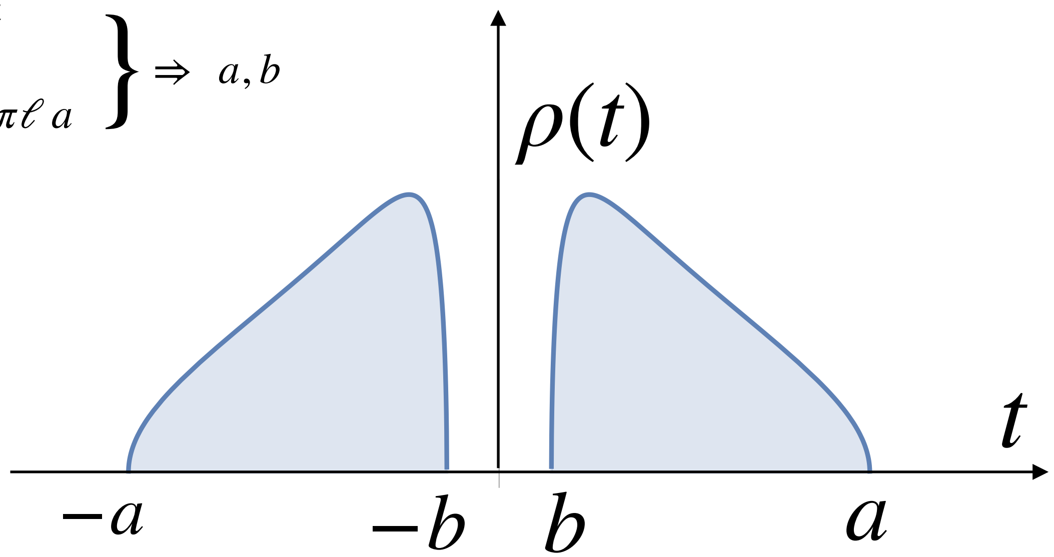
[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

- Density:

$$\rho(t) = \frac{1}{\pi}\frac{\ell\, t}{t^2-\sigma^2}\sqrt{\frac{(a^2-t^2)(t^2-b^2)}{(a^2-\sigma^2)(b^2-\sigma^2)}} + \frac{1}{\pi^2}\frac{t}{a}\sqrt{\frac{t^2-b^2}{a^2-t^2}}\,\Pi\left(\frac{a^2-b^2}{a^2-t^2}\middle| 1-\frac{b^2}{a^2}\right)$$

$$k^2 = 1-(k')^2, \quad k' = \frac{b}{a}$$

$$\left.\begin{aligned} a^2\mathbb{E} - \sigma^2\mathbb{K} &= \pi(2m+\ell)a \\ \sqrt{(a^2-\sigma^2)(b^2-\sigma^2)}\,\mathbb{K} &= \pi\ell\, a \end{aligned}\right\} \Rightarrow a, b$$



This is the density of the Bethe roots that correspond to the Frolov-Tseytlin folded string.

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

- Free energy:

