

Towards precision charm physics with a mixed action

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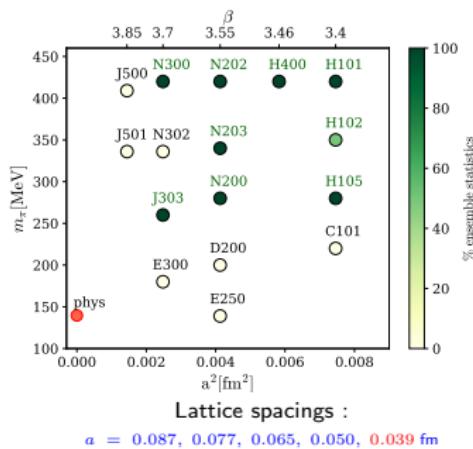
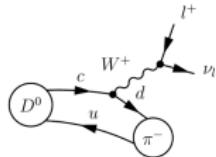
Sea

- CLS $N_f = 2 + 1$
- Wilson regularization
- $O(a)$ -improved

Valence

- $N_f = 2 + 1 + 1$
- Wtm regularization
- Maximal twist

- Tuning valence to maximal twist
 - Automatic $O(a)$ -impr.
- Matching sea and valence
 - Recover unitarity



Observables

- Two-point correlation functions defined as

$$f_{\mathcal{O}\mathcal{O}'}^{qs} = \frac{a^6}{L^3} \sum_{\vec{x}, \vec{y}} \langle \mathcal{O}^{qs}(x_0, \vec{x}) \mathcal{O}'^{qs}(y_0, \vec{y}) \rangle,$$

- Matrix elements and meson masses from a GEVP

RGI charm quark mass [ALPHA Eur. Phys. J.C. 78 (2018) , 387]

$$M_c^{\text{RGI}} = \frac{M}{\overline{m}(\hat{\mu}_{\text{had}})} Z_P^{-1}(g_0^2, \hat{\mu}_{\text{had}}) \mu_c.$$

Decay constants

$$f_{qr}^R = \sqrt{\frac{2}{L^3 m_{qr}^3}} (\mu_q + \mu_r) \langle 0 | P^{qr} | D_{(s)} \rangle$$

Charm quark matching

- Heavy propagators at 3 values of μ_c in the charm region
- Physical value interpolation within chiral-continuum fits through the matching observable

$$\phi_H^{(i)} = \sqrt{8t_0} M_H^{(i)}$$

Matching Conditions

- Flavour average

$$M_H^{(1)} = \frac{1}{3}(2M_D + M_{D_s})$$

- connected η_c

$$M_H^{(2)} = M_{\eta_c}$$

- Spin-flavour average

$$M_H^{(3)} = \frac{1}{4}(M_H^{(1)} + 2M_D^* + M_{D_s}^*)$$

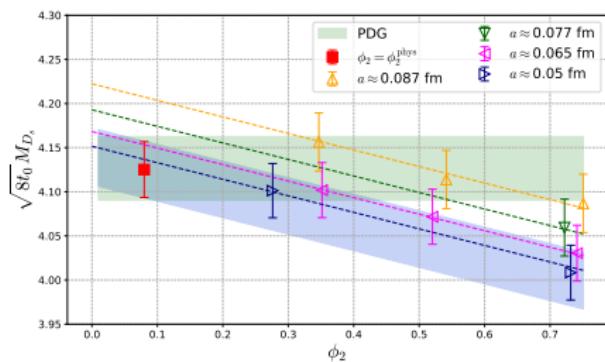
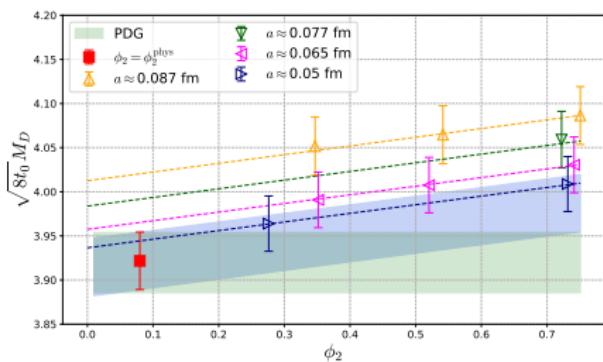
η_c disconnected contributions presumably negligible at current precision
[hep-lat/0404016, 1208.2855, 2005.01845]

Charm matching: check on M_D and M_{D_s}

- Chiral behaviour of $D_{(s)}$ meson masses
- χ^2_{exp} as measure of the effective number of dof [ALPHA, in preparation]

$$\chi^2/\chi^2_{\text{exp}} = 0.77$$

$$\chi^2/\chi^2_{\text{exp}} = 1.22$$



- Matching condition¹: $\phi_H^{(2)} = \sqrt{8t_0} M_{\eta_c}$

¹Neglecting disconnected contributions.

M_c^{RGI} chiral-continuum extrapolations

- Continuum dependence

$$\sqrt{8t_0} M_c^{\text{RGI}}(0, \phi_2, \phi_H^{(i)}) = p_0 + p_1 \phi_2 + p_3 \phi_H^{(i)}$$

$$\phi_2 = 8t_0 m_\pi^2, \quad \phi_H^{(i)} = \sqrt{8t_0} M_H^{(i)}$$

- Cutoff effects

$$c_M(a, \phi_2, \phi_H) = \frac{a^2}{8t_0} (c_1 + \textcolor{red}{c}_2 \phi_2 + \textcolor{red}{c}_3 \phi_H^2) + \frac{a^4}{(8t_0)^2} (\textcolor{red}{c}_4 + \textcolor{red}{c}_5 \phi_H^2 + \textcolor{red}{c}_6 \phi_H^4)$$

