Implications of gradient flow on the static force

The 39th International Symposium on Lattice Field Theory (Lattice 2022)

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Outline

1 Motivation

2 Setup

3 Lattice results

4 Conclusion

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- Interested in the QCD static energy of a quark-antiquark pair $E(r)$
- Given by the Wilson loop

$$E(r) = -\lim_{T\to\infty} \frac{\ln\langle \mathrm{Tr}(W_{r\times T})\rangle}{T}, \qquad W_{r\times T} = P\left\{\exp\left(i\oint_{r\times T} dz_\mu g A_\mu\right)\right\}$$

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→ can be used for precise α_S-running extraction by comparing PT and lattice

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➡ take a derivative of $E(r)$ for the force $F(r) = \partial_r E(r)$

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\begin{aligned}
F(r) &= -\lim_{T\to\infty} \frac{i}{\langle \mathrm{Tr}(W_{r\times T})\rangle} \left\langle \mathrm{Tr}\left(P\left\{ \exp\left(i \oint_{r\times T} dz_\mu g A_\mu \right) \hat{\mathbf{r}} \cdot g\mathbf{E}(\mathbf{r}, t^*) \right\} \right) \right\rangle \\
&= \frac{\langle \mathrm{Tr}\{ P W_{r\times T} g E_j(r, t^*) \} \rangle}{\langle \mathrm{Tr}\{ P W_{r\times T} \} \rangle}
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On the lattice: modifying Wilson loop with a discretized E-field insertion

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- Clover discretization of E:

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> We use **Gradient flow** for targeting the renormalization and the signal to noise ratio problems, new scale: **flowtime** τ_F, **flowradius** $\sqrt{8\tau_F}$, **flowtime ratio** τ_F/r^2

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→ Utilize the implications of the force measurement with gradient flow

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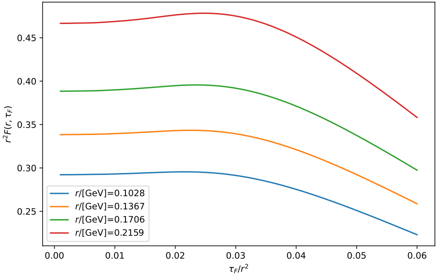
  (Hee Sok Chung et. al. JHEP01(2022)184 (2022))
  - Here: scale $\mu = 1/r$
  - slight difference to $\mu = 1/\sqrt{r^2 + 8\tau_F}$
  - The one-loop behavior should dominate the small τ_F regime



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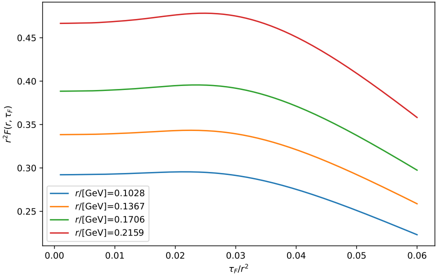

We combine $r^2F(r, \tau_F = 0)$ at well known orders with flow behavior of one loop

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$$r^2 F(r;\tau_F) \approx r^2 F(r;\tau_F = 0) + \frac{\alpha_S^2 C_F}{4\pi} \underbrace{\left[-12\beta_0 - 6C_A c_L\right]}_{8n_f} \frac{\tau_F}{r^2} \qquad c_L = -\frac{22}{3}$$

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→ At small flowtime the force is constant in pure gauge ($n_f = 0$)

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Lattice results: setup and parameters

- Parameters:

N_S	N_T	β	a [fm]	N_{conf}	Label
20	40	6.284	0.060	6000	L20
26	52	6.481	0.046	6000	L26
30	60	6.594	0.040	6000	L30
40	80	6.816	0.030	2700	L40

- Pure gauge configuration produced with overrelaxation and heatbath
- Scale setting with
  $\ln(a/r_0) = -1.6804 - 1.7331(\beta - 6) + 0.7849(\beta - 6)^2 - 0.4428(\beta - 6)^3$
  (1S. Necco & R. Sommer. Nucl. Phys. B622 (2002))
- Gradient flow with **fixed** and **adaptive** solver, with **Symanzik** action

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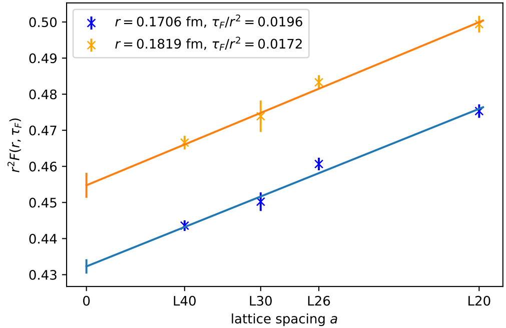
- Working out the continuum limit



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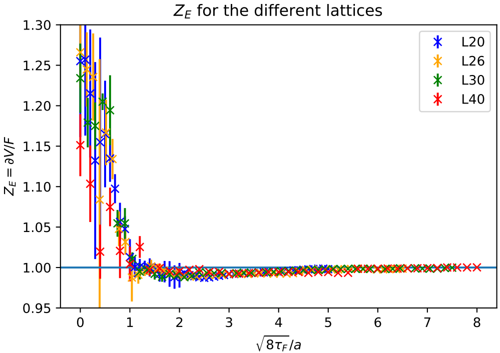