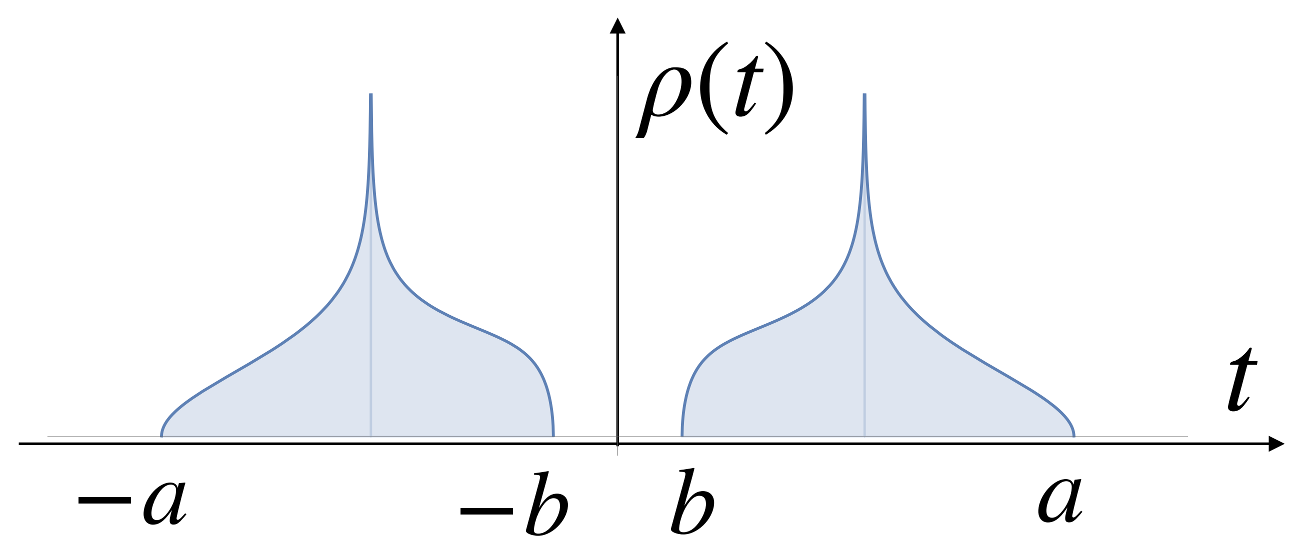
$$\partial_m \mathcal{F} = (2m+\ell)\log\frac{(a^2-b^2)}{4(2m+\ell)^2} + 2\ell\,\mathrm{arctanh}\frac{\sqrt{b^2-\sigma^2}}{\sqrt{a^2-\sigma^2}} - \frac{2\ell\,\sigma^2}{\sqrt{(a^2-\sigma^2)\,(b^2-\sigma^2)}} + \max(|\varphi|, |\sigma)$$

$$\mathbb{E} = E(k^2) = \int_0^{\pi/2} d\theta\,\sqrt{1-k^2\sin^2\theta}, \qquad \mathbb{K} = K(k^2) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \qquad \Pi(\alpha^2|k^2) = \int_0^{\pi/2}\frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

● Solution in **regime II** $(\ell, \sigma, \varphi \sim m, \ \ |\varphi| > |\sigma|)$

- Density: $\rho(t) = \frac{1}{\pi}\frac{\ell\, t}{t^2-\sigma^2}\sqrt{\frac{(a^2-t^2)(t^2-b^2)}{(a^2-\sigma^2)(b^2-\sigma^2)}} + \frac{1}{\pi^2}\frac{t}{a}\sqrt{\frac{t^2-b^2}{a^2-t^2}}\,\Pi\left(\frac{a^2-b^2}{a^2-t^2};\psi\middle|k^2\right)$ $\quad k^2 = 1-\frac{b^2}{a^2}, \quad \psi = \arcsin\frac{\sqrt{a^2-\varphi^2}}{\sqrt{a^2-b^2}}$

$$\left.\begin{aligned} a^2E\left(\psi\middle|k^2\right) - \sigma^2F\left(\psi\middle|k^2\right) &= \pi(2m+\ell)a \\ F\left(\psi\middle|k^2\right)\sqrt{\left(a^2-\sigma^2\right)\left(b^2-\sigma^2\right)} &= \pi\ell a \end{aligned}\right\} \Rightarrow \ a, b$$



- Free energy:

$$\partial_m\mathcal{F} = (2m+\ell)\log\frac{(a^2-b^2)}{4(2m+\ell)^2} + \frac{2\varphi}{\pi}\arctan\frac{\sqrt{a^2-\varphi^2}}{\sqrt{\varphi^2-b^2}} + 2\ell\,\mathrm{arctanh}\frac{\sqrt{b^2-\sigma^2}}{\sqrt{a^2-\sigma^2}} - \frac{2\ell\,\sigma^2}{\sqrt{\left(a^2-\sigma^2\right)\left(b^2-\sigma^2\right)}}$$

$$F(\psi\,|\,k^2) = \int_0^{\psi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \quad E(\psi\,|\,k^2) = \int_0^{\psi} d\theta\,\sqrt{1-k^2\sin^2\theta}$$

incomplete elliptic integrals of first and second kind

$$\Pi(\alpha^2;\psi\,|\,k^2) = \int_0^{\psi}\frac{d\theta}{(1-\alpha^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

