The 39th International Symposium on Lattice Field Theory

Probing the R ratio on the lattice

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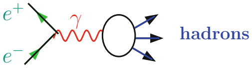
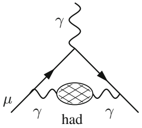
- First historical evidence of $N_c = 3$

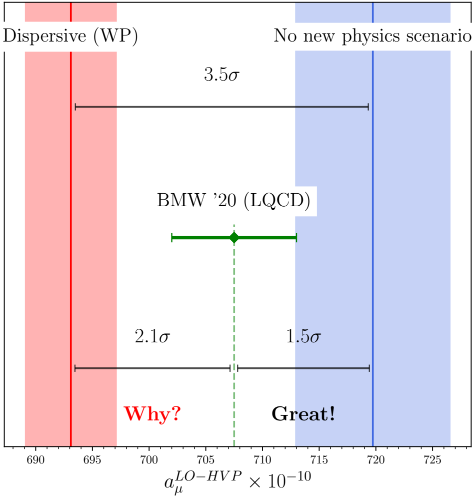
$$R(E) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \to N_c \sum_{i=1} Q_i^2$$





- Crucial to calculation of $(g-2)_\mu$ from **dispersive integral**

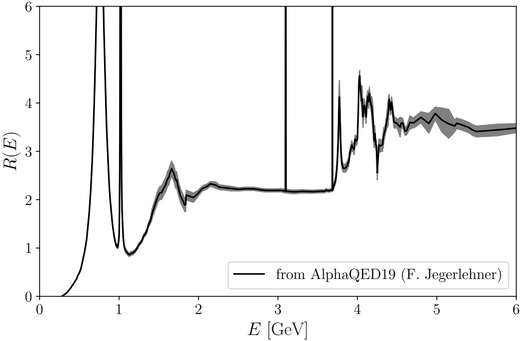
$$a_\mu^{\text{LO-HVP}} = \frac{\alpha_{\text{em}}^2}{3\pi^2} \int_{M_\pi^2}^{\infty} \mathrm{d}s \frac{K(s)}{s} R(s)$$



What do we mean by "probing the R ratio"?

Infinite number of observables

$$\boxed{R[K](E) = \int_0^\infty \mathrm{d}\omega K(\omega, E) R(\omega)}$$



- From $R(E)$ itself

$$R(E) = \int_0^\infty \mathrm{d}\omega \delta(\omega - E) R(\omega)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$K(\omega, E)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

- to the **Euclidean correlator** (Bernecker, Meyer '11)

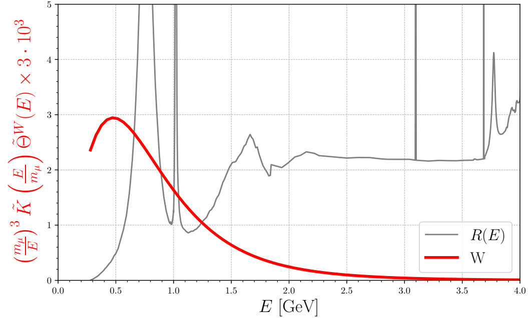
$$C(t) = \frac{1}{12\pi^2} \int_0^\infty \mathrm{d}\omega \, e^{-\omega t} \omega^2 R(\omega)$$

- Each kernel **probes R in a very different way**

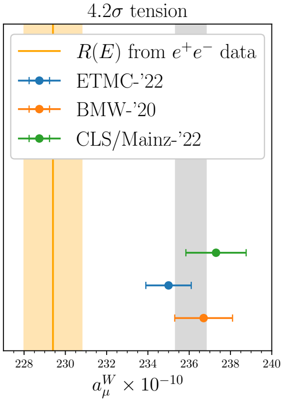
a_μ^W : an example of observable

RBC/UKQCD:

$$a_\mu^W = \frac{2\alpha_{\text{em}}^2}{9\pi^2 m_\mu}\int_{E_{\text{thr}}}^{\infty} \mathrm{d}E \left(\frac{m_\mu}{E}\right)^3 \tilde{K}\left(\frac{E}{m_\mu}\right)\tilde{\Theta}^W(E) R(E)$$



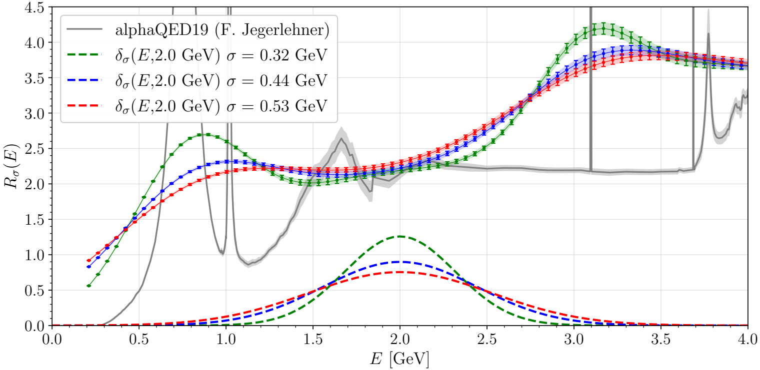
- From LQCD computation of $C(t)$:



- Some observables may suggest other ones to look at

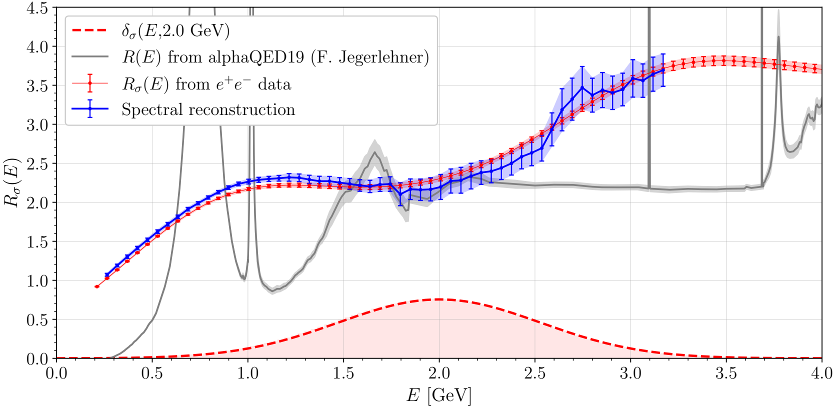
Probe R in the energy regime: smeared $R_\sigma(E)$