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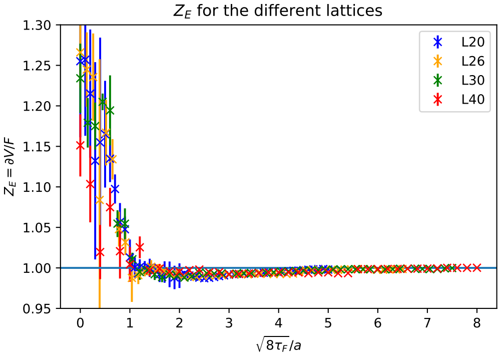

Gradient flow reduces discretization effects of field insertions for $\sqrt{8\tau_F} > a$

Lattice results: Continuum results at large r

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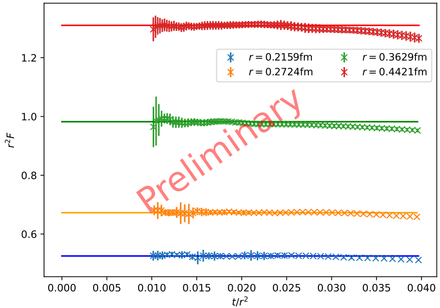

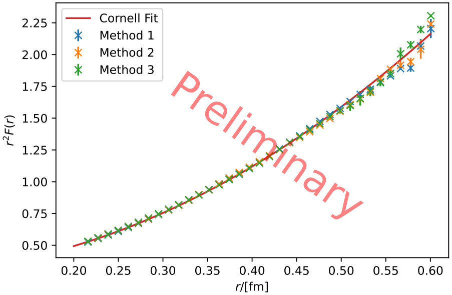

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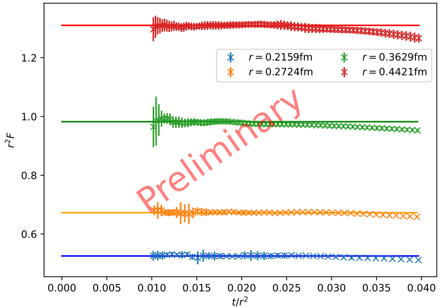

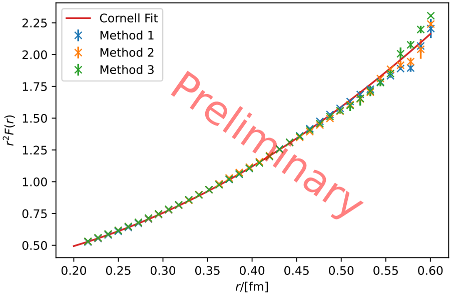

- Cornell fit $r^2F(r) = A + \sigma r^2$
- $\sigma = 5.18 \ldots 5.23$ fm^{-2}, literature: 5.5 fm^{-2}
- $A = 0.2853 \ldots 0.2954$

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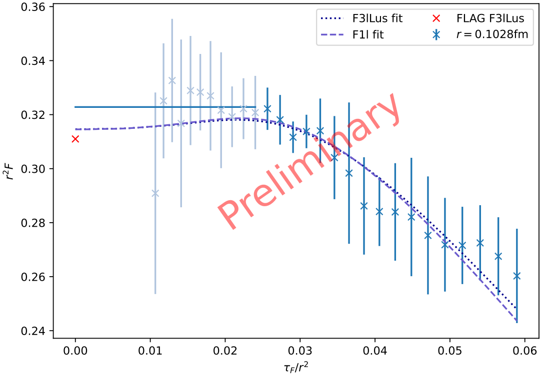

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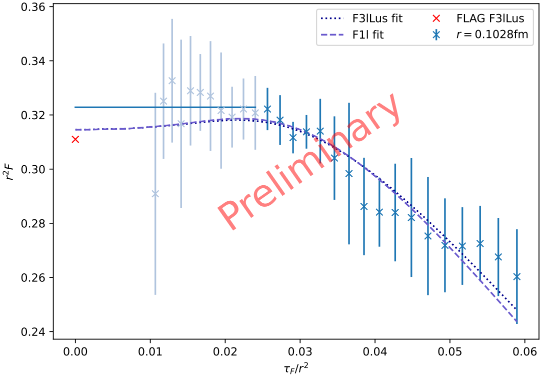

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  F2l: $\Lambda_0 = 0.2731$ GeV
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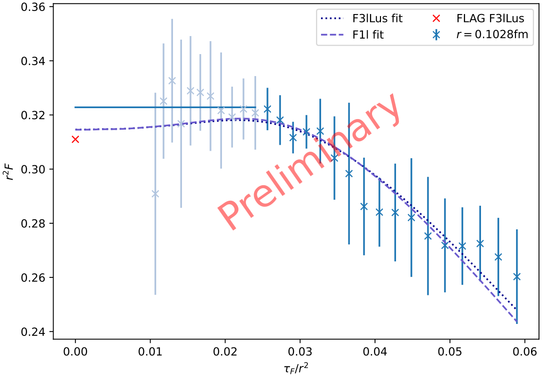

