Observation and results with the gradient flow

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Cutoff effects on the gradient flow for fermions

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Gradient flow

x_μ $t = \text{flow-time}$ $[t] = -2$ $A_\mu(x) = A_\mu^a(x)T^a \rightarrow$ gluon fields

$$\partial_t B_\mu(x,t) = D_\nu G_{\nu\mu}(x,t)$$

$$B_\mu(x,t)|_{t=0} = A_\mu(x)$$

$D_\nu = \partial_\nu + [B_\nu(x,t), \cdot]$

O(a²) cutoff effects analyzed

$G_{\mu\nu}(x,t) = \partial_\mu B_\nu(x,t) - \partial_\nu B_\mu(x,t) + [B_\mu, B_\nu]$

Gradient flow for fermions

$$\partial_t \chi(x,t) = \Delta \chi(x,t) \qquad \partial_t \bar{\chi}(x,t) = \bar{\chi}(x,t) \overleftarrow{\Delta}$$

$$\chi(x, t = 0) = \psi(x)$$

$$\bar{\chi}(x, t = 0) = \bar{\psi}(x)$$

$$x_\mu = (x_0, \mathbf{x}) \qquad t = \text{flow} - \text{time} \qquad [t] = -2$$

$$\Delta = D_{\mu,t} D_{\mu,t} \qquad D_{\mu,t} = \partial_\mu + B_{t,\mu}$$

4+1 Local field theory

$$S = S_{\mathrm{G}} + S_{\mathrm{G,fl}} + S_{\mathrm{F,QCD}} + S_{\mathrm{F,fl}}$$

$$S_{\mathrm{F,fl}} = \int_0^\infty dt \int d^4x \left[\overline{\lambda}(t,x) \left(\partial_t - \Delta\right) \chi(t,x) + \overline{\chi}(t,x) \left(\overleftarrow{\partial}_t - \overleftarrow{\Delta}\right) \lambda(t,x) \right]$$

- Wick contractions

- Renormalization. All order proof for gauge sector

- Chiral symmetry and Ward identities

- Wilson twisted mass

$$S_{\mathrm{F},fl} = \int_0^\infty \mathrm{d}t \int \mathrm{d}^4x \left[\overline{\lambda}(t,x) \left(\partial_t - \Delta \right) \chi(t,x) + \overline{\chi}(t,x) \left(\overleftarrow{\partial}_t - \overleftarrow{\Delta} \right) \lambda(t,x) \right]$$

4+1 chiral symmetry

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$$\begin{cases} \chi(t,x) \to \exp\left\{ i \left(\alpha_V^a \frac{T^a}{2} + \alpha_A^a \frac{T^a}{2} \gamma_5 \right) \right\} \chi(t,x) \\ \\ \overline{\chi}(t,x) \to \overline{\chi}(t,x) \exp\left\{ i \left(-\alpha_V^a \frac{T^a}{2} + \alpha_A^a \frac{T^a}{2} \gamma_5 \right) \right\}. \end{cases}$$

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5

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$$\langle \mathcal{O}_t \delta S \rangle = \langle \delta \mathcal{O}_t \rangle$$

Chiral variation before integrating
the Lagrange multipliers

4+1 chiral symmetry

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Chiral variation before integrating
the Lagrange multipliers

$$\begin{cases} \left\langle \left[\partial_\mu A_{R,\mu}^a(x) - 2m P^a(x) + \widetilde{P}_R^a(0,x) \right] \mathcal{O}_R \left(\{t_0\}, x \right) \right\rangle = 0 \quad t = 0 \\[3ex] \left\langle \left[\partial_s \widetilde{P}^a(s,x) + \partial_\mu \mathcal{A}_\mu^a(s,x) \right] \mathcal{O}_R \left(\{t_0\}, x \right) \right\rangle = 0 \qquad s > 0 \,, \quad s < \{t_0\} \end{cases}$$

$$\widetilde{P}^a(t,x) = \overline{\lambda}(t,x) \frac{T^a}{2} \gamma_5 \chi(t,x) + \overline{\chi}(t,x) \frac{T^a}{2} \gamma_5 \lambda(t,x)$$

