Neutrinoless Double Beta Decay from Lattice QCD: The Short-Distance $\pi^- \to \pi^+ e^- e^-$ Amplitude

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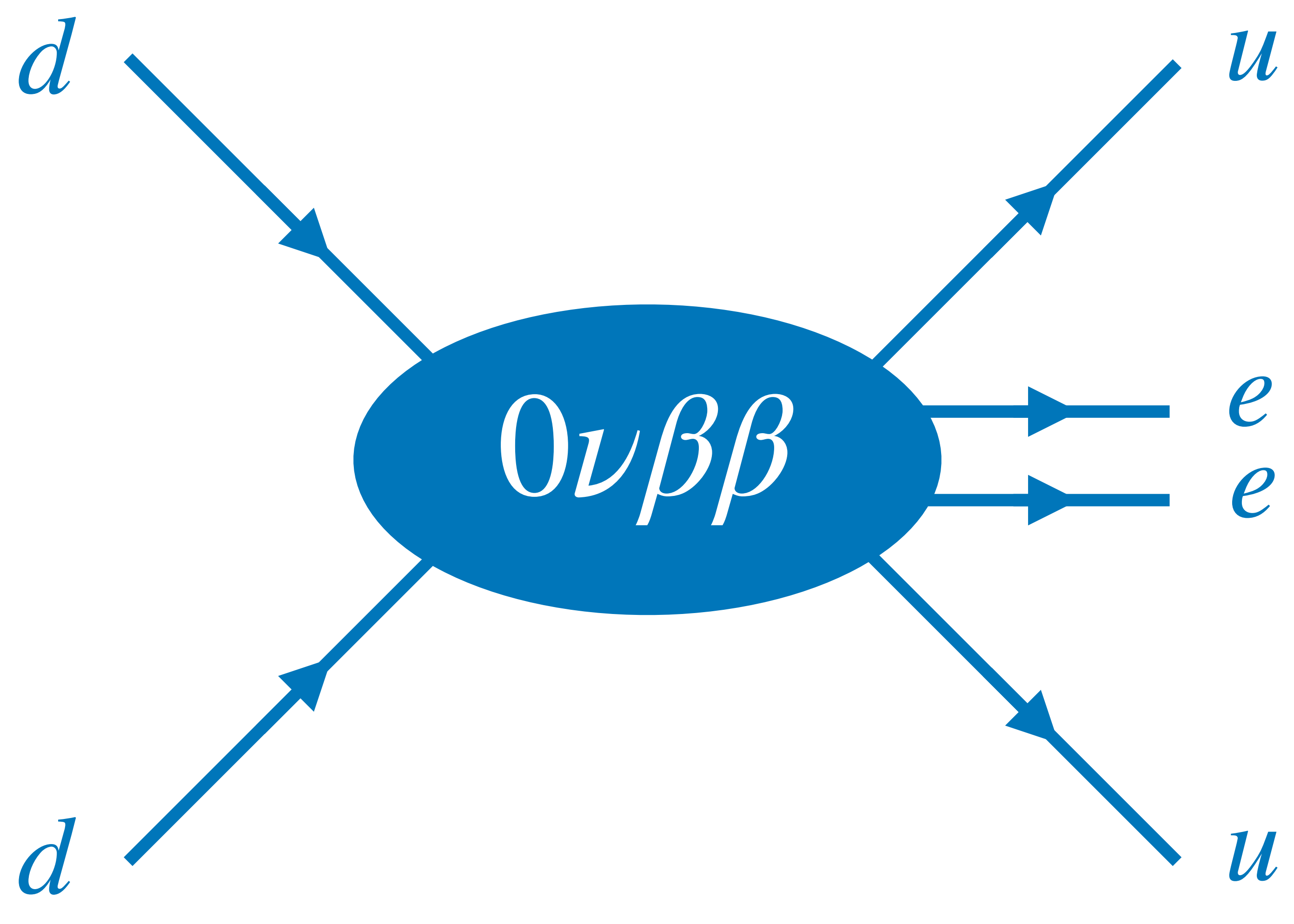
Neutrinoless double β ($0\nu\beta\beta$) decay



- $0\nu\beta\beta$ decay is a hypothetical process:

$$dd \to uue^- e^-,$$

  which, if observed, **would**:
  - Violate lepton number (really $B - L$).
  - Show that neutrinos are Majorana particles.
- Experiments looking for $0\nu\beta\beta$ decay in heavy nuclei (i.e. ^{76}Ge, ^{136}Xe).
  - Direct LQCD calculation of matrix elements in these nuclei not possible.
  - Instead, use LQCD to compute inputs to EFT in the form of low-energy constants (LECs), and use EFT to study nuclear $0\nu\beta\beta$ decay.

$0\nu\beta\beta$ decay mechanisms

- Models are characterized by whether the decay is induced by non-local interactions (**long-distance**) or local interactions (**short-distance**).

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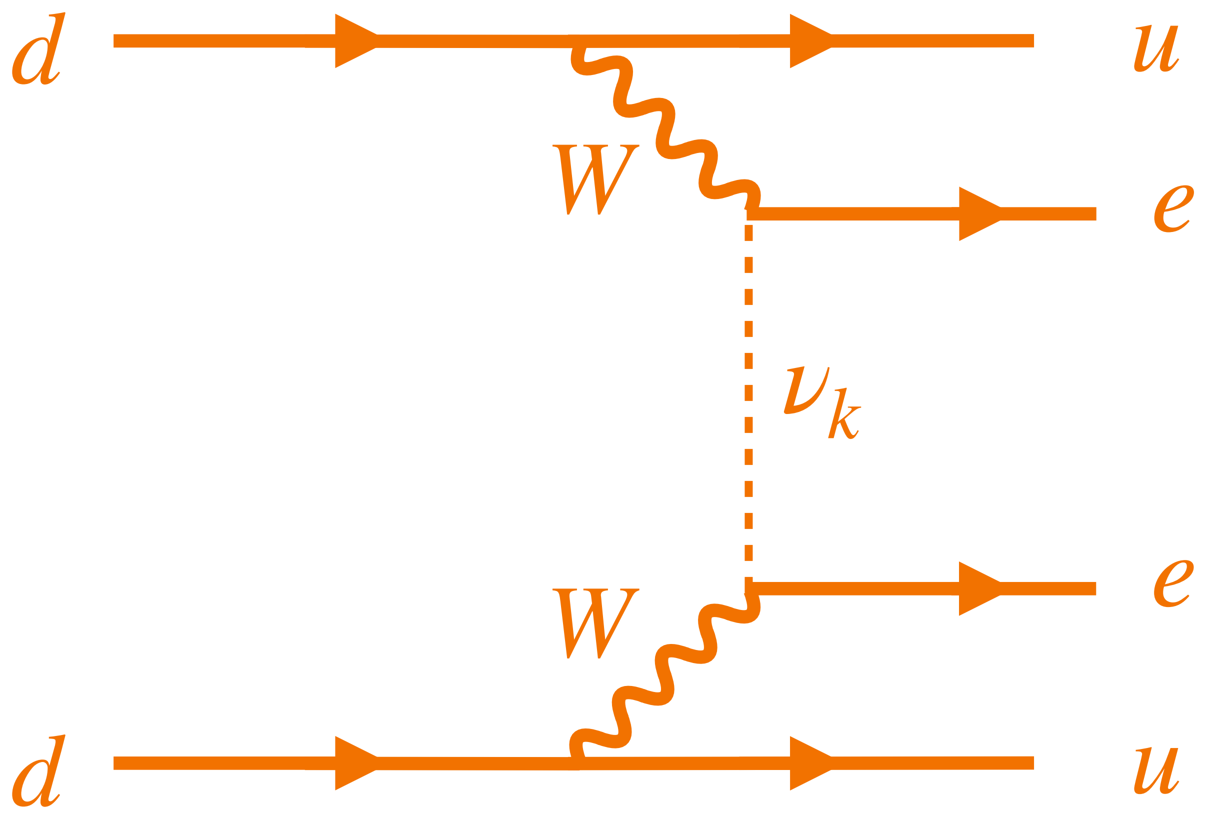
Light Majorana exchange



"long-distance"

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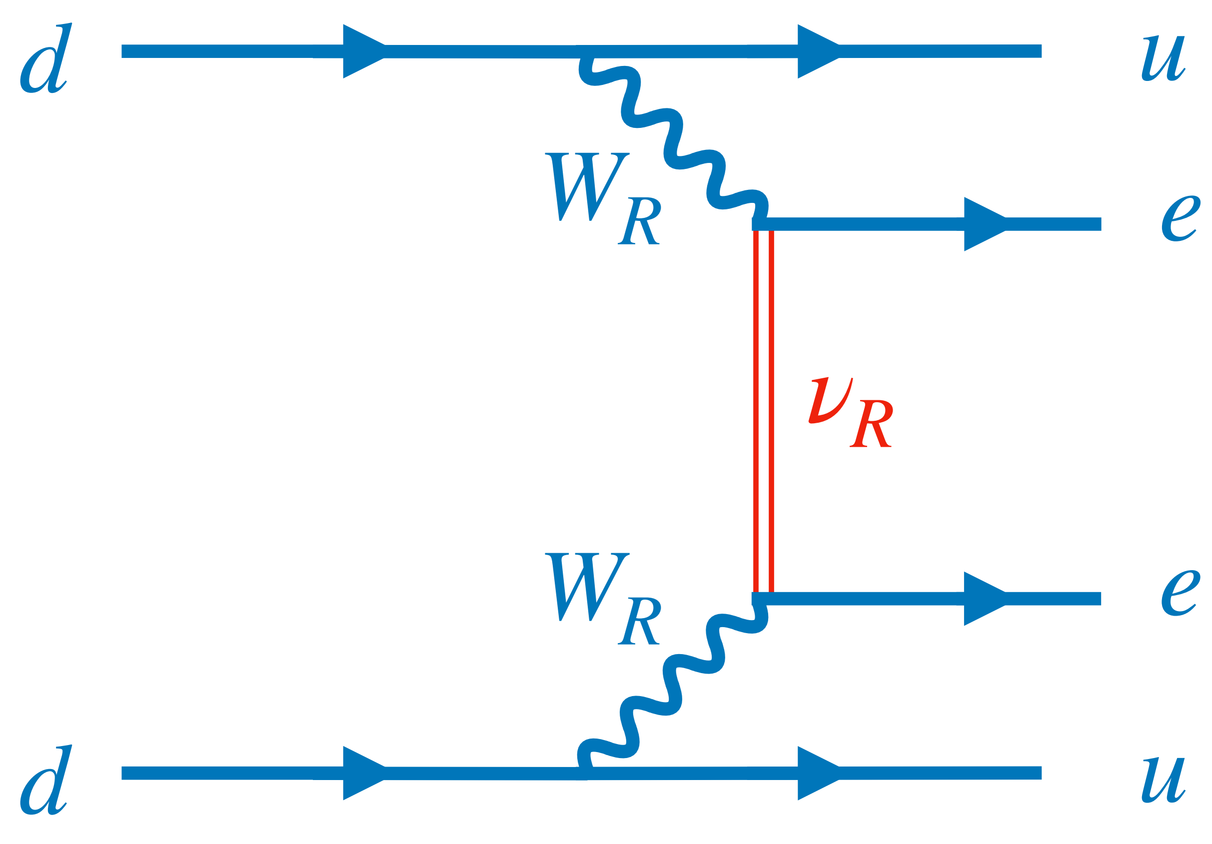
Heavy neutrino exchange



Light Majorana exchange

"long-distance"

"short-distance"

This talk!

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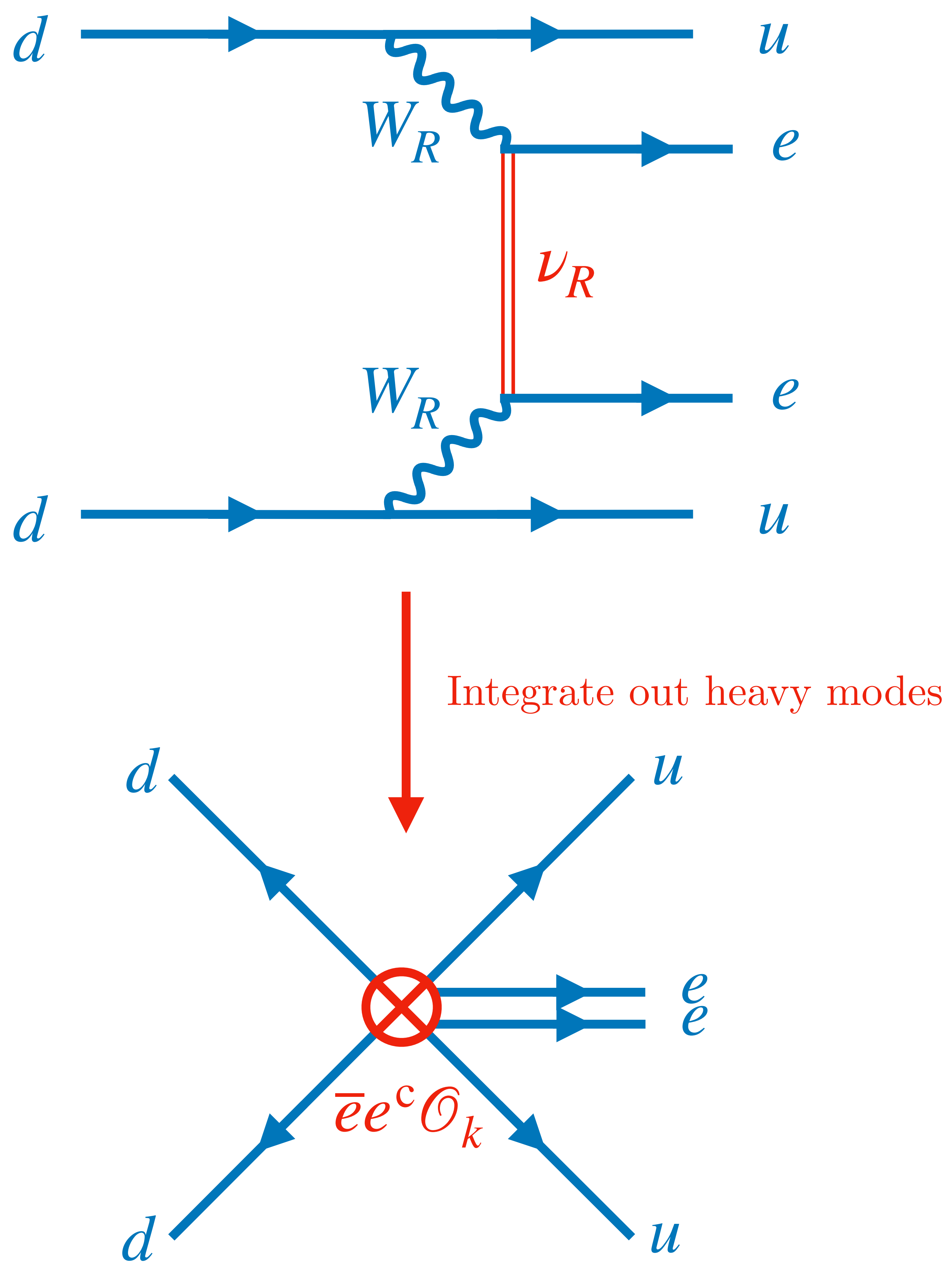

"short-distance"

This talk!

$$\mathscr{L}^{\mathrm{SD}}_{0\nu\beta\beta} = \frac{G_F^2}{\Lambda_{\mathrm{LNV}}} \sum_k c_k \bar{e} e^{\mathrm{c}} \mathcal{O}_k$$