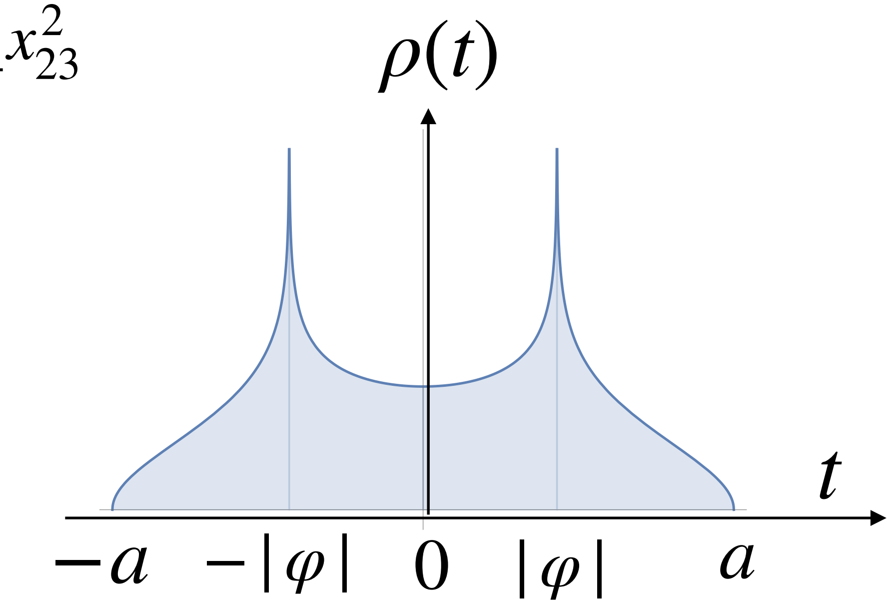
incomplete elliptic integral of third kind

Explicit solution for square fishnet ($\ell = 0$) with $\sigma = 0$ $\Leftrightarrow$ $x_{12}^2 x_{34}^2 = x_{14}^2 x_{23}^2$

$\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = m$ and special kinematics $x_{12}^2 x_{34}^2 = x_{14}^2 x_{23}^2$

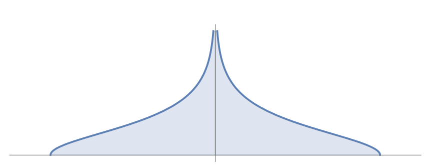
- Density: $b = 0, \;\; a = \sqrt{\varphi^2 + 4\pi^2 m^2}$

$$\rho(t) = \frac{1}{2\pi^2} \log \left| \frac{\sqrt{\varphi^2 + 4\pi^2 m^2 - t^2} + 2\pi m}{\sqrt{\varphi^2 + 4\pi^2 m^2 - t^2} - 2\pi m} \right|$$

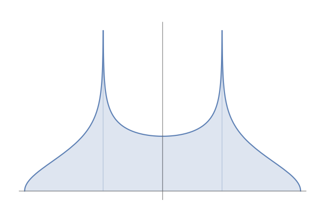


- The free energy can be evaluated in elementary functions::

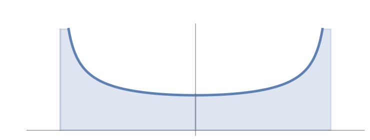
$$\mathcal{F} = m^2 \log\left(\frac{\varphi^2 + 4\pi^2 m^2}{16 m^2}\right) - \frac{\varphi^2}{4\pi^2} \log\left(\frac{\varphi^2 + 4\pi^2 m^2}{\varphi^2}\right) + \frac{2\varphi m}{\pi} \operatorname{arccot}\left(\frac{\varphi}{2\pi m}\right)$$



$\leftarrow$ $\varphi \to 0$



$\rightarrow$ $\varphi \to \infty$



Bulk thermodynamical limit

[Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

"Double light-like limit"

4. Euclidean OPE and light-like limits

- **Euclidean short-distance (OPE) limit** $\left(\hat{\sigma} \to \infty \text{ with } \hat{\varphi} \text{ finite}\right)$

$$\sigma \to \infty \quad \Rightarrow \quad \{U, V\} \to \{0, 1\}$$ [Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

$$|x_{12}|^2, |x_{34}|^2 \sim \sqrt{U}\ |x_{13}|^2 \qquad (U \to 0,\ V \to 1)$$

$$x_{14}^2 x_{23}^2 = x_{13}^2 x_{24}^2$$

OPE limit in the U-channel: $x_1 \sim x_2, \quad x_3 \sim x_4$.