$$R_\sigma(E) = \int_{2M_\pi}^{\infty} d\omega \delta_\sigma(\omega, E) R(\omega) \qquad \delta_\sigma(\omega, E) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(\omega-E)^2}{2\sigma^2}} \quad \rightarrow \quad \text{local probing of } R \text{ in energy}$$



$$\boxed{\text{HLT spectral reconstruction of } \quad R_\sigma(E) \quad \text{from} \quad C(t) = \frac{1}{12\pi^2} \int_{2M_\pi}^{\infty} d\omega\, e^{-\omega t} \omega^2 R(\omega)}$$

$$R_\sigma(E) = \int_{2M_\pi}^{\infty} \mathrm{d}\omega \delta_\sigma(\omega, E) R(\omega) \qquad \delta_\sigma(\omega, E) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(\omega-E)^2}{2\sigma^2}} \qquad \sigma \sim 0.5 \text{ GeV}$$



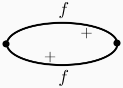
- How did we get it?

ETMC Lattice setup

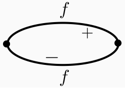
- arXiv:2206.15084 by ETM Collaboration
- S. Bacchio & G. Gagliardi - Talks Lattice 2022;

$$C(t) = -\frac{a^3}{3} \sum_{i=1}^{3} \sum_{\vec{x}} \left\langle \hat{J}_i(x) \hat{J}_i(0) \right\rangle$$

$J_\mu^{f,OS} \propto \bar{\psi}_f^+ \gamma_\mu \psi_f^+$ (RC: Z_V).



$J_\mu^{f,tm} \propto \bar{\psi}_f^+ \gamma_\mu \psi_f^-$ (RC: Z_A)



ensemble	β	V/a^4	a (fm)	$a\mu_\ell$	M_π (MeV)	L (fm)	$M_\pi L$
B64	1.778	$64^3 \cdot 128$	0.07961 (13)	0.00072	140.2 (0.2)	5.09	3.62
B96	1.778	$96^3 \cdot 192$	0.07961 (13)	0.00072	140.1 (0.2)	7.64	5.43
C80	1.836	$80^3 \cdot 160$	0.06821 (12)	0.00060	136.7 (0.2)	5.46	3.78
D96	1.900	$96^3 \cdot 192$	0.05692 (10)	0.00054	140.8 (0.2)	5.46	3.90

- $N_f = 2 + 1 + 1$ dynamical flavours (ℓ, s, c in isosymmetric theory)
- All quark-line **connected** and **disconnected** contributions included
- **Two regularizations** for quark connected contributions: Twisted Mass (TM) and Osterwalder-Seiler (OS)

The **new method** (inspired by **G. Backus and F. Gilbert method '68, '70**):

- M. Hansen, A. Lupo, N. Tantalo - Phys.Rev.D 99 (2019); J. Bulava et. al - JHEP 07 (2022)
- J. Bulava - Talk Lattice 2022, A. Lupo - Talk Lattice 2022, A. Evangelista - Poster Lattice 2022

$$\underbrace{C(t) = \int_0^\infty \mathrm{d}\omega\, e^{-\omega t} \rho(\omega)}_{\text{Avaiable from lattice}}$$

$$\boxed{\rho(\omega) = \frac{\omega^2 R(\omega)}{12\pi^2} \qquad K(\omega, E) = 12\pi^2 \frac{\delta_\sigma(\omega, E)}{\omega^2}}$$

If we could write

$$K^{\text{true}}(\omega, E) \sim K^{\text{rec}}(\omega, E) = \sum_{\tau=1}^{\tau_{\max}} g_\tau(E) \left[e^{-\omega\tau} + e^{-\omega(T-\tau)}\right]$$

then

$$\rho[K](E) = \int_0^\infty \mathrm{d}\omega\, K^{\text{true}}(\omega, E)\rho(\omega) = \sum_{\tau=1}^{\tau_{\max}} g_\tau(E) C(\tau) + \Delta\rho$$

- How can we compute the coefficient vector **g**?

- The vector of coefficients $\boldsymbol{g}$ is obtained by the minimization of

$$W_\lambda[\boldsymbol{g}] = (1-\lambda)\frac{A_\alpha[\boldsymbol{g}]}{A_\alpha[0]} + \lambda B[\boldsymbol{g}]$$

- $A_\alpha[\boldsymbol{g}]$ is the difference between the true and the approximated kernel

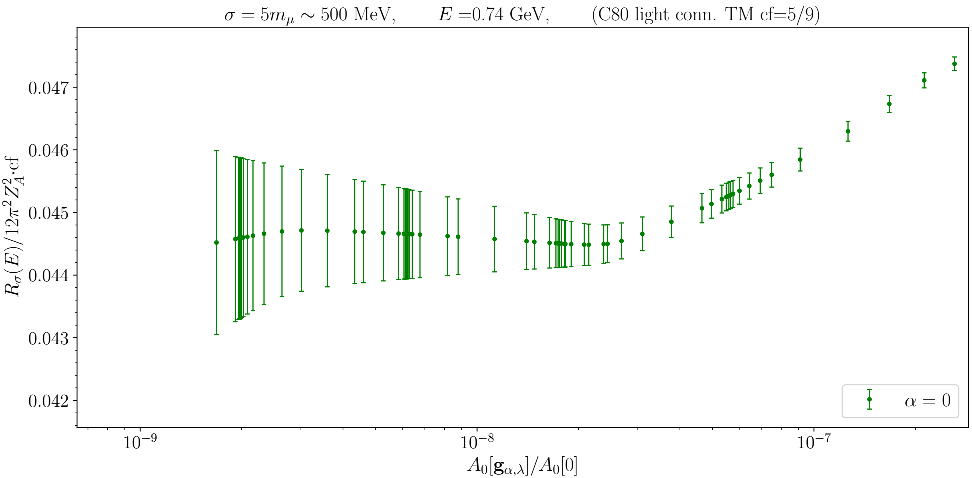
$$A_\alpha[\boldsymbol{g}] = \int_{E_0}^{\infty} \mathrm{d}\omega \left\{ K^{\text{true}}(\omega, E) - K^{\text{rec}}(\omega, E)[\boldsymbol{g}] \right\}^2 \cdot e^{\alpha\omega}$$