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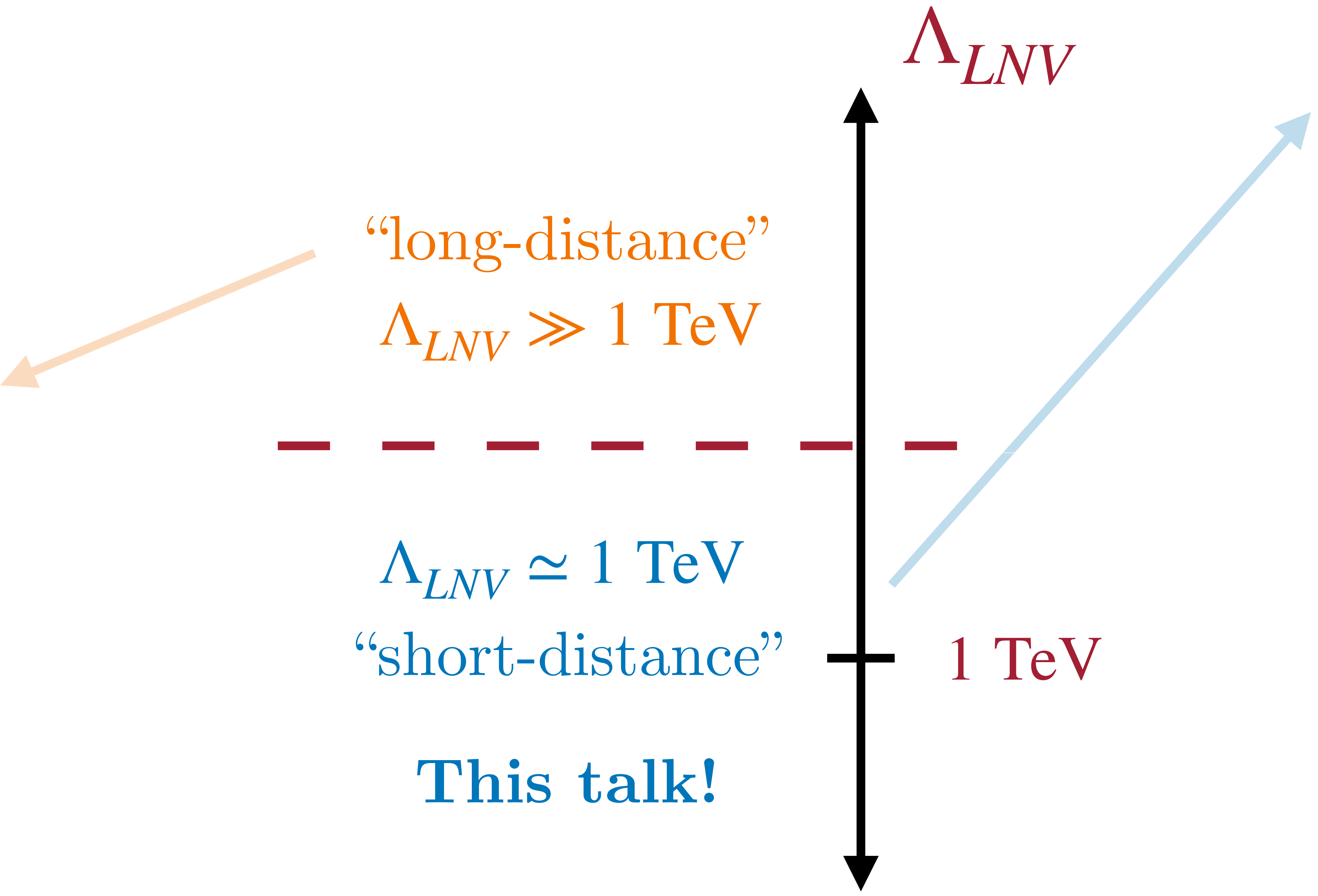
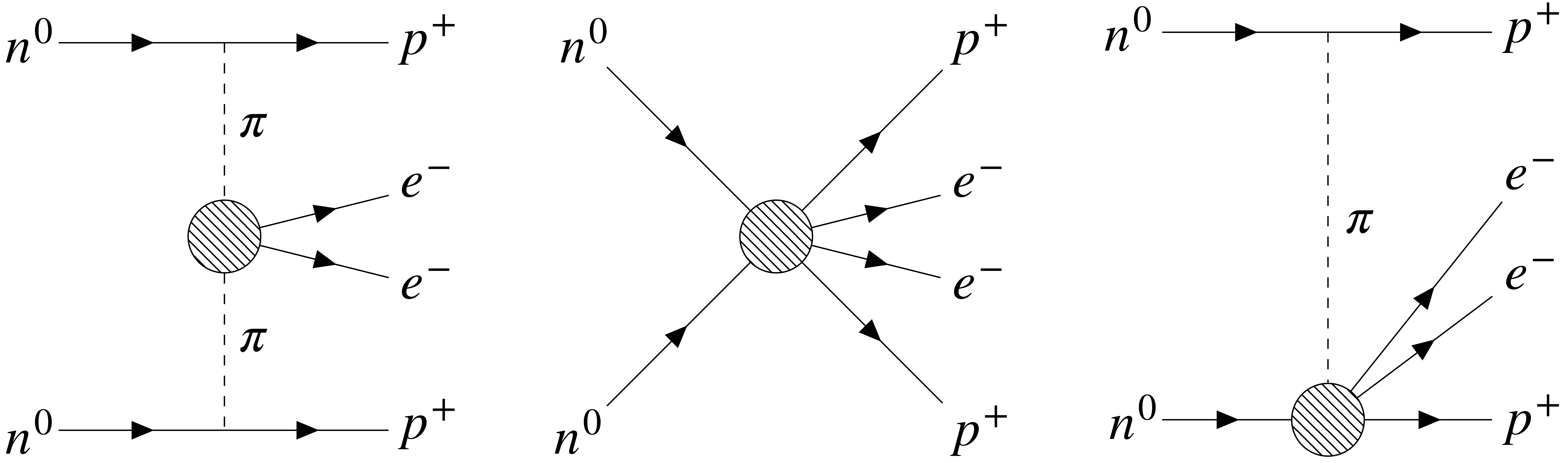


Connection to nuclear $0\nu\beta\beta$

- Nuclear $0\nu\beta\beta$ decay induced in chiral EFT (χEFT) through 3 modes:



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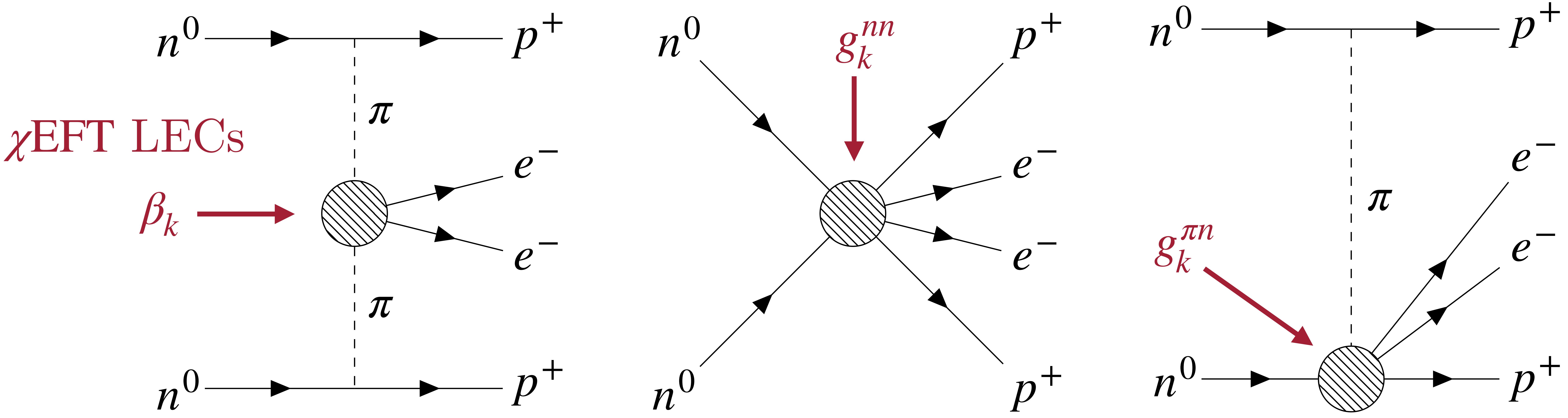

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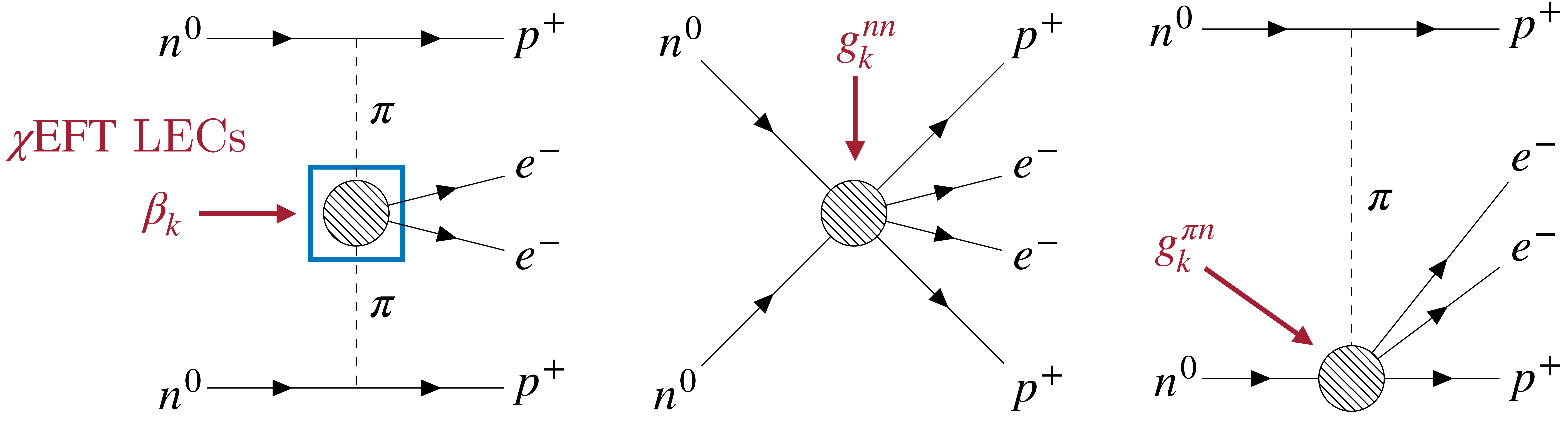

- This work: study $\pi^- \to \pi^+ e^- e^-$ with $m_e = 0$.
- Compute the **pion matrix elements** $\langle \pi^+ | \mathcal{O}_k | \pi^- \rangle$, where $\mathcal{O}_k$ are the LO short-distance operators.

Connection to nuclear $0\nu\beta\beta$

*Anthony Grebe will discuss $n^0 n^0 \to p^+ p^+ e^- e^-$ in the next talk.

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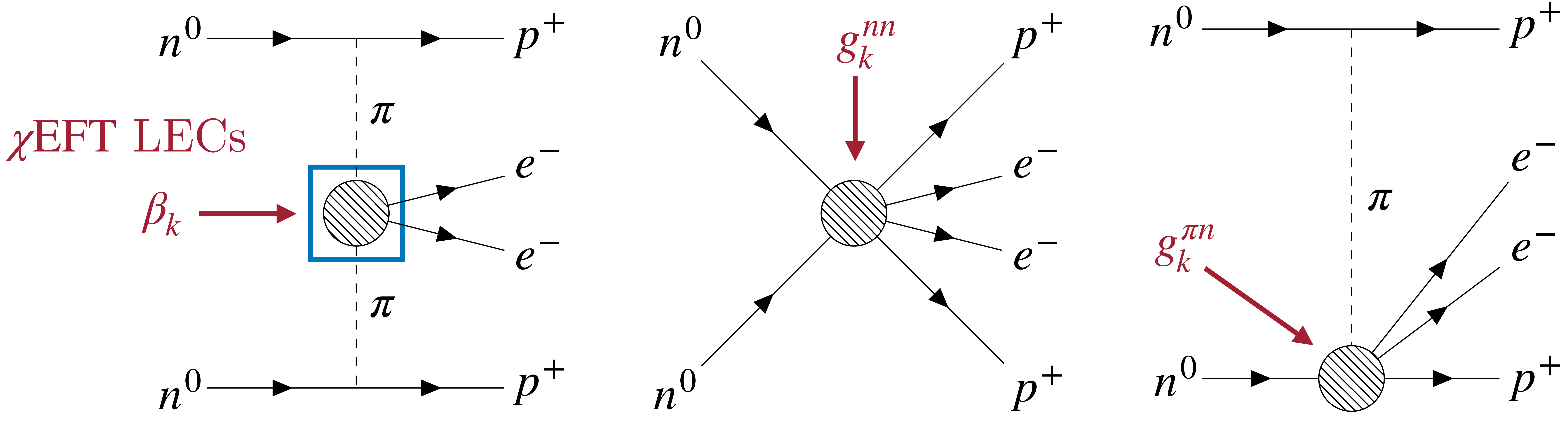

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Short-distance operators for $\pi^- \to \pi^+ e^- e^-$

- Five operators $\mathcal{O}_k$ contribute to the decay $\pi^- \to \pi^+ e^- e^-$ at leading order:

3 different chiral transformation properties.

$$\mathcal{O}_1 = (\bar{u}_L \gamma^\mu d_L)[\bar{u}_R \gamma_\mu d_R]$$

$$\mathcal{O}_{1'} = (\bar{u}_L \gamma^\mu d_L][\bar{u}_R \gamma_\mu d_R)$$

$$\mathcal{O}_2 = (\bar{u}_R d_L)[\bar{u}_R d_L] + (L \leftrightarrow R)$$

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$$\mathcal{O}_3 = (\bar{u}_L \gamma^\mu d_L)[\bar{u}_L \gamma_\mu d_L] + (L \leftrightarrow R)$$

Takahashi Bracket:

$$(A)[B] = A^{aa} B^{bb}$$

$$(A][B) = A^{ab} B^{ba}$$

Lattice setup

C. Allton *et. al.* (RBC/UKQCD Collaboration), Phys. Rev. D 78, 114509 (2008).

- We have used the domain wall fermions and the Iwasaki gauge action.
- This calculation is performed on 5 ensembles with $N_f = 2+1$ flavors:

Ensemble	am_l	am_s	β	$L^3 \times T \times L_s$	a [fm]	m_π [MeV]
24I	0.01	0.04	2.13	$24^3 \times 64 \times 16$	0.1106(3)	432.2(1.4)
	0.005					339.6(1.2)
32I	0.008	0.03	2.25	$32^3 \times 64 \times 16$	0.0828(3)	410.8(1.5)
	0.006					359.7(1.2)
	0.004					302.0(1.1)

- These ensembles have been previously used to compute the long-distance $\pi^- \to \pi^+ e^- e^-$ amplitude by W. Detmold and D. Murphy.

W. Detmold, D. Murphy, hep-lat/2004.07404 (2020).

Extracting $\langle \pi^+ | \mathcal{O}_k | \pi^- \rangle$

$$\chi_\pi(z) = \overline{u}(z)\gamma_5 d(z)$$

$$t_+ = t_- + \Delta t$$

$$C_k(t_-, t_x, t_+) = \sum_{\mathbf{y},\mathbf{x},\mathbf{z}} \langle \chi_\pi^\dagger(\mathbf{y}, t_+) \mathcal{O}_k(\mathbf{x}, t_x) \chi_\pi^\dagger(\mathbf{z}, t_-) \rangle$$

$$C_{2\mathrm{pt}}(\Delta t) = \frac{1}{T} \sum_{t_-=0}^{T-1} \sum_{\mathbf{x},\mathbf{y}} \langle 0 | \chi_\pi(\mathbf{x}, t_+) \chi_\pi^\dagger(\mathbf{y}, t_-) | 0 \rangle$$

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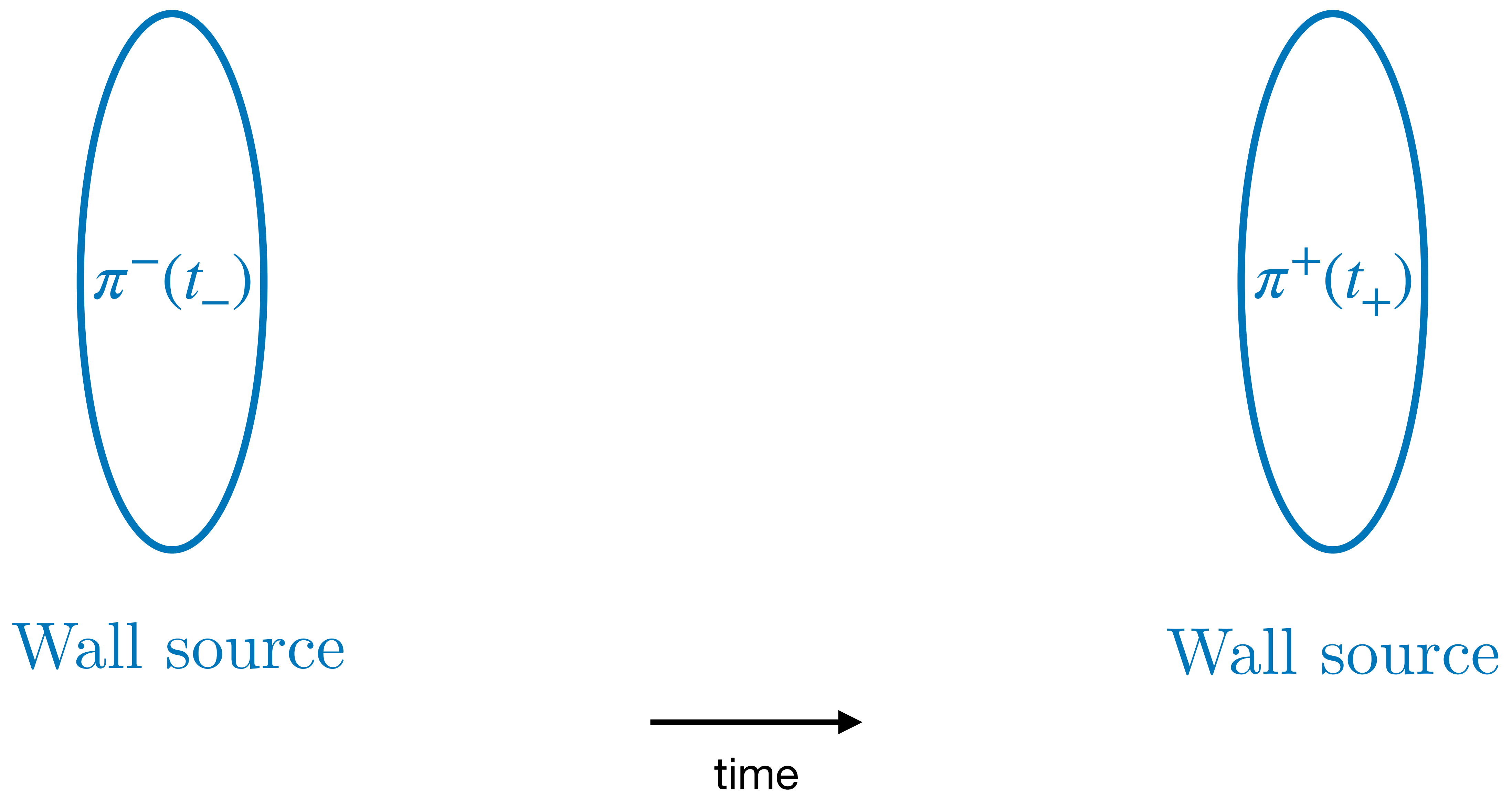