- **Double light-cone, or nul, limit** $\left(\hat{\varphi} \to \infty \text{ with } \hat{\sigma} \text{ finite}\right)$

$$\varphi \to \infty \quad \Rightarrow \quad \{U, V\} \to \{0, 0\}$$

$$x_{12}^2,\ x_{34}^2 \sim \sqrt{U}\ |x_{13}||x_{24}|; \quad x_{14}^2,\ x_{23}^2 \sim \sqrt{V}\ |x_{13}||x_{24}|$$

i.e. Minkowski intervals $x_{12}^2,\ x_{23}^2, x_{34}^2, x_{41}^2$ become simultaneously light-like

Exact solutions:

- **Euclidean short-distance limit** $(\hat{\sigma} \to \infty \text{ with } \hat{\varphi} \text{ fixed}):$ [Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

Asymptotics of the ladder integrals: $$f_k(z,\bar{z}) \to \int_0^\infty (2|\sigma|)^k\, t^{k-1} e^{-t} dt = (2|\sigma|)^k\,(k-1)!$$

$$I^{\rm BD}_{m,n} \to \frac{(2|\sigma|)^{mn}}{\mathcal{N}} \det_{j,k}\left[(j+k+\ell-2)!\right] = \left(\log\frac{1}{U}\right)^{mn} C_{m,n}$$

$$C_{m,n} = \frac{G(m+1)G(n+1)}{G(m+n+1)}, \quad G(m) = 1!2!\ldots(m-2)!$$

Barnes' G-function

- **Double light-cone, or nul, limit** $(\hat{\varphi} \to \infty \text{ with } \hat{\sigma} \text{ fixed}):$

Asymptotics of the ladder integrals: $$f_k(z,\bar{z}) \underset{\varphi \gg k}{\to} 2\int_0^\varphi t^{2k-1} dt = \frac{\varphi^{2k}}{k}$$

$$I^{\rm BD}_{m,n} \to \frac{\varphi^{2m(m+\ell)}}{\mathcal{N}} \times \det\left[\frac{1}{i+j-1+n-m}\right]_{i,j=1,\ldots,m} = \frac{\varphi^{2mn}}{\mathcal{N}} \times \mathcal{N}\,(C_{m,n})^2$$

$$= C_{m,n}\left(\log\frac{1}{U}\right)^{mn} \times C_{m,n}\left(\log\frac{1}{V}\right)^{mn}$$

What happens with the spectral density in these limits?

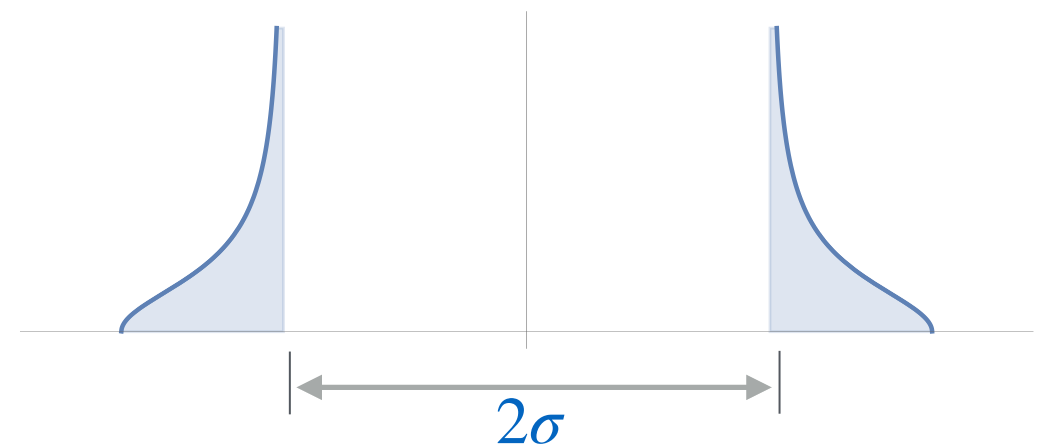
- $m \to \infty$ asymptotics of exact solution in Euclidean OPE and double light-cone limits matches $\hat{\sigma} \to \infty$ and $\hat{\varphi} \to \infty$ limits of the saddle-point solution