Combined models

- 64 models \times 3 matching conditions

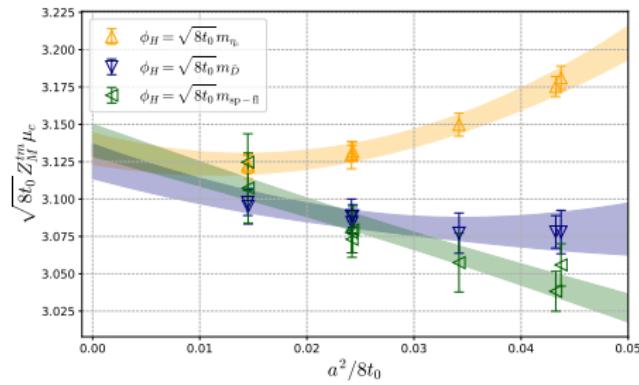
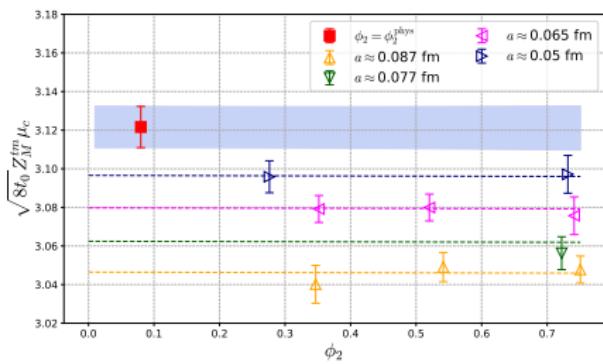
$$\sqrt{8t_0} M_c^{\text{RGI}}(a, \phi_2, \phi_H) = \sqrt{8t_0} M_c^{\text{RGI}}(0, \phi_2, \phi_H) + c_M(a, \phi_2, \phi_H)$$

$$\sqrt{8t_0} M_c^{\text{RGI}}(a, \phi_2, \phi_H) = \sqrt{8t_0} M_c^{\text{RGI}}(0, \phi_2, \phi_H) \times (1 + \tilde{c}_M(a, \phi_2, \phi_H))$$

M_c^{RGI} chiral-continuum extrapolation

Summary of the best fits

- Flavour average and η_c matching compatible
- Spin flavour not well under control (ongoing investigation)
 - issues with vector states \rightarrow poor $\chi^2/\chi^2_{\text{exp}}$

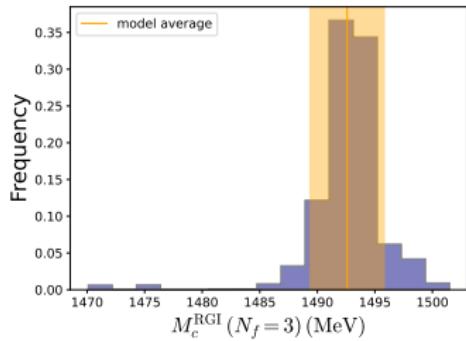
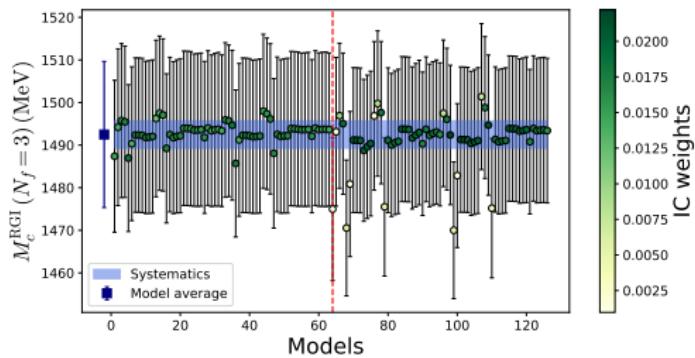


- flavour av. chiral behaviour

M_c^{RGI} results

Model Average [Phys.Rev.D 103 (2021) 114502, ALPHA: JHEP 05 (2021) 288]

- Information Criteria (IC) to perform the model average
- Estimation of the systematics



$$M_c^{\text{RGI}}(N_f = 3) = 1.492(17)(3) \text{ GeV}$$

M_c^{RGI} results

- RGI $\rightarrow \overline{\text{MS}}$ (5-loops p.t.)

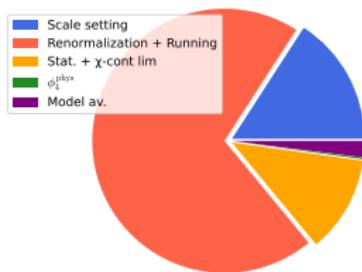
[Comput. Phys. Commun., vol. 133, pp. 43-65, 2000]

- $N_f = 3 \rightarrow 4$ matching

- $\Lambda_{\overline{\text{MS}}}^{N_f=3} = 338(13) \text{ MeV}$

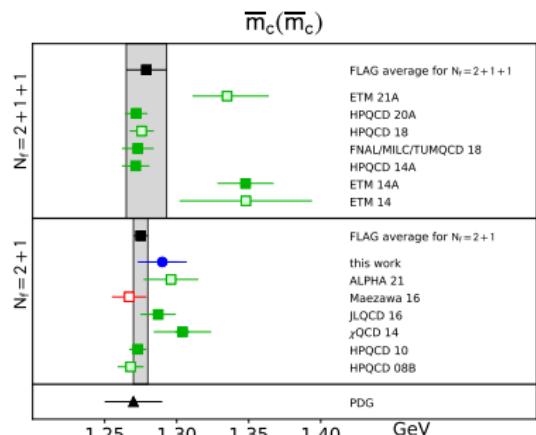
[FLAG: 2111.09849]

$M_c^{\text{RGI}}(N_f=3)$



$$M_c^{\text{RGI}} = \frac{M}{\overline{m}(\hat{\mu}_{\text{had}})} Z_P^{-1}(g_0^2, \hat{\mu}_{\text{had}}) \mu_c.$$

[ALPHA: Eur.Phys.J.C 78 (2018) 5, 387]



$$M_c^{\text{RGI}}(N_f = 3) = 1.492(17)(3) \text{ GeV}$$

$$\overline{m}_c(\mu = \overline{m}_c, N_f = 4) = 1.290(11)(13)_\Lambda \text{ GeV}$$