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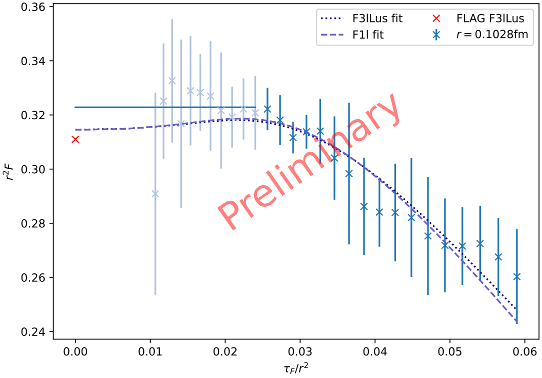

→ Constant fit is too naive for small r

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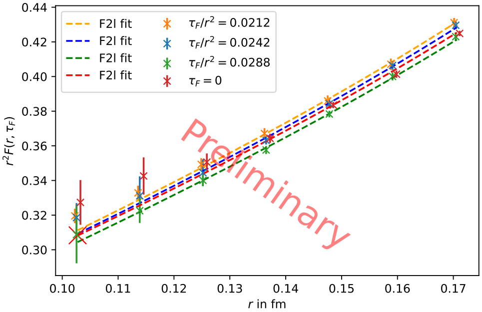
- Results (in GeV):

$\frac{\tau_F}{r^2} =$	0	0.0212	0.0242	0.0288
F2l	0.2626	0.262	0.2595	0.2560
F3lLus	0.2520	0.2524	0.2498	0.2470



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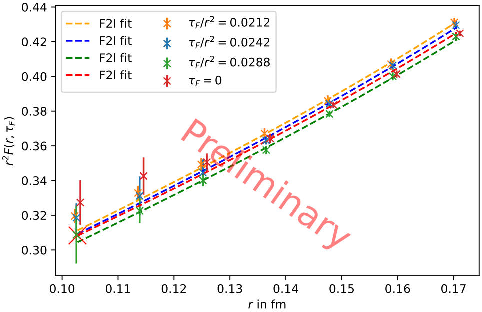

Λ_0 varies little, but is within the FLAG error

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  - Gradient flow improves qualitatively the signal to noise ratio
  - Good preparation for future applications in NREFTs
- For the future:
  - Proper Λ extraction
  - Go to finer lattices
  - Other operators with field insertions
  - Extend to dynamical fermions

Thank you for your attention!

r_0 scale flow dependence

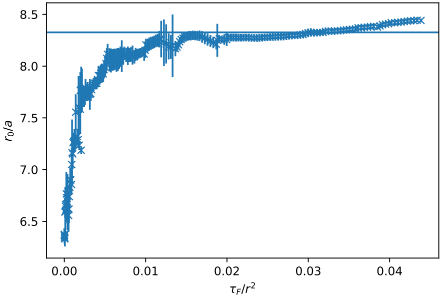

Figure 1 The r_0-scale in lattice units for the L20 lattice. The horizontal line corresponds to the expected value.

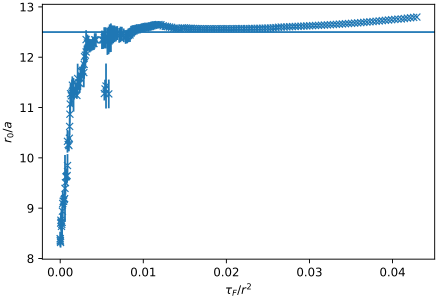

Figure 2 The r_0-scale in lattice units for the L30 lattice. The horizontal line corresponds to the expected value.

Tree level behavior

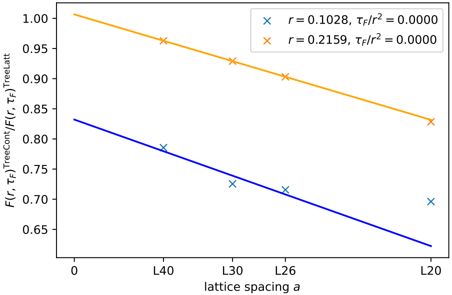

Figure 3 The ratio of the tree level forces in continuum and on the lattice at zero flowtime.

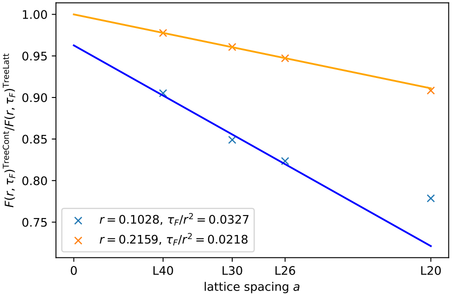

Figure 4 The ratio of the tree level forces in continuum and on the lattice at finite flowtime.