Nonperturbative renormalisation and the QCD Lambda parameter via the gradient flow

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Renormalisation on the lattice

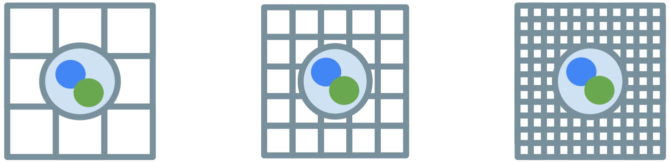
Renormalisation on the lattice poses challenges beyond those encountered in textbook discussions

Renormalisation procedures should be nonperturbative

★ Parameters of QCD Lagrangian typically renormalised by tuning physical parameters

$$a_1 m_{u/d}^{(1)} \to a_2 m_{u/d}^{(2)} \to a_3 m_{u/d}^{(3)}$$

$$m_\pi = m_\pi^{\rm phys}$$



★ Composite operators require different approaches

- connect hadronic and perturbative scales
- matched to the MS-bar scheme via perturbation theory

E.g. Rome-Southampton method *NPB* 445 (1995) 81
or Schrödinger functional *NPB* 384 (1992) 168

Aim: apply the gradient flow as a nonperturbative renormalisation procedure for local composite operators

c.f. Monahan & Orginos, arXiv:1311.2310

Hasenfratz, Monahan, Rizik, Shindler & Witzel, arXiv:2201.9740

Gradient flow and real-space renormalisation group

Block-spin transformation provides a real-space averaging or blocking procedure

★ leaves partition function invariant

★ modifies parameters of the action and expectation values of operators

$$\Phi_b(x_b) = f(\phi)$$
$$g_i \to g_i^{(b)}$$

In vicinity of a fixed point, scaling operators and two-point functions transform as

$$\tilde{\mathcal{O}} = b^{d+\gamma}\mathcal{O} \qquad G_{\mathcal{O}}(g_i, x_0) = b^{-2(d+\gamma)} G_{\mathcal{O}}(g_i^{(b)}, x_0/b)$$

$$G_{\mathcal{O}}(g_i, x_0) = \langle \mathcal{O}(0)\mathcal{O}(x_0)\rangle$$
$$x_0 \gg b$$

Provides definition of the anomalous dimension of the operator/two-point function

$$\Delta_{\mathcal{O}}(g_i^{(b)}) = b\frac{\mathrm{d}}{\mathrm{d}b}\log G_{\mathcal{O}}(g_i^{(b)}, x_0/b)$$

Gradient flow provides natural tool for blocking - smears fields over a region ~ $\sqrt{8\tau}$

$$\Phi_b(x_b) = b^{-d_\phi - \eta/2}\phi(bx_b;\tau)$$

Carosso, Hasenfratz & Neil, PRL 121 (2018) 201601

Narayanan & Neuberger, JHEP 0603 064
Lüscher, JHEP 1008 071
Lüscher, JHEP 04 (2013) 123

Renormalisation scheme: some preliminaries

Define the bare two-point functions

$$G_{\mathcal{O}}(x_4;\tau) = \int \mathrm{d}^3\mathbf{x}\,\langle \mathcal{O}(\mathbf{x},x_4;\tau)P_{\mathcal{O}}(x)\rangle$$

$$G_V(x_4;\tau) = \frac{1}{3}\sum_{j=1}^{3}\int \mathrm{d}^3\mathbf{x}\,\langle V_j(\mathbf{x},x_4;\tau)P_V(x)\rangle$$

and introduce the bare double ratio

$$\overline{R}_{\mathcal{O}}(x_4;\tau) = \frac{R_{\mathcal{O}}(x_4;\tau=0)}{R_{\mathcal{O}}(x_4;\tau)}$$

$$R_{\mathcal{O}}(x_4;\tau) = \frac{G_{\mathcal{O}}(x_4;\tau)}{G_V(x_4;\tau)}$$

which renormalises as

$$\overline{R}^{\mathrm{R}}_{\mathcal{O}}(x_4;\tau) = \frac{Z_{\mathcal{O}}}{Z_V}\overline{R}_{\mathcal{O}}(x_4;\tau)$$

c.f. Monahan & Orginos, arXiv:1311.2310

Renormalisation scheme

Define the gradient flow scheme by imposing the renormalisation condition

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu)\overline{R}_{\mathcal{O}}(x_4;\tau)\Big|_{\substack{\mu^2\tau = c \\ x_4^2 \gg \tau/c}} = \overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)$$

$$\overline{R}_{\mathcal{O}}(x_4;\tau) = \frac{R_{\mathcal{O}}(x_4;\tau=0)}{R_{\mathcal{O}}(x_4;\tau)}$$