$f_{D_{(s)}}$ chiral-continuum extrapolation

Talk by: J. Frison: Wilson and tmWilson semileptonics

- Global fit between f_D and f_{D_s}
- Continuum dependence with χ PT chiral logs

[Nucl. Phys. B, vol. 380, pp. 369-376, 1992, Phys. Rev. D, vol. 46, pp. 3929-3936, 1992]

$$\begin{aligned} f_D &= p_0 + p_1 \phi_2 + \frac{p_2}{\sqrt{\phi_H}} + \textcolor{red}{p}_3 \left(3\mu_\pi + 2\mu_K + \frac{1}{3}\mu_\eta \right) \\ f_{D_s} &= p_0 + 2p_1(\phi_4 - \phi_2) + \frac{p_2}{\sqrt{\phi_H}} + \textcolor{red}{p}_3 \left(4\mu_K + \frac{4}{3}\mu_\eta \right) \end{aligned}$$

- Cutoff effects similar to M_c^{RGI} $\mu_\pi = \phi_2 \log(\phi_2), \dots$

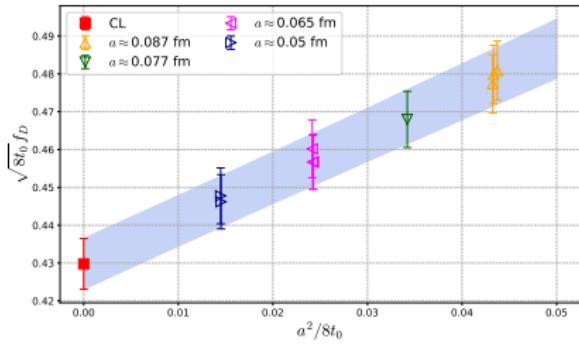
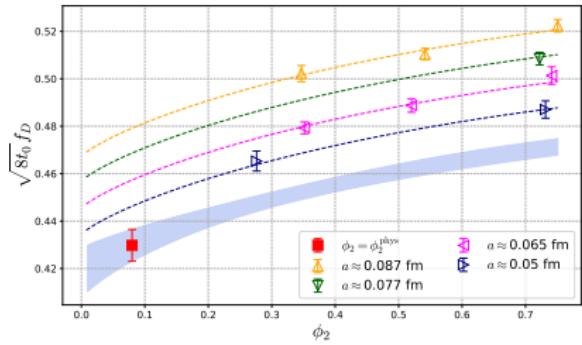
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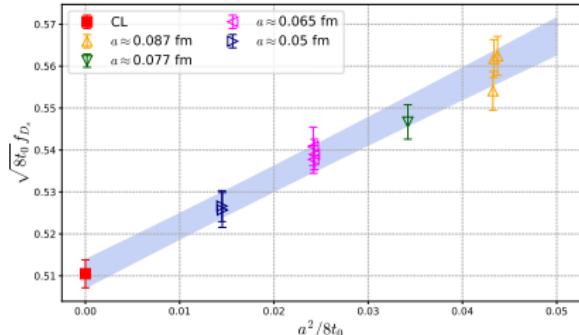
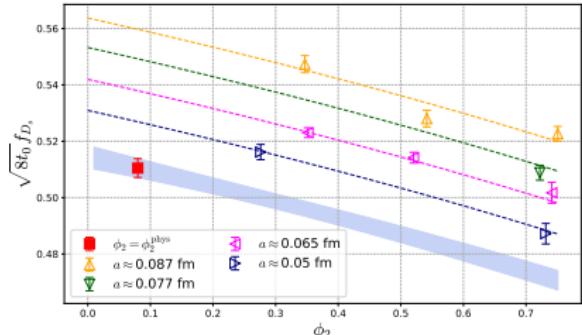
$$\sqrt{8t_0} f_{D_{(s)}}(a, \phi_2, \phi_H) = \sqrt{8t_0} f_{D_{(s)}}(0, \phi_2, \phi_H) + c_f(a, \phi_2, \phi_H)$$

$f_{D(s)}$ chiral-continuum extrapolation

- f_D $\chi^2/\chi^2_{\text{exp.}} = 0.6$



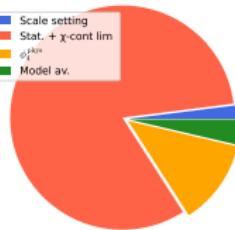
- f_{D_s} $\chi^2/\chi^2_{\text{exp.}} = 0.6$



Results comparison

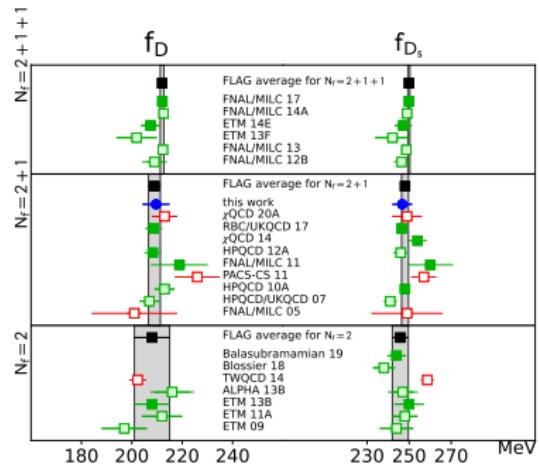
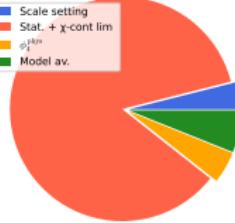
f_D

- Scale setting
- Stat. + χ -cont lim
- $\phi_1^{14/9}$
- Model av.



f_{D_s}

- Scale setting
- Stat. + χ -cont lim
- $\phi_1^{14/9}$
- Model av.



$$f_D = 209.6(5.2)(1.7) \text{ MeV}$$

$$\text{fit : } f_{D_s}/f_D = 1.1665(95)_{\text{stat.}}$$

$$f_{D_s} = 246.7(4.7)(1.1) \text{ MeV}$$

$$\text{ratio : } f_{D_s}/f_D = 1.177(16)$$

Conclusions and outlook

Summary

- Precision M_c and $f_{D_{(s)}}$ results from tmQCD mixed action with fully non-pert. $\mathcal{O}(a)$ -improvement of the observables
 - $M_c^{\text{RGI}}(N_f = 3) = 1.492(17)(3)$ GeV
 - $f_D = 209.6(5.2)(1.7)$ MeV
 - $f_{D_s} = 246.7(4.7)(1.1)$ MeV
- M_c^{RGI} dominated by non-pert. renormalisation and RG running
- $f_{D_{(s)}}$ dominated by chiral-cont. extrap. → room for improvement

Future

- Ongoing extension across more ensembles, by including the finest lattice spacings $a = 0.039\text{fm}$ and physical point ensembles
- Push towards heavier masses to tackle the B sector

Thank You!