- **Euclidean short-distance limit** $(\hat{\sigma} \to \infty \text{ with } \hat{\varphi} \text{ fixed}):$ [Basso-Dixon-Kosower-Krajenbrink-Zhong, 2021]

$$a \approx \sigma + (\sqrt{m} + \sqrt{n})^2,\ b \approx \sigma + (\sqrt{m} - \sqrt{n})^2, \quad n = m + \ell$$

$$\mathcal{F} = mn\left[\tfrac{3}{2} + \log(2\sigma)\right]$$

$$+\tfrac{1}{2}m^2\log(m) + \tfrac{1}{2}n^2\log n - \tfrac{1}{2}(m+n)^2\log(m+n)$$

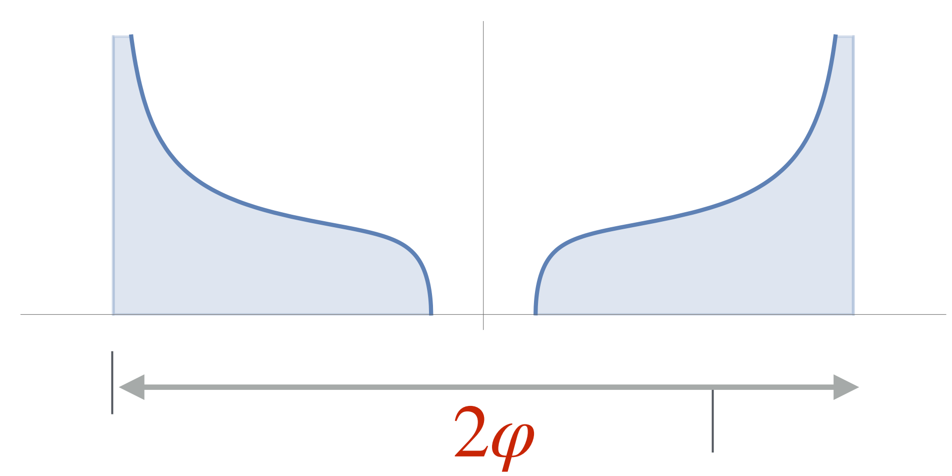


- **Double light-cone, or nul, limit** $(\hat{\varphi} \to \infty \quad \text{with} \quad \sigma \text{ fixed}):$

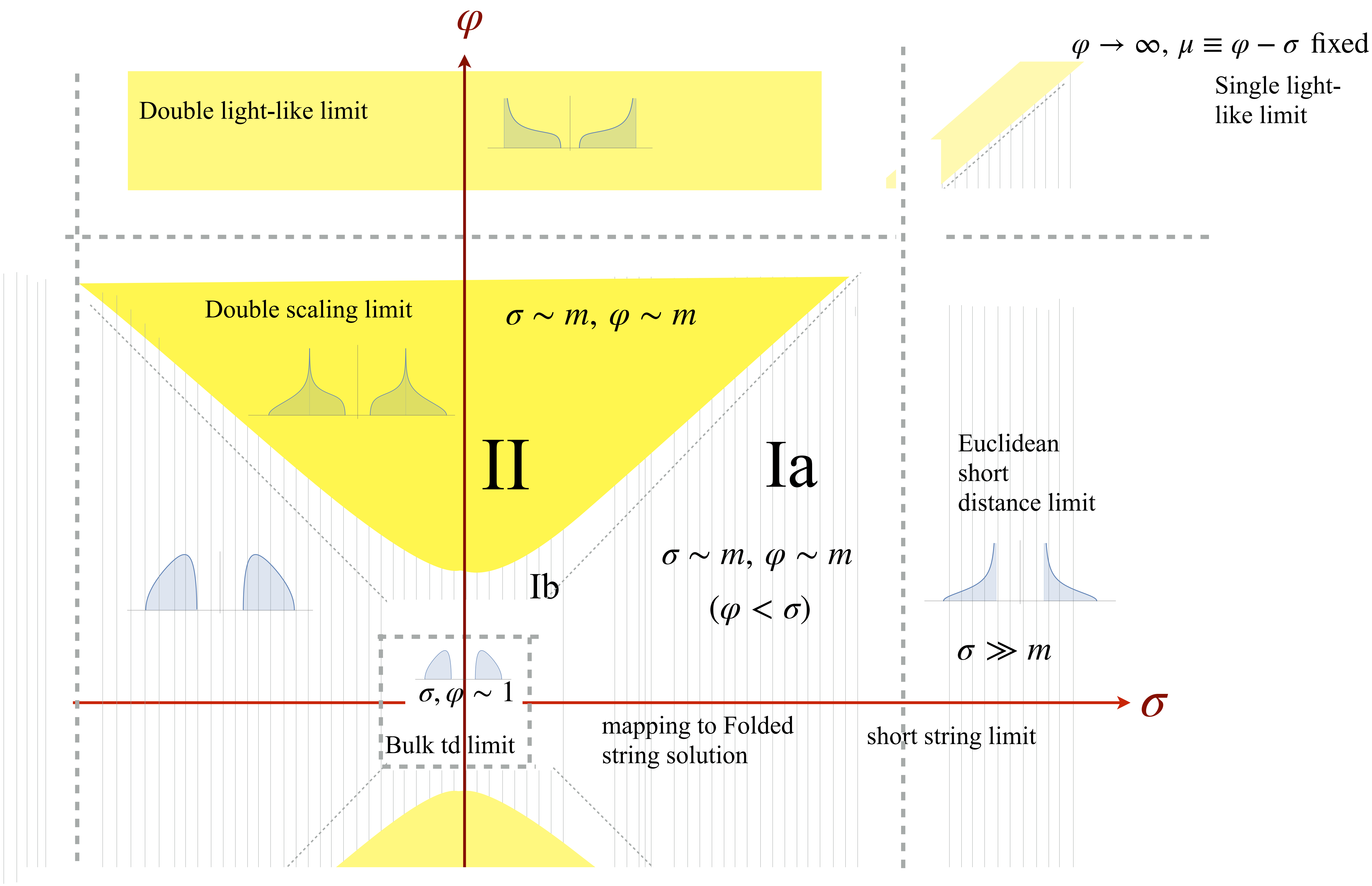
$$a \approx \varphi,\ b \approx \frac{n-m}{n+m}\varphi$$