Discretization 4+1 local field theory

$$S = S_{\mathrm{G}} + S_{\mathrm{G,fl}} + S_{\mathrm{F,QCD}} + S_{\mathrm{F,fl}}$$

Standard discretization for the gauge action

Wilson-type discretization for the quarks

$$S_{\mathrm{F,fl}} = \epsilon \sum_{n \geq 0} a^4 \sum_{x} \left[\overline{\lambda}(x,t) \left(\partial_t - \nabla^2 \right) \chi(x,t) + \overline{\chi}(x,t) \left(\overleftarrow{\partial}_t - \overleftarrow{\nabla}^2 \right) \lambda(x,t) \right]$$

$$\partial_t \chi(x,t) = \frac{1}{\epsilon} \left(\chi(x,t+\epsilon) - \chi(x,t) \right) \qquad \Delta = \nabla^*_\mu \nabla_\mu$$

$$S_{\mathrm{eff}} = S + aS_1 + O(a^2) \qquad S_1 = \int d^4x \sum_i \mathcal{O}_i$$

Flow equations on the lattice are classically O(a) improved + loop diagrams do not contribute for large t ==> S_1 contains only boundary terms

Lüscher: 2013

6

O(a) cutoff effects

$$S_{\text{eff}} = S + aS_1 + O(a^2) \qquad S_1 = \int d^4x \sum_i \mathcal{O}_i$$

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$$\mathcal{O}_2 = \overline{\psi}D^2_\mu\psi + \overline{\psi}\overleftarrow{D}^2_\mu\psi$$

$$\mathcal{O}_3 = m\text{Tr}\left[G_{\mu\nu}G_{\mu\nu}\right]$$

$$\mathcal{O}_4 = m\overline{\psi}\left[\gamma_\mu D_\mu - \gamma_\mu\overleftarrow{D}_\mu\right]\psi$$

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Bhattacharya, Gupta, Lee,
Sharpe, Wu: 2006

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$$\mathcal{O}_9 = \left.\partial_t\overline{\chi}\chi\right|_{t=0}$$

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9 fields – 4 conditions ==> 5 independent operators

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9

Discrete gradient flow equation

$$\partial_t \chi(x,t) = \frac{1}{\epsilon} \left(\chi(x, t+\epsilon) - \chi(x,t) \right) \qquad \Delta = \nabla^*_\mu \nabla_\mu$$

$$S_{\mathrm{F,fl}} = \epsilon \sum_{n \geq 0} a^4 \sum_x \left[\overline{\lambda}(x,t) \left(\partial_t - \nabla^2 \right) \chi(x,t) + \overline{\chi}(x,t) \left(\overleftarrow{\partial}_t - \overleftarrow{\nabla}^2 \right) \lambda(x,t) \right]$$

$$= a^4 \sum_x \left[\overline{\lambda}(x,0)\chi(x,\epsilon) - \overline{\lambda}(x,0)\chi(x,0) - \epsilon\overline{\lambda}(x,0)\Delta\chi(x,0) + \overline{\chi}(x,\epsilon)\lambda(x,0) - \overline{\chi}(x,0)\lambda(x,0) - \epsilon\overline{\chi}(x,0)\overleftarrow{\Delta}\lambda(x,0) \right] + \cdots$$

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$$\chi(x,0) = \psi(x) \qquad\qquad\qquad \overline{\chi}(x,0) = \overline{\psi}(x)$$

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$$\mathcal{O}_7 = m \left(\overline{\lambda}\psi + \overline{\psi}\lambda \right)$$

$$\mathcal{O}_8 = \overline{\lambda}\gamma_\mu D_\mu \psi - \overline{\psi}\gamma_\mu \overleftarrow{D}_\mu \lambda$$