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Related talks:

- A. Sáez: scale setting and matching
- G. Herdoíza: light quark masses
- J. Frison: Wilson and tmWilson semileptonics

Wtm Mixed Action

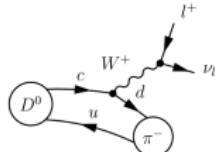
Talks by: A. Sáez: scale setting and matching o G. Herdoíza: light quark masses

- Control of systematics in heavy quark computations
- Different regularizations between sea and valence
 - Matching the two sectors to recover unitarity
- CLS ensembles:

$$0.05 \text{ fm} \lesssim a \lesssim 0.087 \text{ fm}, \quad 260 \text{ MeV} \lesssim M_\pi \lesssim 420 \text{ MeV}$$

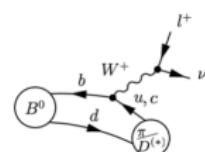
Sea

- CLS $N_f = 2 + 1$
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Valence

- $N_f = 2 + 1 + 1$
- Wtm regularization
- Maximal twist



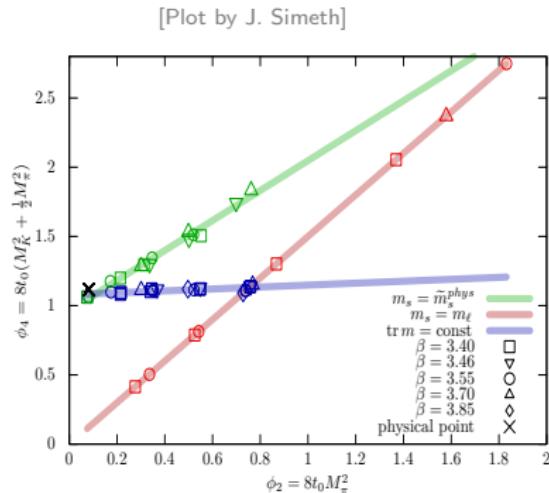
Sea sector - CLS ensembles

[Lüscher and Schaefer, JHEP 1107 036

Bruno et al. JHEP 1502 043 - 1712.04884 - 2003.13359]

- Lüscher-Weisz tree-level improved gauge action
- $N_f = 2 + 1$ non-perturbatively $\mathcal{O}(a)$ -improved Wilson fermions
- Open boundary conditions in time
 - Avoid topological freezing
[Lüscher and Schaefer, 1206.2809]
 - Small a relevant for heavy physics
- Chiral trajectory $\text{tr}(M_{\text{sea}}) = 2m_l + m_s = \text{const}$
- Mass corrections to ensure $\Phi_4 = \Phi_4^{\text{phys}}$ LCP
[M. Bruno, T. Korzec, S. Schaefer, 1608.08900]

$$\langle \mathcal{O} \rangle \rightarrow \langle \mathcal{O} \rangle + \sum_{i=1}^{N_f} \Delta m_{q,i} \frac{d\langle \mathcal{O} \rangle}{dm_{q,i}}$$



Lattice spacings :

$a = 0.087, 0.077, 0.065, 0.050, 0.039 \text{ fm}$

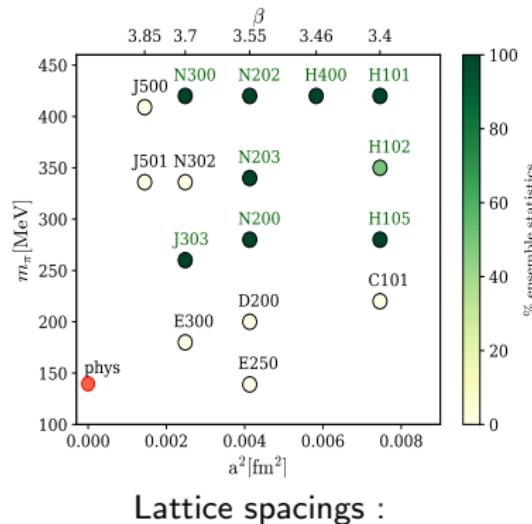
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$$\langle \mathcal{O} \rangle \rightarrow \langle \mathcal{O} \rangle + \sum_{i=1}^{N_f} \Delta m_{q,i} \frac{d\langle \mathcal{O} \rangle}{dm_{q,i}}$$



Wilson twisted mass Dirac operator

$$D_{tm} = \frac{1}{2} [\gamma_\mu (\nabla_\mu^\star + \nabla_\mu) - a \nabla_\mu^\star \nabla_\mu] + \frac{i}{4} ac_{SW} \sigma_{\mu\nu} \hat{F}_{\mu\nu} + \mathbf{m}_0 + i\gamma_5 \boldsymbol{\mu}_0$$

- Maximal twist $\omega = \pi/2$

$$\mathbf{m}_0 = \mathbb{1} \mathbf{m}_{\text{cr}} \quad \boldsymbol{\mu}_0 = \text{diag}(\mu_l, -\mu_l, -\mu_s, +\mu_c)$$

Advantages

- Automatic $O(a)$ improv.
- $\mu_R = Z_P^{-1} \mu_0$
- Renormalised decay const.

