

Finite-Size Effects of the HVP Contribution to the Muon $g - 2$ with C^* Boundary Conditions

2022-08-11

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In collaboration with A. Patella, Humboldt University

Lattice 2022, Universität Bonn

Outline

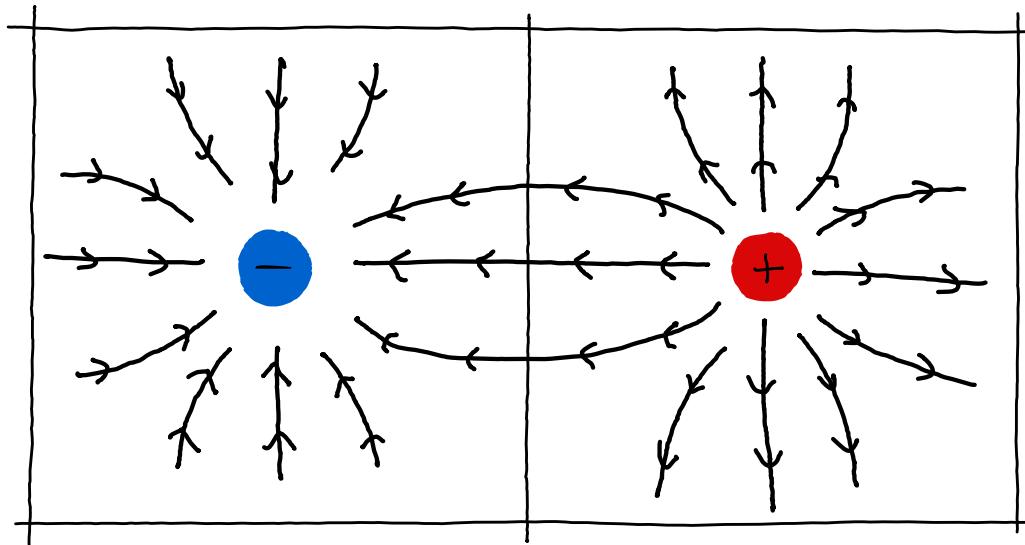
Motivation

Finite-Size Effects in pure QCD

Results and Conclusion

C^* Boundary Conditions Reduce FV Effects

[Kronfeld and Wiese 1991; Polley and Wiese 1991]



... see also [228] by Uwe-Jens Wiese 9th Aug. at 5:30 pm,

Time-Window Observables

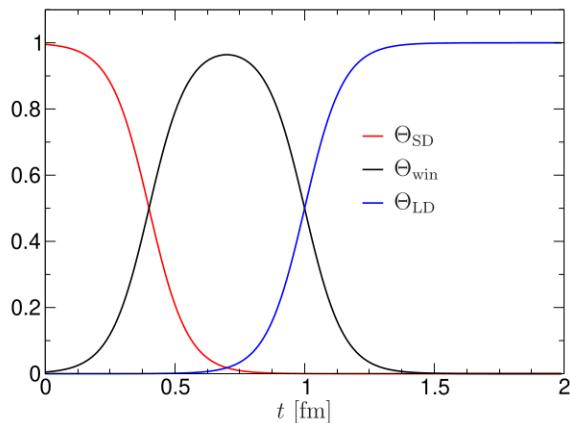


Figure: [Colangelo et al. 2022]