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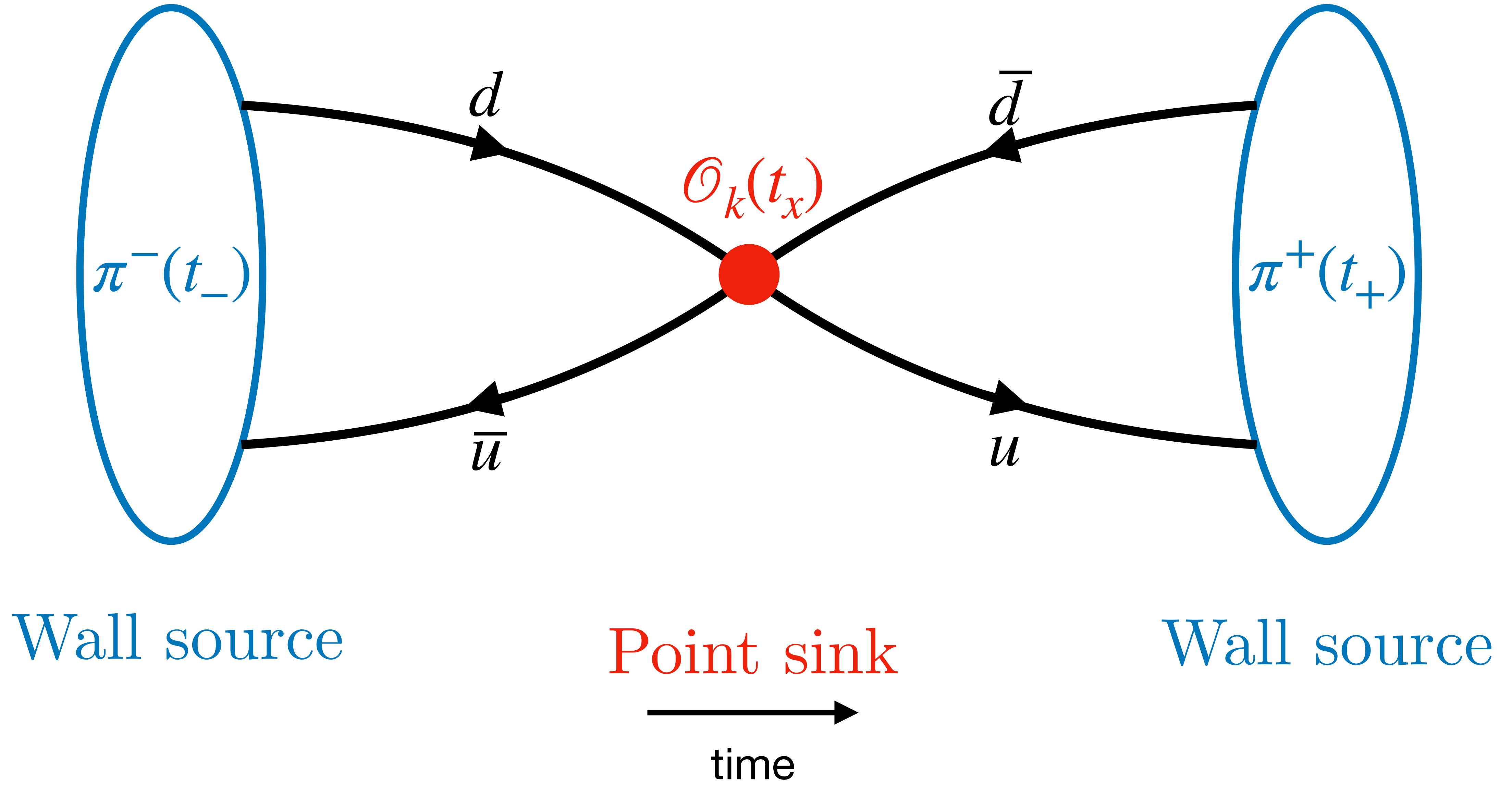

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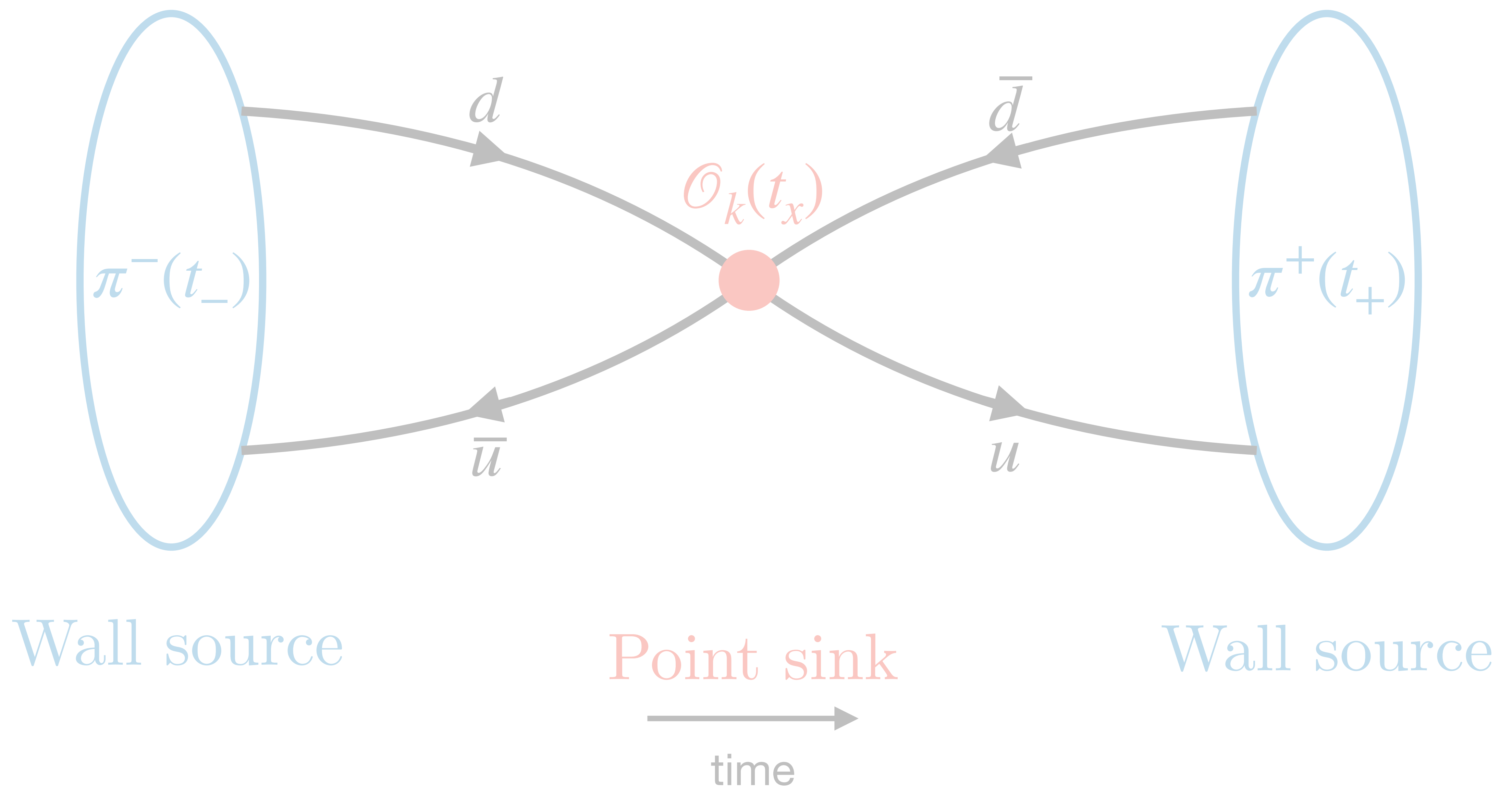

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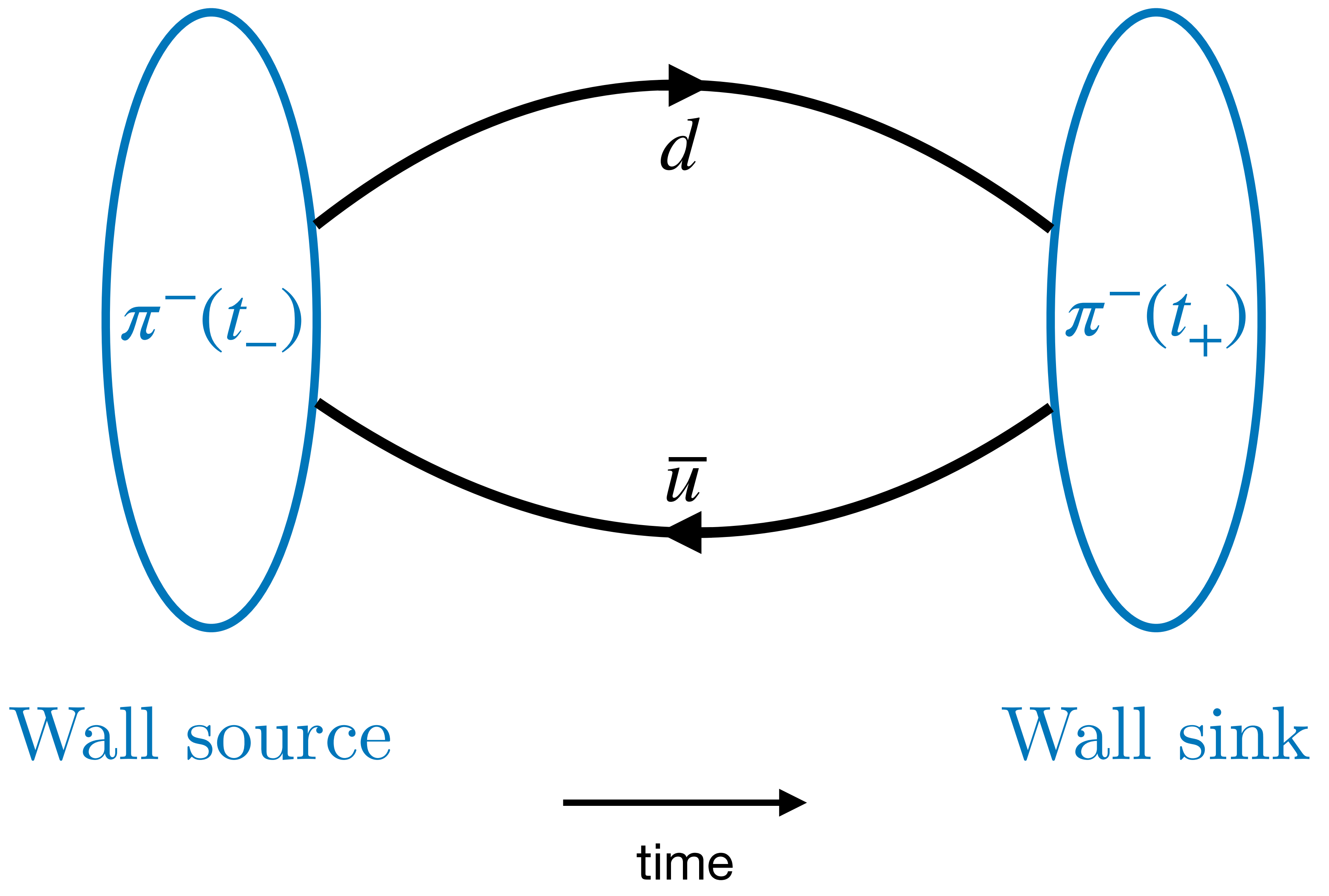

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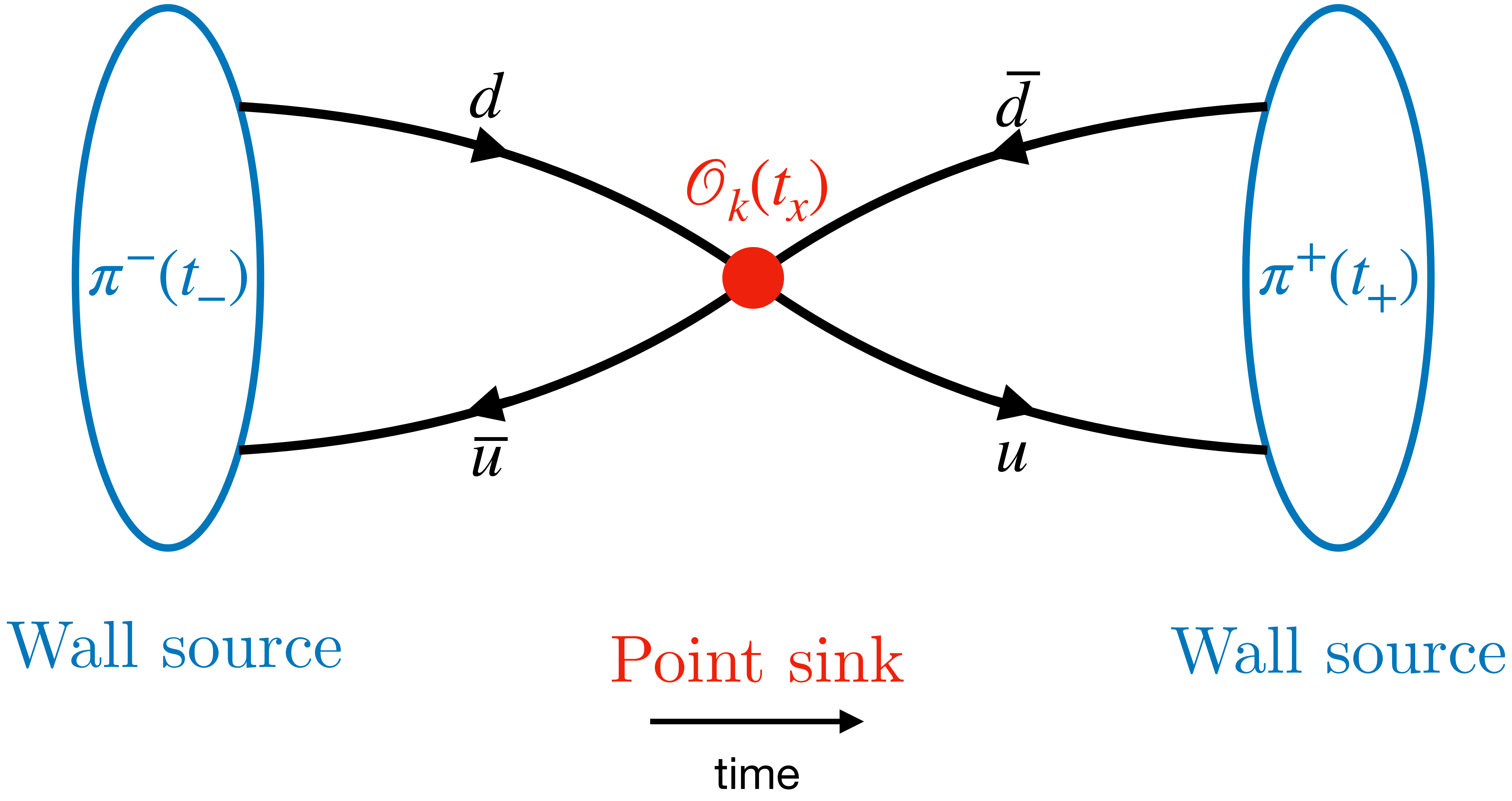
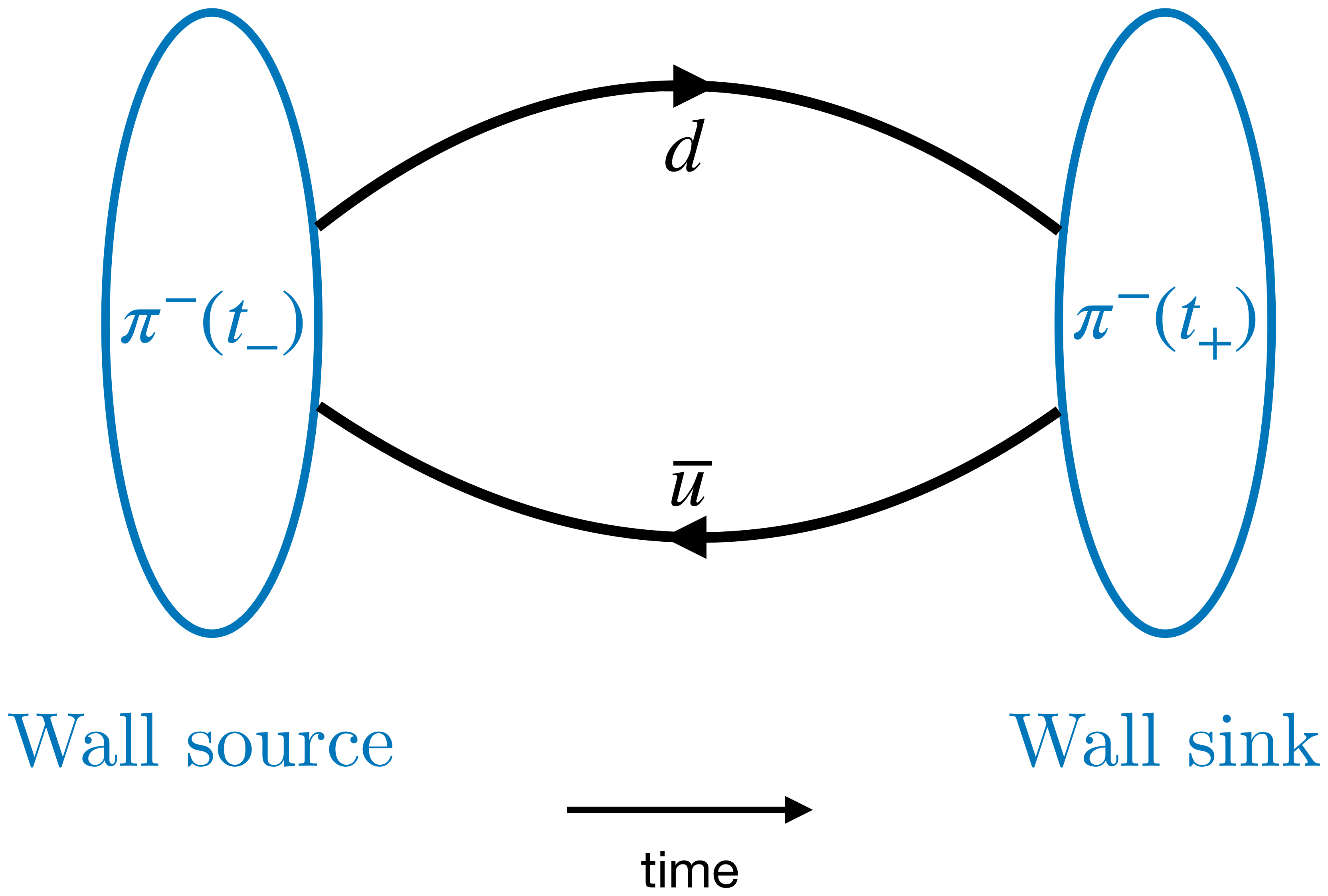
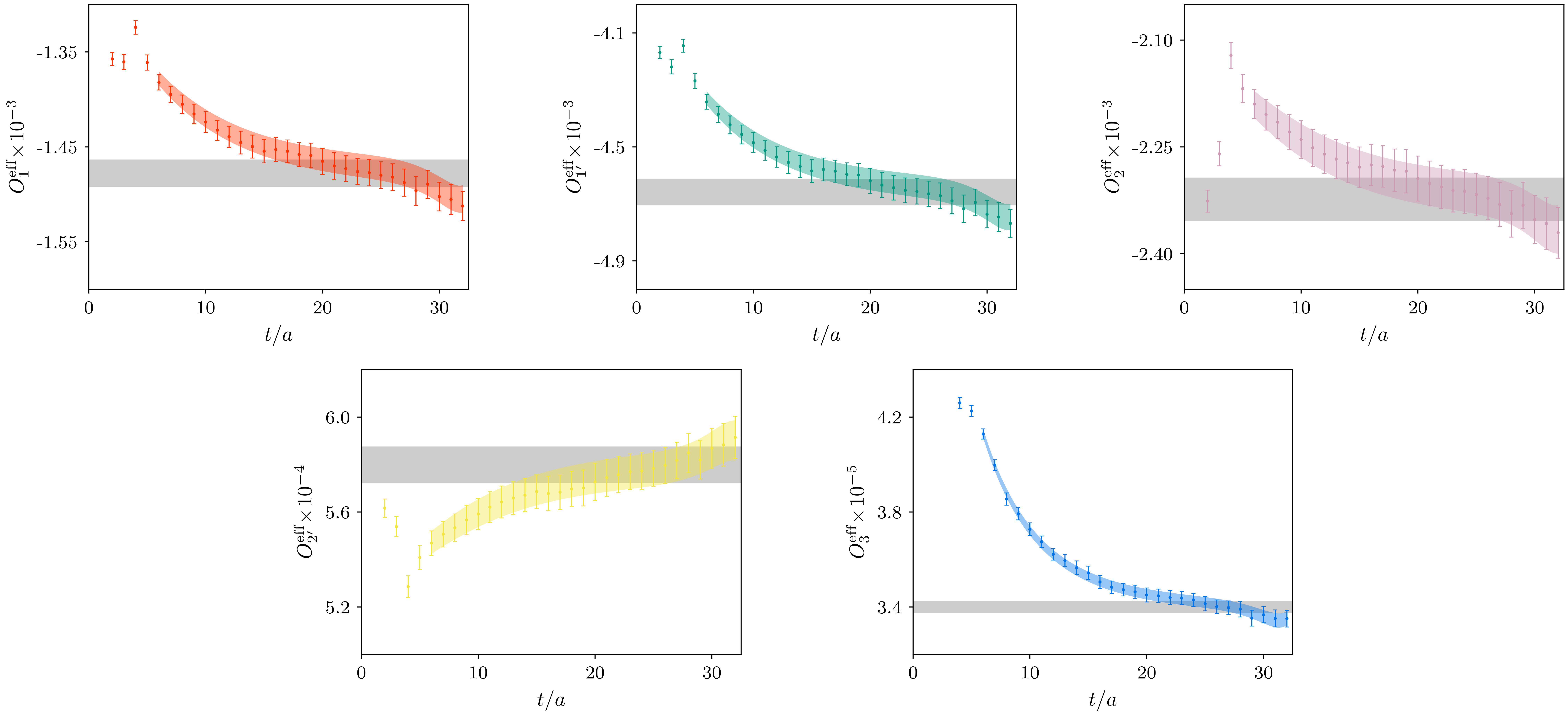