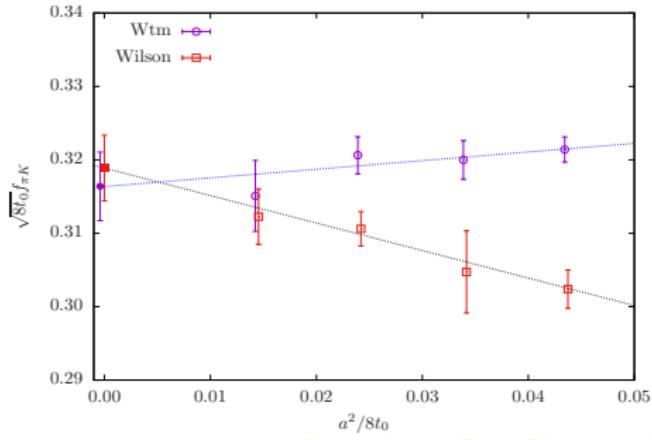
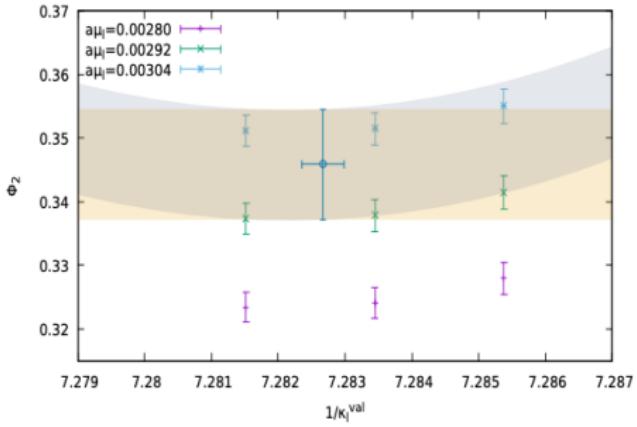
Drawbacks - $O(a)$

- $O(ag_0^4 \text{tr } M_{\text{sea}})$ effects
- \cancel{P}
- \cancel{X}

Matching conditions [1711.06017, 1812.05458, 1903.00286]

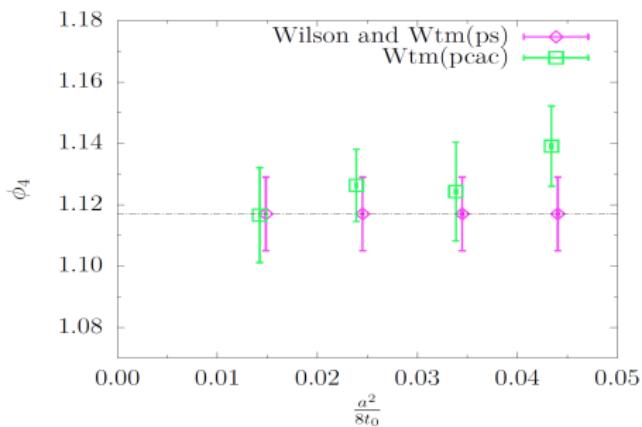
Talks by: A. Sáez: scale setting and matching ◦ G. Herdoíza: light quark masses

- Achieve **maximal twist** imposing standard light quark mass to vanish
- Match the sea and valence quark masses to recover unitarity
 - Consider a tuning grid in the $(\kappa_l|_v, \mu_{0,l}, \mu_{0,s})$ hyperplane
 - Impose $\Phi_2|_v = 8t_0 M_\pi^2 = \Phi_2|_s$, $\Phi_4|_v = 8t_0(M_K^2 + \frac{1}{2}M_\pi^2) = \Phi_4|_s$

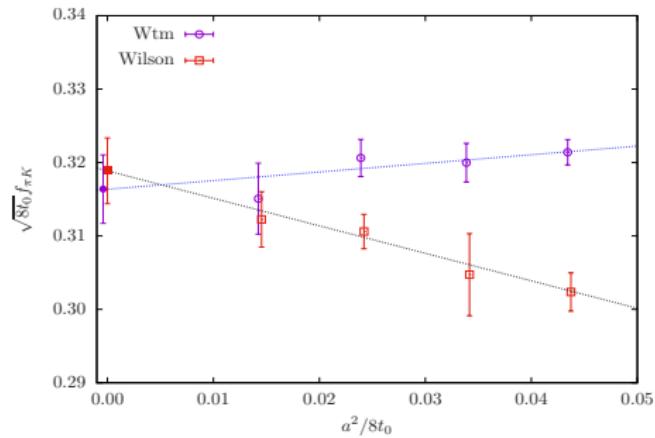


Continuum-limit scaling

- $\phi_4 \equiv 8t_0(\frac{1}{2}M_\pi^2 + M_K^2)$



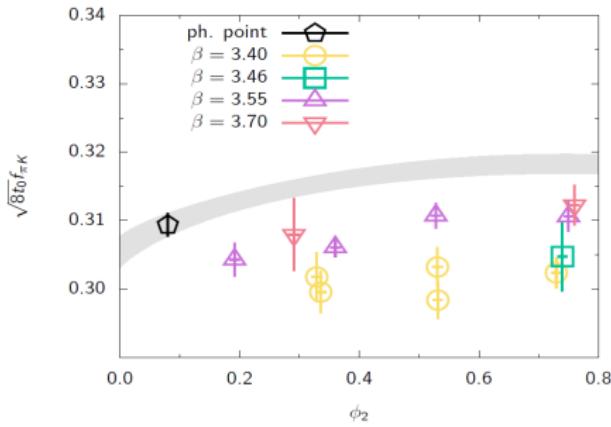
- $f_{\pi K} \equiv \frac{2}{3}(\frac{1}{2}f_\pi + f_K)$