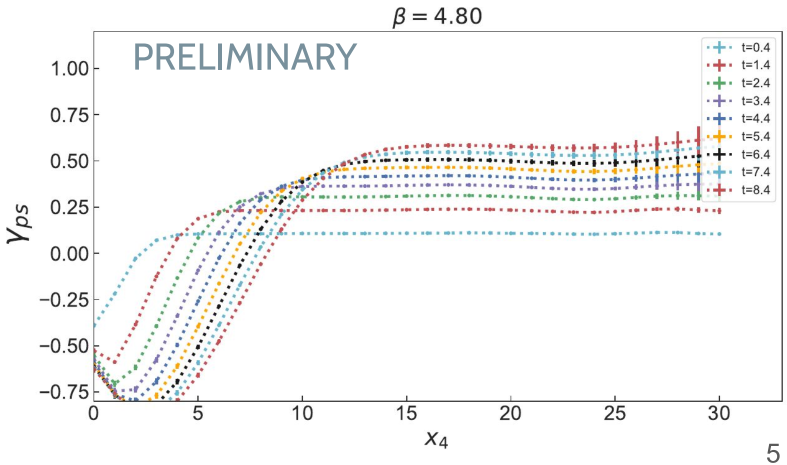
$$R_{\mathcal{O}}(x_4;\tau) = \frac{G_{\mathcal{O}}(x_4;\tau)}{G_V(x_4;\tau)}$$

which allows us to extract

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu) = \frac{\overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)}{\overline{R}_{\mathcal{O}}(x_4;\tau)}\Bigg|_{\substack{\mu^2\tau = c \\ x_4^2 \gg \tau/c}}$$

We further define the (nonperturbative) anomalous dimension

$$\gamma_{\mathcal{O}} = -2\tau\frac{\mathrm{d}}{\mathrm{d}\tau}\log Z_{\mathcal{O}}^{\mathrm{GF}}(\mu)$$



Procedure: a schematic outline

1. Calculate the renormalisation parameter nonperturbatively

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu) = \left.\frac{\overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)}{\overline{R}_{\mathcal{O}}(x_4;\tau)}\right|_{\substack{\mu^2\tau = c \\ x_4^2 \gg \tau/c}}$$

2. Calculate the anomalous dimension nonperturbatively

$$\gamma_{\mathcal{O}} = -2\tau \frac{\mathrm{d}}{\mathrm{d}\tau} \log Z_{\mathcal{O}}^{\mathrm{GF}}(\mu)$$

   to move from low to high scales

3. Match to the MS-bar scheme using perturbation theory

Procedure: in a little more detail

1. Calculate renormalisation parameter at fixed bare coupling

$$\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu) = \left.\frac{\overline{R}_{\mathcal{O}}^{(\mathrm{tree})}(x_4;\tau)}{\overline{R}_{\mathcal{O}}(x_4;\tau)}\right|_{\substack{\mu^2\tau = c \\ x_4^2 \gg \tau/c}}$$

2. Calculate anomalous dimension
   a. Compute ratio of two-point functions at fixed bare coupling

$$R_{\mathcal{O}}(\tau) = \left.\frac{G_{\mathcal{O}}(x_4;\tau)}{G_V(x_4;\tau)}\right|_{x_4^2 \gg \tau}$$

   b. Determine numerical logarithmic derivative at specific flow time (which fixes renormalised coupling)

$$\gamma_{\mathcal{O}}^{\mathrm{GF}}\left(g_{\mathrm{GF}}^2(\tau,\beta)\right) = -\frac{2\tau}{\epsilon}\log\frac{R_{\mathcal{O}}(\tau+\epsilon)}{R_{\mathcal{O}}(\tau)}$$

   c. Take continuum limit as limit of infinite flow time at fixed renormalised coupling (in gradient flow scheme)
3. Determine the beta function on the same ensembles
4. Numerically integrate renormalisation group equations to relate low and high energy regimes