$$O_k^{\mathrm{eff}}(t) \equiv 2m_\pi \frac{C_k(0,t,2t)}{C_{2\mathrm{pt}}(2t) - \frac{1}{2}C_{2\mathrm{pt}}(T/2)e^{m_\pi(2t-T/2)}} \xrightarrow{T \gg t \gg 0} \langle \pi^+ | \mathcal{O}_k | \pi^- \rangle$$

Bare $O_k^{\text{eff}}(t)$ on $32^3 \times 64$, $am_\ell = 0.004$ ensemble



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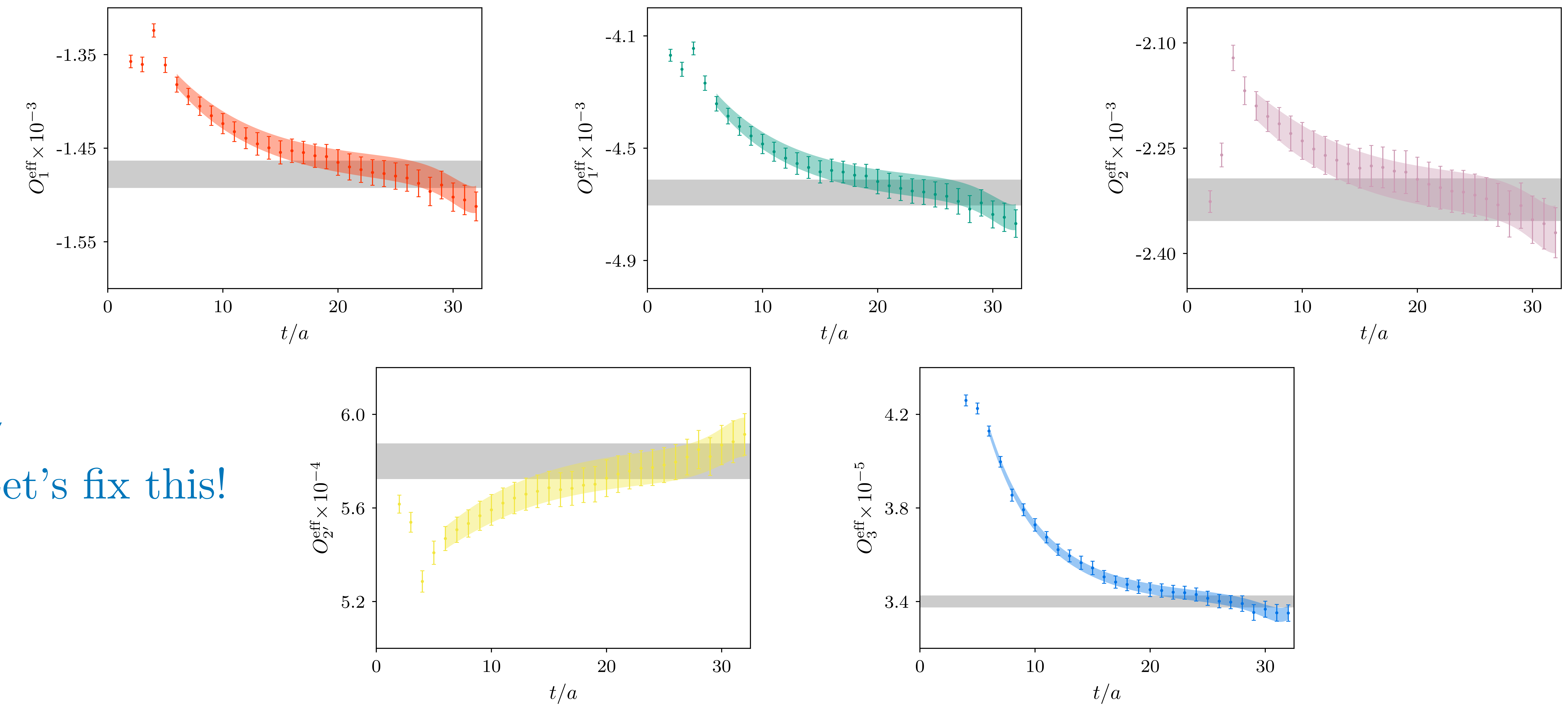

Let's fix this!

Renormalization

- Renormalize matrix elements in $\overline{\text{MS}}$ at 3 GeV.
- Compute in RI/sMOM scheme and perturbatively match to $\overline{\text{MS}}$.
- Operators with the same quantum numbers **mix** under renormalization.

$$\mathcal{O}_k^{\overline{\text{MS}}}(x;\mu^2,a) = Z_{k\ell}^{\overline{\text{MS}}}(\mu^2,a)\ \mathcal{O}_\ell^{(0)}(x;a)$$

P. A. Boyle *et. al.*, JHEP 10, 054 (2017).

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Diagonals: order 1 numbers
Off-diagonals: small

$$\begin{pmatrix} \boxed{*} & * & 0 & 0 & 0 \\ * & \boxed{*} & 0 & 0 & 0 \\ 0 & 0 & \boxed{*} & * & 0 \\ 0 & 0 & * & \boxed{*} & 0 \\ 0 & 0 & 0 & 0 & \boxed{*} \end{pmatrix}$$

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Renormalization coefficients in $\overline{\text{MS}}$

$$Z^{\overline{\text{MS}}}\left(\mu^2 = 9\ \text{GeV}^2, a = 0.11\ \text{fm}\right) = \begin{pmatrix} 0.6068(29) & -0.07630(43)0 & 0 & 0 & 0 \\ -0.06168(46) & 0.5563(26) & 0 & 0 & 0 \\ 0 & 0 & 0.5219(25) & -0.02778(33) & 0 \\ 0 & 0 & 0.00800(19) & 0.6768(32) & 0 \\ 0 & 0 & 0 & 0 & 0.5290(257) \end{pmatrix}$$

$$Z^{\overline{\text{MS}}}\left(\mu^2 = 9\ \text{GeV}^2, a = 0.08\ \text{fm}\right) = \begin{pmatrix} 0.6727(46) & -0.08926(60) & 0 & 0 & 0 \\ -0.05425(40) & 0.5567(39) & 0 & 0 & 0 \\ 0 & 0 & 0.5379(37) & -0.01399(26) & 0 \\ 0 & 0 & 0.03968(35) & 0.7780(54) & 0 \\ 0 & 0 & 0 & 0 & 0.5993(54) \end{pmatrix}$$

Chiral extrapolation

A. Nicholson *et al.*,
Phys. Rev. Lett. 121, 172501 (2018).

- $\langle \pi^+ | \mathcal{O}_k^{\overline{\mathrm{MS}}} | \pi^- \rangle$ evaluated at finite a, L, and heavier-than-physical quark mass.
- Use functional model $\mathcal{F}_k$ for $\langle \pi^+ | \mathcal{O}_k^{\overline{\mathrm{MS}}} | \pi^- \rangle$ computed in χEFT, where (α_k, β_k, c_k) determine the χEFT LECs.

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Finite volume artifacts.

$\Lambda_\chi^2 = 8\pi^2 f_\pi^2$

$$\mathcal{F}_1(m_\pi, f_\pi, a, L; \alpha_1, \beta_1, c_1) = \frac{\beta_1 \Lambda_\chi^4}{(4\pi)^2}\left[1 + \epsilon_\pi^2(\log \epsilon_\pi^2 - 1 + c_1 - f_0(m_\pi L) + 2 f_1(m_\pi L)) + \alpha_1 a^2\right]$$

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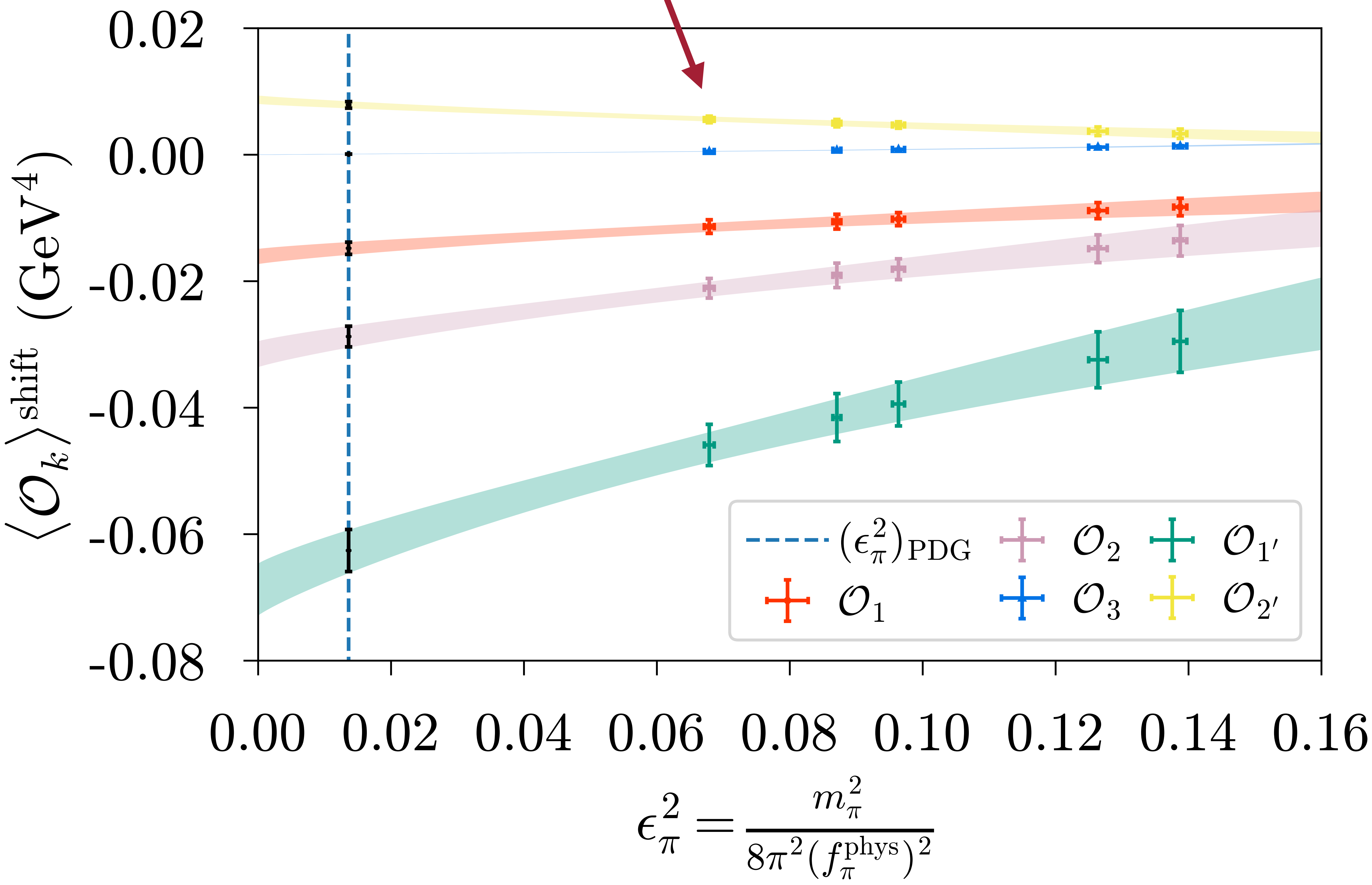
- Fits to $\mathcal{F}_k$ for (α_k, β_k, c_k) performed with least-squares minimization.

$$\langle \mathcal{O}_k \rangle^{\text{shift}} = \langle \pi^+ | \mathcal{O}_k^{\overline{\text{MS}}} | \pi^- \rangle - \mathcal{F}_k(m_\pi, f_\pi, a, L; \alpha_k, \beta_k, c_k)$$
$$+\mathcal{F}_k(m_\pi, f_\pi^{(\text{phys})}, 0, \infty; \alpha_k, \beta_k, c_k)$$



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$\mathcal{O}_1$	-0.01479(96)	-1.42(10)	0.02
$\mathcal{O}_{1'}$	-0.0626(33)	-6.04(35)	0.04
$\mathcal{O}_2$	-0.0287(16)	-2.78(17)	0.69
$\mathcal{O}_{2'}$	0.00788(52)	0.765(55)	0.11
$\mathcal{O}_3$	0.0001008(33)	0.702(27)	0.03