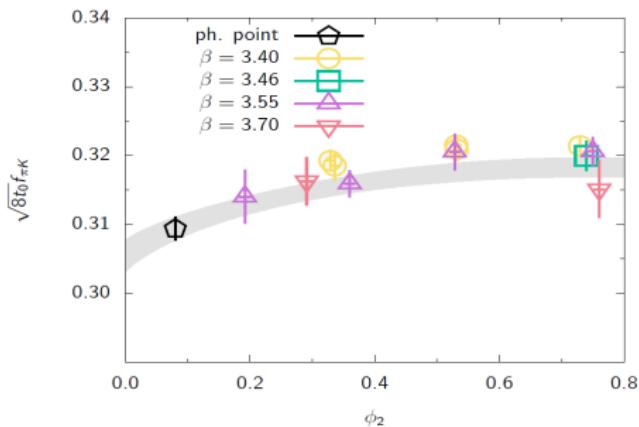
Symmetric point ensemble: $m_l = m_s$, $M_\pi = M_K = 420\text{MeV}$

Chiral-continuum extrapolation of $f_{\pi K}$

Wilson

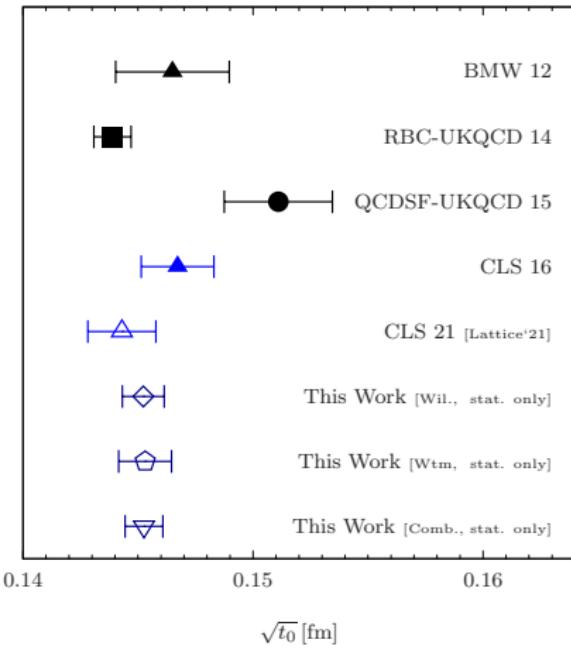


Wilson twisted mass



- Ongoing study on systematic effects:
 - chiral extrapolation
 - excited states contamination
 - discretization effects

Scale setting [2112.06696] PRELIMINARY



- uncertainty in $\sqrt{t_0} \sim 1\%$
- physical input: M_π, M_K, f_π, f_K [FLAG & PDG]
- further studies of the systematics ongoing

The Generalized Eigenvalue Problem

[M. Lüscher, U. Wolff, Nucl. Phys. B339 (1990) 222-252]

- Consider a set of operators $\{\hat{O}_i\}$, then combine different interpolators to get a matrix of Euclidean space correlation functions

$$C_{ij}(t) := \langle O_i(t) O_j^\dagger(0) \rangle = \sum_{n=1}^{\infty} e^{-E_n t} \psi_{ni} \psi_{nj}^*, \quad i, j = 1, \dots, N$$
$$\psi_{ni} \equiv (\psi_n)_i = \langle 0 | \hat{O}_i | n \rangle \quad E_n < E_{n+1}$$

- The GEVP is defined as

$$C(t)v_k(t, t_0) = \lambda_k(t, t_0) C(t_0)v_k(t, t_0), \quad k = 1, \dots, N, \quad t > t_0$$

Assuming that only N states contribute:

$$\lambda_k^{(0)}(t, t_0) = e^{-E_k(t-t_0)}, \quad v_k(t, t_0) \rightsquigarrow \psi_m$$

GEVP set-up

- We consider the following interpolating fields of different Dirac structures in the twisted basis

$$P = \bar{\psi} \gamma_5 \psi \quad A_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi, \quad \mu = 0, \dots, 3$$

$$C_{PP}(t) = \begin{bmatrix} \langle P(t)P(0) \rangle & \langle P(t+\tau)P(0) \rangle \\ \langle P(t)P(-\tau) \rangle & \langle P(t+\tau)P(-\tau) \rangle \end{bmatrix} \quad C_{VV}(t) = \begin{bmatrix} \langle A_k(t)A_k(0) \rangle & \langle A_k(t+\tau)A_k(0) \rangle \\ \langle A_k(t)A_k(-\tau) \rangle & \langle A_k(t+\tau)A_k(-\tau) \rangle \end{bmatrix}$$

- Then we solve the associated GEVP for the pseudoscalar-pseudoscalar and vector-vector matrix of correlators

$$C_{PP}(t)v_l^P(t, t_0) = \lambda_l^P(t, t_0)C_{PP}(t_0)v_l^P(t, t_0)$$

$$C_{VV}(t)v_l^V(t, t_0) = \lambda_l^V(t, t_0)C_{VV}(t_0)v_l^V(t, t_0)$$

Observables

- Observables extracted by solving the GEVP

$$C(\textcolor{blue}{t})v_k(t, t_0) = \lambda_k(t, t_0)C(t_0)v_k(t, t_0), \quad k = 1, \dots, N, \quad t > t_0$$

where

$$C_{ij}(t) := \langle O_i(t)O_j^\dagger(0) \rangle = \sum_{n=1}^{\infty} e^{-E_n t} \psi_{ni} \psi_{nj}^*, \quad i, j = 1, \dots, N$$

$$P = \bar{\psi} \gamma_5 \psi \quad A_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi, \quad \mu = 0, \dots, 3$$

Effective masses extracted from eigenvalues $\lambda_k(t, t_0)$

$$aE_n^{\text{eff}}(t, t_0) = \log \frac{\lambda_n(t, t_0)}{\lambda_n(t+a, t_0)} + O(e^{-(E_m - E_n)t})$$

Matrix elements extracted from eigenvectors $v_k(t, t_0)$

$$f_{ps} = \sqrt{\frac{2}{m_p^3 L^3}} (\mu_q + \mu_c) |\langle 0 | P^{c,q} | ps \rangle|, \quad f_v \epsilon_k = Z_A \sqrt{\frac{2}{m_v L^3}} |\langle 0 | A_k | v \rangle|$$

More on matrix elements

[Alpha Collaboration, hep/lat0902.1265v2]