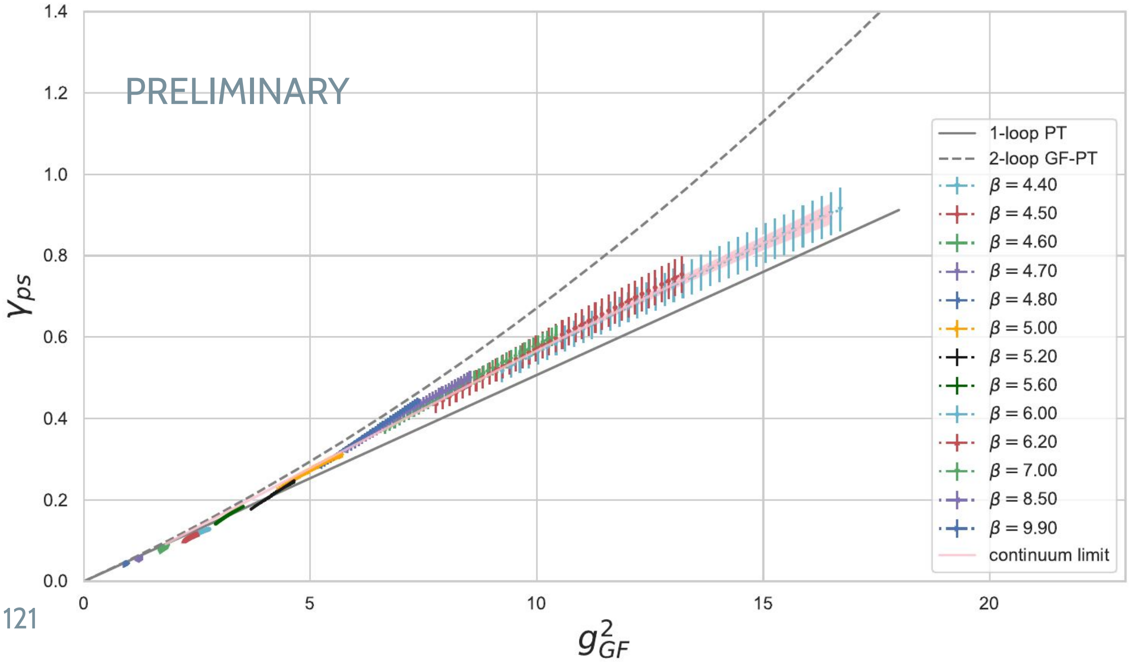
$$\frac{\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu_{\mathrm{UV}})}{\overline{Z}_{\mathcal{O}}^{\mathrm{GF}}(\mu_{\mathrm{IR}})} = \exp\left[\int_{g(\mu_{\mathrm{IR}})}^{g(\mu_{\mathrm{UV}})} \mathrm{d}g' \, \frac{\gamma_{\mathcal{O}}^{\mathrm{GF}}(g')}{\beta_{\mathcal{O}}^{\mathrm{GF}}(g')}\right]$$

5. Match results in high energy regime to the MS-bar scheme at fixed order in perturbation theory

Preliminary nonperturbative results: pseudoscalar anomalous dimension

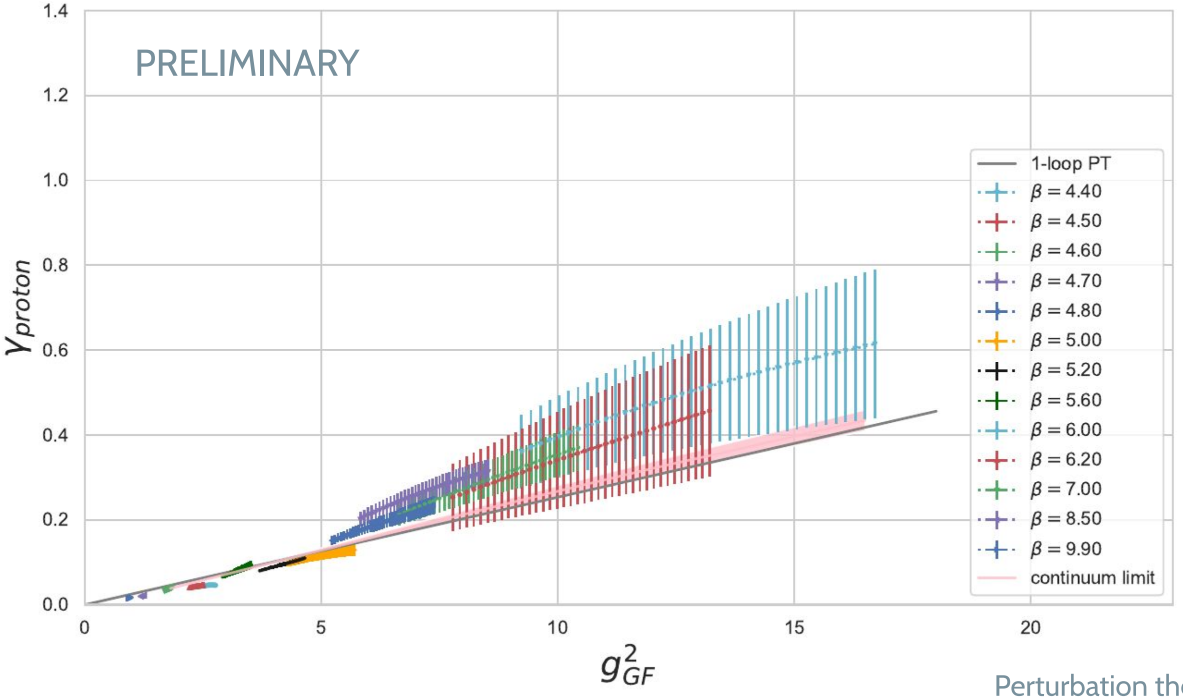
Demonstrate the nonperturbative procedure

