A Quenched Exploration of Heavy Quarks Moments and their Perturbative Expansion

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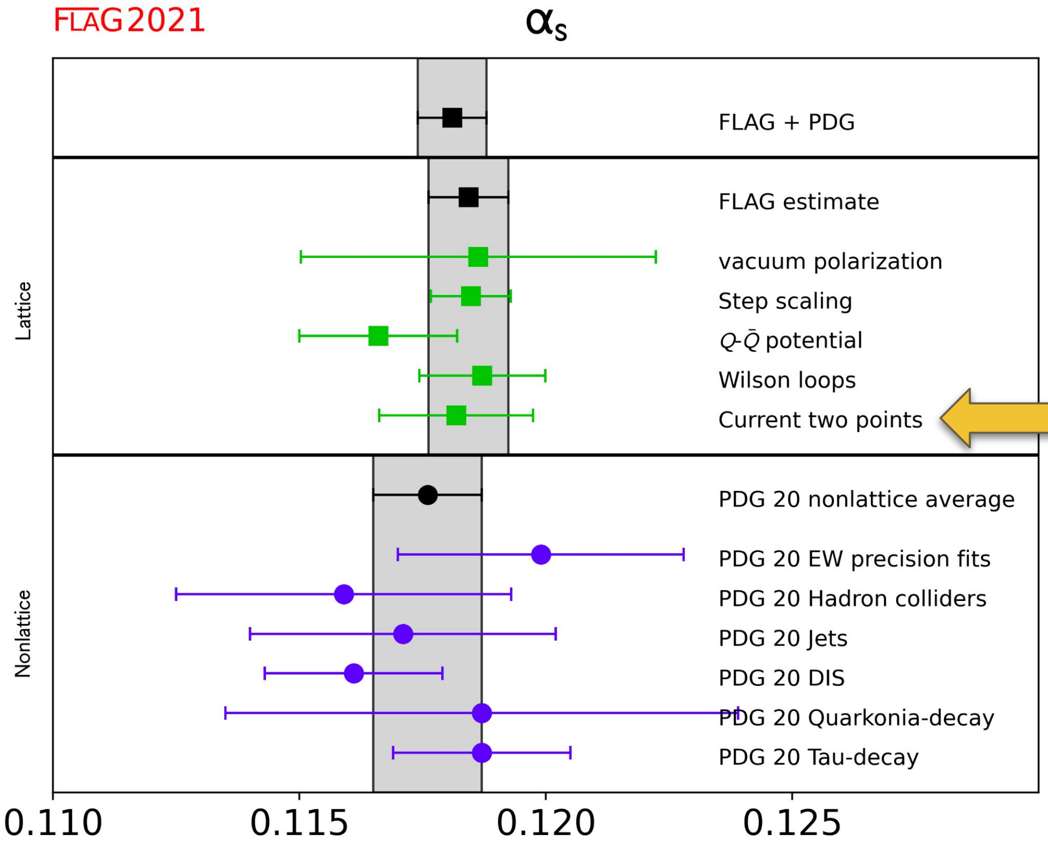
Thanks to N. Husung, T. Korzec,
S. Schaefer, B. Strassberger.

The Strong Coupling



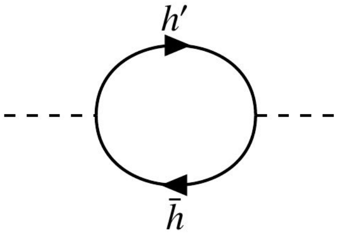
Study **systematics** of **“moments method”** in quenched theory

Definition of Moments

- Moments method, pioneered by Bochkarev, de Forcrand [*hep-lat/9505025*] and HPQCD in 2008 [*hep-lat/0805.2999*].
- The observables are derivatives of the vacuum polarization with heavy quarks (*h, h'*) at CoM energy $q^2 = 0$.
- $m \leftrightarrow$ scale of observable.

$$\Pi(q^2, m) = i \int \mathrm{d}^4x\, e^{iq\cdot x} \langle 0| \mathscr{T} \left\{ J^\dagger(x,m) J(0,m) \right\} |0\rangle$$

$$\mathcal{M}_n(m) = \frac{1}{n!} \left(\frac{\partial}{\partial q^2} \right)^n \Pi(q^2, m) \bigg|_{q^2=0} \qquad [\mathcal{M}_n] = \mathrm{En.}^{4-n}$$



