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The gradient flow formulation of the electroweak Hamiltonian

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The effective electroweak Hamiltonian

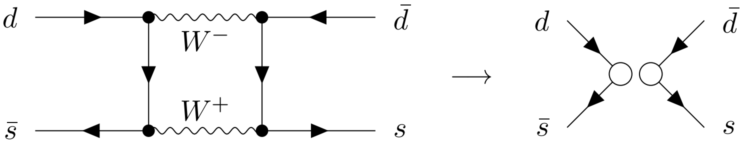
- Observables in flavor physics often computed with effective Hamiltonian of electroweak interactions

$$\mathcal{H}_{\mathrm{eff}} = -\left(\frac{4G_{\mathrm{F}}}{\sqrt{2}}\right)^{x} V_{\mathrm{CKM}} \sum_i C_i \mathcal{O}_i$$

with four-fermion operators like

$$\mathcal{O}^{|\Delta S|=2} = \left(\bar{s}\gamma_\mu(1-\gamma_5)d\right)\left(\bar{s}\gamma_\mu(1-\gamma_5)d\right)$$

for $K^0 - \bar{K}^0$ mixing:



- Wilson coefficients $C_i(\mu)$ obtained from perturbative matching to Standard Model at $\mu = \mu_W \sim M_W$
- V_{CKM}: relevant entries of the CKM matrix, e.g. $V_{is}^* V_{id} V_{js}^* V_{jd}$ with $i, j = c, t$

Computing observables

- Flavor observables mostly at low energies

$\Rightarrow$ Use renormalization group equations to evolve down to appropriate scale to avoid large logarithms

- Schematically for Kaon mixing:

$$\langle \bar{K}^0 | \mathcal{H}_{\text{eff}}^{|\Delta S|=2} | K^0 \rangle \approx C(\mu_W) U(\mu_W, \mu \sim M_K) \langle \bar{K}^0 | \mathcal{O}^{|\Delta S|=2}(\mu \sim M_K) | K^0 \rangle$$

- Running with $U(\mu_W, \mu)$ determined by anomalous dimension γ of $\mathcal{O}^{|\Delta S|=2}$
- Matrix element $\langle \bar{K}^0 | \mathcal{O}^{|\Delta S|=2}(\mu \sim M_K) | K^0 \rangle$ nonperturbative

$\Rightarrow$ Compute on lattice

Complications

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^{x} V_{\text{CKM}} \sum_i C_i \mathcal{O}_i$$

- While $\mathcal{H}_{\text{eff}}$ is scheme independent, C_i and $\mathcal{O}_i$ are not:
  1. Explicitly depend on renormalization scale μ
  2. Depend on scheme used for γ_5
  3. In dimensional regularization $\mathcal{O}_i$ mix with evanescent operators, which vanish in $D = 4$, but their choice affects the finite pieces in C_i

$\Rightarrow$ Scheme matching between lattice and perturbative results is a source of uncertainty

The gradient flow

- Introduce parameter *flow time* $t \geq 0$ [Narayanan, Neuberger 2006; Lüscher 2009; Lüscher 2010]
- *Flowed fields* in $D + 1$ dimensions obey differential *flow equations*:

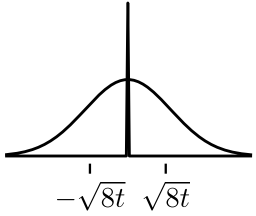
$$\partial_t \Phi(t, x) = - \left. \frac{\delta S[\phi(x)]}{\delta \phi(x)} \right|_{\Phi(t,x)} \sim D_x \Phi(t, x) \quad \text{with} \quad \Phi(t, x)|_{t=0} = \phi(x)$$

- Flow equation drives flowed fields to minimum of action
- Flow equation similar to the heat equation (thermodynamics)

$$\partial_t u(t, \vec{x}) = \alpha \Delta u(t, \vec{x}) \quad \text{with} \quad \Delta = \sum_i \partial_{x_i}^2$$