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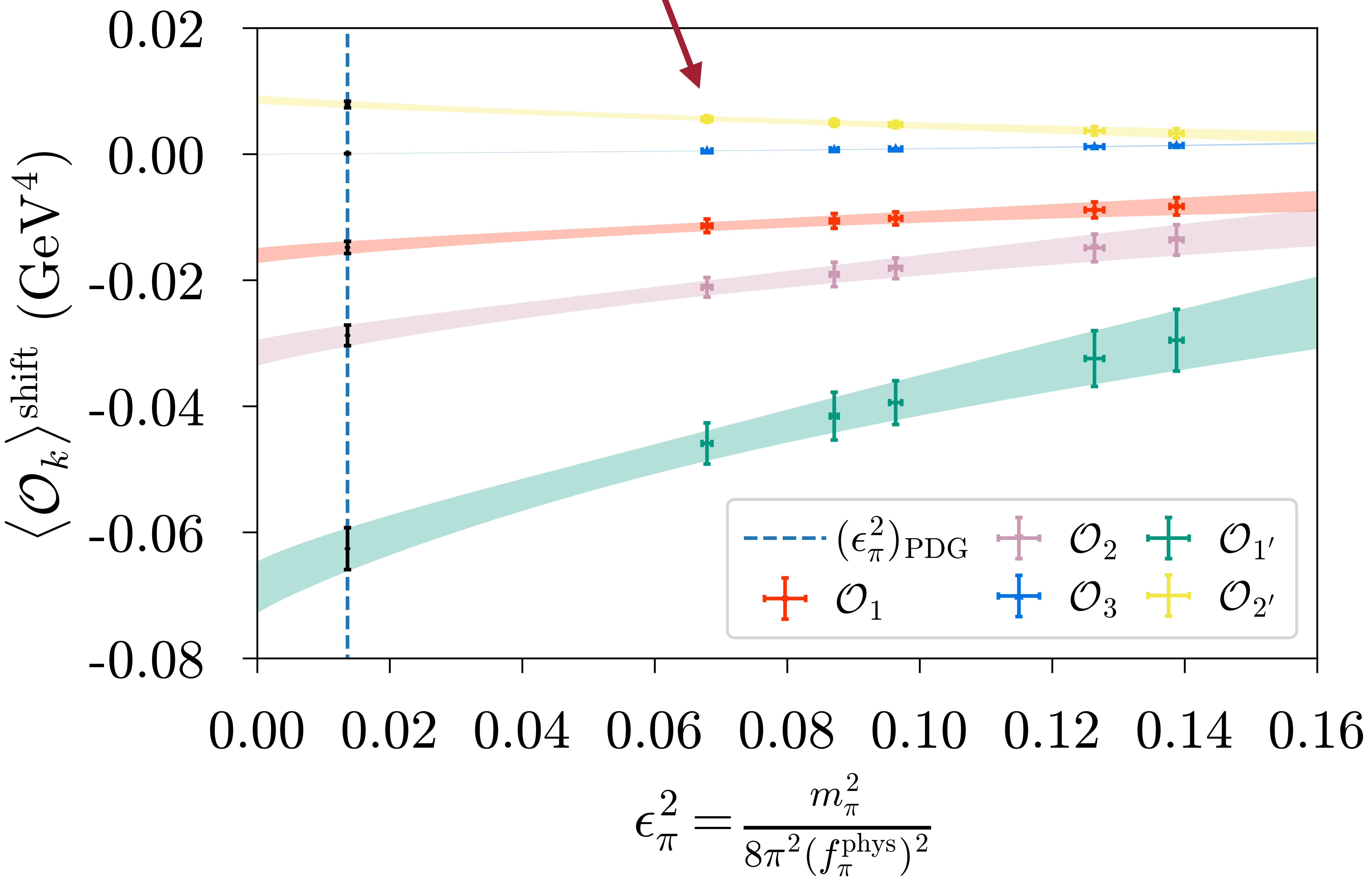

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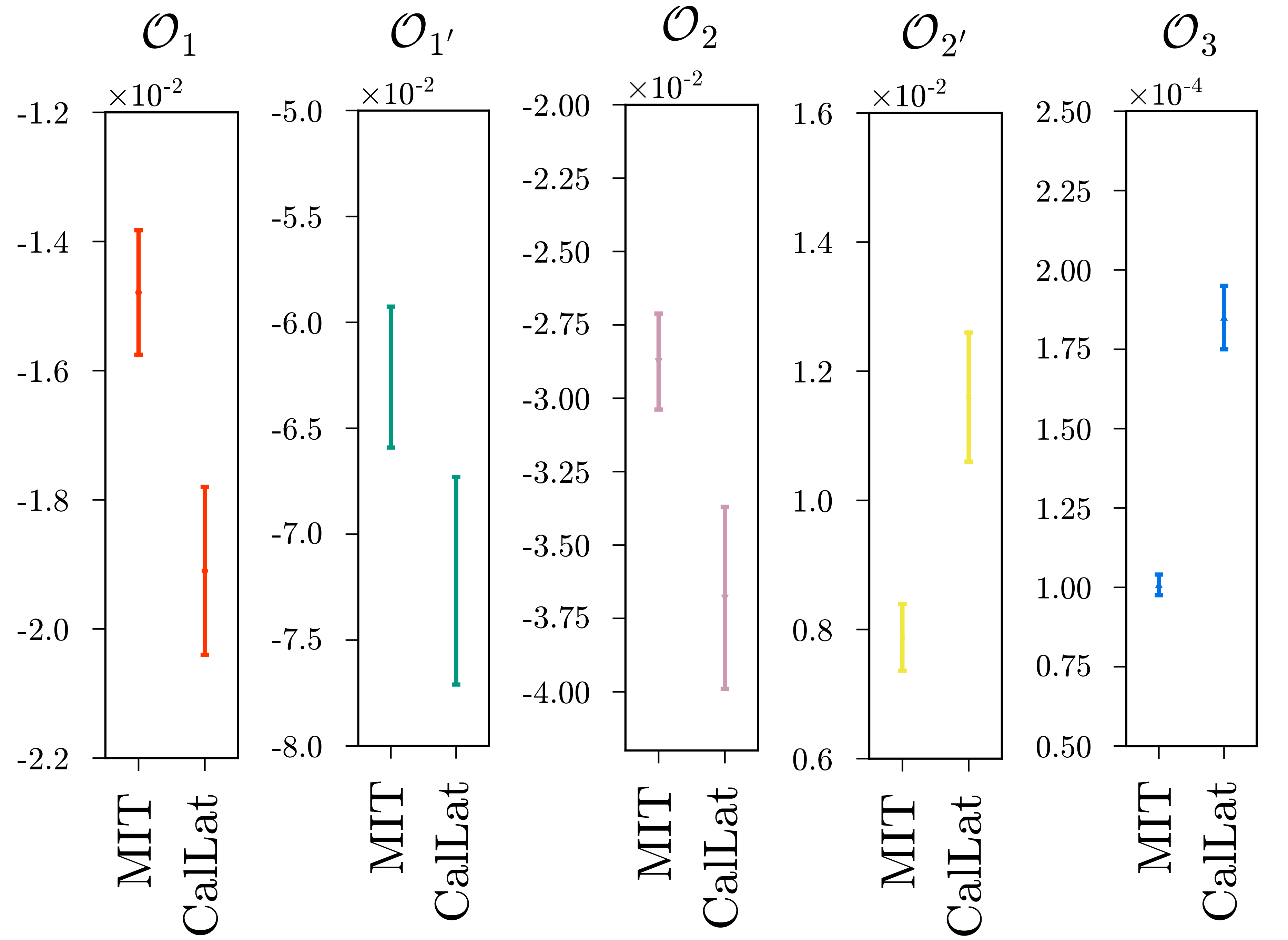

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- How do they compare?

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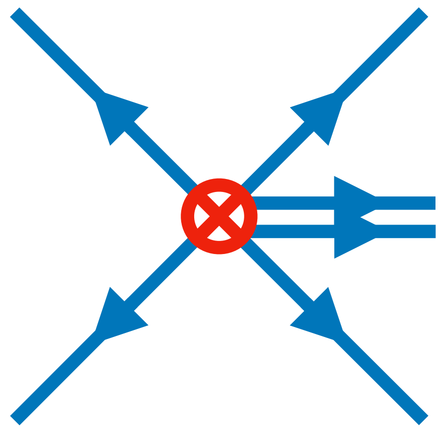

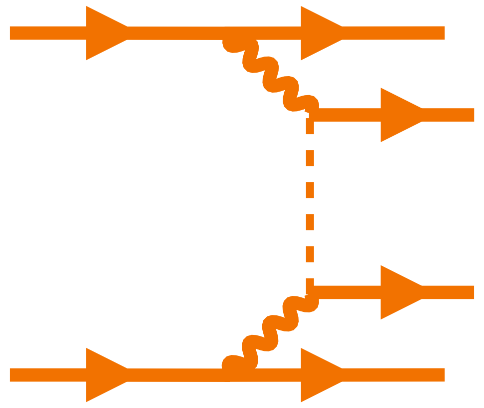

$$= \frac{1}{\Lambda_{\mathrm{LNV}}\, m_{\beta\beta}} \frac{\left|\sum_k c_k \langle \pi^+ | \mathcal{O}_k | \pi^- \rangle\right|}{|M^{0\nu}|}$$

W. Detmold, D. Murphy, hep-lat/2004.07404 (2020).

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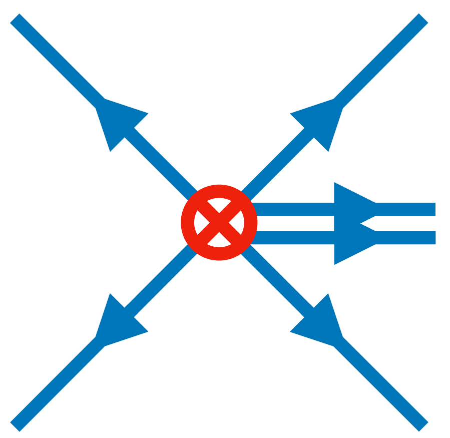
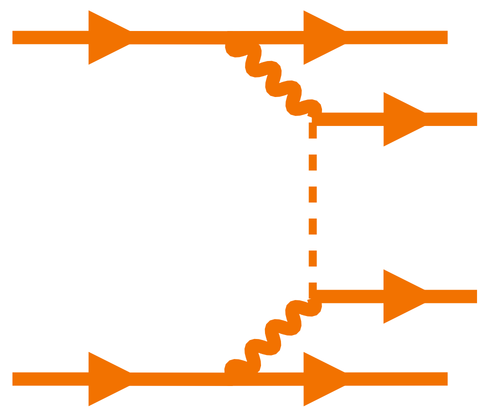
$$\boxed{\text{Seesaw: } m_{\beta\beta} = c\frac{v^2}{\Lambda_{\rm LNV}} \sim \frac{c}{G_F \Lambda_{\rm LNV}}}$$

Wilson coefficient

$$\frac{\mathscr{A}_{\rm SD}}{\mathscr{A}_{\rm LD}} = \frac{\sum_k}{}$$





$$= \frac{1}{\Lambda_{\rm LNV}\, m_{\beta\beta}} \frac{|\sum_k c_k \langle \pi^+ | \mathscr{O}_k | \pi^- \rangle|}{|M^{0\nu}|}\Bigg|_{\rm seesaw}$$

$$= G_F \frac{|\sum_k c_k \langle \pi^+ | \mathscr{O}_k | \pi^- \rangle|}{c\,|M^{0\nu}|}$$

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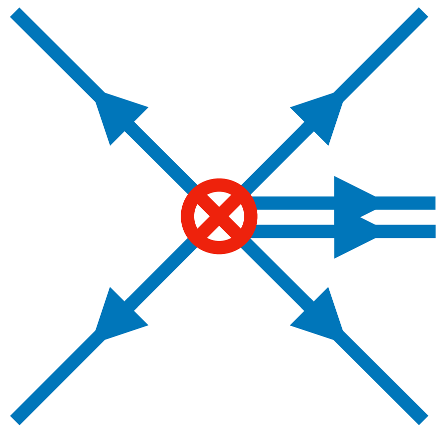
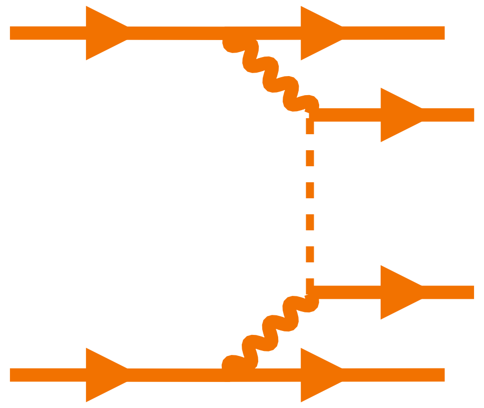
$$\frac{\mathscr{A}_{\text{SD}}}{\mathscr{A}_{\text{LD}}} =$$

$$\sum_k$$





$$= \frac{1}{\Lambda_{\text{LNV}}\, m_{\beta\beta}} \frac{|\sum_k c_k \langle \pi^+ | \mathcal{O}_k | \pi^- \rangle|}{|M^{0\nu}|}\Bigg|_{\text{seesaw}}$$

$$= G_F \frac{|\sum_k c_k \langle \pi^+ | \mathcal{O}_k | \pi^- \rangle|}{c\,|M^{0\nu}|}\Bigg|_{c_k \approx c} \sim 10^{-4}$$

W. Detmold, D. Murphy, hep-lat/2004.07404 (2020).

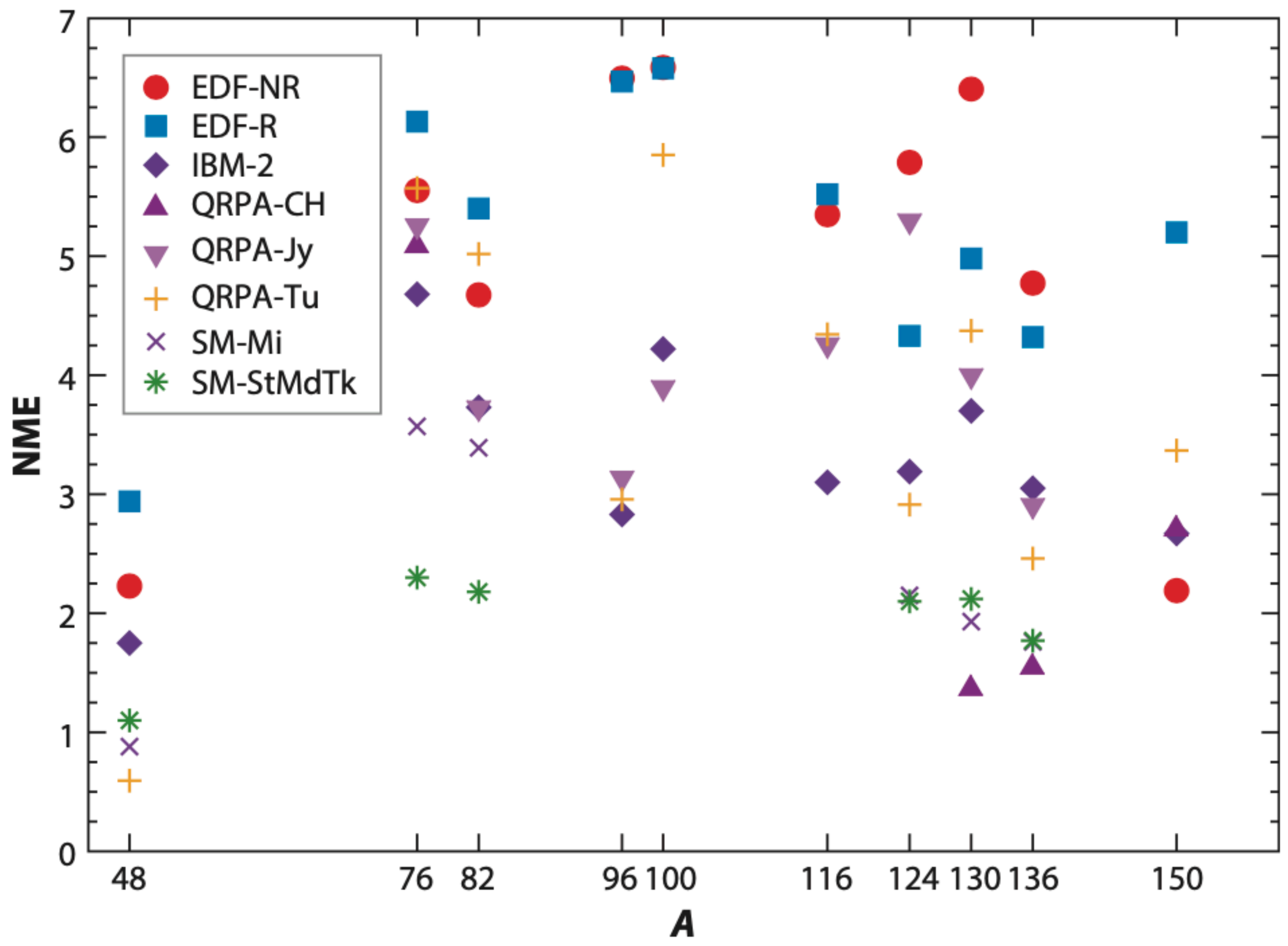
Summary and outlook

- For the 5 leading order short-distance operators $\mathcal{O}_k$, we have computed:
  - Pion matrix elements $\langle \pi^+ | \mathcal{O}_k | \pi^- \rangle$.
  - The χEFT LECs β_k.
- First computation of $\langle \pi^+ | \mathcal{O}_k | \pi^- \rangle$ with domain-wall valence and sea quarks.
- Completes the $\pi^- \to \pi^+ e^- e^-$ computation of hep-lat/2004.07404 (2020).
- Remaining short-distance LECs, g_k^{nn} and $g_k^{\pi n}$, need to be computed to fully parameterize the decay in χEFT.