Time-Window Observables

continuum
extrapolation

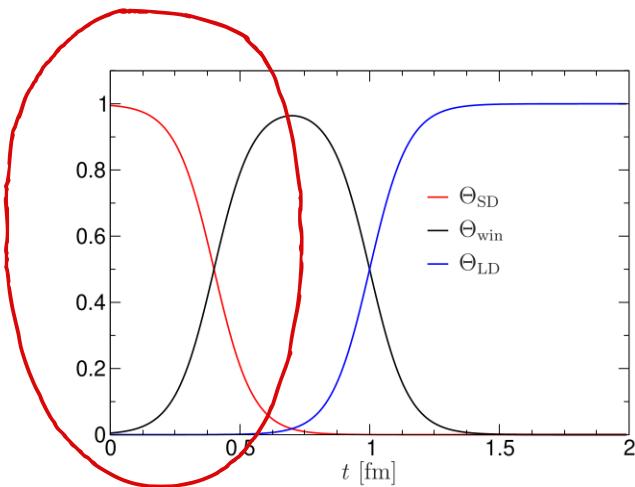


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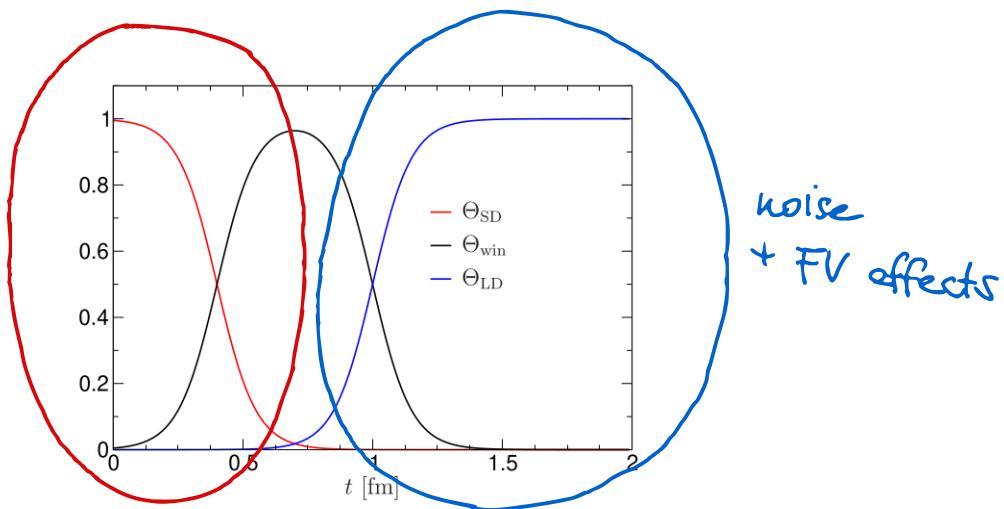


Figure: [Colangelo et al. 2022]

Time-Window Observables

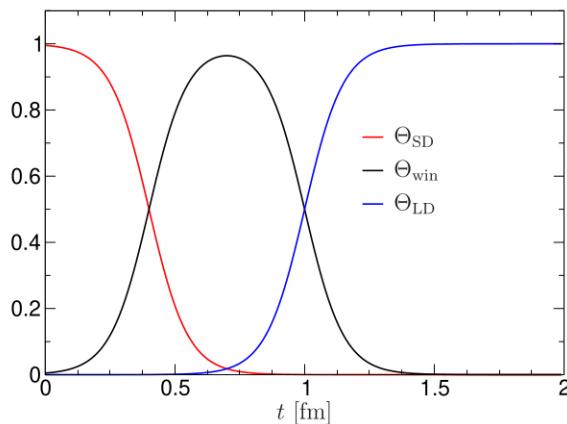


Figure: [Colangelo et al. 2022]

- ▶ Consistency checks for intermediate-time window or windows of different hadronic decay channels

C^* Boundary Conditions

[Lucini et al. 2016]

$$\Psi_f(x + L\hat{e}_i) = \Psi_f^c(x) = C^{-1}\bar{\Psi}_f^T(x)$$

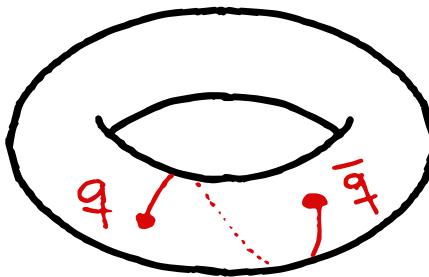
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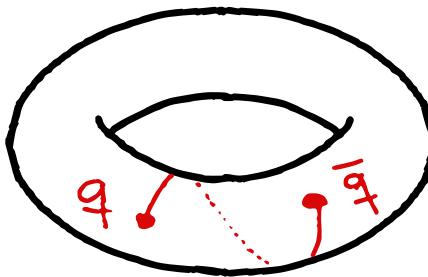