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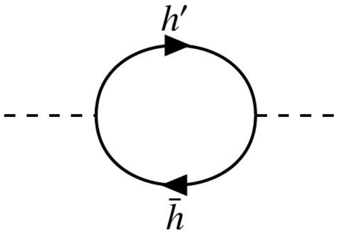

Object known to high (4-loop) orders in <u>PT</u>
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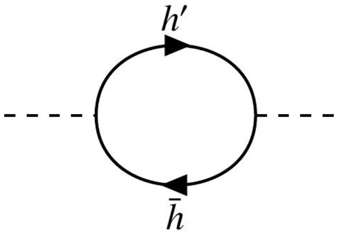

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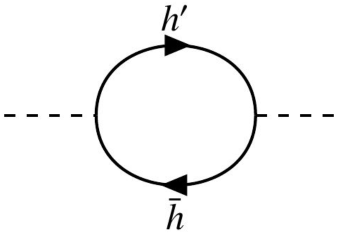

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★ At maximal twist: $Z_P Z_\mu = 1$

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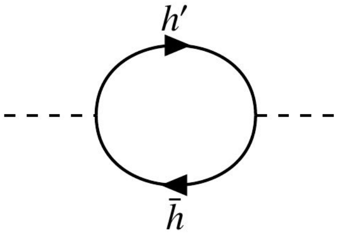

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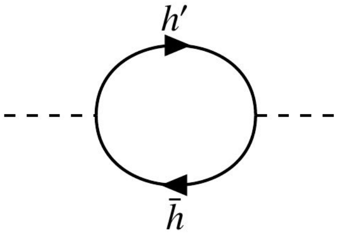

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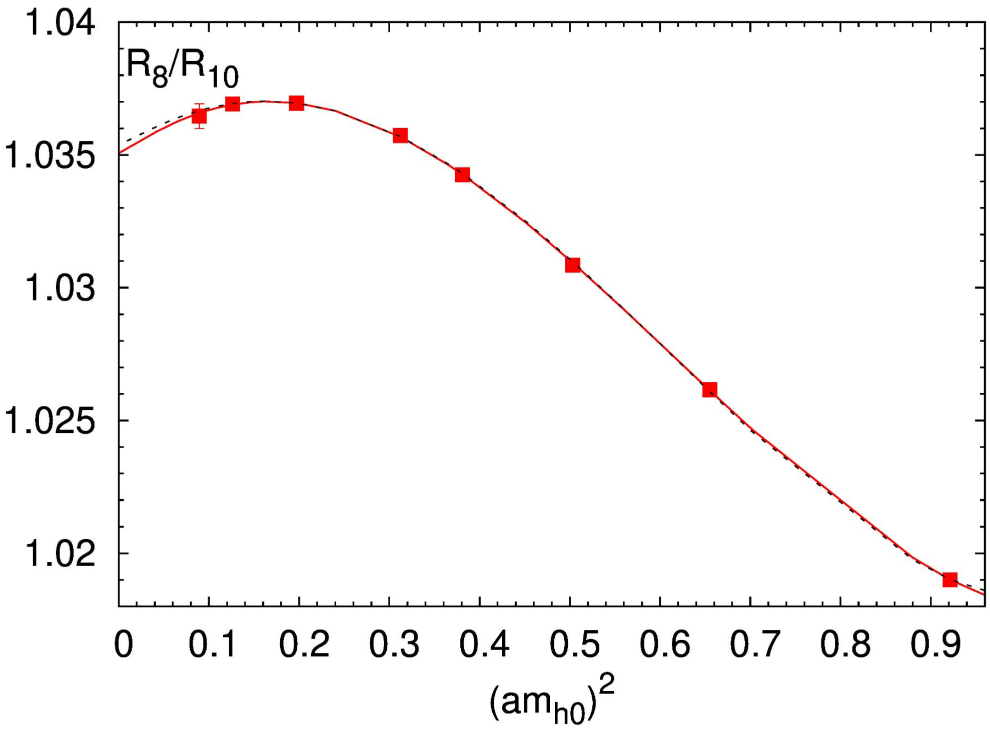
$$G(t) \underset{t\to 0}{\sim} \frac{1}{|t|^3} \implies \text{for } n > 3 \quad \exists \lim_{t\to 0} \left\{ G(t)\, t^n \right\} \Rightarrow \boxed{n = 4,\ 6,\ 8,\ 10, \ldots}$$