- Fields at positive flow time smeared out with smearing radius $\sqrt{8t}$

$\Rightarrow$ Regulates divergencies



sketch of smearing

Flow equations of QCD

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Gluon flow equation [Narayanan, Neuberger 2006; Lüscher 2010]

$$\partial_t B^a_\mu = \mathcal{D}^{ab}_\nu G^b_{\nu\mu} \quad \text{with} \quad B^a_\mu(t,x)\big|_{t=0} = A^a_\mu(x)$$

$$\mathcal{D}^{ab}_\mu = \delta^{ab}\partial_\mu - f^{abc}B^c_\mu, \qquad G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + f^{abc}B^b_\mu B^c_\nu$$

Flow equations of QCD

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$$\partial_t B_\mu^a = \mathcal{D}_\nu^{ab} G_{\nu\mu}^b \quad \text{with} \quad B_\mu^a(t,x)\big|_{t=0} = A_\mu^a(x)$$

$$\mathcal{D}_\mu^{ab} = \delta^{ab}\partial_\mu - f^{abc} B_\mu^c, \qquad G_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + f^{abc} B_\mu^b B_\nu^c$$

Quark flow equation [Lüscher 2013]

$$\partial_t \chi = \Delta \chi \quad \text{with} \quad \chi(t,x)|_{t=0} = \psi(x)\,,$$
$$\partial_t \bar{\chi} = \bar{\chi} \overleftarrow{\Delta} \quad \text{with} \quad \overline{\chi}(t,x)|_{t=0} = \overline{\psi}(x)$$

$$\Delta = (\partial_\mu + B_\mu^a T^a)(\partial_\mu + B_\mu^b T^b), \qquad \overleftarrow{\Delta} = (\overleftarrow{\partial}_\mu - B_\mu^a T^a)(\overleftarrow{\partial}_\mu - B_\mu^b T^b)$$

Lagrangian

- Write Lagrangian for the gradient flow as [Lüscher, Weisz 2011; Lüscher 2013]

$$\mathcal{L} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_B + \mathcal{L}_\chi,$$

$$\mathcal{L}_{\mathrm{QCD}} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{n_\mathrm{f}} \bar{\psi}_f (\not{D}^{\mathrm{F}} + m_f) \psi_f + \dots$$

- Construct flowed Lagrangian using Lagrange multiplier fields $L^a_\mu(t,x)$ and $\lambda_f(t,x)$:

$$\mathcal{L}_B = -2 \int_0^\infty \mathrm{d}t \, \mathrm{Tr} \left[L^a_\mu T^a \left(\partial_t B^b_\mu T^b - \mathcal{D}^{bc}_\nu G^c_{\nu\mu} T^b \right) \right], \qquad \partial_t B^a_\mu = \mathcal{D}^{ab}_\nu G^b_{\nu\mu}$$

$$\mathcal{L}_\chi = \sum_{f=1}^{n_\mathrm{f}} \int_0^\infty \mathrm{d}t \left(\bar{\lambda}_f \left(\partial_t - \Delta \right) \chi_f + \bar{\chi}_f \left(\overleftarrow{\partial_t} - \overleftarrow{\Delta} \right) \lambda_f \right), \qquad \partial_t \chi = \Delta \chi, \;\; \partial_t \bar{\chi} = \bar{\chi} \overleftarrow{\Delta}$$

$\Rightarrow$ Flow equations automatically fulfilled

$\Rightarrow$ QCD Feynman rules + gradient-flow Feynman rules (complete list in [Artz, Harlander, **FL**, Neumann, Prausa 2019])

Flowed operator product expansion

- Flowed operators $\tilde{\mathcal{O}}_j(t)$ do not require renormalization [Lüscher, Weisz 2011]
- Small flow-time expansion [Lüscher, Weisz 2011] :

$$\tilde{\mathcal{O}}_i(t,x) = \sum_j \zeta_{ij}(t)\mathcal{O}_j(x) + O(t)$$

- Invert to express operators through flowed operators [Suzuki 2013; Makino, Suzuki 2014; Monahan, Orginos 2015]:

Flowed OPE

$$T = \sum_i C_i \mathcal{O}_i = \sum_{i,j} C_i \zeta_{ij}^{-1}(t)\tilde{\mathcal{O}}_j(t) \equiv \sum_j \tilde{C}_j(t)\tilde{\mathcal{O}}_j(t)$$

- Gradient-flow definition of T valid both in perturbation theory and on lattice
- First used to define the energy-momentum tensor of QCD on the lattice [Suzuki 2013; Makino, Suzuki 2014; Harlander, Kluth, **FL** 2018] , now many more (potential) applications

Flowed OPE for the electroweak Hamiltonian

- Write electroweak Hamiltonian as

$$\mathcal{H}_{\mathrm{eff}} = -\left(\frac{4G_{\mathrm{F}}}{\sqrt{2}}\right)^{x} V_{\mathrm{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_{\mathrm{F}}}{\sqrt{2}}\right)^{x} V_{\mathrm{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j$$

- Current-current operators and flowed counterparts:

$$\begin{aligned}
\mathcal{O}_1 &= -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right) &\Rightarrow\quad \tilde{\mathcal{O}}_1 &= -\mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu T^a \chi_{4,\mathrm{L}}\right)\\
\mathcal{O}_2 &= \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right) &\Rightarrow\quad \tilde{\mathcal{O}}_2 &= \mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu \chi_{4,\mathrm{L}}\right)
\end{aligned}$$

- No operator mixing through renormalization for $\tilde{\mathcal{O}}_i$

$\Rightarrow$ Combine <u>without</u> scheme matching between perturbation theory and lattice:
  - C_i known perturbatively through (N)NLO (depending on process)
  - ζ_{ij}^{-1} has to be computed, some first results in [Suzuki, Taniguchi, Suzuki, Kanaya 2020; Harlander, **FL** 2022]
  - $\langle \tilde{\mathcal{O}}_j \rangle$ to be computed on the lattice

Method of projectors

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- Define projectors [Gorishny, Larin, Tkachov 1983; Gorishny, Larin 1987]

$$P_k[\mathcal{O}_i] \equiv D_k \langle 0|\mathcal{O}_i|k\rangle \stackrel{!}{=} \delta_{ik} + O(\alpha_s)$$

- Apply to small flow-time expansion:

$$P_k[\tilde{\mathcal{O}}_i(t)] = \sum_j \zeta_{ij}(t) P_k[\mathcal{O}_j]$$

- $\zeta_{ij}(t)$ only depend on t

$\Rightarrow$ Set all other scales to zero

$\Rightarrow$ No perturbative corrections to $P_k[\mathcal{O}_j]$, because all loop integrals are scaleless

"Master formula"

$$\zeta_{ij}(t) = P_j[\tilde{\mathcal{O}}_i(t)]\Big|_{p=m=0}$$

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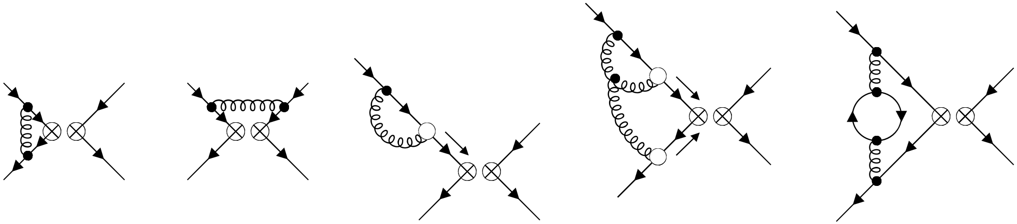
Projectors and example diagrams