- Tree-level improved Symanzik gauge configurations with N_f = 2 stout-smeared Möbius DWF
- 24^3x64 and 32^3x64 volumes
- Apply Wilson kernel for gradient flow
- For $\beta < 4.70$ we use finite bare quark masses of am_q = 0.010 and 0.005
- We observe very small volume and quark mass dependence



Perturbation theory: Artz et al., JHEP 06 (2019) 121

Preliminary nonperturbative results: proton anomalous dimension

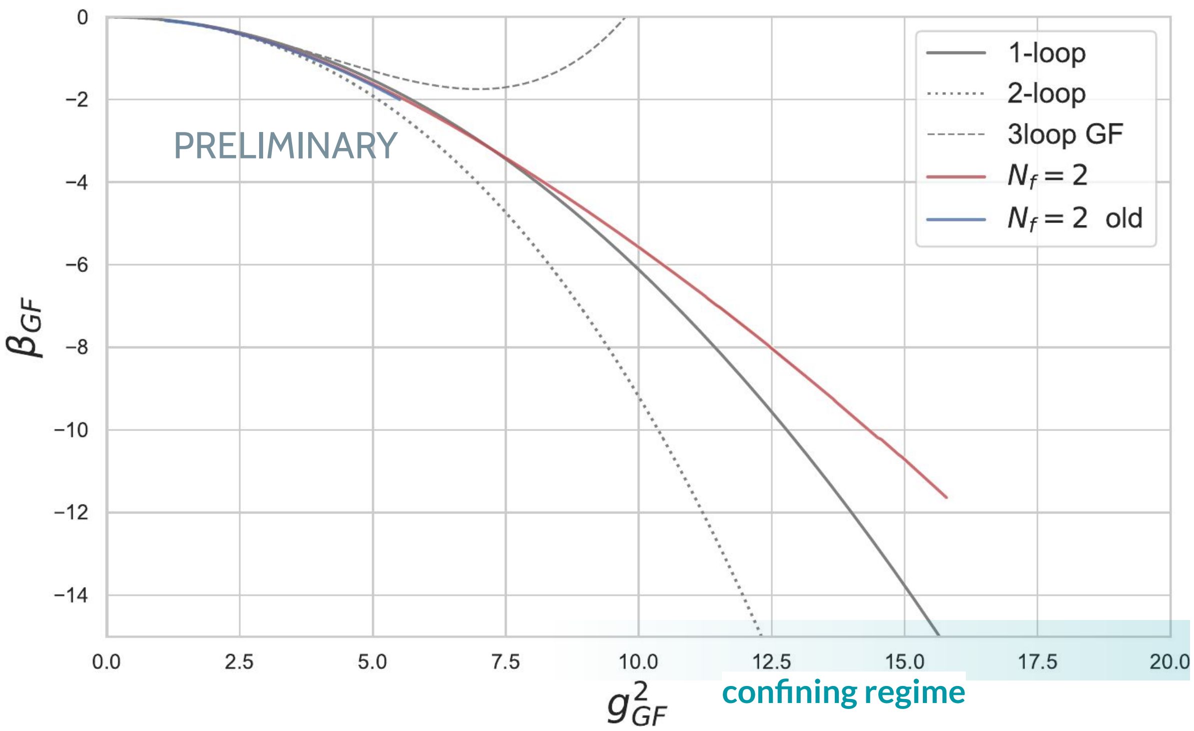


Perturbation theory: Gracey et al., PRD 97 (2018) 116018

N.B. Running of three-quark operators relevant to matching proton decay calculations to phenomenology