Disclaimer: Study the Systematics

Different discretization: (highly improved) staggered quarks

- N_f=2+1
- M=2M_c

Petreczky, Weber, *arXiv:hep-lat/1901.06424*



- ❑ Very <u>difficult extrapolation</u>, no range with just ~ a^2 behavior
- ❑ Do quenched study, configurations are pure Yang Mills and smaller a can be reached

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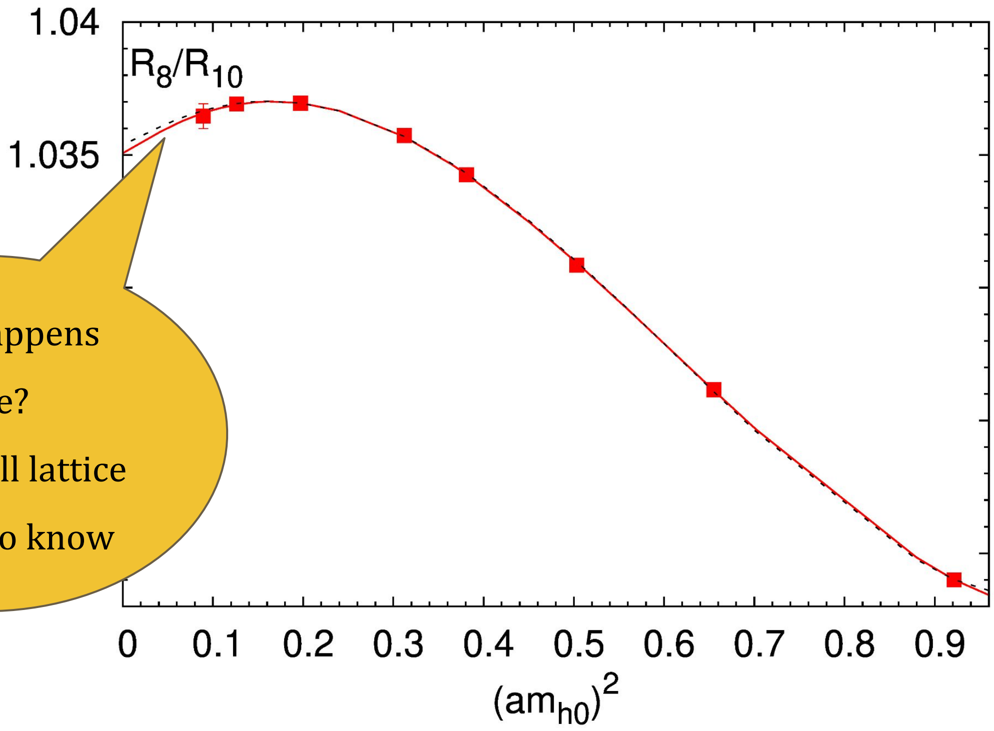

What happens here?

Need small lattice spacing to know

Lattice Setup

1. **Plaquette** gauge action.
2. P.b.c. in space, **open b.c. in time** to avoid **frozen topological charge** at small *a*.
3. Full twist doublet, with **non-perturbative c_{sw}** to reduce cutoff effects.
4. Stochastic evaluation of trace and sum over space with **U(1) noise sources**.
5. **Source** placed at **1 fm from boundary**, checked absence of boundary effects:

   Physical **Volume L≃2 fm**, time direction about **T≃6 fm**.
6. Full twist, set **κ to its critical value** [1].
7. Autocorrelation analysis done with **Γ-method**.
8. Scale set through **gradient flow** t_0 [2].
9. Line of constant physics: fix M_{RGI} in t_0 units, *Z* and running factors from [3].