- Projectors for $\mathcal{O}_1$ and $\mathcal{O}_2$ (schematically):

$$P_1[\mathcal{O}] = -\frac{1}{16T_R^2 N_A} \mathrm{Tr}_{\text{line 1}} \mathrm{Tr}_{\text{line 2}} \left\langle 0 \right| \left(\psi_{4,\mathrm{L}} T^b \gamma_\nu \bar{\psi}_{3,\mathrm{L}}\right) \left(\psi_{2,\mathrm{L}} T^b \gamma_\nu \bar{\psi}_{1,\mathrm{L}}\right) \mathcal{O} \left|0\right\rangle \Big|_{p=m=0},$$

$$P_2[\mathcal{O}] = \frac{1}{16N_c^2} \mathrm{Tr}_{\text{line 1}} \mathrm{Tr}_{\text{line 2}} \left\langle 0 \right| \left(\psi_{4,\mathrm{L}} \gamma_\nu \bar{\psi}_{3,\mathrm{L}}\right) \left(\psi_{2,\mathrm{L}} \gamma_\nu \bar{\psi}_{1,\mathrm{L}}\right) \mathcal{O} \left|0\right\rangle \Big|_{p=m=0}$$

- Sample diagrams:



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Results

- Physical matching matrix $(\zeta^{-1})_{\mathrm{PP}}$:

$$(\zeta^{-1})_{11}(t) = 1 + a_s\left(4.212 + \frac{1}{2}L_{\mu t}\right) + a_s^2\left[22.72 - 0.7218\,n_f + L_{\mu t}\,(16.45 - 0.7576\,n_f) + L_{\mu t}^2\left(\frac{17}{16} - \frac{1}{24}\,n_f\right)\right],$$

$$(\zeta^{-1})_{12}(t) = a_s\left(-\frac{5}{6} - \frac{1}{3}L_{\mu t}\right) + a_s^2\left[-4.531 + 0.1576\,n_f + L_{\mu t}\left(-3.133 + \frac{5}{54}\,n_f\right) + L_{\mu t}^2\left(-\frac{13}{24} + \frac{1}{36}n_f\right)\right],$$

$$(\zeta^{-1})_{21}(t) = a_s\left(-\frac{15}{4} - \frac{3}{2}L_{\mu t}\right) + a_s^2\left[-23.20 + 0.7091\,n_f + L_{\mu t}\left(-15.22 + \frac{5}{12}\,n_f\right) + L_{\mu t}^2\left(-\frac{39}{16} + \frac{1}{8}\,n_f\right)\right],$$

$$(\zeta^{-1})_{22}(t) = 1 + a_s\,3.712 + a_s^2\left[19.47 - 0.4334\,n_f + L_{\mu t}\,(11.75 - 0.6187\,n_f) + \frac{1}{4}\,L_{\mu t}^2\right]$$

- $a_s = \alpha_s(\mu)/\pi$ renormalized in $\overline{\mathrm{MS}}$ scheme and $L_{\mu t} = \ln 2\mu^2 t + \gamma_E$
- Set $N_c = 3$, $T_R = \frac{1}{2}$, and transcendental coefficients replaced by floating-point numbers

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Application to flavor physics

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i C_i \mathcal{O}_i = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_i \tilde{C}_i \tilde{\mathcal{O}}_i$$

- Flowed Wilson coefficients $\tilde{C}_i$ and flowed operators $\tilde{\mathcal{O}}_j$ individually completely scheme independent:
  - Formally independent of renormalization scale μ
  - Do not depend on scheme used for γ_5
  - Do not depend on choice of evanescent operators

$\Rightarrow$ $\tilde{C}_i$ and $\langle \tilde{\mathcal{O}}_j \rangle$ can be computed in different schemes, e.g. perturbatively and on the lattice

- Perturbative ingredients C_i and ζ_{ij}^{-1} have to be computed in the same scheme, but this is no major problem

Status

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\rm F}}{\sqrt{2}}\right)^x V_{\rm CKM} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \; :$$