Discrete gradient flow equation

$$\partial_t \chi(x,t) = \frac{1}{\epsilon} \left(\chi(x, t+\epsilon) - \chi(x,t) \right) \qquad \Delta = \nabla_\mu^* \nabla_\mu$$

$$S_{\mathrm{F,fl}} = \epsilon \sum_{n \geq 0} a^4 \sum_x \left[\overline{\lambda}(x,t) \left(\partial_t - \nabla^2 \right) \chi(x,t) + \overline{\chi}(x,t) \left(\overleftarrow{\partial}_t - \overleftarrow{\nabla}^2 \right) \lambda(x,t) \right]$$

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$$\chi(x, t=0) = \left(1 + a\frac{1}{2}c_\lambda \gamma_\mu D_\mu + \frac{1}{2}ac_m m \right) \psi(x)$$

$$\overline{\chi}(x, t=0) = \overline{\psi}(x) \left(1 - a\frac{1}{2}c_\lambda \gamma_\mu \overleftarrow{D}_\mu + \frac{1}{2}ac_m m \right)$$

Tree-level calculation - warmup

$$\widetilde{S}_{\mathrm{W}}(p) = \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)} \qquad M(p) = m + \frac{1}{2}a\hat{p}^2 \qquad \mathring{p}_\mu = \frac{1}{a}\sin(ap_\mu) \qquad \hat{p}_\mu = \frac{2}{a}\sin(\frac{ap_\mu}{2})$$

Tree-level calculation - warmup

$$\widetilde{S}_{\mathrm{W}}(p) = \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)} \qquad M(p) = m + \frac{1}{2}a\hat{p}^2 \qquad \mathring{p}_\mu = \frac{1}{a}\sin(ap_\mu) \qquad \hat{p}_\mu = \frac{2}{a}\sin(\frac{ap_\mu}{2})$$

$$m \to m(1 + \frac{1}{2}am) \qquad S(x,y) \to \int \frac{d^4p}{(2\pi)^4} \mathrm{e}^{ip(x-y)} \frac{-i\ \not{p} + m}{p^2 + m^2}(1 - am) + \frac{1}{2}a\delta^{(4)}(x-y) + O(a^2)$$

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$$m \to m(1 + \frac{1}{2}am) \qquad S(x,y) \to \int \frac{d^4p}{(2\pi)^4}\mathrm{e}^{ip(x-y)}\frac{-i\,\slashed{p} + m}{p^2 + m^2}(1 - am) + \frac{1}{2}a\delta^{(4)}(x-y) + O(a^2)$$

$$\begin{cases} \psi \to (1 + ab_\psi m)\psi \\ \overline{\psi} \to \overline{\psi}(1 + ab_\psi m) \end{cases} \qquad b_\psi^{(0)} = \frac{1}{2}$$

$$\psi \to [1 + ac_q(\gamma_\mu D_\mu + m)]\,\psi$$

$$c_q^{(0)} = -\frac{1}{4}$$

Tree-level calculation
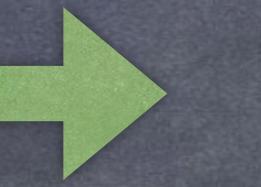
$$m \to m(1 + \frac{1}{2}am)$$

$$\widetilde{S}_{\mathrm{W}}(p; t, s) = \mathrm{e}^{-\hat{p}^2(t+s)} \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)}$$

Tree-level calculation

$$m \to m(1 + \frac{1}{2}am)$$

$$\widetilde{S}_{\mathrm{W}}(p; t, s) = \mathrm{e}^{-\hat{p}^2(t+s)} \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)} \quad \Longrightarrow \quad \mathrm{e}^{-p^2(t+s)} \left(1 + \frac{a^2 p^4}{12}\right) \frac{-i \not{p} + m + \frac{1}{2}am^2 + \frac{1}{2}ap^2}{p^2 + m^2}(1 - am) + \cdots$$