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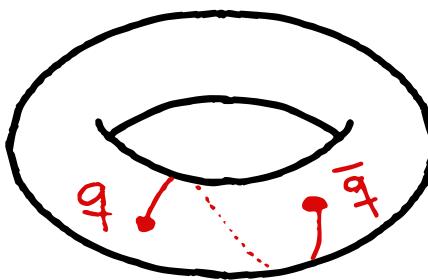
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- ▶ BC violate charge & flavor conservation

$$A_\mu(x + L\hat{e}_i) = A_\mu^c(x) = -A_\mu(x)$$

Consequences from Antiperiodicity of $A_\mu(x)$

Estimator HVP (zero-momentum projection?):

$$G(x_0 | T, L) = -\frac{1}{3} \int_{V_L} \langle j_k(x) j_k(0) \rangle_{T,L} \quad (1)$$

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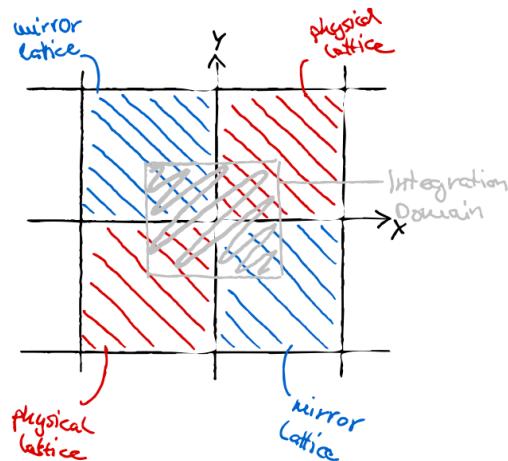
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Choose:

$$V_L = \left(-\frac{L}{2}, \frac{L}{2}\right)^3 \times (0, T) \quad (2)$$



Structure of Finite-Size Effects: PBC

[Hansen and Patella 2020]

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$$e^{-M_\pi L} + e^{-\sqrt{2}M_\pi L} + e^{-\sqrt{3}M_\pi L} + e^{-2M_\pi L} \dots$$

Finite-Volume Effects

Structure of Finite-Size Effects: PBC

[Hansen and Patella 2020]

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$$+ e^{-M_\pi T} + \dots + e^{-M_\pi \sqrt{L^2 + T^2}} + \dots$$

Finite-Time Effects
→ Subleading!

Structure of Finite-Size Effects: PBC

[Hansen and Patella 2020]

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$$+ e^{-M_K L} + \dots$$

Higher Mass Hadrons
→ subleading

Finite-Size Effects: PBC

[Hansen and Patella 2020]

$$\Delta G_L(x_0) = - \sum_{\mathbf{n} \neq \mathbf{0}} \int \frac{dp_3}{2\pi} \frac{e^{-|\mathbf{n}|L\sqrt{M_\pi^2 + p_3^2}}}{24\pi|\mathbf{n}|L}$$

$$\int \frac{dk_3}{2\pi} \cos(k_3 x_0) \operatorname{Re} T(-k_3^2, -p_3 k_3)$$

$$+ \mathcal{O}(e^{-\sqrt{2+\sqrt{3}}M_\pi L})$$

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pion Compton
scattering amplitude

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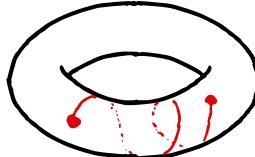
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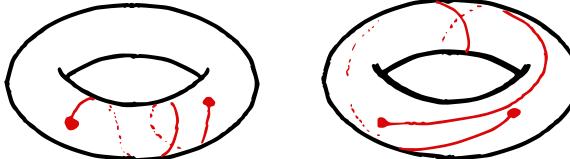
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- ▶ Charged case removed by factor $\frac{1+(-1)^{\langle n \rangle}}{2}$

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- ▶ Proportional to pion formfactor \rightarrow zero for uncharged case, also for periodic case