- Kaon mixing ($|\Delta S| = 2$):
  - C_i: NLO [Buchalla, Buras, Lautenbacher 1995 and references therein] with two of three contributions known through NNLO [Brod, Gorbahn 2010 + 2012]
  - ζ_{ij}^{-1}: NNLO [Harlander, **FL** 2022]
- Mass difference in neutral B-meson mixing ($|\Delta B| = 2$):
  - C_i: NLO [Buchalla, Buras, Lautenbacher 1995 and references therein]
  - ζ_{ij}^{-1}: NNLO [Harlander, **FL** 2022]
- Non-leptonic $|\Delta F| = 1$ decays:
  - C_i: NNLO [Bobeth, Misiak, Urban 2000; Gorbahn, Haisch 2004]
  - ζ_{ij}^{-1}: NNLO, but without penguins yet [Harlander, **FL** 2022]
- $\langle \tilde{\mathcal{O}}_j \rangle$ not computed yet, some first exploratory studies underway (as far as I know)

Conclusions and outlook

Conclusions:

- Flowed operator product expansion can be used to match lattice results to perturbative schemes like $\overline{\text{MS}}$ without complicated scheme matching between perturbation theory and the lattice
- We computed the matching matrix $\overline{\text{MS}} \leftrightarrow$ GF for the current-current operators of the electroweak Hamiltonian through NNLO

Outlook:

- Non-trivial comparison of matching matrix with NLO result of [Suzuki, Taniguchi, Suzuki, Kanaya 2020] (different basis and different scheme for γ_5) should be done
- Extension to penguin operators for $|\Delta F| = 1$
- Extension to full Hamiltonian of other processes like $|\Delta B| = 2$
- Matrix elements from lattice simulations to be computed
- Comparison to traditional approaches with schemes like RI-(S)MOM to be studied once matrix elements available

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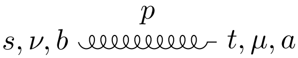
Flowed propagators and flow lines

$$\mathcal{L}_B = -2\int_0^\infty \mathrm{d}t\,\mathrm{Tr}\left[L_\mu^a T^a \left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b\right)\right]$$

- Combined Feynman rule for the (flowed) gluon propagator $\left\langle \widetilde{B}_\mu^a(t,p)\widetilde{B}_\nu^b(s,q)\right\rangle$:



$$= \delta^{ab}\frac{1}{p^2}\delta_{\mu\nu}\, e^{-(t+s)p^2}$$

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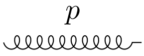
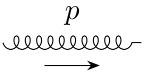


$$s,\nu,b \qquad t,\mu,a \;=\; \delta^{ab}\frac{1}{p^2}\delta_{\mu\nu}\, e^{-(t+s)p^2}$$

- No squared L_μ^a in $\mathcal{L}_B \Rightarrow$ no propagator
- Instead, there is a mixed propagator $\left\langle \widetilde{B}_\mu^a(t,p)\widetilde{L}_\nu^b(s,q)\right\rangle$ called *flow line*:



$$s,\nu,b \qquad t,\mu,a \;=\; \delta^{ab}\,\theta(t-s)\delta_{\mu\nu}\, e^{-(t-s)p^2}$$

- Directed towards increasing flow time

Flow vertices

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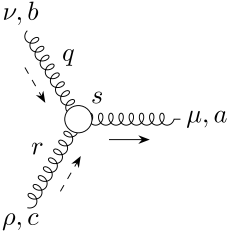
$$\mathcal{L}_B = -2\int_0^\infty \mathrm{d}t\,\mathrm{Tr}\left[L_\mu^a T^a \left(\partial_t B_\mu^b T^b - \mathcal{D}_\nu^{bc} G_{\nu\mu}^c T^b\right)\right]$$

- Example:



$$= -\mathrm{i}gf^{abc}\int_0^\infty \mathrm{d}s\left(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu\right)$$

- Integral restricted by $\theta(t-s)$ from outgoing flow line

Renormalization

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- QCD renormalization of QCD parameters like α_s and quark masses
- Flowed gluon fields do not require renormalization [Lüscher 2010; Lüscher, Weisz 2011]
- Flowed quark fields have to be renormalized: $\chi^{\mathrm{R}} = Z_\chi^{1/2} \chi^{\mathrm{B}}$ [Lüscher 2013]