Matrix elements are extracted from the effective operator

$$\hat{\mathcal{A}}_n^{\text{eff}}(t, t_0) = e^{-\hat{H}t} \hat{\mathcal{Q}}_n^{\text{eff}}(t, t_0), \quad |n\rangle = \hat{\mathcal{A}}^\dagger |0\rangle, \quad \hat{H} |n\rangle = E_n |n\rangle$$

$$\hat{\mathcal{Q}}_n^{\text{eff}}(t, t_0) = R_n \left(\hat{O}, v_n(t, t_0) \right)$$

$$R_n = (v_n(t, t_0), C(t)v_n(t, t_0))^{-1/2} \frac{\lambda_n(t_0 + t/2, t_0)}{\lambda_n(t_0 + t, t_0)}$$

Corrections to the large time asymptotic behaviour are parametrized by

$$e^{-\hat{H}t} \hat{\mathcal{Q}}_n^{\text{eff}}(t, t_0)^\dagger |0\rangle = |n\rangle + \sum_{n'=1}^{\infty} \pi_{nn'}(t, t_0) |n'\rangle$$

More on matrix elements

[Alpha Collaboration, hep/lat0902.1265v2]

They can show that

$$\pi_{nn'} = O(e^{-\Delta E_{N+1,n} t_0}) \quad \text{at } t - t_0 = \text{const}$$

Thus, matrix elements of a local operator \hat{X} can be computed via

$$\langle 0 | \hat{Q}_n^{\text{eff}} e^{-\hat{H}t} \hat{X} e^{-\hat{H}t} (\hat{Q}_n^{\text{eff}})^\dagger | 0 \rangle = \langle n | \hat{X} | n' \rangle + O(e^{-\Delta E_{N+1,n} t_0})$$

And the amplitude between a vacuum and the state $|n\rangle$ is then given by

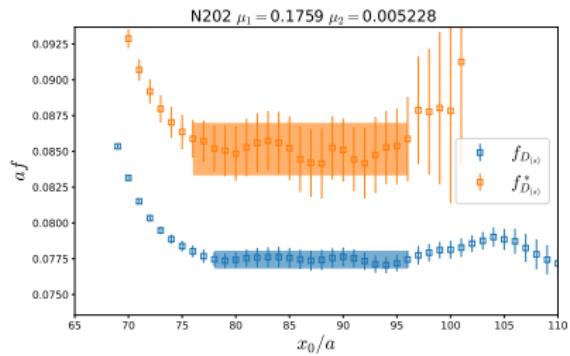
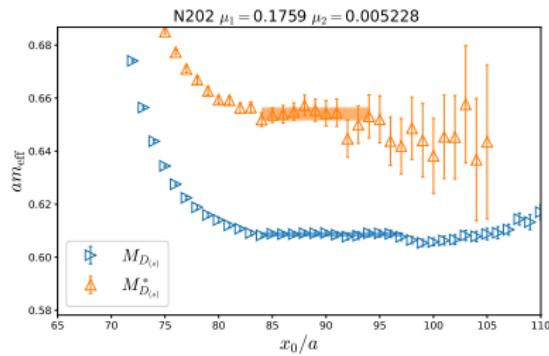
$$p_n^{\text{eff}} = \langle 0 | \hat{Q}_n^{\text{eff}} e^{-\hat{H}t} \hat{X} | 0 \rangle = R_n(v_n(t, t_0), C_X), \quad (C_X)_j = \langle O_j(0) X(t) \rangle$$

If \hat{X} denotes the time component of an axial current, the decay constant of the associated ground state meson is

$$p_1^{\text{eff}}(t, t_0) = \langle 0 | \hat{X} | 1 \rangle$$

Charmed meson masses and decays

- $M_\pi = M_K \approx 420$ MeV



Mass corrections and line of constant physics

[M. Bruno, T. Korzec, S. Schaefer, 1608.08900]

- Chiral trajectory defined in terms of $\Phi_4 \equiv 8t_0^2(m_K^2 + \frac{1}{2}m_\pi^2)$
- Achieve $\Phi_4 = \Phi_4^{\text{phys}}$ in each ensemble by employing quark mass shift via a Taylor expansion

$$\frac{d\langle \mathcal{O} \rangle}{dm_{q,i}} = \left\langle \frac{\partial \mathcal{O}}{\partial m_{q,i}} \right\rangle - \left\langle \mathcal{O} \frac{\partial \mathcal{S}}{\partial m_{q,i}} \right\rangle + \langle \mathcal{O} \rangle \left\langle \frac{\partial \mathcal{S}}{\partial m_{q,i}} \right\rangle$$

- The shift is performed at the level of the observables by

$$\langle \mathcal{O} \rangle \rightarrow \langle \mathcal{O} \rangle + \sum_{i=1}^{N_f} \Delta m_{q,i} \frac{d\langle \mathcal{O} \rangle}{dm_{q,i}}$$

Model average

- Information Criteria à la Akaike (IC) as measure for the best fit

[ALPHA: JHEP 05 (2021) 288]

$$\text{IC}(M_i) = \frac{\chi_i^2}{\chi_{i,\text{exp}}^2} (N - k_i) + 2k_i + \frac{2k_i^2 + 2k_i}{N - k_i - 1}, \quad w_i^{\text{IC}} \propto e^{(-\frac{1}{2}\text{IC}(M_i))}$$

Model average [Jay, Neil: Phys.Rev.D 103 (2021) 114502]

$$\langle M_c \rangle = \sum_{i=1}^M w_i \langle M_c \rangle_i$$

Estimate the systematics

$$\sigma_{M_c}^2 = \sum_{i=1}^M w_i \langle M_c \rangle_i^2 - \left(\sum_{i=1}^M w_i \langle M_c \rangle_i \right)^2$$