- $B[\boldsymbol{g}]$ is a noise regulator (reflecting the statistical variance)

$$B[\boldsymbol{g}] \propto \boldsymbol{g}^T \cdot \hat{\text{Cov}} \cdot \boldsymbol{g}$$

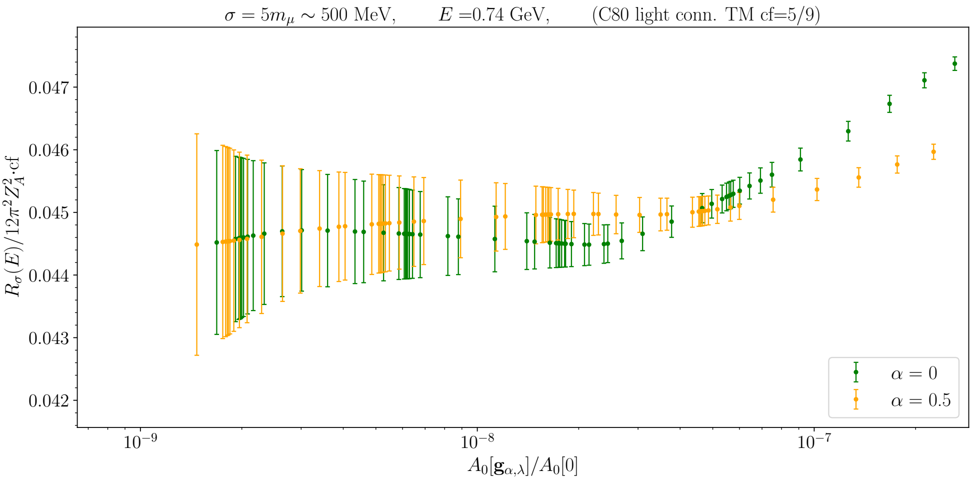
- α improves the convergence and allows for a better stability ($\alpha < 2$ to have convergent integrals)
- λ is a trade-off parameter
- We look for **stability with respect** $\lambda \Leftrightarrow \dfrac{A_0[\boldsymbol{g}_{\boldsymbol{\alpha},\boldsymbol{\lambda}}]}{A_0[0]}$

Stability of the reconstruction



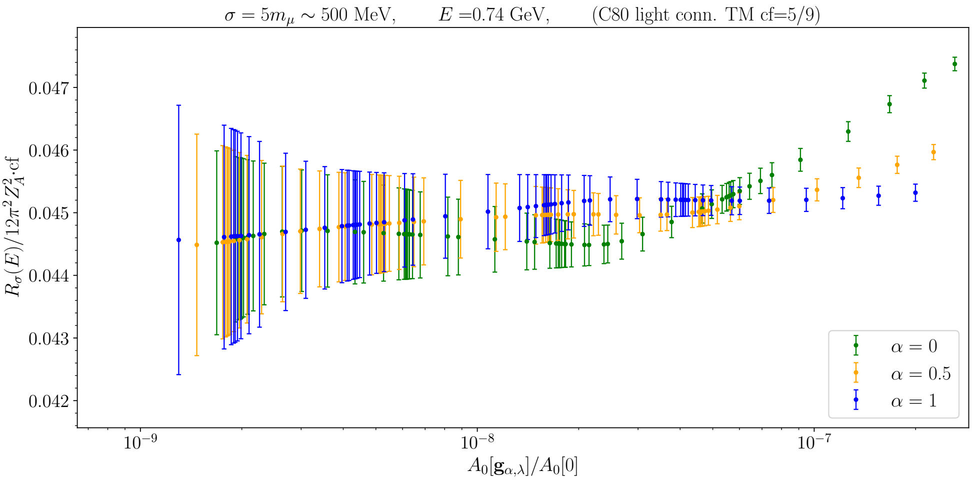
- Vertical bars correspond to statistical error only

Stability of the reconstruction



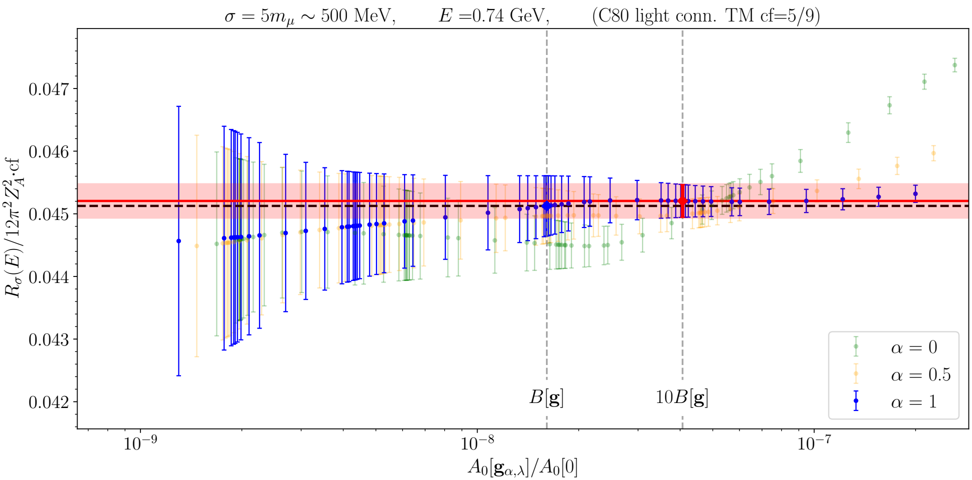
- The stability of $R(E)$ wrt variations of $A_0[g]/A_0[0]$ is enhanced upon increasing α