FV Effects: Results

Table: $-\Delta a_\mu(L) \times 10^{10}$

$M_\pi L$	C [*] BC	PBC
4	9.74(1.6)	22.4(3.1)
5	3.25(0.23)	10.0(0.4)
6	1.027(0.034)	4.42(0.06)
7	0.311(0.005)	1.924(0.009)
8	0.0909(0.0008)	0.826(0.001)

Table: $-100 \times \Delta a_\mu(L)/a_\mu$

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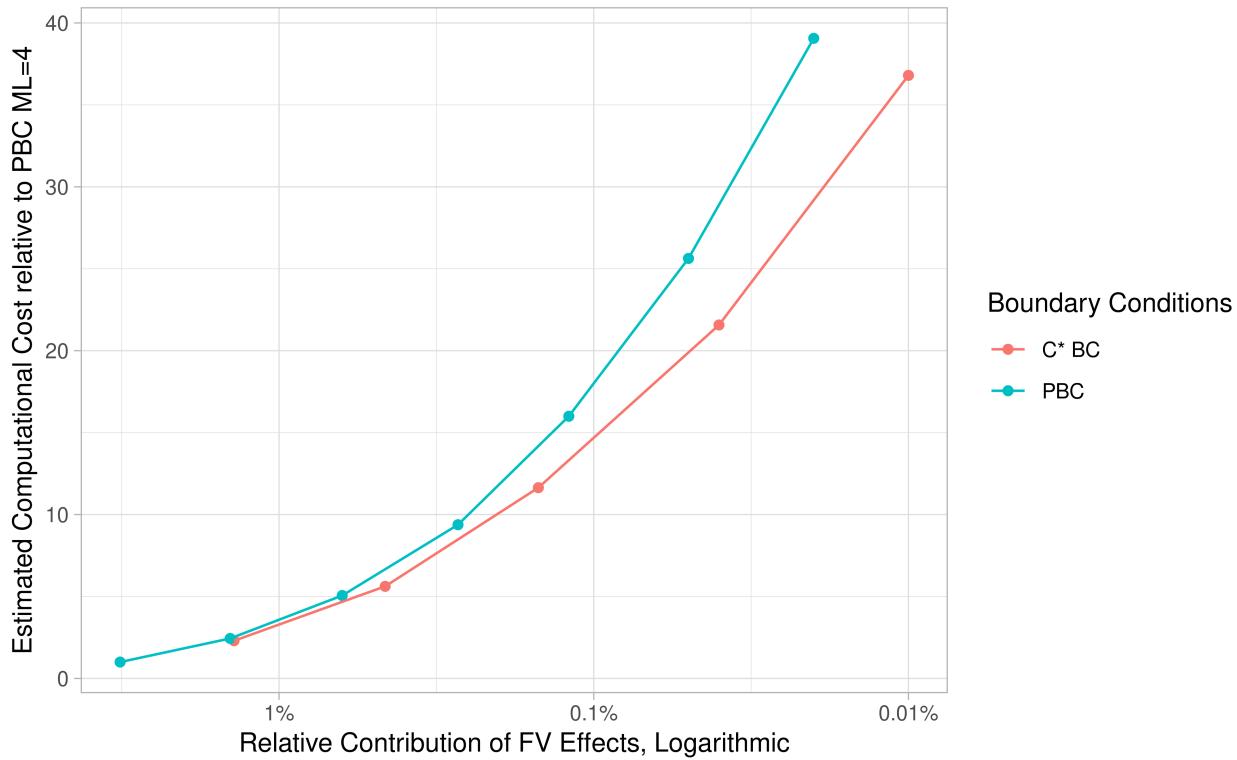
- ▶ C* BC FV effects vanish faster than for periodic

PBC from [Hansen and Patella 2020]

Computational Cost of Precision

$C^* \text{ BC} \sim 2.3 \text{ PBC}$

$\text{cost} \sim L^4$



Conclusion and Related Work

- ▶ Muon $g - 2$ calculations are underway: See contributions
406 Paola Tavella, Poster Session B
343 Roman Gruber, 11th Aug. at 9 am
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- ▶ Work on FV effects of IB breaking contributions is underway