Tree-level calculation

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➡

$$\mathrm{e}^{-p^2(t+s)} \left(1 + \frac{a^2 p^4}{12}\right) \frac{-i\,\not{p} + m + \frac{1}{2}am^2 + \frac{1}{2}ap^2}{p^2 + m^2}(1 - am) + \cdots$$

$$\begin{cases} \chi \to (1 + \frac{a}{2}b_\chi m)\chi \\ \overline{\chi} \to \overline{\chi}(1 + \frac{a}{2}b_\chi m) \end{cases} \qquad b_\chi^{(0)} = 1$$

$$\partial_\mu^* \partial_\mu \to \partial_\mu^* \partial_\mu \left(1 - \frac{a^2}{12}\partial_\mu^* \partial_\mu\right)$$

Battelli, Sint: 2022

Tree-level calculation

$$m \to m(1 + \frac{1}{2}am)$$

$$\widetilde{S}_{\mathrm{W}}(p; t, s) = \mathrm{e}^{-\hat{p}^2(t+s)} \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)} \quad \Rightarrow \quad \mathrm{e}^{-p^2(t+s)} \left(1 + \frac{a^2 p^4}{12}\right) \frac{-i \not{p} + m + \frac{1}{2}am^2 + \frac{1}{2}ap^2}{p^2 + m^2}(1 - am) + \cdots$$

$$\begin{cases} \chi \to (1 + \frac{a}{2} b_\chi m) \chi \\ \overline{\chi} \to \overline{\chi}(1 + \frac{a}{2} b_\chi m) \end{cases} \qquad b_\chi^{(0)} = 1$$

$$\partial_\mu^* \partial_\mu \to \partial_\mu^* \partial_\mu \left(1 - \frac{a^2}{12} \partial_\mu^* \partial_\mu\right)$$

Battelli, Sint: 2022

$$\widetilde{S}_{\mathrm{W}}(p; t, s) \to \frac{\mathrm{e}^{-p^2(t+s)}}{p^2 + m^2} + \frac{1}{2}a\mathrm{e}^{-p^2(t+s)} + \cdots$$

Tree-level calculation

$$m \to m(1 + \frac{1}{2}am)$$

$$\widetilde{S}_{\mathrm{W}}(p; t, s) = \mathrm{e}^{-\hat{p}^2(t+s)} \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)} \quad \Longrightarrow \quad \mathrm{e}^{-p^2(t+s)} \left(1 + \frac{a^2 p^4}{12}\right) \frac{-i \not{p} + m + \frac{1}{2}am^2 + \frac{1}{2}ap^2}{p^2 + m^2}(1 - am) + \cdots$$

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Battelli, Sint: 2022

$$\widetilde{S}_{\mathrm{W}}(p; t, s) \to \frac{\mathrm{e}^{-p^2(t+s)}}{p^2 + m^2} + \frac{1}{2}a\mathrm{e}^{-p^2(t+s)} + \cdots$$

$$K^\dagger K \to c_{\mathrm{fl}}^{(0)} = \frac{1}{2}$$

Lüscher: 2013

Tree-level calculation

$$\chi(x, t=0) = \left(1 + a\frac{1}{2}c_\lambda \gamma_\mu D_\mu + \frac{1}{2}ac_m m\right)\psi(x)$$

$$\overline{\chi}(x, t=0) = \overline{\psi}(x)\left(1 - a\frac{1}{2}c_\lambda \gamma_\mu \overleftarrow{D}_\mu + \frac{1}{2}ac_m m\right)$$

$$\widetilde{S}_{\mathrm{W}}(p; t, s) = \mathrm{e}^{-\hat{p}^2(t+s)}\frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)}$$