Heavy mass dependence of M_c

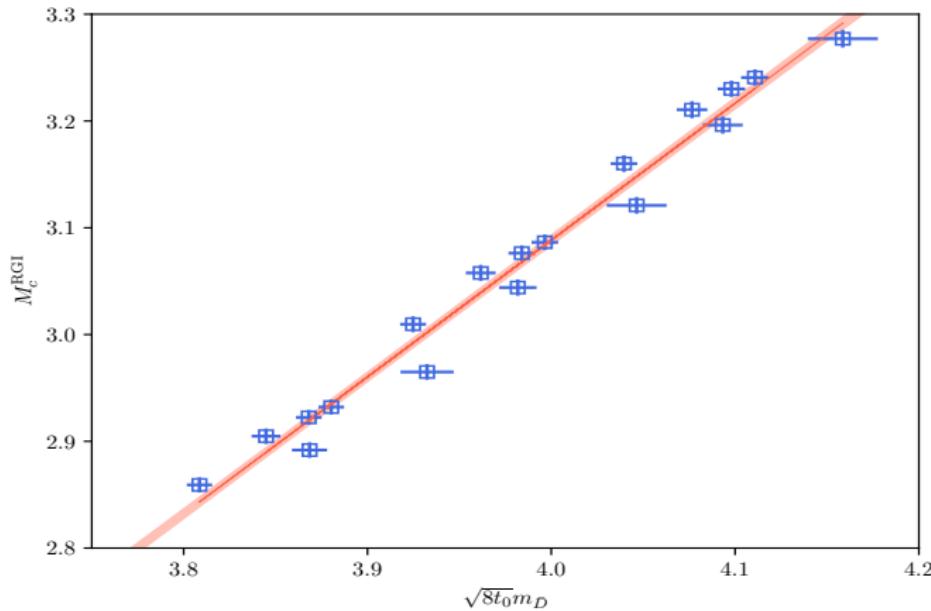


Figure: Charm mass dependence on the dimensionless heavy meson mass $\phi_H = \sqrt{8t_0}m_H$.

M_c^{RGI} error contributions

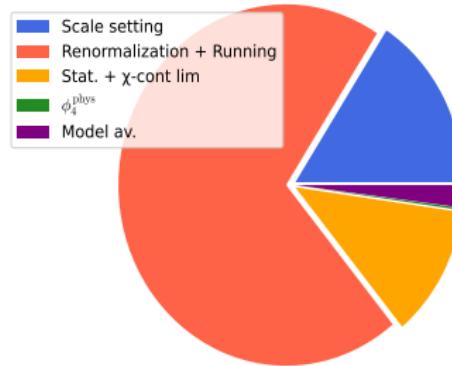
- **ADerrors.jl** by Alberto Ramos <https://gitlab.ift.uam-csic.es/alberto/aderrors.jl>

MC data analysis with:

- Γ -method [Wolff, hep-lat/03060174; Bruno, Sommer, in preparation]
- Automatic Differentiation [A. Ramos, 1809.01289]
- $\chi^2_{\text{exp.}}$ fitting routine [Bruno, Sommer, in preparation]

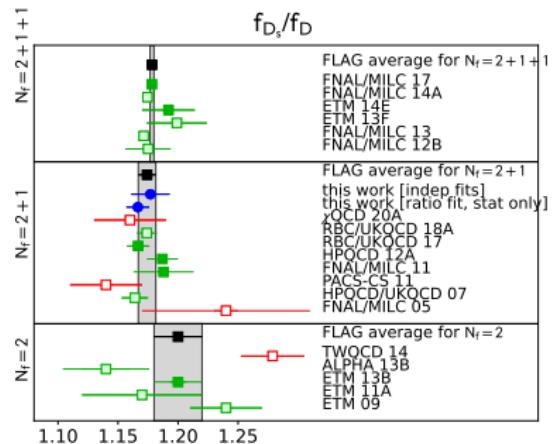
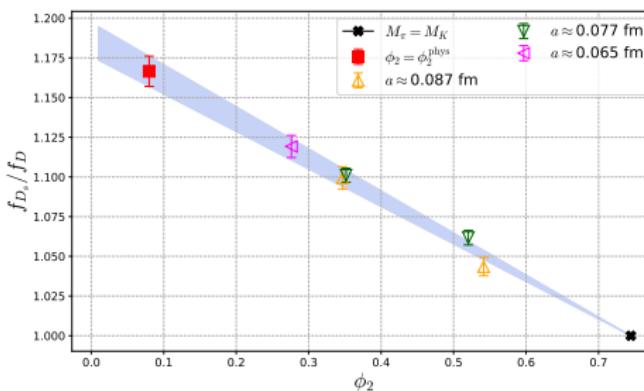
- **Juobs.jl** by J. Ugarrio and AC <https://gitlab.ift.uam-csic.es/jugarrio/juobs>

$$M_c^{\text{RGI}}(N_f=3)$$



f_{D_s}/f_D results PRELIMINARY

$$\frac{f_{D_s}}{f_D}(a, \phi_2) = 1 + p_1 \left(\phi_4 - \frac{3}{2} \phi_2 \right) + c_1 \frac{a^2}{8t_0}, \quad \chi^2/\chi^2_{\text{exp}} = 0.51$$



from fit : $f_{D_s}/f_D = 1.1665(95)_{\text{stat}}$.

from ratio : $f_{D_s}/f_D = 1.177(16)$

Decay constants - functional forms

$$f_D = p_0 + p_1 \phi_2 + \frac{p_2}{\sqrt{\phi_H}} + \textcolor{red}{p}_3 \left(3\mu_\pi + 2\mu_K + \frac{1}{3}\mu_\eta \right)$$

$$f_{D_s} = p_0 + 2p_1(\phi_4 - \phi_2) + \frac{p_2}{\sqrt{\phi_H}} + \textcolor{red}{p}_3 \left(4\mu_K + \frac{4}{3}\mu_\eta \right)$$

$$\mu_\pi = \phi_2 \log(\phi_2)$$

$$\mu_K = \left(\phi_4 - \frac{1}{2}\phi_2 \right) \log \left(\phi_4 - \frac{1}{2}\phi_2 \right)$$

$$\mu_\eta = \left(\frac{4}{3}\phi_4 - \phi_2 \right) \log \left(\frac{4}{3}\phi_4 - \phi_2 \right)$$