References |

-  Campos, Isabel et al. (2020). “openQ*D code: a versatile tool for QCD+QED simulations”. In: *The European Physical Journal C* 80.3. ISSN: 1434-6052. DOI: 10.1140/epjc/s10052-020-7617-3. URL: <http://dx.doi.org/10.1140/epjc/s10052-020-7617-3>.
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-  Hansen, Maxwell T. and Agostino Patella (2020). “Finite-volume and thermal effects in the leading-HVP contribution to muonic (g_2)”. In: *JHEP* 10, p. 029. DOI: 10.1007/JHEP10(2020)029. arXiv: 2004.03935 [hep-lat].

References II

-  Kronfeld, Andreas S. and U. J. Wiese (1991). “ $SU(N)$ gauge theories with C periodic boundary conditions. 1. Topological structure”. In: *Nucl. Phys. B* 357, pp. 521–533. DOI: [10.1016/0550-3213\(91\)90479-H](https://doi.org/10.1016/0550-3213(91)90479-H).
-  Lucini, Biagio et al. (2016). “Charged hadrons in local finite-volume QED+QCD with C boundary conditions”. In: *JHEP* 02, p. 076. DOI: [10.1007/JHEP02\(2016\)076](https://doi.org/10.1007/JHEP02(2016)076). arXiv: [1509.01636 \[hep-th\]](https://arxiv.org/abs/1509.01636).
-  Polley, L. and U. J. Wiese (1991). “Monopole condensate and monopole mass in $U(1)$ lattice gauge theory”. In: *Nucl. Phys. B* 356, pp. 629–654. DOI: [10.1016/0550-3213\(91\)90380-G](https://doi.org/10.1016/0550-3213(91)90380-G).

Backup

Finite-Size Propagator: PBC

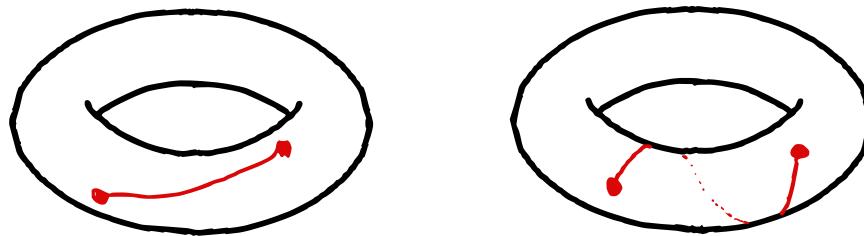
[Hansen and Patella 2020]

$$\Delta_{T,L}(x) = \Delta_\infty(x) + \sum_{n \in \mathbb{Z}^4 \setminus \{0\}} \Delta_\infty(x + L n)$$
$$L = \text{diag}(T, L, L, L)$$

Finite-Size Propagator: PBC

[Hansen and Patella 2020]