Backup slides

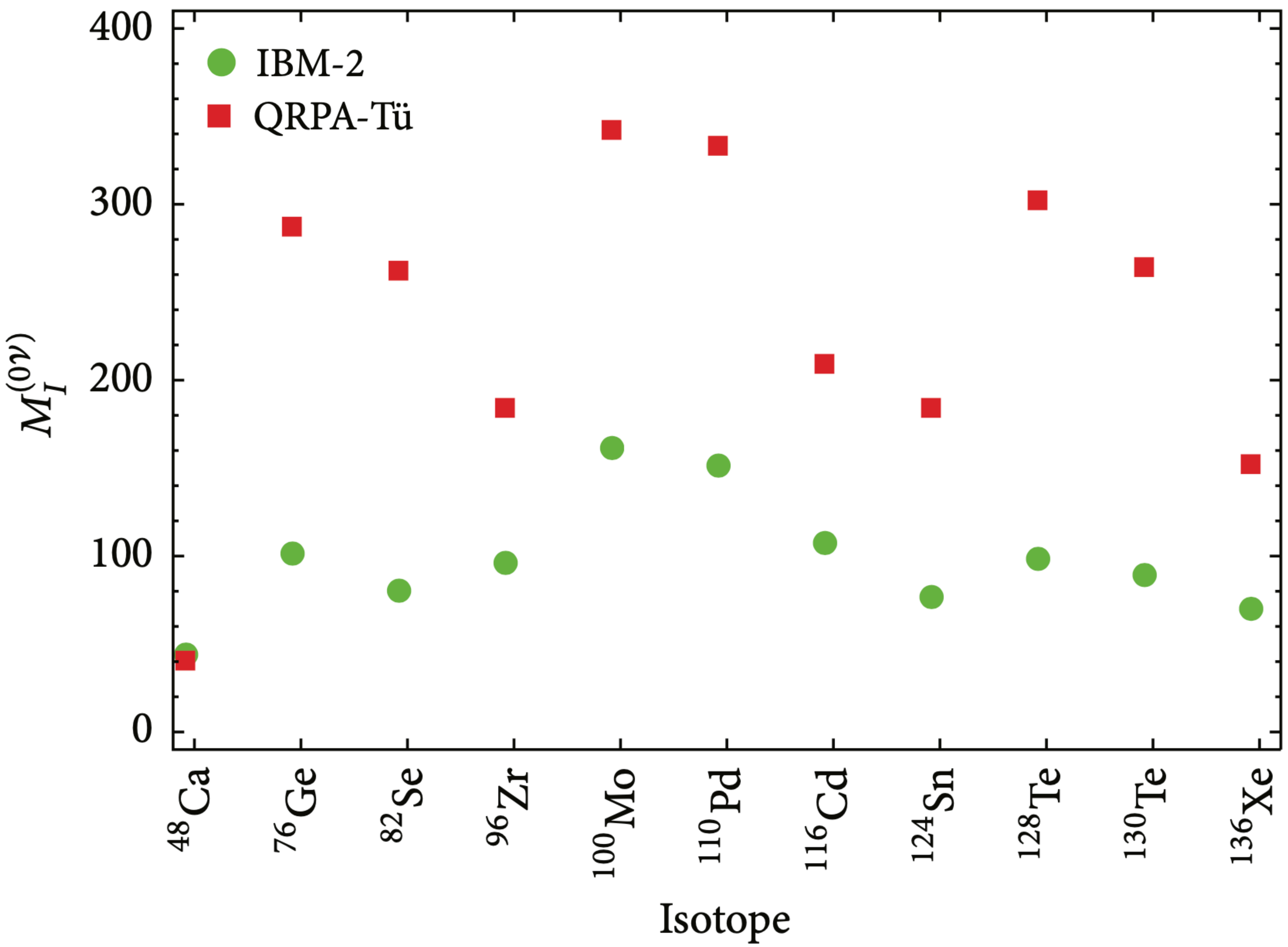
Nuclear matrix elements (many-body)

Long-distance NMEs



Dolinski *et. al.*, $0\nu\beta\beta$: Status and Prospects [nucl-ex/1902.04097]

Short-distance NMEs (heavy neutrino exchange)



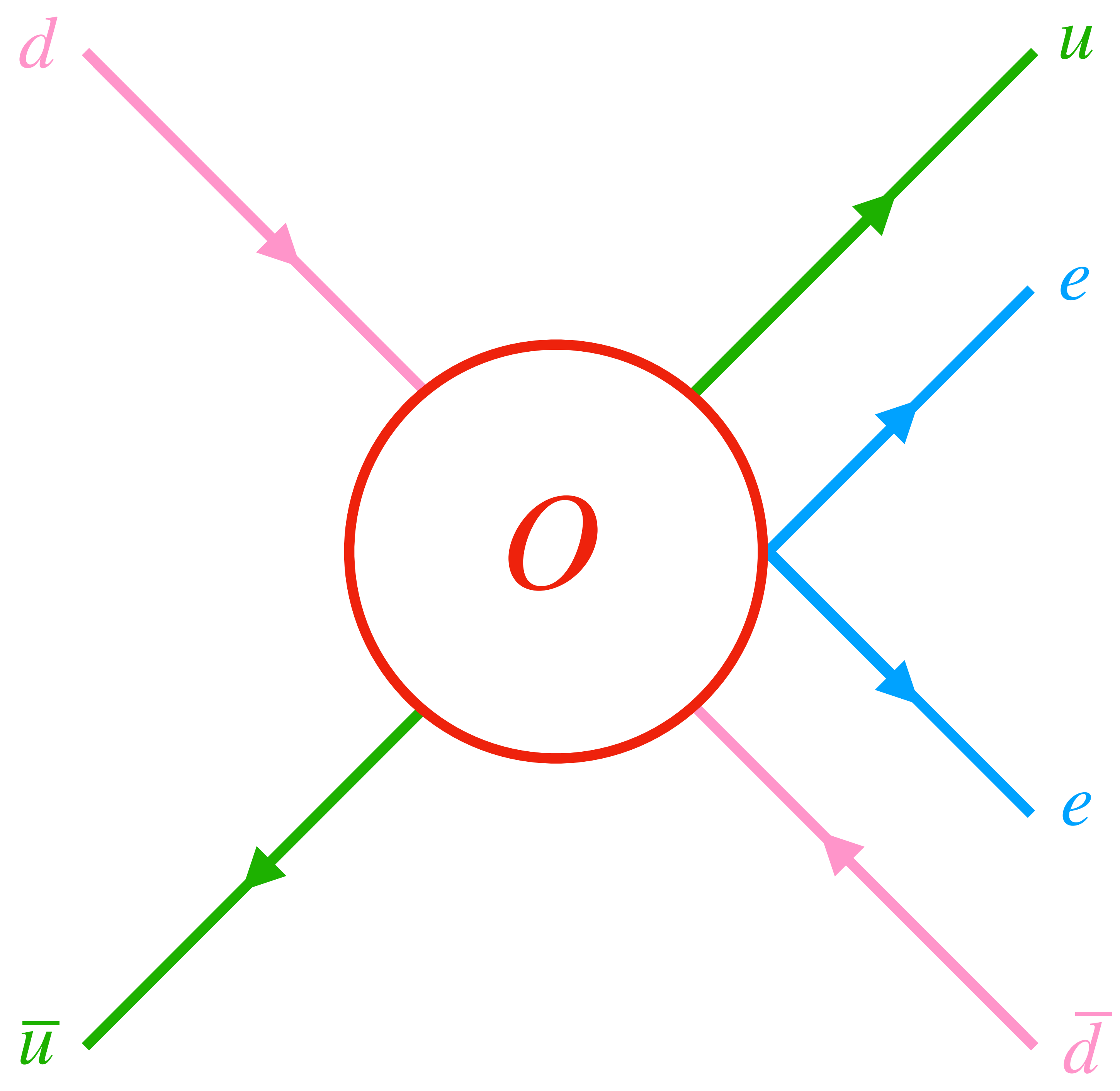
Dell'Oro *et. al.*, $0\nu\beta\beta$: 2015 Review [hep-ph/1601.07512]

Operators for short-distance $0\nu\beta\beta$

- Classify operators O constructed from SM fields with $[O] > 4$ which can contribute to $0\nu\beta\beta$. Schematically:

$$(2\text{ u fields}) \times (2\text{ d fields}) \times (2\text{ e fields}) \implies [O] \geq 9$$

- Operators must be Lorentz invariant and obey SM gauge symmetries, including $U(1)_{\mathrm{EM}}$.
- 4-quark part of vector operators match onto $\pi(\partial^\mu \pi)\bar{e}\gamma_\mu\gamma_5 e^c + \mathrm{h.c.}$, which is suppressed by powers of m_e (and set to 0 in this calculation).
- Only positive parity operators contribute.



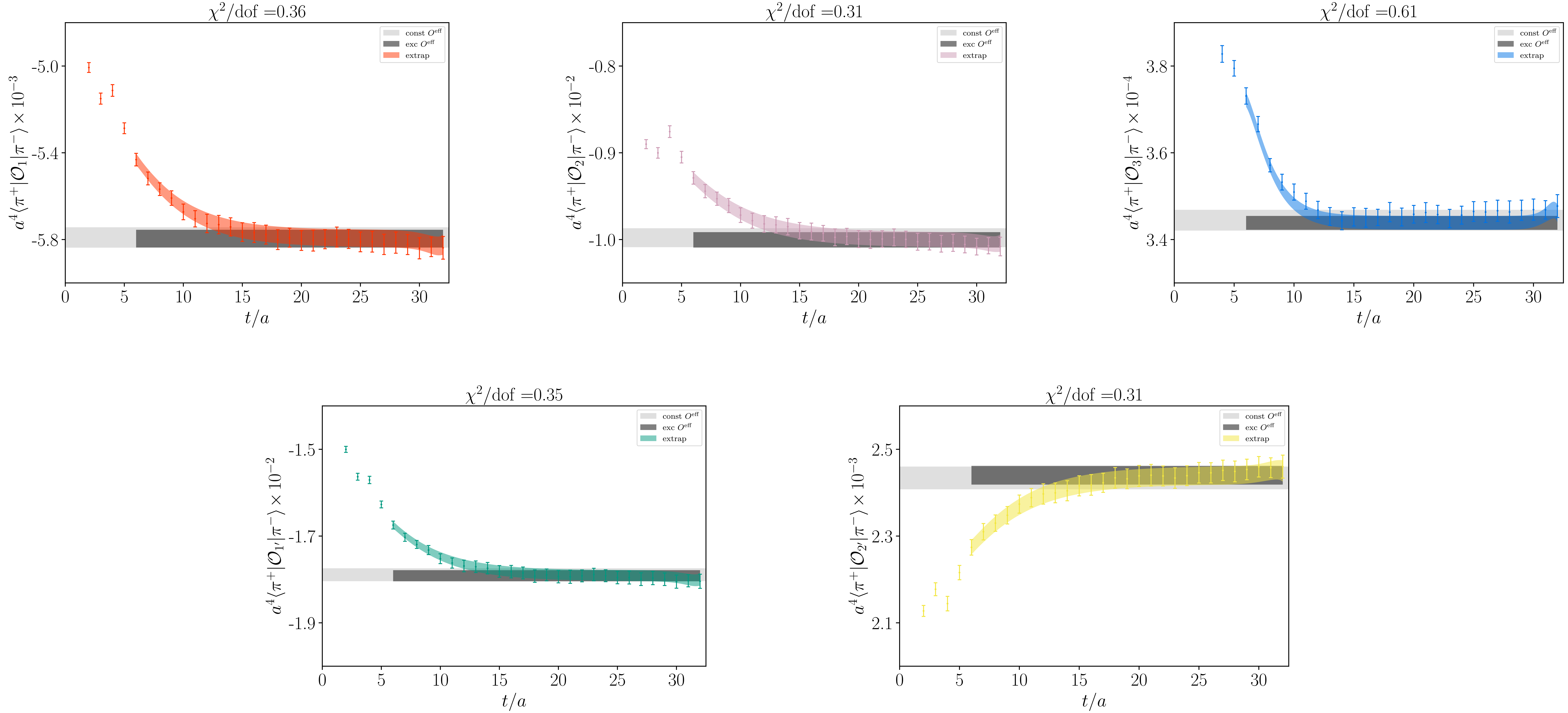
Excited state fits

- Functional model for excited states:

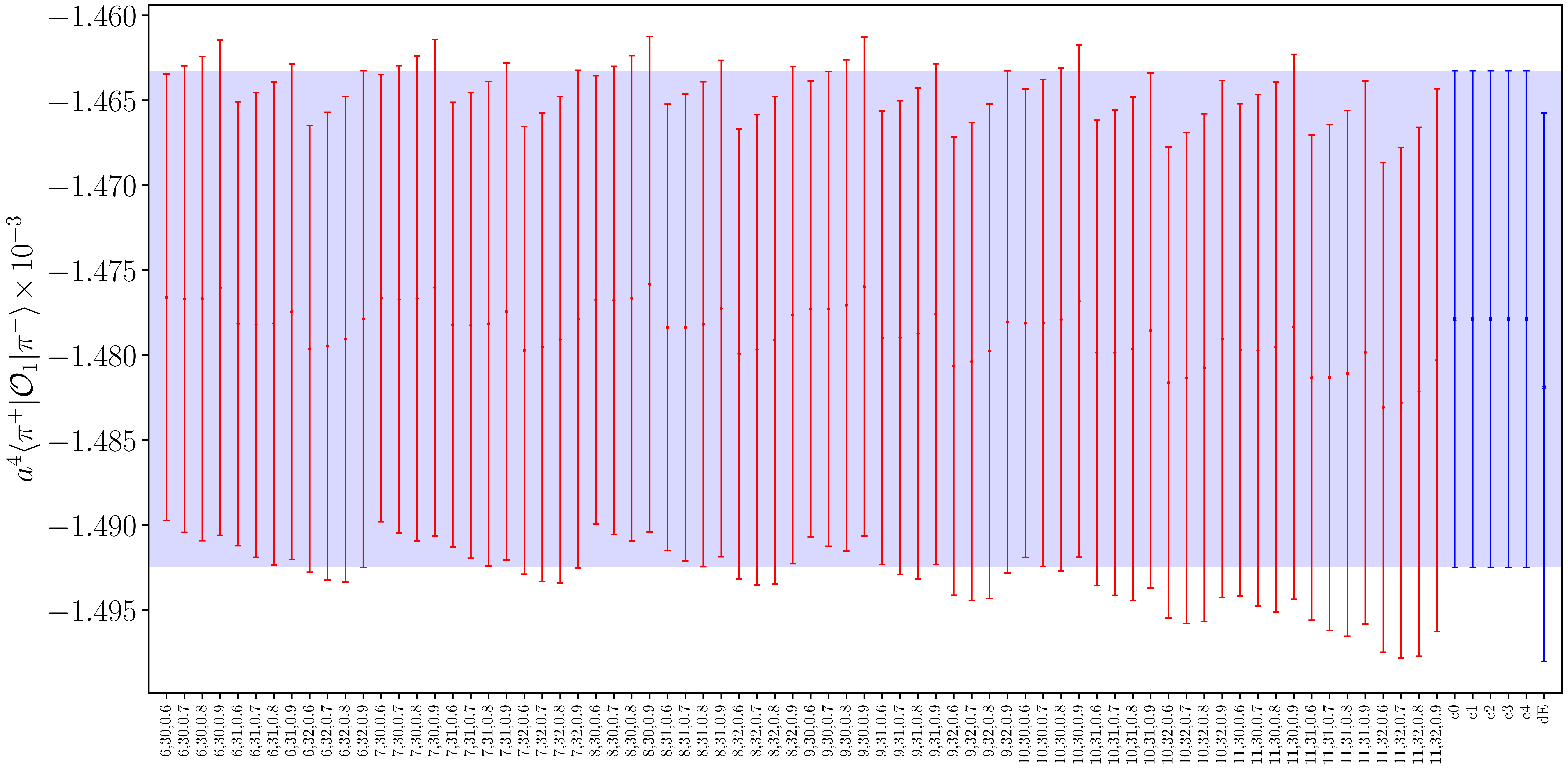
$$f_k(t; \langle \mathcal{O}_k \rangle, m^{(k)}, \Delta^{(k)}, A_i^{(k)}) \equiv \langle \mathcal{O}_k \rangle + A_1^{(k)} e^{-\Delta^{(k)} t} + A_2^{(k)} e^{-(m^{(k)}+\Delta)(T-2t)} - A_3^{(k)} e^{-2\Delta^{(k)} t} - A_4^{(k)} e^{-(m^{(k)}+\Delta)T+2(2m^{(k)}+\Delta^{(k)})t},$$

  - Bayesian least-squares fit on range $[t_{\mathrm{min}}, t_{\mathrm{max}}]$ with parameters $m^{(k)} \sim N(m_\pi, \delta m_\pi)$, $\Delta^{(k)} \sim N(2m_\pi, m_\pi)$, $A_k^{(k)} \sim N(0.0, 0.1)$
  - Covariance matrix obtained from sample covariance via linear shrinkage with parameter λ
  - Statistically indistinguishable results under variation of $t_{\mathrm{min}} \in [6, 11]$, $t_{\mathrm{max}} \in [30, 32]$, and $\lambda \in \{0.6, 0.7, 0.8, 0.9\}$

Comparison to constant fit on 24I, $am_\ell = 0.01$



Stability plot for $\langle \mathcal{O}_1 \rangle$, 32I/$am_\ell = 0.004$



Non-perturbative renormalization (NPR)

- The lattice comes equipped with a UV regulator: a^{-1}.
- Correlation functions computed on the lattice are of bare operators.
- Work in **NPR basis** to simplify calculation.