Stability of the reconstruction

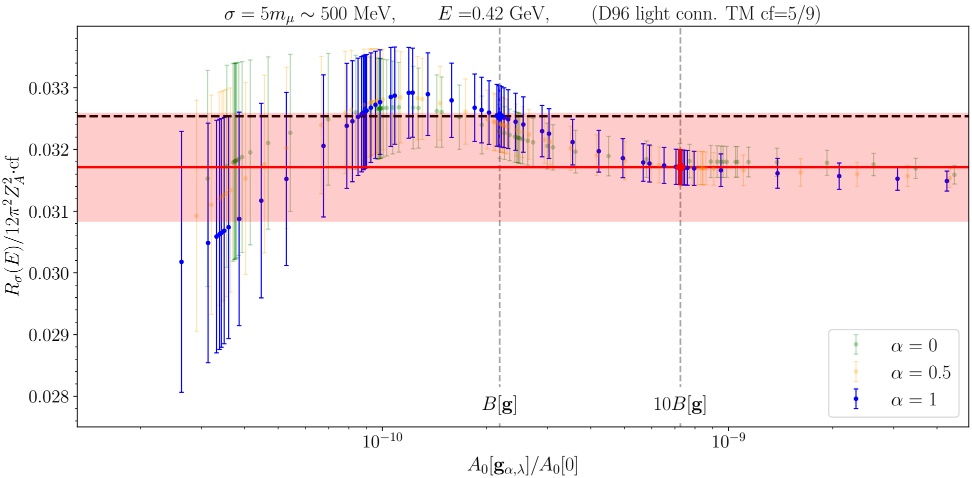


- difference between $\alpha = 2^-$ and $\alpha = 1$ results found to be negligible $\rightarrow$ saturation level reached

Stability of the reconstruction: final result and error estimation



$$R_\sigma(E) = R_\sigma(E)\big|_{10B} \qquad \Delta_{\text{final}} = \sqrt{\Delta_{\text{stat}}^2 + \Delta_{\text{syst}}^2} \qquad \Delta_{\text{syst}} = \left| R_\sigma(E)\big|_{10B} - R_\sigma(E)\big|_{B} \right| \qquad \alpha = 1$$

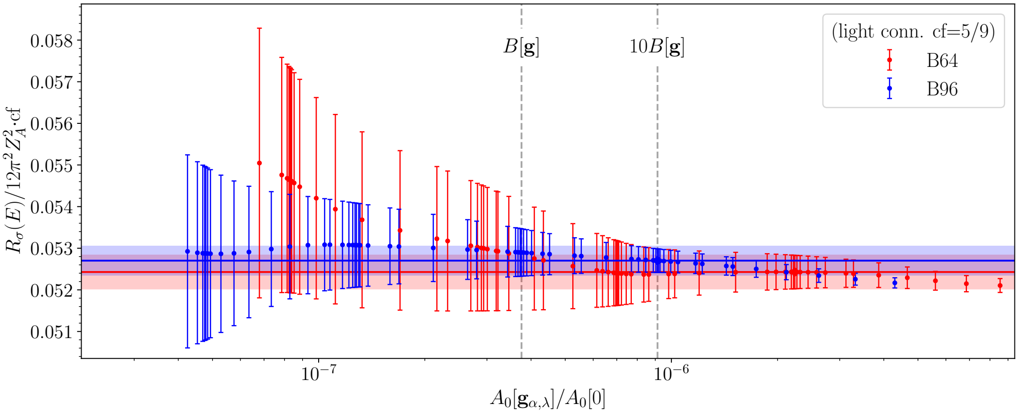


- Systematic uncertainty under change of $A_0[\mathbf{g}]/\mathbf{A_0}[\mathbf{0}]$ reliably taken into account

$\Rightarrow$ **conservative error estimation**

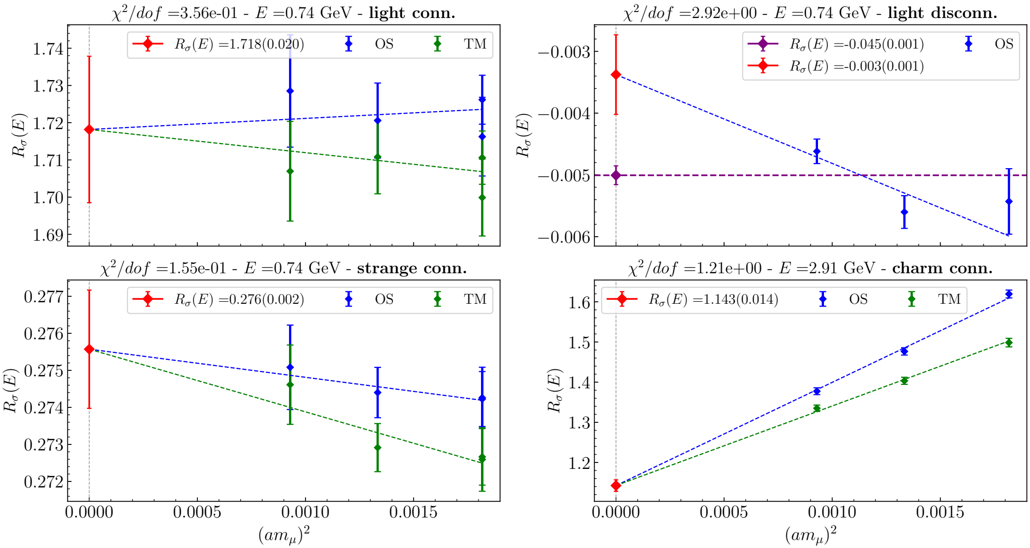
Same lattice spacing but different volume (5.1 fm and 7.6 fm)

- The results from data at the two different L are consistent within the quoted errors ($E = 1.06$ GeV, $\sigma \sim 500$ MeV)