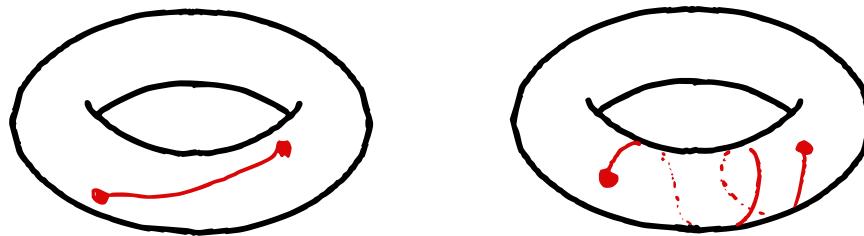
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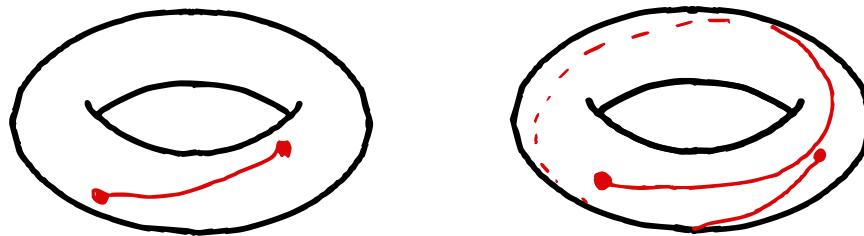
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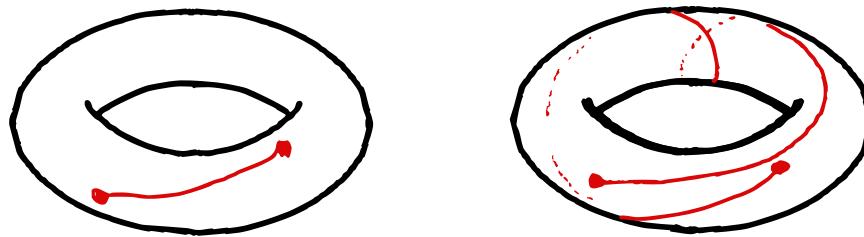
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C-parity basis

$$\pi^3(x) = \pi^0(x), \quad \pi^\pm = \frac{\pi^1(x) \pm i\pi^2(x)}{\sqrt{2}} \quad (1)$$

$$\Delta_{T,L}^3(x) = \Delta_{T,L}^1(x) = \sum_{n \in \mathbb{Z}^4} \Delta_\infty(x + L n) \quad (2)$$

$$\Delta_{T,L}^2(x) = \sum_{n \in \mathbb{Z}^4} (-1)^{\langle n \rangle} \Delta_\infty(x + L n) \quad (3)$$

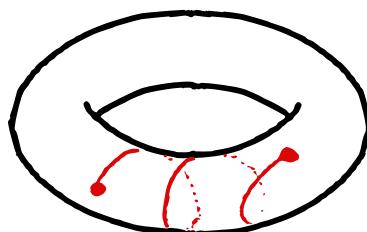
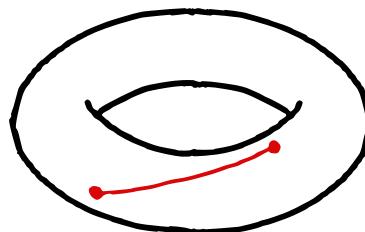
Finite-Size Propagator: C^* BC

$$\Delta_{T,L}^q(x) = \Delta_\infty(x) + \sum_{n \in \mathbb{Z}^4 \setminus \{0\}} \frac{1 + (-1)^{q\langle n \rangle}}{2} \Delta_\infty(x + L n)$$

$$\langle n \rangle = \sum_i n_i \bmod 2$$

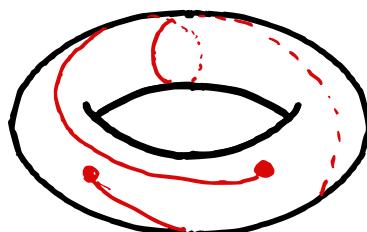
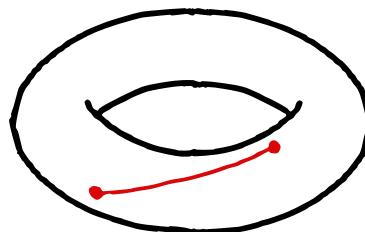
Finite-Size Propagator: C^* BC

$$\Delta_{T,L}^q(x) = \Delta_\infty(x) + \sum_{n \in \mathbb{Z}^4 \setminus \{0\}} \frac{1 + (-1)^{q\langle n \rangle}}{2} \Delta_\infty(x + L n)$$