$\Rightarrow$ χ thus acquire anomalous dimension and are not scheme independent

- "Physical" scheme: Ringed fermions $\mathring{\chi} = \mathring{Z}_\chi^{1/2} \chi^{\mathrm{B}}$ [Makino, Suzuki 2014]:

$$\mathring{Z}_\chi = -\frac{2N_c}{(4\pi t)^2 \left. \langle \bar{\chi}^{\mathrm{B}} \overleftrightarrow{\not{D}} \chi^{\mathrm{B}} \rangle \right|_{m=0}}$$

$\Rightarrow$ $\mathring{\chi}$ formally independent of renormalization scale μ

- $\mathring{Z}_\chi$ available through NNLO [Artz, Harlander, **FL**, Neumann, Prausa 2019]

Renormalization

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$$\mathring{Z}_\chi = -\frac{2N_\mathrm{c}}{(4\pi t)^2 \left.\langle \bar{\chi}^\mathrm{B} \overleftrightarrow{\not{D}} \chi^\mathrm{B} \rangle\right|_{m=0}}$$

$\Rightarrow$ $\mathring{\chi}$ formally independent of renormalization scale μ

- $\mathring{Z}_\chi$ available through NNLO [Artz, Harlander, **FL**, Neumann, Prausa 2019]
- Composite operators do not require renormalization [Lüscher, Weisz 2011]

$\Rightarrow$ No operator mixing through renormalization

Automatized calculation

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- `qgraf` [Nogueira 1991]: Generate Feynman diagrams
- `q2e` and `exp` [Harlander, Seidensticker, Steinhauser 1998; Seidensticker 1999]: Assign diagrams to topologies and prepare `FORM` code
- `FORM` [Vermaseren 2000; Kuipers, Ueda, Vermaseren, Vollinga 2013]: Insert Feynman rules, perform tensor reduction, Dirac traces, color algebra, and expansions
- Generate system of equations employing integration-by-parts-like relations [Tkachov 1981; Chetyrkin, Tkachov 1981] with in-house `Mathematica` code
- `Kira` [Maierhöfer, Usovitsch, Uwer 2017; Klappert, **FL**, Maierhöfer, Usovitsch 2020] $\oplus$ `FireFly` [Klappert, **FL** 2019; Klappert, Klein, **FL** 2020]: Solve system to express all integrals through master integrals
- Master integrals already computed in [Harlander, Kluth, **FL** 2018]

Operator basis

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- Operator basis depends on the process under consideration
- We focus on the current-current operators
- Operator basis not unique even for the same process, but different bases related by basis transformations
- CMM basis [Chetyrkin, Misiak, Münz 1997]:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right),$$
$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right)$$

with

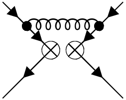
$$\psi_{\mathrm{R/L}} = P_\pm \psi = \tfrac{1}{2}(1 \pm \gamma_5)\psi$$

- Advantage of CMM basis: can use anticommuting γ_5

Evanescent operators

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$$

- In dimensional regularization, loop corrections produce additional non-reducible γ structures:



$$\Rightarrow \qquad \left(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\right)\otimes\left(\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\right)$$

- These contributions have to be attributed to *evanescent* operators like [Buras, Weisz 1990; Dugan, Grinstein 1991; Herrlich, Nierste 1995]

$$E_2^{(1)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3}\psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_2 \qquad \text{with} \qquad \gamma_{\mu_1\cdots\mu_n} \equiv \gamma_{\mu_1}\cdots\gamma_{\mu_n}$$

- Algebraically, they are of $O(\epsilon)$ and vanish for $D \to 4$
- Nonetheless required to renormalize the physical operators
- Renormalization has to take care of finite pieces from $\frac{1}{\epsilon}$ (poles) $\times\,\epsilon$ (operators)
- Every loop order introduces more evanescent operators

Complete operator basis

- Physical operators:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right),$$
$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right)$$