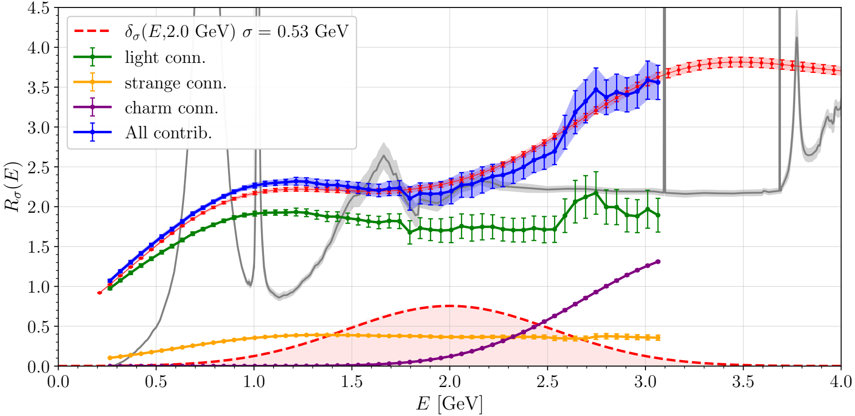


$R_\sigma(E)$: continuous extrapolation

- Combined $(am_\mu)^2$-linear extrapolation to zero lattice spacing - ($\sigma \sim 500$ MeV)

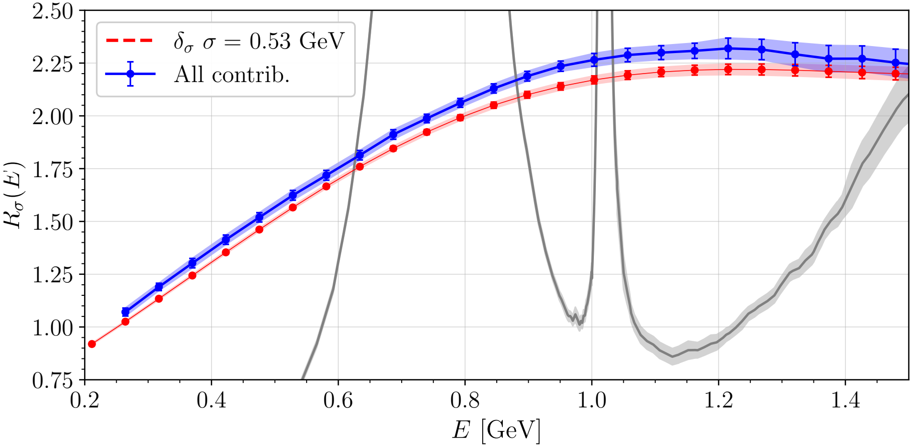


$R_\sigma(E)$ from e^+e^- data

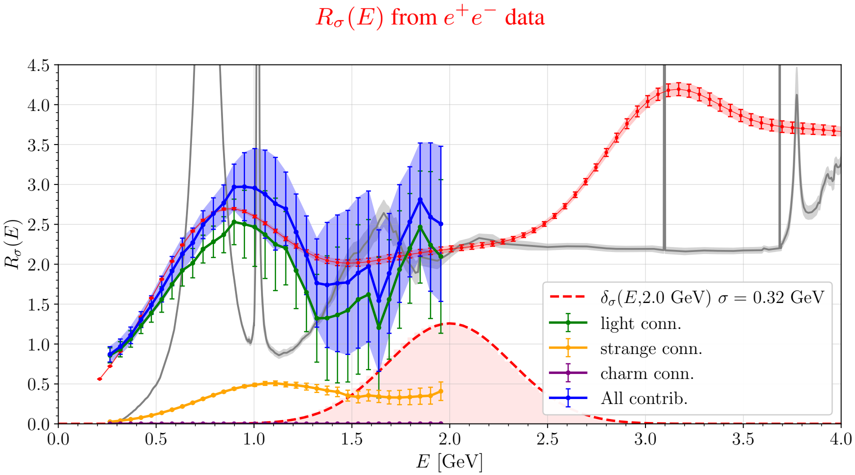


- Uncertainty coming mostly from light quark contributions, strange & charm ones are very precise
- Disconnected contributions are tiny and cannot be appreciated on this scale

$R_\sigma(E)$: preliminary results, let's zoom in ...



- Spectral reconstruction seems to deviate significantly from e^+e^- data in line width a_μ^W calculation but ...
- a **more careful study of the error on data-driven results is needed** (correlations, up-to-date datasets):
  - Here a naive estimate of them is shown based on a public database (AlphaQED19).
  - It would be interesting to have data-driven $R_\sigma(E)$ values based on other databases (**KNT, DHMZ**).



- **Large errors** from light contributions invalidate accurate reconstruction for $\sigma < 0.3$ GeV
- Any possible discrepancy is now masked by the reconstruction uncertainty

Conclusions and future perspectives

- The R ratio can be investigated by computing any related observable $R[K](E) = \int_0^\infty \mathrm{d}\omega K(\omega, E) R(\omega)$
- We focused on the **smearing** by a Gaussian of width $\sigma \rightarrow R_\sigma(E) \rightarrow$ important for comparison with $R_\sigma(E)^{\text{exp}}$
- $R_\sigma(E)$ is computed from the **Euclidean** lattice correlator via a **new spectral reconstruction method (HLT)**
- Results for $R_\sigma(E)$ are given with reliable errors based on a careful reconstruction stability analysis
- Results at $\sigma < 300$ MeV are currently inaccurate due to limitations in volume size and statistics but **feasible in future**
- A deviation from e^+e^- data, if confirmed, would be in line with the a_μ^W computation from the same ETMC correlators

Next on our to-do list:

- A solid estimate of the experimental errors and a detailed comparison with our data
- Exploration of the sensitivity to other kernels
- Extension to **larger statistics** (a second ensemble with $L = 7.5$ fm) and possibly to **even larger size** L in order to reduce the error and explore smaller σ

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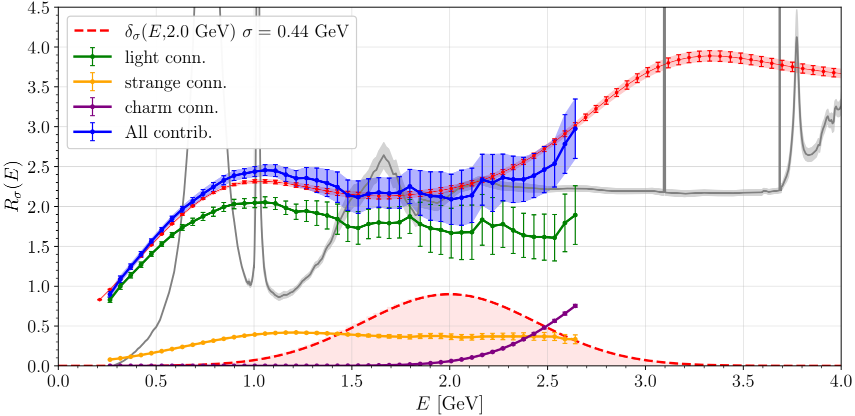
Thank you for the attention!