Tree-level calculation

$$\chi(x, t = 0) = \left(1 + a\frac{1}{2}c_\lambda \gamma_\mu D_\mu + \frac{1}{2}ac_m m \right) \psi(x)$$

$$\overline{\chi}(x, t = 0) = \overline{\psi}(x) \left(1 - a\frac{1}{2}c_\lambda \gamma_\mu \overleftarrow{D}_\mu + \frac{1}{2}ac_m m \right)$$

$$\widetilde{S}_{\mathrm{W}}(p; t, s) = \mathrm{e}^{-\hat{p}^2(t+s)} \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)}$$

Tree-level calculation

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$$\overline{\chi}(x, t = 0) = \overline{\psi}(x)\left(1 - a\frac{1}{2}c_\lambda\gamma_\mu\overleftarrow{D}_\mu + \frac{1}{2}ac_m m\right)$$

$$\widetilde{S}_{\mathrm{W}}(p; t, s) = \mathrm{e}^{-\hat{p}^2(t+s)}\frac{-i\gamma_\mu\mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)}$$

$$\widetilde{S}_{\mathrm{I}}(p; t, s) = \mathrm{e}^{-\hat{p}^2(t+s)}\left(1 + \frac{1}{2}ac_\lambda^{(0)}i\gamma_\mu\mathring{p}_\mu + \frac{1}{2}ac_m^{(0)}m\right)\frac{-i\gamma_\mu\mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)}\left(1 + \frac{1}{2}ac_\lambda^{(0)}i\gamma_\mu\mathring{p}_\mu + \frac{1}{2}ac_m^{(0)}m\right)$$

Tree-level calculation

$$\chi(x, t = 0) = \left(1 + a \frac{1}{2} c_\lambda \gamma_\mu D_\mu + \frac{1}{2} a c_m m \right) \psi(x)$$

$$\overline{\chi}(x, t = 0) = \overline{\psi}(x) \left(1 - a \frac{1}{2} c_\lambda \gamma_\mu \overleftarrow{D}_\mu + \frac{1}{2} a c_m m \right)$$

$$\widetilde{S}_{\mathrm{W}}(p; t, s) = \mathrm{e}^{-\hat{p}^2 (t+s)} \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)}$$

$$\widetilde{S}_{\mathrm{I}}(p; t, s) = \mathrm{e}^{-\hat{p}^2 (t+s)} \left(1 + \frac{1}{2} a c_\lambda^{(0)} i\gamma_\mu \mathring{p}_\mu + \frac{1}{2} a c_m^{(0)} m \right) \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M^2(p)} \left(1 + \frac{1}{2} a c_\lambda^{(0)} i\gamma_\mu \mathring{p}_\mu + \frac{1}{2} a c_m^{(0)} m \right)$$

$$\widetilde{S}_{\mathrm{I}}(p; t, s) = \mathrm{e}^{-p^2 (t+s)} \frac{-i\gamma_\mu p_\mu + m}{p^2 + m^2} + O(a^2) \qquad c_\lambda^{(0)} = -\frac{1}{2} \qquad c_m^{(0)} = \frac{1}{2}$$

O(a) improved gradient flow for fermions

$$\partial_t \chi(x,t) = \Delta \left(1 - \frac{a^2}{12}\Delta\right) \chi(x,t) \qquad \partial_t \bar{\chi}(x,t) = \bar{\chi}(x,t) \left(1 - \frac{a^2}{12}\overleftarrow{\Delta}\right) \overleftarrow{\Delta}$$

$$\chi(x, t=0) = \left(1 + a\frac{1}{2}c_\lambda \gamma_\mu D_\mu + \frac{1}{2}ac_m m\right) \psi(x)$$

$$\overline{\chi}(x, t=0) = \overline{\psi}(x) \left(1 - a\frac{1}{2}c_\lambda \gamma_\mu \overleftarrow{D}_\mu + \frac{1}{2}ac_m m\right)$$