- Evanescent operators through NNLO (also from [Chetyrkin, Misiak, Münz 1997]):

$$E_1^{(1)} = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_1,$$
$$E_2^{(1)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \psi_{4,\mathrm{L}}\right) - 16\mathcal{O}_2,$$
$$E_1^{(2)} = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \psi_{4,\mathrm{L}}\right) - 20E_1^{(1)} - 256\mathcal{O}_1,$$
$$E_2^{(2)} = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \psi_{4,\mathrm{L}}\right) - 20E_2^{(1)} - 256\mathcal{O}_2$$

Flowed operator basis

- Flowed physical operators:

$$\mathcal{O}_1 = -\left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu T^a \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu T^a \psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_1 = -\mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu T^a \chi_{4,\mathrm{L}}\right)$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu \psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu \psi_{4,\mathrm{L}}\right) \quad \Rightarrow \quad \tilde{\mathcal{O}}_2 = \mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_\mu \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_\mu \chi_{4,\mathrm{L}}\right)$$

- Flowed evanescent operators:

$$\begin{aligned}
\tilde{E}_1^{(1)} &= -\mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} T^a \chi_{4,\mathrm{L}}\right) - 16\tilde{\mathcal{O}}_1\,,\\
\tilde{E}_2^{(1)} &= \mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3} \chi_{4,\mathrm{L}}\right) - 16\tilde{\mathcal{O}}_2\,,\\
\tilde{E}_1^{(2)} &= -\mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} T^a \chi_{4,\mathrm{L}}\right) - 20\tilde{E}_1^{(1)} - 256\tilde{\mathcal{O}}_1\,,\\
\tilde{E}_2^{(2)} &= \mathring{Z}_\chi^2 \left(\bar{\chi}_{1,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \chi_{2,\mathrm{L}}\right)\left(\bar{\chi}_{3,\mathrm{L}}\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5} \chi_{4,\mathrm{L}}\right) - 20\tilde{E}_2^{(1)} - 256\tilde{\mathcal{O}}_2
\end{aligned}$$

- Note: Since flowed operators do not have to be renormalized, the flowed evanescent operators actually vanish and could be dropped
- Keeping them allows us to check our results

Matching matrix and renormalization

- Small-flow-time expansion for operators of electroweak Hamiltonian:

$$\begin{pmatrix} \tilde{\mathcal{O}}(t) \\ \tilde{E}(t) \end{pmatrix} \asymp \zeta^{\mathrm{B}}(t) \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}$$

$$\text{with} \qquad \mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)^{\mathrm{T}}, \qquad E = (E_1^{(1)}, E_2^{(1)}, E_1^{(2)}, E_2^{(2)})^{\mathrm{T}}$$

- Since regular operators are divergent, $\zeta^{\mathrm{B}}(t)$ is divergent as well
- Regular operators renormalized through [Buras, Weisz 1990; Dugan, Grinstein 1991; Herrlich, Nierste 1995]

$$\begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}} = Z \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix} \equiv \begin{pmatrix} Z_{\mathrm{PP}} & Z_{\mathrm{PE}} \\ Z_{\mathrm{EP}} & Z_{\mathrm{EE}} \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}$$

- Renormalized $\zeta(t)$:

$$\begin{pmatrix} \tilde{\mathcal{O}}(t) \\ \tilde{E}(t) \end{pmatrix} \asymp \zeta^{\mathrm{B}}(t) Z^{-1} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}} \equiv \zeta(t) \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}} \equiv \begin{pmatrix} \zeta_{\mathrm{PP}}(t) & \zeta_{\mathrm{PE}}(t) \\ \zeta_{\mathrm{EP}}(t) & \zeta_{\mathrm{EE}}(t) \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}^{\mathrm{R}}$$

Treatment of γ_5 (I)

- In dimensional regularization,

$$\{\gamma_\mu, \gamma_5\} = 0$$

is incompatible with the trace requirement

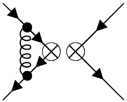
$$\mathrm{Tr}(\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5) \neq 0 \xrightarrow[D\to 4]{} 4\mathrm{i}\epsilon_{\mu\nu\rho\sigma}$$