NPR operator basis

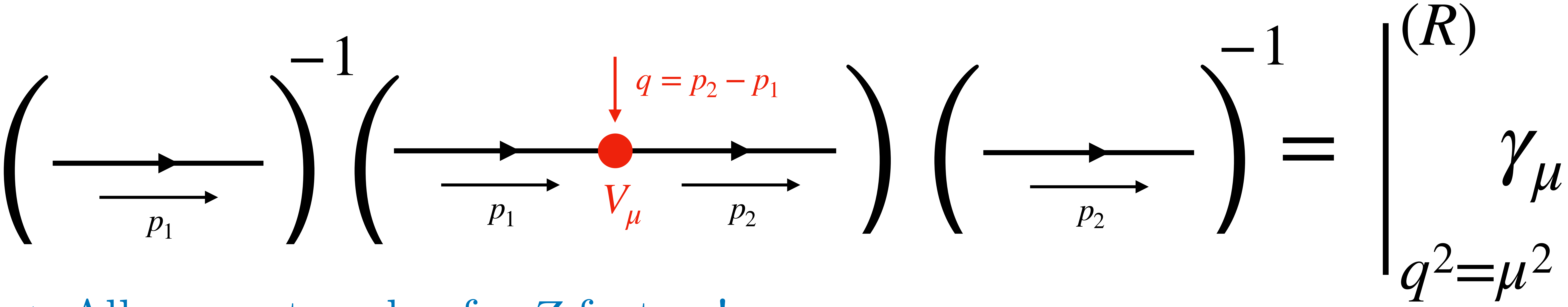
$$Q_1 = 2[\mathcal{O}_3]_+ = VV + AA$$
$$Q_2 = 4[\mathcal{O}_1]_+ = VV - AA$$
$$Q_3 = -2[\mathcal{O}_1']_+ = SS - PP$$
$$Q_4 = 2[\mathcal{O}_2]_+ = SS + PP$$
$$Q_5 = 4[\mathcal{O}_2']_+ + 2[\mathcal{O}_2]_+ = TT$$

$VV = (\bar{u}\gamma_\mu d)[\bar{u}\gamma^\mu d]$

RI/sMOM scheme

- Renormalization condition at scale μ: For an operator with $n-1$ quark fields, impose that its **renormalized**, amputated n-point function equals its tree level value at kinematical point $p_1^2 = p_2^2 = (p_2 - p_1)^2 = \mu^2$.

- Example: vector current $V_\mu(x) = \overline{q}(x)\gamma_\mu q(x)$:



$\Longrightarrow$ Allows us to solve for Z factors!

RI/sMOM details

- RI/sMOM renormalization coefficients computed from the following correlation functions

$$(G_n)^{\alpha\beta\gamma\delta}_{abcd}(q;a,m_\ell) \equiv \frac{1}{V}\sum_x \sum_{x_1,\ldots,x_4} e^{i(p_1\cdot x_1 - p_2\cdot x_2 + p_1\cdot x_3 - p_2 \cdot x_4 + 2q\cdot x)}\langle 0 \,|\, \overline{d}^\delta_d(x_4) u^\gamma_c(x_3) Q_n(x) \overline{d}^\beta_b(x_2) u^\alpha_a(x_1) \,|\, 0\rangle$$

$$(\Lambda_n)^{\alpha\beta\gamma\delta}_{abcd}(q) \equiv (S^{-1})^{\alpha\alpha'}_{aa'}(p_1)(S^{-1})^{\gamma\gamma'}_{cc'}(p_1)(G_n)^{\alpha'\beta'\gamma'\delta'}_{a'b'c'd'}(q)(S^{-1})^{\beta'\beta}_{b'b}(p_2)(S^{-1})^{\delta'\delta}_{d'd}(p_2),$$

$$F_{mn}(q;a,m_\ell) \equiv (P_n)^{\beta\alpha\delta\gamma}_{badc}(\Lambda_m)^{\alpha\beta\gamma\delta}_{abcd}(q;a,m_\ell)$$

$$S(p;a,m_\ell) = \frac{1}{V}\sum_{x,y} e^{ip\cdot(x-y)}\langle 0 \,|\, q(x)\overline{q}(y) \,|\, 0\rangle$$

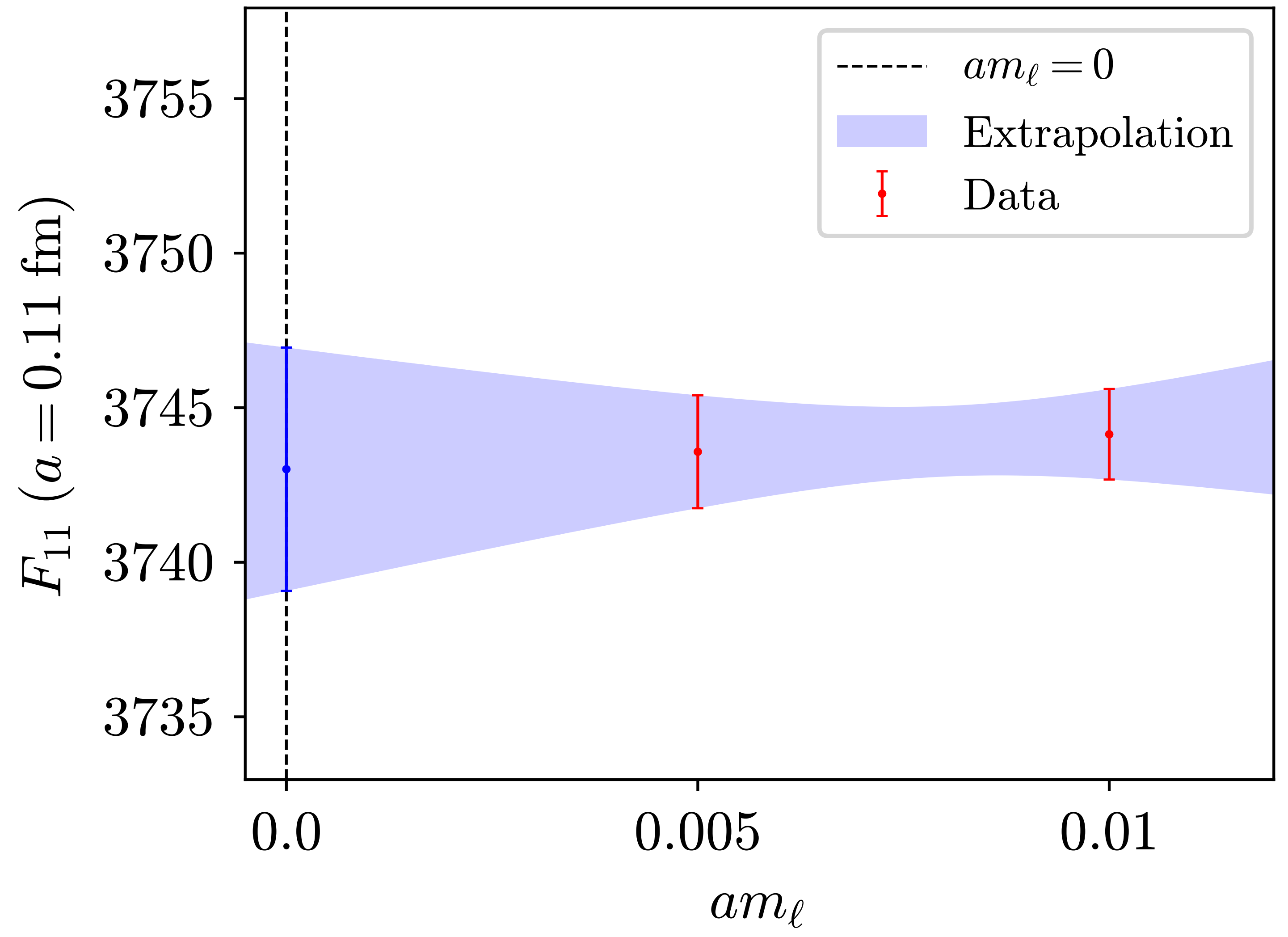
Projectors onto tree-level structure of Λ

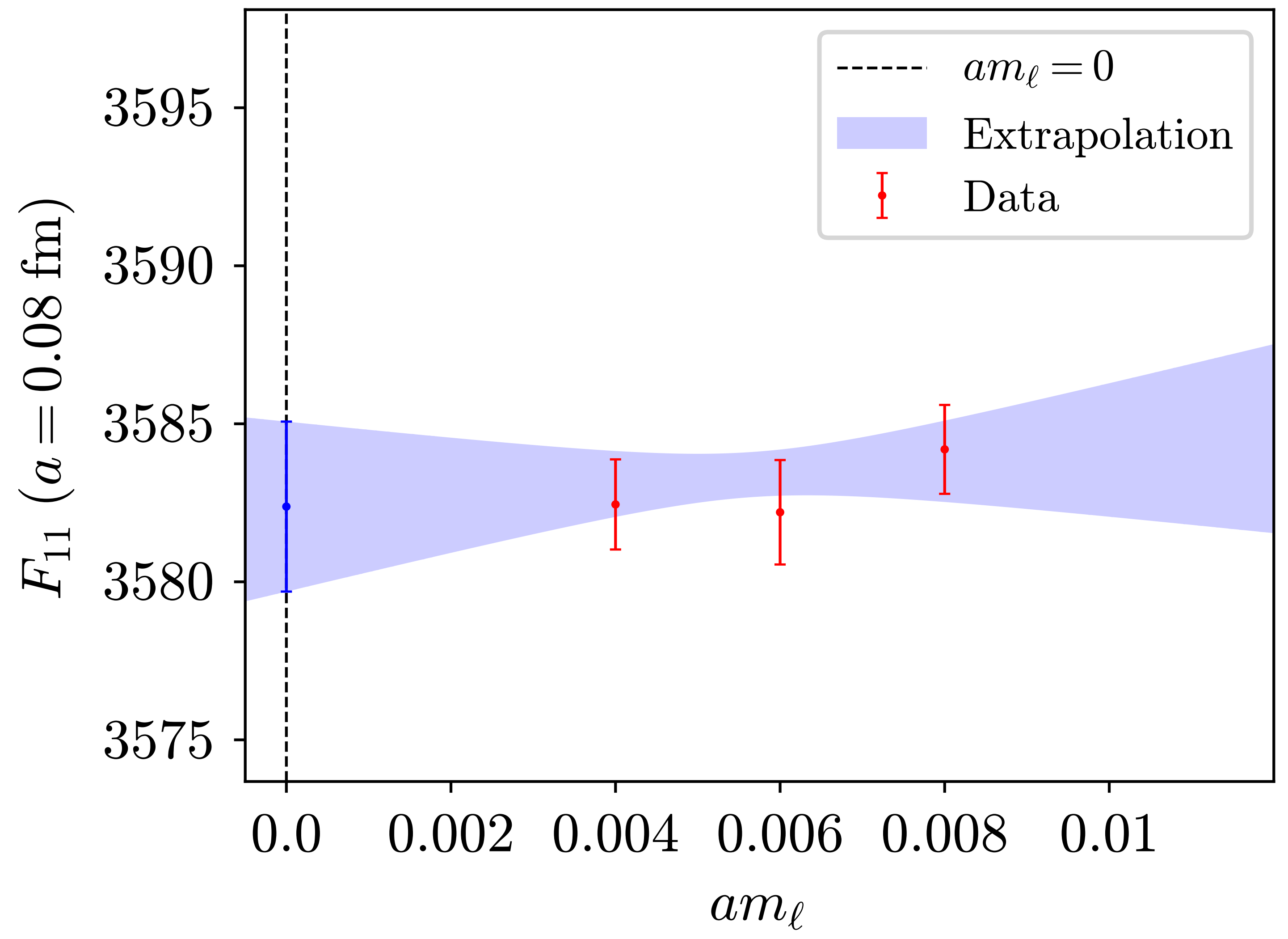
Chiral limit of renormalization coefficients

- $F_{nm}(q; a, m_\ell)$ must be extrapolated to $m_\ell \to 0$ to determine $F_{nm}(q; a)$
- Perform a linear extrapolation to $m_\ell \to 0$, including correlations with other renormalization coefficients computed on each ensemble: quark field Z_q, vector current Z_V, axial current Z_A
- Extract Z^{RI}_{nm} as

$$Z^{\mathrm{RI};Q}_{nm}(\mu^2; a)\bigg|_{\mathrm{sym}} = \left(Z^{\mathrm{RI}}_q(\mu^2; a)\right)^2 \left[F^{(\mathrm{tree})}_{nr} F^{-1}_{rm}(q; a)\right]\bigg|_{\mathrm{sym}}$$

Tree-level value of $F_{nm}(q; a)$





Matching to $\overline{\text{MS}}$

- Must match to a scheme useful for phenomenology: $\overline{\text{MS}}$

RI/sMOM

μ

$\overline{\text{MS}}$

$Z_{ij}^{\text{RI}}(\mu_1)$ μ_1 $Z_{ij}^{\overline{\text{MS}}}(\mu_1)$

Match to $\overline{\text{MS}}$ at μ_1 with **matching coefficients** computed perturbatively @ NLO in α_s.

$$\mu \frac{d}{d\mu} \frac{Z_{ij}}{Z_V^2} = -\gamma_{ik}(\alpha_s) \frac{Z_{kj}}{Z_V^2}(\mu)$$

Compute Z_{ij}^{RI} at some scale μ_0 with

$$\Lambda_{\text{QCD}} \ll \mu_0 \ll \pi a^{-1}$$

$Z_{ij}^{\text{RI}}(\mu_0)$ μ_0

$\mu_1 = 3$ GeV
$\mu_0^{24\text{I}} = 2.64$ GeV $\mu_0^{32\text{I}} = 2.65$ GeV

Perturbative

Minimize discretization artifacts

Chiral extrapolation

$$\Sigma \mapsto L\Sigma R^\dagger$$

- Use χEFT to extrapolate to the physical point.
- Write each operator $\mathcal{O}_k$ as a function of the meson field $\Sigma = \exp(2i\pi^a t^a/F)$ by promoting τ^+ to a spurion.

$$\mathcal{O}_1 = (\overline{q}_L\tau^+\gamma^\mu q_L)[\overline{q}_R\tau^+\gamma_\mu q_R] \longrightarrow \mathrm{Tr}[\Sigma^\dagger\tau^+_{LL}\Sigma\tau^+_{RR}] \longrightarrow \mathrm{Tr}[\Sigma^\dagger\tau^+\Sigma\tau^+]$$

$$\tau^+_{LL} \mapsto L\tau^+_{LL}L^\dagger \qquad\qquad \tau^+_{RR} \mapsto R\tau^+_{RR}R^\dagger$$

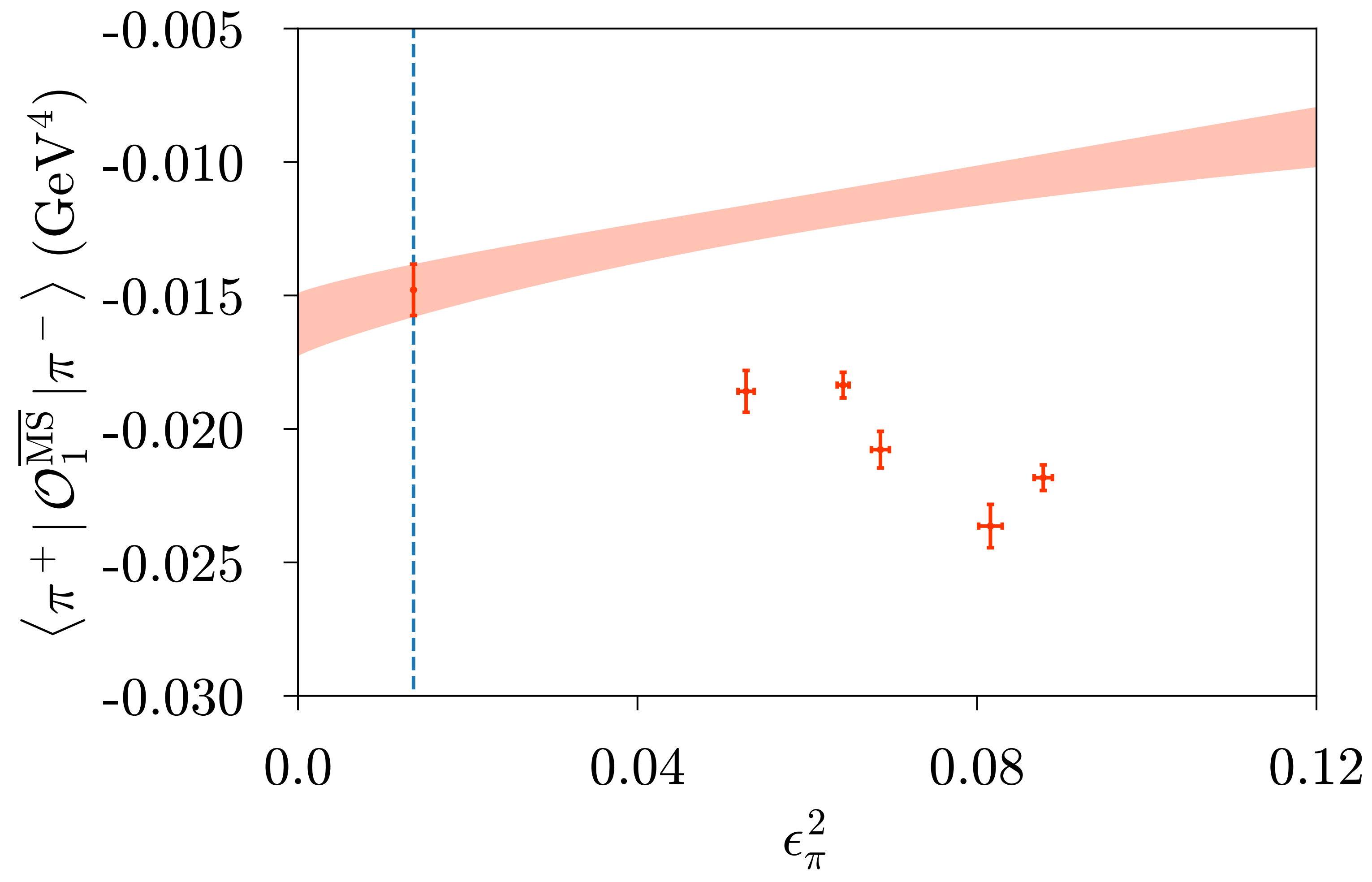
- Spurion analysis yields three independent operator structures:

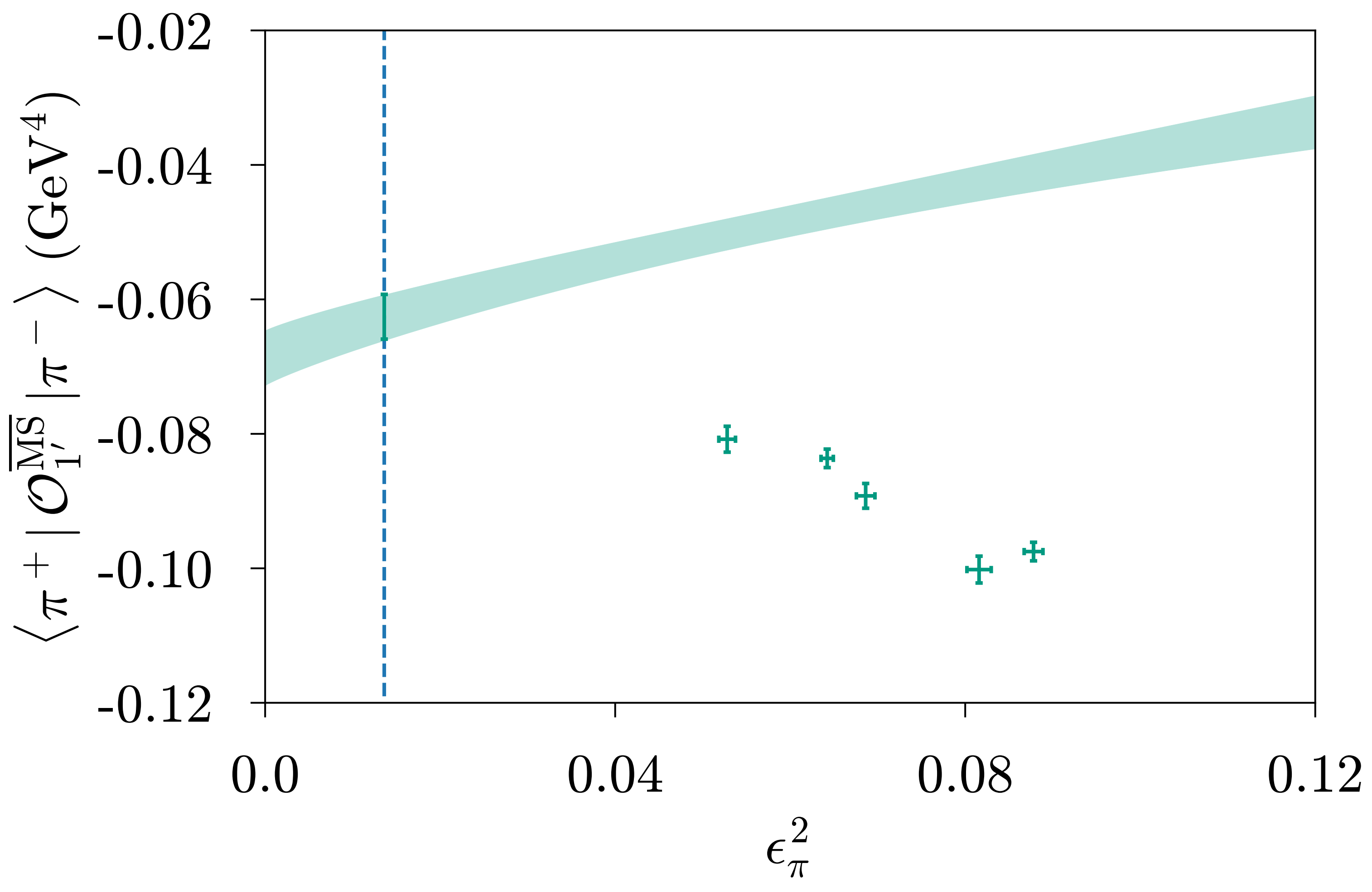
$$\mathcal{O}_1, \mathcal{O}_1' \sim \mathrm{Tr}[\Sigma^\dagger\tau^+\Sigma\tau^+] \qquad\qquad \mathcal{O}_2, \mathcal{O}_2' \sim \mathrm{Tr}[\Sigma\tau^+\Sigma\tau^+] + \mathrm{h.c.}$$

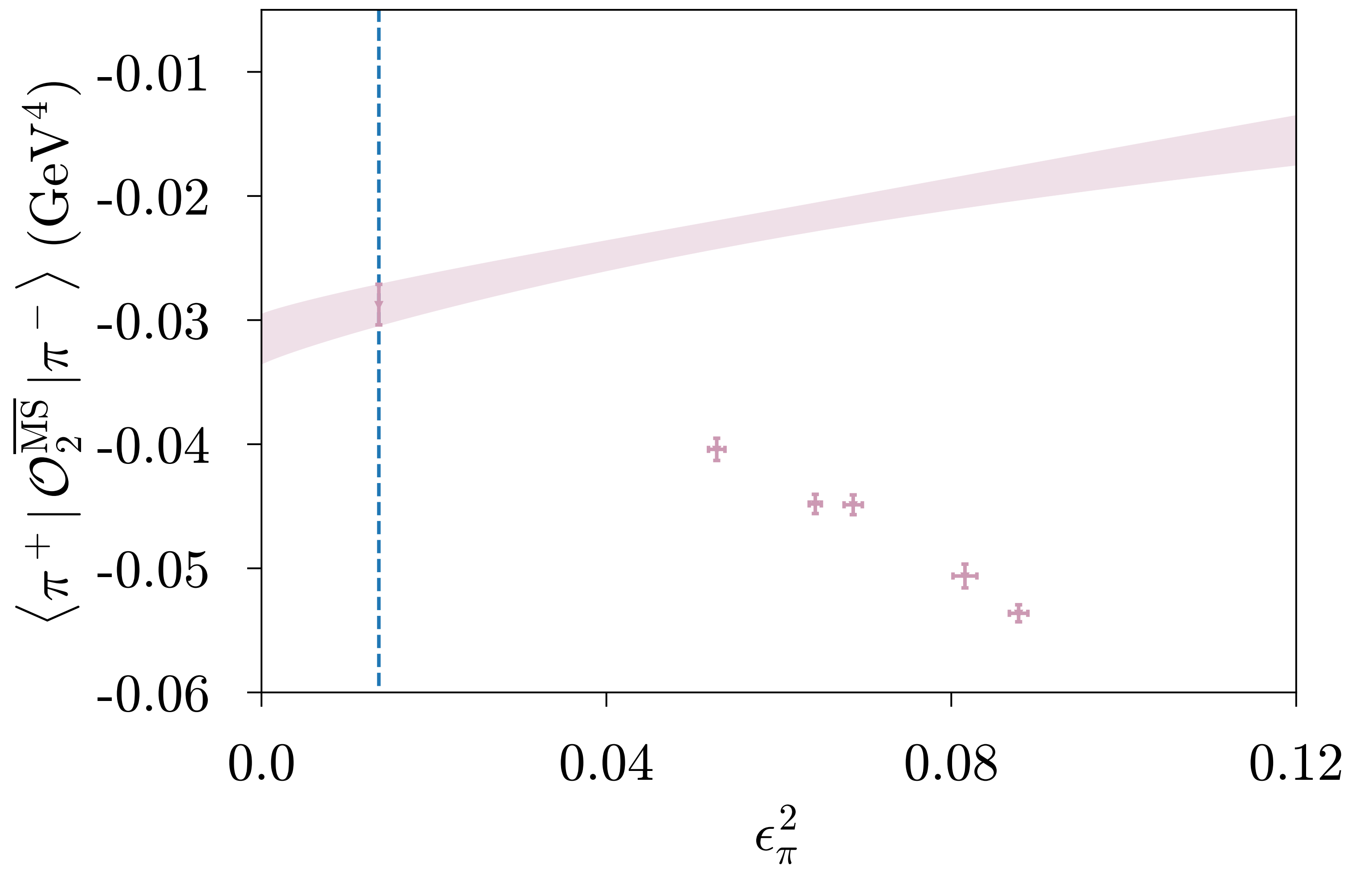
$$\mathcal{O}_3 \sim \mathrm{Tr}[L_\mu\tau^+L^\mu\tau^+] + \mathrm{h.c.}$$

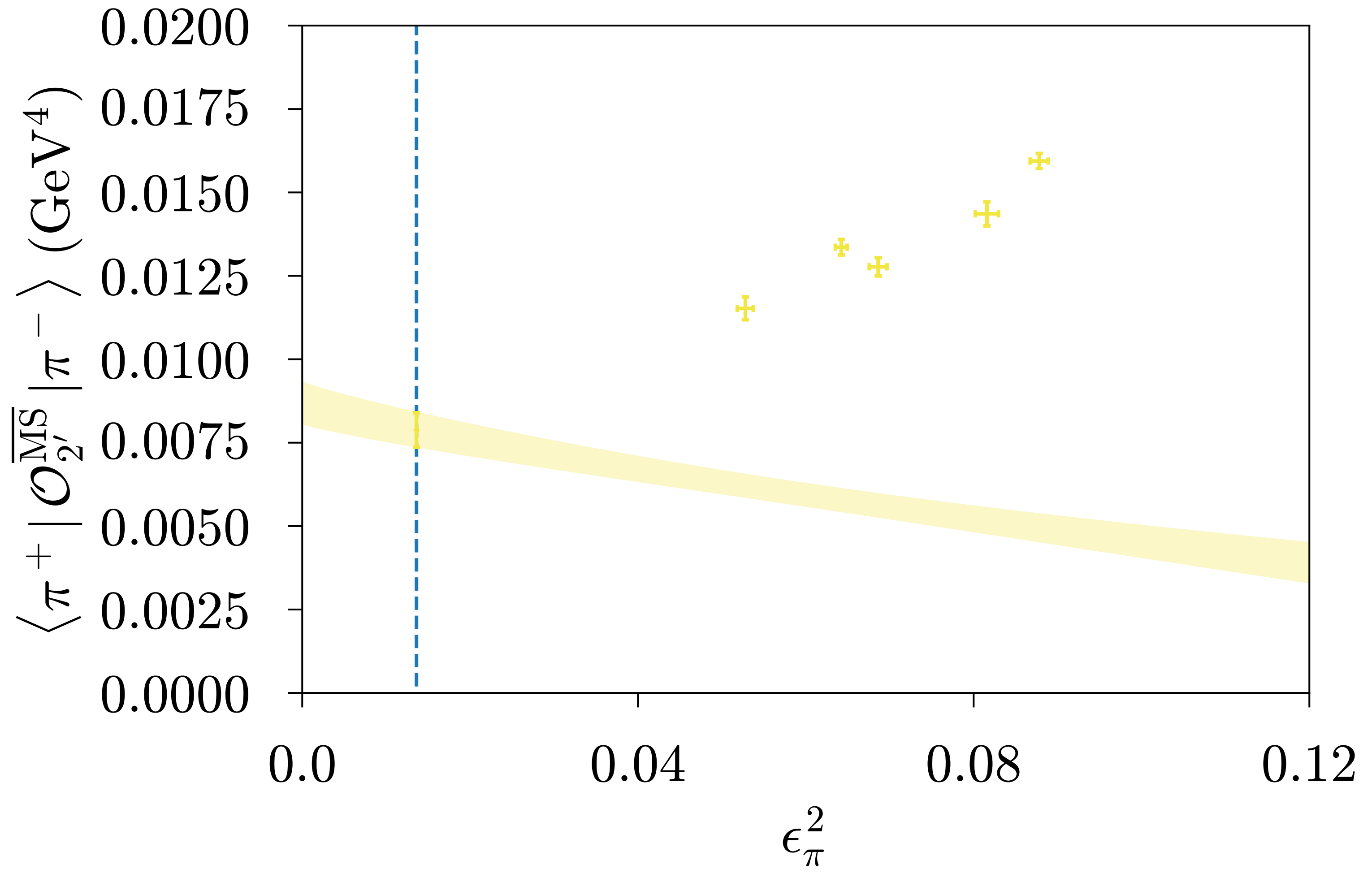
$$L_\mu \equiv \Sigma\partial_\mu\Sigma^\dagger$$

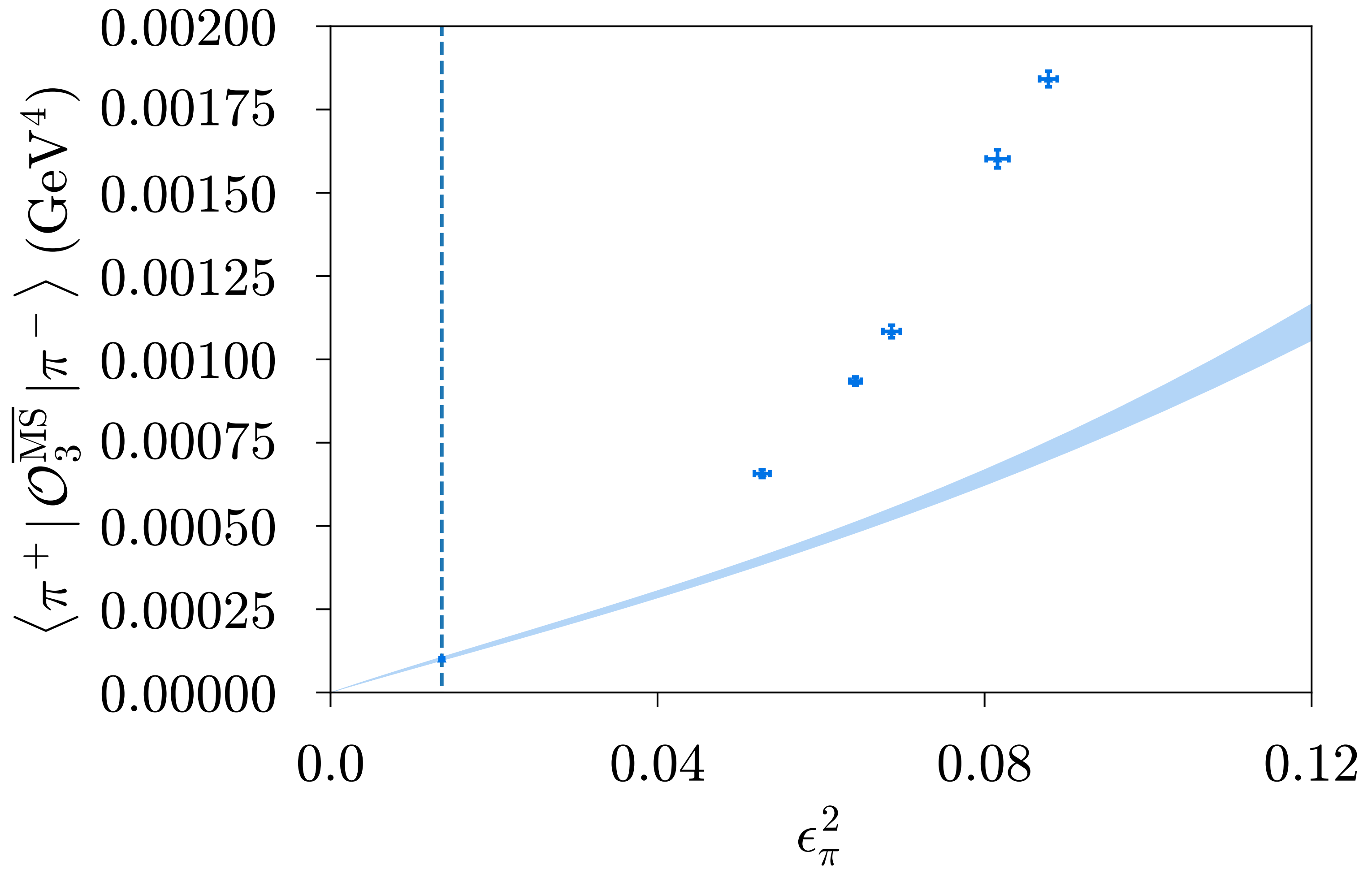
Chiral Extrapolation (unshifted)











Chiral Extrapolation (shifted)