Backup slides

$R_\sigma(E)$: preliminary results

$R_\sigma(E)$ from e^+e^- data



- Still a tension between our results and e^+e^- data
- As σ decreases the reconstruction becomes more challenging

Backup: the functional $A_\alpha[\boldsymbol{g}]$

$$W_\lambda[\boldsymbol{g}] = (1-\lambda)\frac{A_\alpha[\boldsymbol{g}]}{A_\alpha[0]} + \lambda B[\boldsymbol{g}] \qquad K^{\text{rec}}(\omega, E) = \sum_{\tau=1}^{\tau_{\max}} g_\tau(K, E, \cdots)e^{-\tau\omega} \tag{1}$$

where

$$A_\alpha[\boldsymbol{g}] = \int_{E_0}^{\infty} \mathrm{d}\omega \, \left\{K^{\text{true}}(\omega, E) - K^{\text{rec}}(\omega, E)\right\}^2 e^{\alpha\omega} = \boldsymbol{g}^T \cdot \hat{A}\boldsymbol{g} - 2\boldsymbol{g}^T \cdot \boldsymbol{f} + A_\alpha[0] \tag{2}$$

$$\rho[K]^{\text{true}}(E) - \rho[K]^{\text{rec}}(E) = \int_{E_0}^{\infty} \mathrm{d}\omega\rho(\omega) \left[K^{\text{true}}(\omega, E) - K^{\text{rec}}(\omega, E)\right] \tag{3}$$

- $\rho(\omega)$ in general increases as a power of the energy (Axiomatic QFT)
- $\left[K(\omega, E) - K^{\text{rec}}(\omega, E)\right]$ is forced to decrease exponentially thanks to $e^{\alpha\omega}$ with $\alpha > 0$

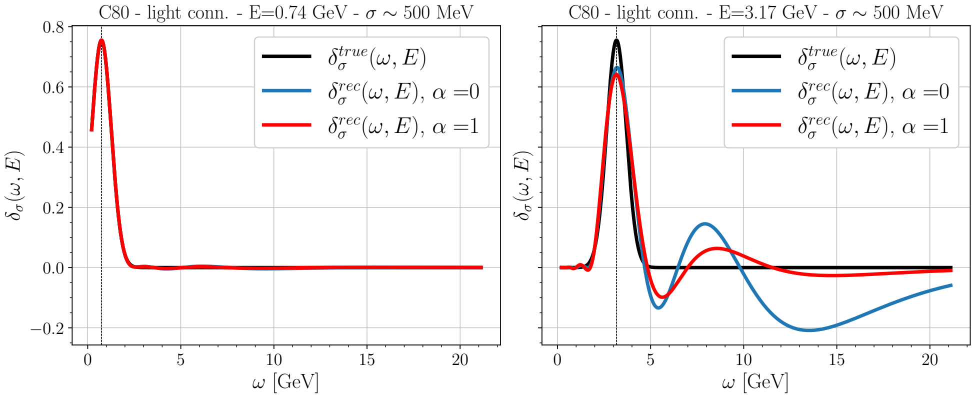
$$\boldsymbol{g}^T \cdot \hat{A}\boldsymbol{g} = \sum_{\tau_1,\tau_2=1}^{\tau_{\max}} g_{\tau_1} g_{\tau_2} \frac{e^{\omega(\alpha-\tau_1-\tau_2)}}{\alpha-\tau_1-\tau_2}\Bigg|_{E_0}^{\infty} \quad \text{convergent if} \quad \alpha < \tau_1 + \tau_2 < 2 \tag{4}$$

$$-2\boldsymbol{g}^T \cdot \boldsymbol{f} = \sum_{\tau=1}^{\tau_{\max}} g_\tau \int_{E_0}^{\infty} \mathrm{d}\omega \, e^{\omega(\alpha-\tau)} K(\omega, E) \tag{5}$$

As a measure of the reconstruction accuracy we consider

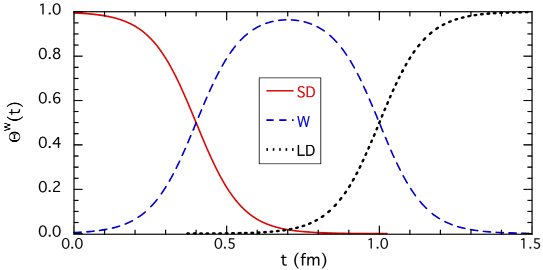
$$\boxed{\frac{A_0[g_{\alpha,\lambda}]}{A_0[0]} = \frac{\int_{E_0}^{\infty} \mathrm{d}\omega \left\{K(\omega, E) - \boldsymbol{g}_{\alpha,\lambda} \cdot \boldsymbol{b}\right\}^2}{\int_{E_0}^{\infty} \mathrm{d}\omega \left\{K(\omega, E)\right\}^2}} \quad B[\boldsymbol{g}] = \sum_{\tau,\tau'=1}^{\tau_{\max}} g_\tau \frac{\mathrm{Cov}(C(\tau), C(\tau'))}{C(1)^2} g_{\tau'} E^{-2[g]} \tag{6}$$

Kernel reconstruction



- The effect of α is a strong suppression of $\delta^{\text{rec}}(\omega, E) - \delta^{\text{true}}(\omega, E)$
- The kernel reconstruction is less accurate at high energies

Backup: a_μ time-distance windows



$$a_\mu^w = 2\alpha_{\rm em}^2 \int_0^\infty {\rm d}t\, t^2 K(m_\mu t)\Theta^w(t)C(t) \qquad\qquad a_\mu^{\rm HVP} = a_\mu^{\rm SD} + a_\mu^{\rm W} + a_\mu^{\rm LD}$$