Preliminary nonperturbative results: beta function

Perturbation theory:
Shrock & Rytov, PRD 83 (2011) 056011 and refs. within
Harlander & Neumann, JHEP 06 (2016) 161



“Old” results use zero quark mass: limited to deconfined regime

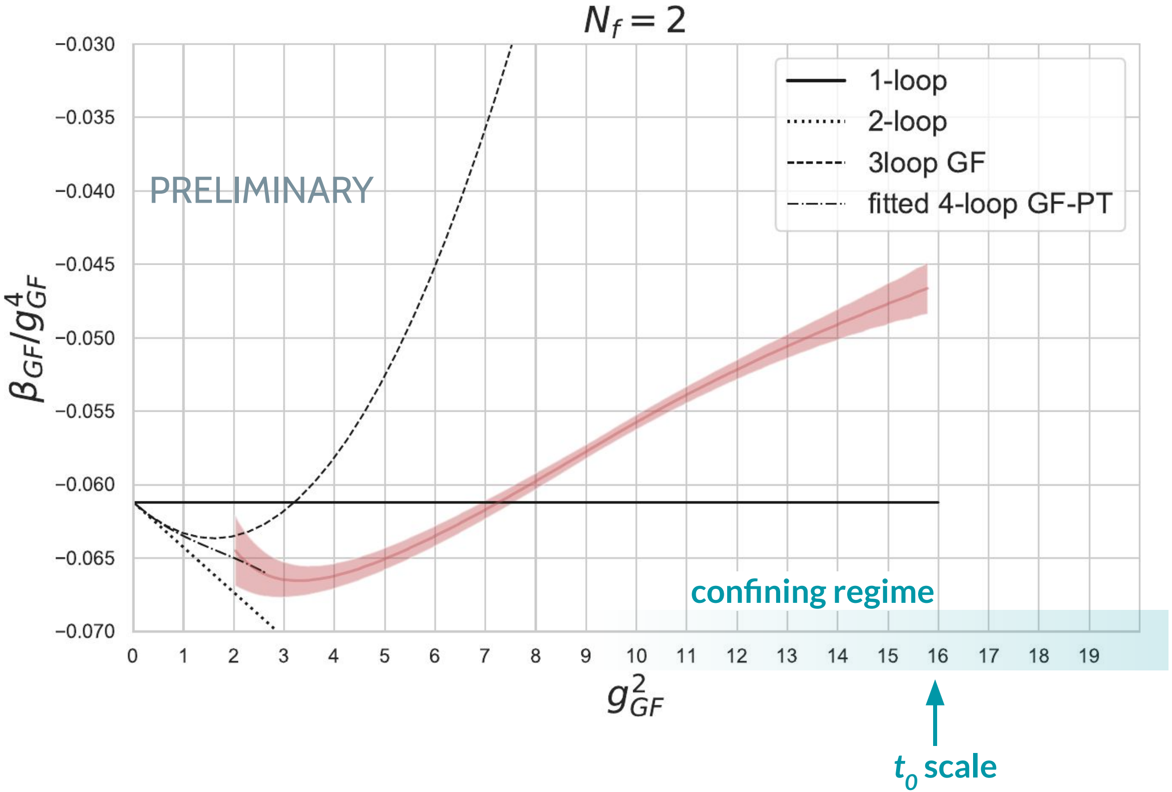
Hasenfratz & Witzel, PRD 101 (2020) 034514

New results include simulations in the confining regime to span region of stronger couplings

c.f nf = 0 study by Schierholz & Nakamura 2201.12875

Preliminary nonperturbative results: beta function

Perturbation theory:
Shrock & Rytov, PRD 83 (2011) 056011 and refs. within
Harlander & Neumann, JHEP 06 (2016) 161



Gradient flow scheme known to be less than optimal in weak coupling regime

See Fodor et al., JHEP 1211 (2012) 007
and, e.g. Bruno et al., PRL 119 (2017) 102001