$$\mathcal{F} = mn\left[3 + 2\log(\varphi)\right]$$

$$+m^2\log(m) + n^2\log n - (m+n)^2\log(m+n)$$



Phase diagram:



● Saddle-point equations as Bethe-Yang equations

At large argument, the derivative of the potential is approximated by a piecewise linear function:

$$V'(t) \underset{t\to\infty}{\to} \operatorname{sgn}(t)\,\theta\left(|t| - |\varphi|\right) - \frac{2t}{\sigma^2 - t^2}\ell, \qquad t \in \mathbb{R}$$

$2m$ Bethe roots $\{t_1, \ldots, t_{2n}\}$:

1) BAE $$\frac{2\ell\, t_j}{t_j^2 - \sigma^2} + \sum_{k\neq j}^{2m} \frac{2}{t_j - t_k} = n_j \in \mathbb{Z} \qquad (j = 1,\ldots,2m)$$

2) Symmetry $$t_j = -\, t_{2m-j+1},\, n_j = -\, n_{2m-j+1}$$

3) $V'(t) \;\Rightarrow$ $$n_j = 1 \text{ if } t > \varphi, \quad n_j = 0 \text{ if } \sigma < |t| < \varphi\,;\, n_j = -1 \text{ if } t < -\varphi$$

Two choices for the “Bethe numbers”:

Regime I. If $|\varphi| \leq |\sigma|$, then $n_j = \operatorname{sign}(t_j), \;\; j = 1,\ldots,2m$

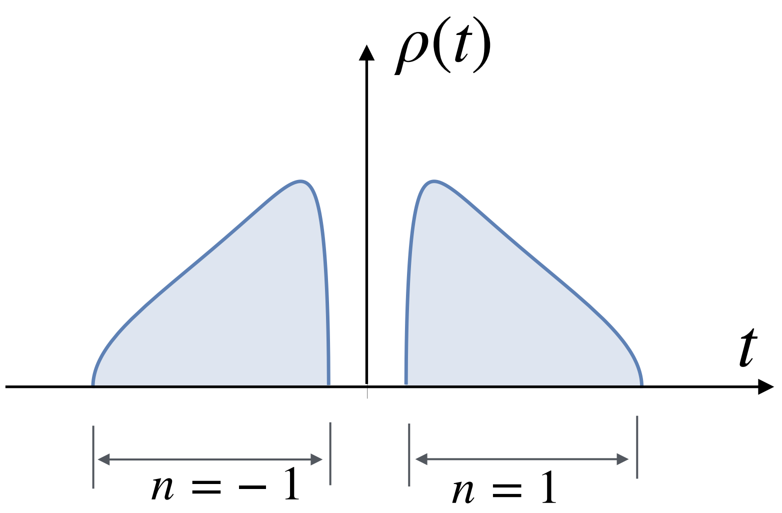
Regime II. If $|\varphi| > |\sigma|$, then $n_j = \operatorname{sign}(t_j)$ if $|t_j| > |\varphi|$ and $n_j = 0$ if $|t_j| < |\varphi|$, $j = 1,\ldots,2m$.