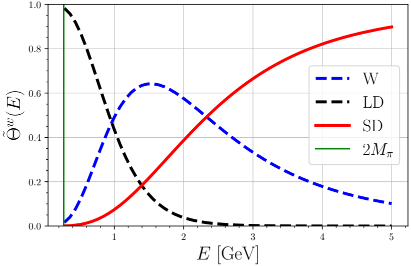
$$\Theta^{SD}(t) = 1 - \frac{1}{1+e^{-2(t-t_0)/\Delta}}$$

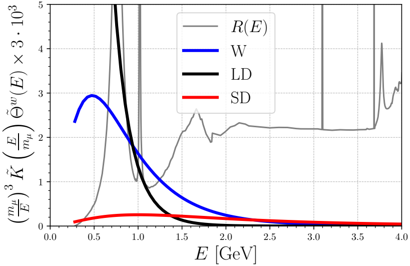
$$\Theta^{W}(t) = \frac{1}{1+e^{-2(t-t_0)/\Delta}} - \frac{1}{1+e^{-2(t-t_1)\Delta}}$$

$$\Theta^{LD}(t) = 1 - \frac{1}{1+e^{-2(t-t_1)/\Delta}}$$

$t_0 = 0.4$ fm, $t_1 = 1$ fm, $\Delta = 0.15$ fm (arXiv:1801.07224)

Backup: windows in the energy regime





Deviations of $R(E)$ data with respect to SM predictions expected in the low and/or intermediate energy region

$$a_\mu^w = \frac{2\alpha_{\rm em}^2}{9\pi^2 m_\mu} \int_{E_{\rm thr}}^{\infty} {\rm d}E \left(\frac{m_\mu}{E}\right)^3 \tilde{K}\left(\frac{E}{m_\mu}\right) \tilde{\Theta}^w(E) R(E) \qquad \tilde{\Theta}^w(E) = \frac{\int_0^\infty {\rm d}t\, t^2 e^{-Et} K(m_\mu t)\Theta^w(t)}{\int_0^\infty {\rm d}t\, t^2 e^{-Et} K(m_\mu t)}$$

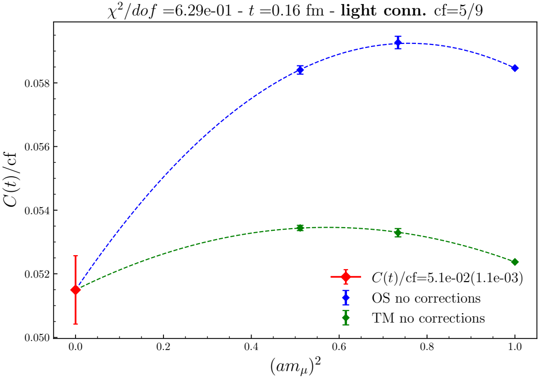
obs.(HVP-LO)	ETMC-22	BMW-20	latt. "aver."	WP-proc.('22)	KNT('19-'22)
a) $a_\mu^{\rm SD}\ 10^{10}$	69.33(29)	–	–	68.4(5)	*68.44(48)*
b) $a_\mu^{\rm W}\ 10^{10}$	235.0(1.1)	236.7(1.4)	236.08(74)	229.4(1.4)	*229.51(87)*
c) $a_\mu^{\rm HVP}\ 10^{10}$	–	707.5(5.5)	–	693.0(3.9)	*692.78(2.42)*

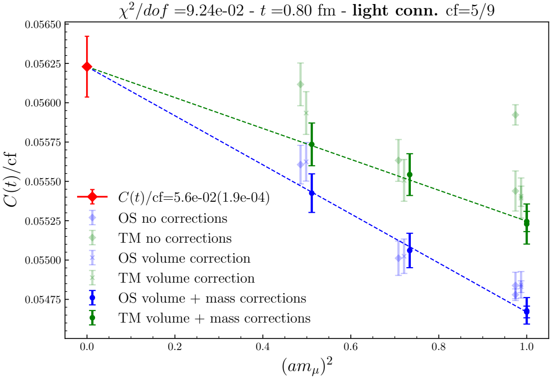
Table from R. Frezzotti ICHEP 2022

Backup: continuous extrapolation of the correlator

$$C\left((am_\mu)^2\right) = C_0 + C_1^r(am_\mu)^2 + C_2^r(am_\mu)^4 + C_{M_\pi}\left(\frac{aM_\pi}{aM} - \frac{M_\pi^{\text{phys}}}{M}\right) + C_L^r\left(e^{-M_\pi L} - e^{-M_\pi^{\text{phys}} L_\star}\right)$$

- $C_2^r(am_\mu)^4$, mass and volume corrections are included depending on the time region. $r = [TM, OS]$
- Correlators are first interpolated at physical times $t = \left(1 + \frac{1}{2}\right) \cdot a_B$ ($a_B = 0.0796$ fm - coarser lattice spacing)
- Continuous extrapolation is more challenging around 0.5 fm where both a^4 and finite volume effects show up





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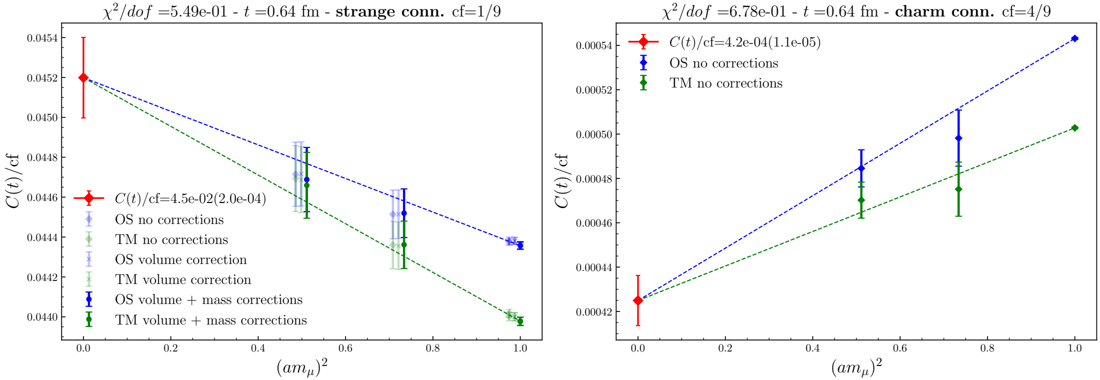
- $L_\star = L_{B64} = 5.09$ fm, $M = 2.48$ GeV