See also:
Chik Him Wong's talk,
Thursday 12:10 (Algorithms)

$N_f = 2$ Lambda parameter

Lambda parameter is a fundamental parameter of QCD

- characterises the nonperturbative energy scale at which the strong coupling constant diverges
- "fixes" the normalisation of the running coupling
- generated by dimensional transmutation
- dominant error in theoretical uncertainty in value of strong coupling constant at M_Z

Our nonperturbative calculation of the beta function gives access to the Lambda parameter

$$\Lambda_{\mathrm{QCD}} = \mu \cdot \frac{e^{-1/(2b_0 g_s^2(\mu))}}{(b_0 g_s^2(\mu))^{b_1/(2b_0^2)}} \exp\left[-\int_0^{g_s(\mu)} \mathrm{d}x \left(\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right)\right]$$

$$\beta(x) \sim -b_0 x^3 - b_1 x^5 + \cdots$$

Our preliminary calculation provides proof-of-principle results

Preliminary nonperturbative results: Lambda parameter

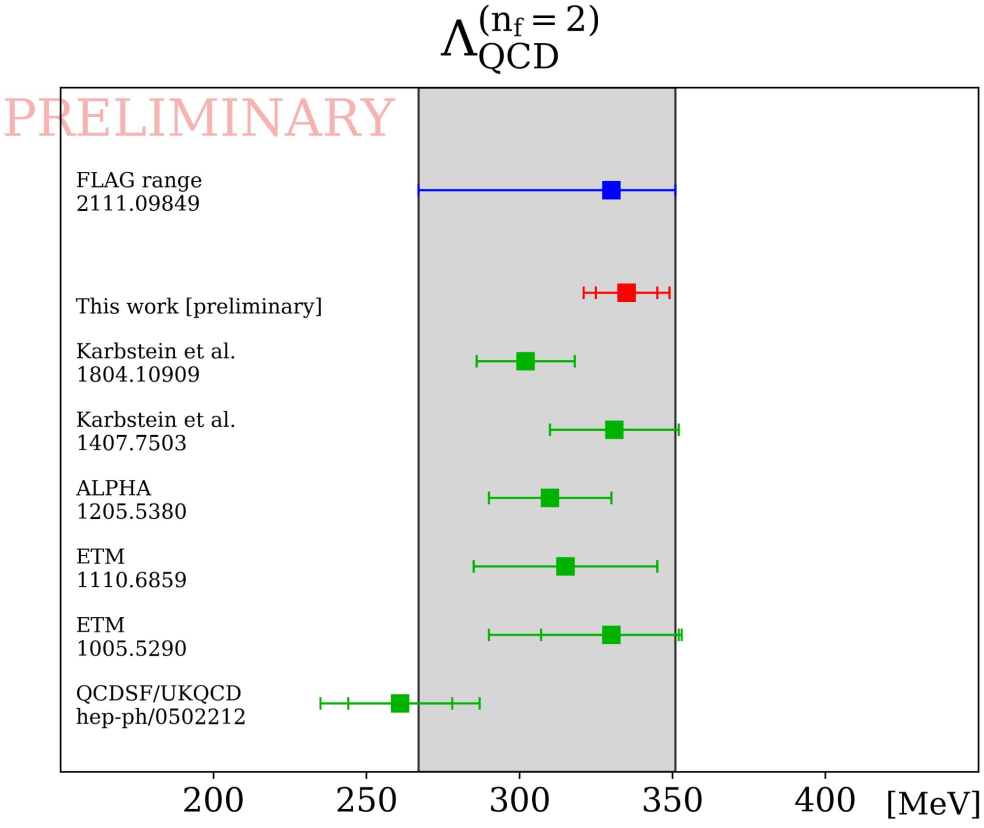
We obtain our **preliminary** value

$$\Lambda_{\rm QCD}^{(n_f=2)} = 335(10)_{\rm stat.}(10)_{\rm sys.}\ {\rm MeV}$$

Note:

- Systematic uncertainties dominated by weak coupling regime, where the gradient flow scheme has largest statistical uncertainties
- We do not include (likely small) systematic contributions from the chiral and infinite volume extrapolations

N.B. FLAG assume r_0 = 0.472 fm if no r_0 scale giv



Conclusions

Gradient flow provides controlled, continuous smearing (or blocking procedure) for fields on the lattice

Applied the gradient flow scheme to renormalise local composite operators

- ★ Nonperturbative
- ★ Gauge-invariant
- ★ Provides nonperturbative step-scaling procedure
- ★ Defined for both small- and large-volume regimes

Determined mass and proton anomalous dimensions in the continuum

Calculated

$$\Lambda_{\mathrm{QCD}}^{(n_f=2)} = 335(10)_{\mathrm{stat.}}(10)_{\mathrm{sys.}}\ \mathrm{MeV}$$

Thank you!