- The qualitative form of the solution for the spectral density in the two regimes:

Regime I. $|\varphi| \le |\sigma|$ ("Mode numbers" $n_j = \text{sign}(t_j)$)

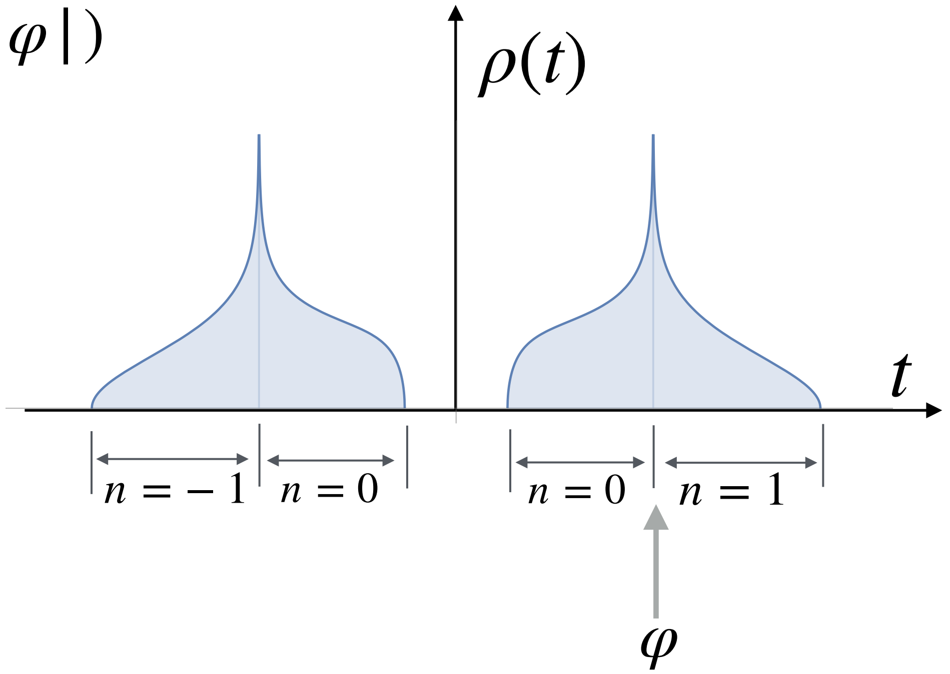
For large number of magnons this is the finite-zone solution for the the Frolov-Tseytlin folded string rotating in $\text{AdS}_3 \times S^1$ with $\{S, J\} = \{2m, \ell\}$

[Basso et al, 2021]



Regime II. $|\varphi| > |\sigma|$ ("Mode numbers" $n_j = \text{sign}(t_j)\, \theta(|t_j| - |\varphi|)$)

Not a finite-gap solution: the two groups of roots (with mode numbers 1 and 0 respectively) do not repel but attract. Logarithmic cusp of the spectral density observed at the collision point.



- Note that these fictive magnons have nothing to do with the original mirror magnons.

- Results compatible with existence of holographic dual.
  Saddle-point equations = Bethe equations for some magnons in t-space.

  However, not clear how to interpret the “unphysical” mode numbers in regime II.
  — Problem still open.

- Curious factorisation observed in the light-cone limit where the result is a product of two factors associated with the direct and with the cross channels

$$I^{\text{BD}}_{m,n} = C_{m,n}\left(\log\frac{1}{U}\right)^{mn} \times C_{m,n}\left(\log\frac{1}{V}\right)^{mn}$$

— There is interpretation of the OPE limit in terms of hopping magnons (“stampedes”) [Olivucci-Vieira, 2022]. Seems that similar description is possible in the light-like limit as well [Enrico]. If so, how the above factorisation appears?

Thank you!