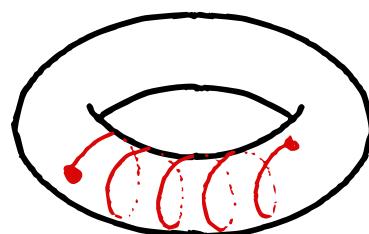
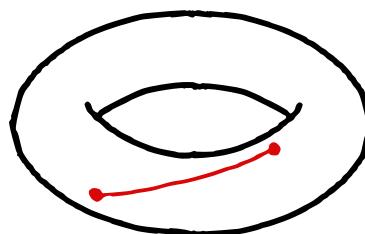
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The Charged Pion Compton Scattering Amplitude

$$T^q(k^2, k \cdot p) = i \lim_{\mathbf{p}' \rightarrow \mathbf{p}} \int d^4x e^{ikx}$$

$$\langle \pi^q(\mathbf{p}') | \hat{T} \{ J_\rho(x) J^\rho(0) \} | \pi^q(\mathbf{p}) \rangle$$

The Charged Pion Compton Scattering Amplitude

[Hansen and Patella 2020]

$$T^q(-k_3^2, -p_3 k_3) = \lim_{p'_3 \rightarrow p_3} \langle \pi^q(p'_3 \hat{e}_3) | J_\rho(0) \hat{O} J^\rho(0) | \pi^q(p_3 \hat{e}_3) \rangle$$

$$\hat{O} = \frac{(2\pi)^3 \delta(\hat{P}_1) \delta(\hat{P}_2) \delta(\hat{P}_3 - p_3 - k_3)}{\hat{H} - \sqrt{\hat{p}_3^2 + M_\pi^2} - i\varepsilon}$$

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Decomposition

[Hansen and Patella 2020]

$$\mathbb{1} = |\Omega\rangle \langle \Omega|$$

$$+ \sum_{q=\{0,\pm 1\}} \int \frac{d^3\ell}{(2\pi)^3} \frac{1}{2E(\ell)} |\pi^q(\ell)\rangle \langle \pi^q(\ell)|$$

$$+ \theta(\hat{M} - 2M_\pi)$$

Decomposition

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$$1 = |\Omega\rangle \langle \Omega| \quad - \text{vacuum contribution}$$

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*one-pion
/ contribution*

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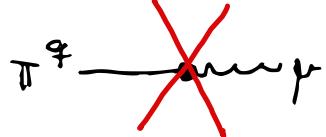
Decomposition

$$T^q = T_{\text{vac}}^q + T_{1\pi}^q + T_{\text{MP}}^q$$

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$+ T_{\text{RP}} \rightarrow \chi\text{PT}$

$\rightarrow \text{small}$

The Pole Contribution is Zero for Uneven Numbers of Translations

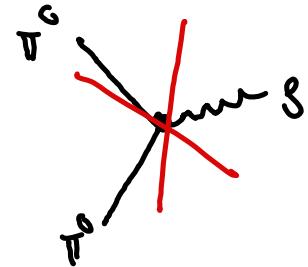
Charged Case

$$\frac{1 + (-1)^{q\langle n \rangle}}{2}$$

The Pole Contribution is Zero for Uneven Numbers of Translations

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Uncharged Case

$$T_{1\pi,\text{pole}}^0 \propto | \langle \pi^0(\ell') | J_\rho(0) | \pi^0(\ell) \rangle |^2 = 0$$

Pole Contribution

Table: $-100 \times \frac{\Delta a(L)}{a_\mu}$

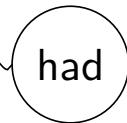
$M_\pi L$	$ n = \sqrt{2}$	2	$\sqrt{6}$	$2\sqrt{2}$	Sum	PBC
4	1.16	0.104	0.0944	0.0128	1.38	3.17
5	0.428	0.0199	0.0112	0.00103	0.461	1.42
6	0.141	0.00349	0.00124	0.0000764	0.146	0.630
7	0.0433	0.000582	0.000130	$< 10^{-5}$	0.0440	0.274
8	0.0128	0.0000936	0.0000132	$< 10^{-5}$	0.0129	0.118

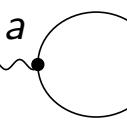
PBC from [Hansen and Patella 2020]

All-orders Expansion in EFT

[Hansen and Patella 2020]

$$G(x_0) = \text{had} = (1) + (2) + \dots$$

(1) 

(2) 

(3) 