[1] Lüscher, Sint, Sommer, Weisz, Wolff. [*arXiv:hep-lat/9609035*]
[2] Lüscher, [*arXiv:hep-lat/1006.4518*]
[3] Capitani, Lüscher, Sommer, Wittig, [*hep-lat/9810063]*

Measurements

Run Name	β	$l^3 \times t$	N_{cnfg}	t_0/a^2	$a[fm]$	$\tau_{\text{int}}(t_0)[\text{cfg}]$
q_beta616	6.1628	$32^3 \times 96$	128	5.1604(98)	0.071	0.78
q_beta628	6.2885	$36^3 \times 108$	137	7.578(22)	0.059	1.37
q_beta649	6.4956	$48^3 \times 144$	109	13.571(50)	0.044	1.55
sft4	6.7859	$64^3 \times 192$	200	29.390(98)	0.030	1.00
sft5	7.1146	$96^3 \times 320$	80	67.74(23)	0.020	0.55
sft6	7.3600	$128^3 \times 320$	98	124.21(91)	0.015	1.03
sft7	7.700	$192^3 \times 480$	31	286.3(4.7)	0.010	–

Gauge run details, $l = L/a$, $t = T/a$.

[Large volume sft ensembles of unprecedented size from: Husung, Krah, Sommer *arXiv:hep-lat/1711.01860]*

Measure for a range of masses:

★ vary the mass → vary the scale of α → study variation of truncated part in PT

$$M/M_{\text{charm}} \simeq 3.48,\ 2.32,\ 1.55,\ 1.16,\ 0.77$$
$$M_{RGI} \simeq 5.75\,,3.83\,,2.56\,,1.92\,,1.28\,\text{GeV}$$

Normalization and Ratios

- ★ Divide by **analytical finite volume and *a* tree-level** (TL), evaluated at $m_* = \overline{m}_{\overline{\rm MS}}(m_*)$ to suppress cutoff effects.

$$R_n(\sqrt{8t_0}M_{RGI}, aM_{RGI}) = \begin{cases} \dfrac{\mathcal{M}_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{\mathcal{M}_n^{TL}(a\mu_{tm})}, & n = 4 \\ \left(\dfrac{\mathcal{M}_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{\mathcal{M}_n^{TL}(a\mu_{tm})}\right)^{\frac{1}{n-4}}, & n > 4 \end{cases}$$

- ★ For *n > 4* take ratios of moments:
  - get rid of strong mass dependence and mitigate some error sources: $[\mathcal{M}_n] = \mathrm{En.}^{4-n} \xrightarrow{n>4} [R_n] = \mathrm{En.}^{-1}$
  - invert equation below for *α* at a given scale $\mu_s = s\overline{m}_{\overline{\rm MS}}(\mu_s)$
  - varying M_{RGI} varies the scale at which the coupling is computed, changing the size of the truncated term
  - also varying *s* varies the size of the truncated term → other handle to turn.

$$\lim_{a\to 0} \frac{R_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{R_{n+2}(\sqrt{8t_0}M_{RGI}, aM_{RGI})} = 1 + \sum_{i=1}^{3} c_n^{(i)}(\mu/\overline{m}_{\overline{\rm MS}}(\mu))\,\alpha^i_{\overline{\rm MS}}(\mu) + O(\alpha^4)$$

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★ For $n > 4$

○ get

○ inv

○ var

○ also

***R*$_4$: (1) naturally adimensional, (2) the most short distance observable**

Has largest cutoff effects, partially understood: just discussed by Rainer **Sommer** in previous talk. See also plenary by **Husung** on Saturday 13th, 8:50 AM.