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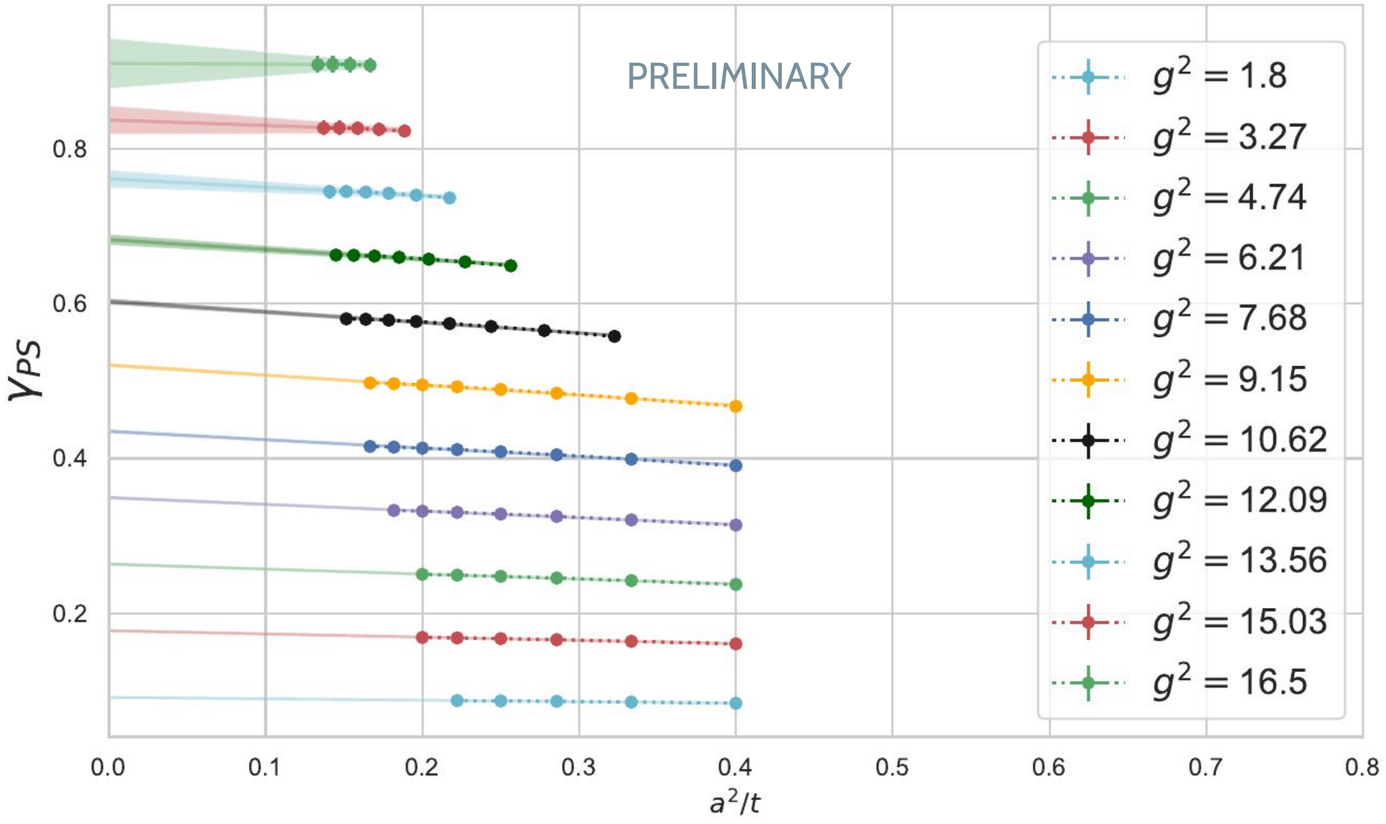
Continuum extrapolation