- Different prescriptions for γ_5 (NDR, ’t Hooft-Veltmann, DREG) lead to different results for scheme-dependent quantities like Wilson coefficients

Treatment of γ_5 (II)

$$P_2[\mathcal{O}] = \frac{1}{16N_c^2}\,\mathrm{Tr}_{\text{line }1}\mathrm{Tr}_{\text{line }2}\,\langle 0|\left(\psi_{4,\mathrm{L}}\gamma_\nu\bar{\psi}_{3,\mathrm{L}}\right)\left(\psi_{2,\mathrm{L}}\gamma_\nu\bar{\psi}_{1,\mathrm{L}}\right)\mathcal{O}|0\rangle\Big|_{p=m=0}$$

$$\mathcal{O}_2 = \left(\bar{\psi}_{1,\mathrm{L}}\gamma_\mu\psi_{2,\mathrm{L}}\right)\left(\bar{\psi}_{3,\mathrm{L}}\gamma_\mu\psi_{4,\mathrm{L}}\right)$$



- The quarks in our operators cannot annihilate due to different flavors

$\Rightarrow$ No γ_5 in traces produced by loop corrections

- Define external quarks in projectors to be left-handed, anticommute γ_5 from operator, and use $P_L^2 = P_L = \frac{1}{2}(1-\gamma_5)$

$\Rightarrow$ No traces with γ_5, simply use naively anticommuting γ_5

- Note: CMM basis avoids γ_5 in traces also for penguin operators ($|\Delta F| = 1$) [Chetyrkin, Misiak, Münz 1997]

Checks

$$\zeta^{-1} = Z(\zeta^{\mathrm{B}})^{-1} = \begin{pmatrix} (\zeta^{-1})_{\mathrm{PP}} & (\zeta^{-1})_{\mathrm{PE}} \\ (\zeta^{-1})_{\mathrm{EP}} & (\zeta^{-1})_{\mathrm{EE}} \end{pmatrix}$$

- Finite after α_{s} + field renormalization and with Z from [Chetyrkin, Misiak, Münz 1997; Gambino, Gorbahn, Haisch 2003; Gorbahn, Haisch 2004]
- $(\zeta^{-1})_{\mathrm{EP}} = O(\epsilon)$
- Independent of QCD gauge parameter
- Non-trivial basis transformation to non-mixing basis of [Buras, Gorbahn, Haisch, Nierste 2006] leads to diagonal ζ^{-1}

Evolving $\tilde{C}_i$

$$\mathcal{H}_{\text{eff}} = -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i,j} C_i \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j \equiv -\left(\frac{4G_{\text{F}}}{\sqrt{2}}\right)^x V_{\text{CKM}} \sum_{i} \tilde{C}_i \tilde{\mathcal{O}}_i$$

- SM matching done at $\mu_W \sim M_W$, lattice calculation done at small $\mu \sim \sqrt{1/t}$
- Avoid large logarithms by either:

1. - Evolve regular Wilson coefficients C_i down to $\mu \sim \sqrt{1/t}$ with the known RGE:

$$C_i(\mu) = \sum_j C_j(\mu_W) U_{ji}(\mu_W, \mu)$$

2. - Construct flowed Wilson coefficents $\tilde{C}_i$ at $\mu \sim \sqrt{1/t}$
   - Construct flowed Wilson coefficents $\tilde{C}_i$ at $\mu \sim M_W$
   - Use the flowed anomalous dimension

$$\tilde{\gamma}(t) = (t\partial_t \zeta(t))\zeta^{-1}(t) \qquad \text{defined through} \qquad t\partial_t \tilde{\mathcal{O}}(t) = \tilde{\gamma}(t)\tilde{\mathcal{O}}(t)$$

   to evolve to t large enough for lattice calculation using

$$t\partial_t \tilde{C}_i(t) = -\sum_j \tilde{C}_j(t)\tilde{\gamma}_{ji}(t)$$

- Compatibility of both methods to be studied