Here: let's look at results for **ratios of moments with n > 4**

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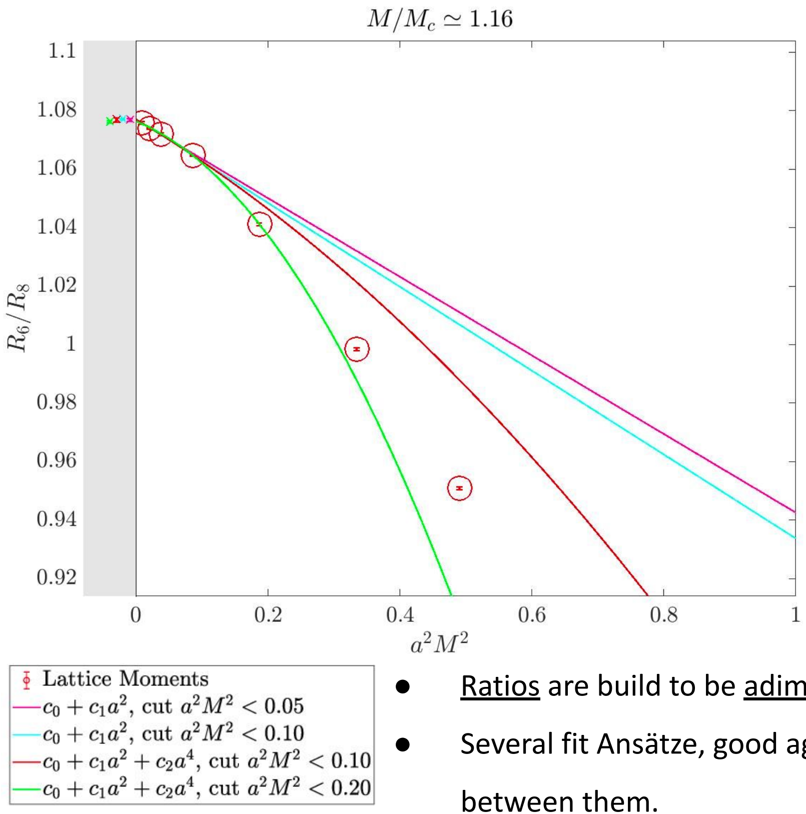
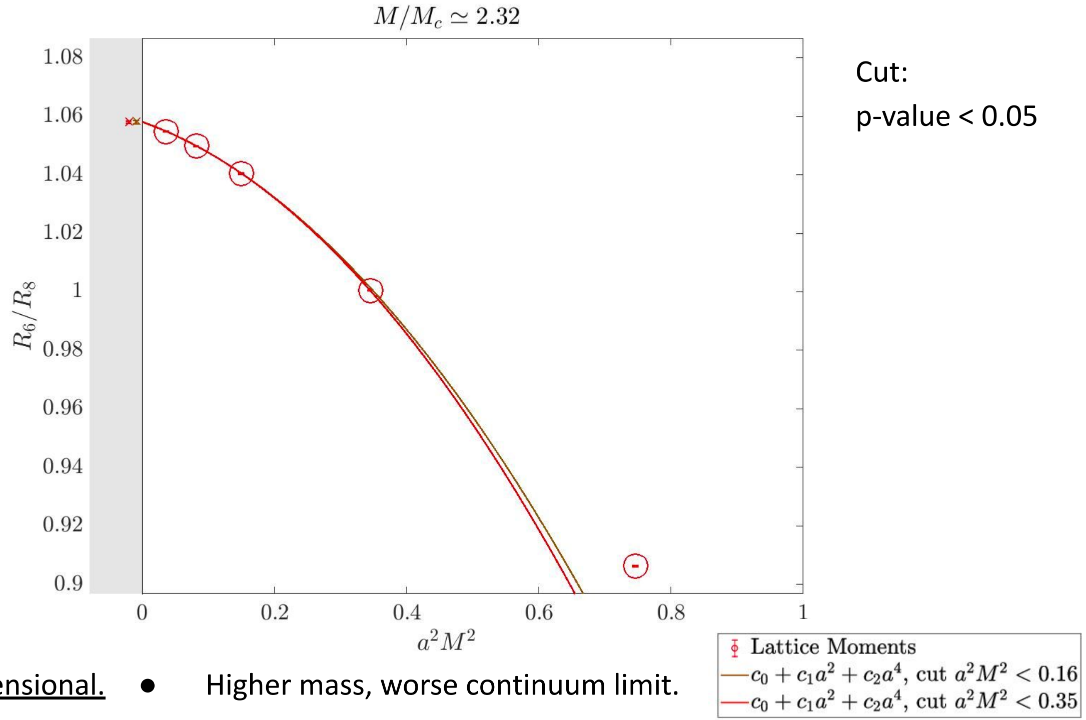
Continuum Extrapolations: R_6/R_8