Gradient flow

Continuous one parameter mapping - evolves fields to classical minimum

$$\partial_\tau B_\mu = D_\nu G_{\nu\mu} + \alpha_0 D_\mu \partial_\nu B_\nu \qquad D_\mu G_{\nu\sigma} = \partial_\mu G_{\nu\sigma} + [B_\mu, G_{\nu\sigma}]$$

$$\partial_\tau \chi = D_\nu D^\nu \chi - \alpha_0 \partial_\nu B_\nu \chi \qquad D_\mu \chi = \partial_\mu \chi + B_\mu \chi \qquad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$$

Provides controlled, continuous smearing

- ★ Gauge invariant
- ★ Nonperturbative
- ★ Renormalised correlation functions remain finite, up to a multiplicative wavefunction renormalisation

Solving the flow equations at leading order

$$\widetilde{B}_\mu(p) = e^{-p^2\tau} \widetilde{A}_\mu(p) + \mathcal{O}(g) \qquad B_\mu|_{\tau=0} = A_\mu$$

$$\widetilde{\chi}(p) = e^{-p^2\tau} \widetilde{\psi}(p) + \mathcal{O}(g) \qquad \chi|_{\tau=0} = \psi$$

Preliminary perturbative calculations

For any bilinear i, we will need the NLO calculation for i and the vector:

$$\tilde{\Gamma}^{(1)}_{i,i} = \frac{C_2(F)}{(4\pi)^2}\tilde{\Gamma}^{(0)}_{i,i}\left\{(B_i^2-4)\left[\frac{1}{\epsilon}+\log(8\pi t)\right] - 6\log 3 + 20\log 2 + 2B_i^2 - 2B_i - 2 + \mathcal{O}(\epsilon)\right\},$$

where $B_i = (2,-2,1,-1,0)$ and

$$\tilde{\Gamma}^{(0)}_{ij}(t,0) = \begin{cases} k_S^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4d}{d(d-2)} + \mathcal{O}(m), & i,j = S,S \\ k_P^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4(d-8)}{d(d-2)} + \mathcal{O}(m), & i,j = P,P \\ -k_V^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4(d-2)}{d(d-2)}\delta_{\mu\nu} + \mathcal{O}(m), & i,j = V,V \quad (\gamma_\mu,\gamma_\nu) \\ k_A^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4(d-6)}{d(d-2)}\delta_{\mu\nu} + \mathcal{O}(m,\hat{\delta}), & i,j = A,A \quad (\gamma_\mu\gamma_5,\gamma_\nu\gamma_5) \\ k_T^2\frac{\dim(F)}{(4\pi)^2 t}(8\pi t)^{2-d/2}\frac{4(d-4)}{d(d-2)}\delta_\mu^{[\rho}\delta_\nu^{\sigma]} + \mathcal{O}(m), & i,j = T,T \quad (\sigma_{\mu\nu},\sigma_{\rho\sigma}) \end{cases}.$$

Renormalizing only the coupling, so that $g_0^2 = \mu^{2\epsilon}g^2 + \mathcal{O}(g^4)$, we have

$$\tilde{\Gamma}_{i,i} = \tilde{\Gamma}^{(0)}_{i,i}\left\{1 + g^2\frac{C_2(F)}{(4\pi)^2}\left[(B_i^2-4)\left(\frac{1}{\epsilon}+\log(2\bar{\mu}^2 t)+\gamma_E\right) - 6\log 3 + 20\log 2 + 2B_i^2 - 2B_i - 2 + \mathcal{O}(\epsilon)\right] + \mathcal{O}(g^4)\right\}.$$

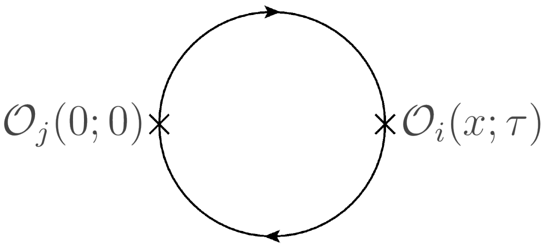
Using

$$\partial_t\frac{\tilde{\Gamma}_{i,i}}{\tilde{\Gamma}^{(0)}_{i,i}} = g^2\frac{C_2(F)}{(4\pi)^2}\frac{B_i^2-4}{t},$$

We have

$$\gamma_P = -2t\partial_t\log[R_P(t)/R_P^{(0)}(t)] = -6g^2\frac{C2F}{(4\pi)^2} + \mathcal{O}(g^4).$$

Tree level



One loop