Backup: mixed-action setup

From G. Gagliardi - SchwingerFest 2022

Local and renormalizable mixed action employed
[Frezzotti and Rossi (2004)] :

$$S = S_{YM}[U] + S_{q,sea}[\Psi_\ell, \Psi_h, U] + S_{q,val}[q_f^\eta, U] + S_{q,ghost}[\phi_f^\eta, U]$$

- Gluonic sector: improved Iwasaki action $S_{YM}[U]$ (not detailed here).
- Fermionic sector: sea quark action $S_{g,sea}$ written in terms of degenerate quark doublet $\Psi_\ell^t = \{u_{sea}, d_{sea}\}$, and heavy non-degenerate doublet $\Psi_h^t = \{c_{sea}, s_{sea}\}$.
- Fermionic sector: valence quark action $S_{g,val}$ written in terms of quark fields q_f^η, where $f = u, d, s, c$, and the replica index η runs from 1 to 3 with $r_f^\eta = (-1)^{\eta+1}$.
- Ghost sector $S_{q,ghost}$ introduced to cancel contribution of $S_{q,val}$ to fermionic determinant.
- Sea and valence quark masses are matched to renormalized ones (see arXiv:1807.00495 and arXiv:2206.15084)

Backup: mixed-action setup

From G. Gagliardi - SchwingerFest 2022

Sea quark action

$$S_{q,sea} = a^4 \sum_x \Big\{ \bar{\Psi}_\ell(x) \left[\gamma \cdot \tilde{\nabla} + \mu_\ell - i\gamma_5 \tau^3 W_{cr}^{clov.} \right] \Psi_\ell$$

$$+ \bar{\Psi}_h(x) \left[\gamma \cdot \tilde{\nabla} + \mu_\sigma + \mu_\delta \tau^3 - i\gamma_5 \tau^1 W_{cr}^{clov.} \right] \Psi_h \Big\}$$

Valence quark action

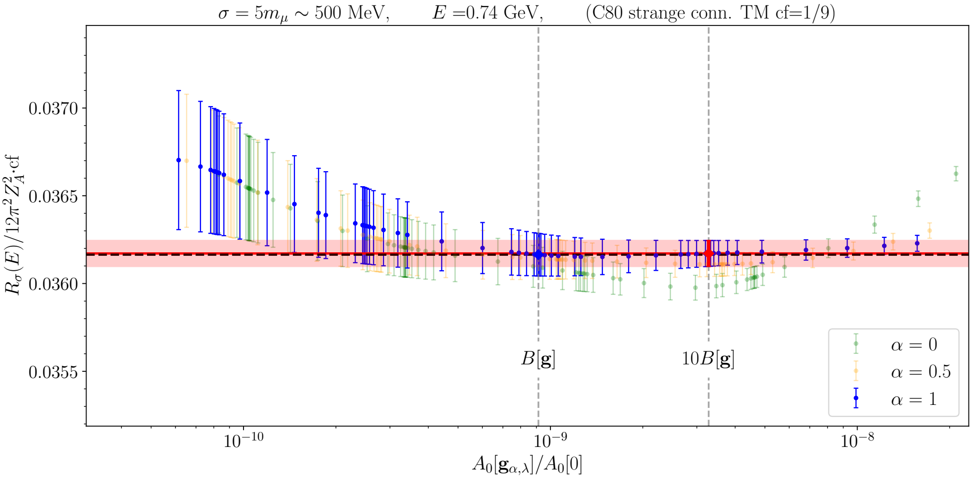
$$S_{q,val} = a^4 \sum_x \bar{q}_f^\eta(x) \left[\gamma \cdot \tilde{\nabla} + m_f - \underbrace{r_{f,\eta}}_{(-1)^{\eta+1}} i\gamma_5 W_{cr}^{clov.} \right] q_f^\eta(x)$$

Critical Wilson-clover operator

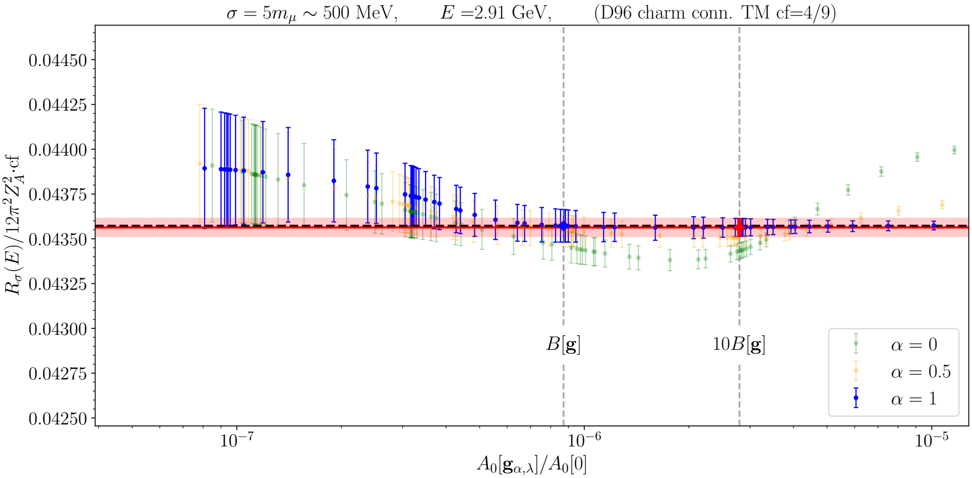
$$W_{cr}^{clov.} = -\frac{a}{2} \nabla^* \cdot \nabla + m_{cr} + \frac{c_{SW}}{32} \gamma_\mu \gamma_\nu \left[Q_{\mu\nu} - Q_{\nu\mu} \right]$$

Strange quarks

- Strange contributions are more precise than light ones



- Charm correlators are less noisy $\rightarrow$ very accurate result



Same lattice spacing but different volume (5.1 fm and 7.6 fm)

- The results from data at the two different L are consistent within the quoted errors   ($E = 1.06$ GeV, $\sigma \sim 500$ MeV)