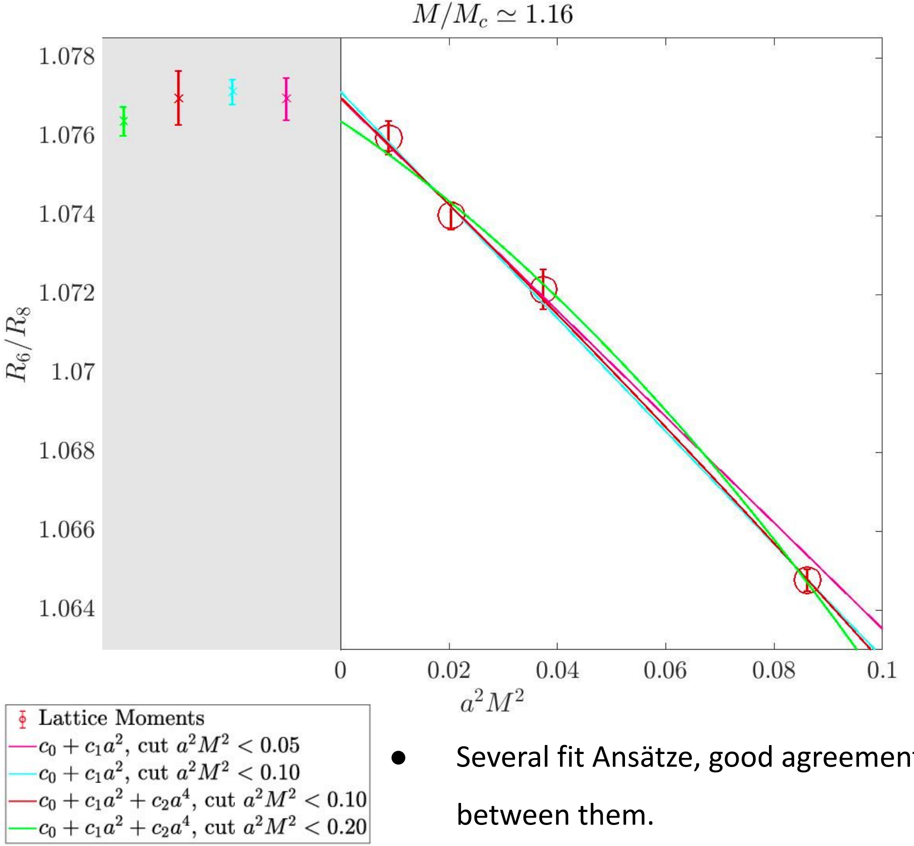




Cut:
p-value < 0.05

- Ratios are build to be adimensional.
- Several fit Ansätze, good agreement between them.
- Higher mass, worse continuum limit.
- Errors from tuning (Z_P, t_0/a^2, κ) computed using continuum PT for corresponding derivatives.

R_6/R_8 zoom in:



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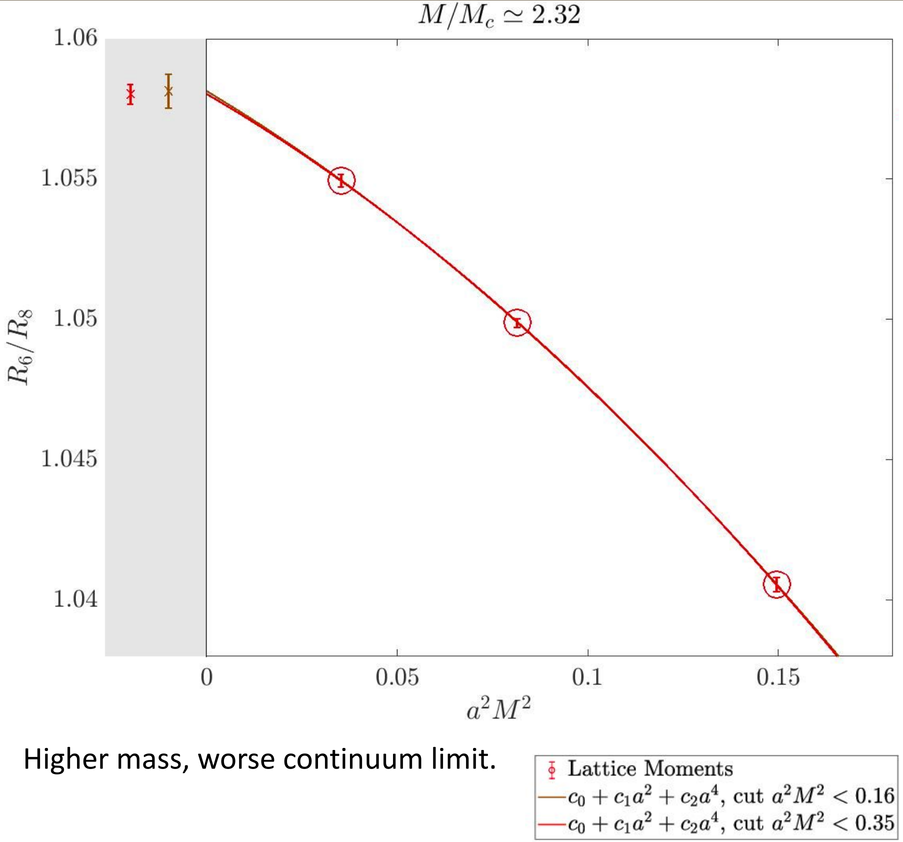

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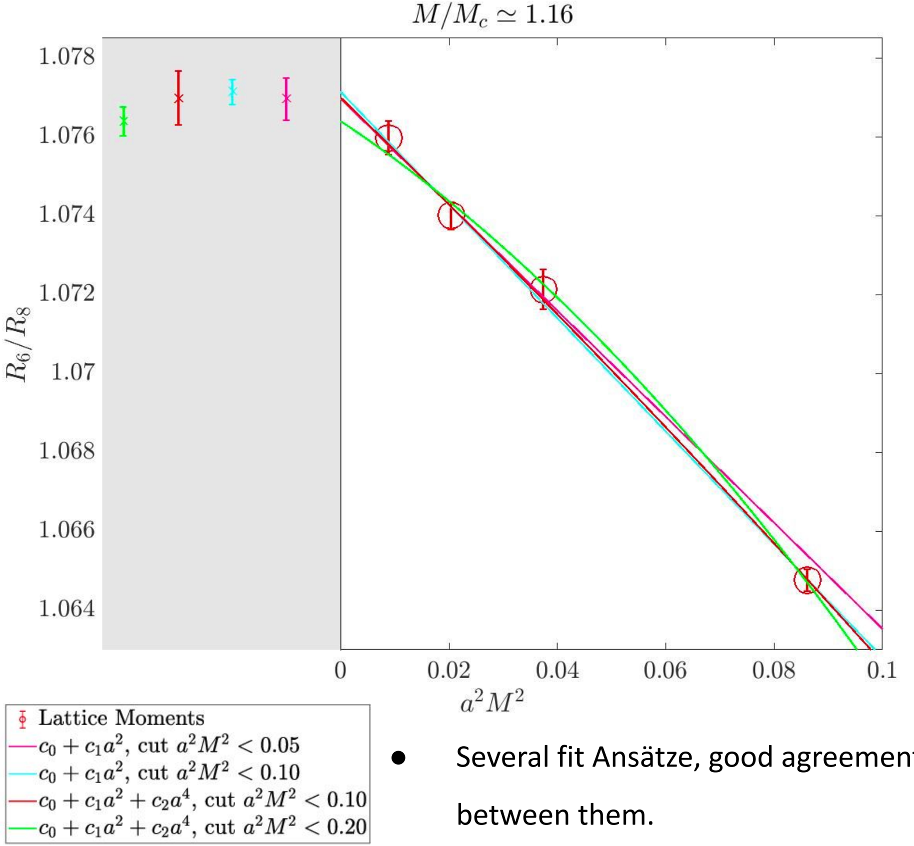

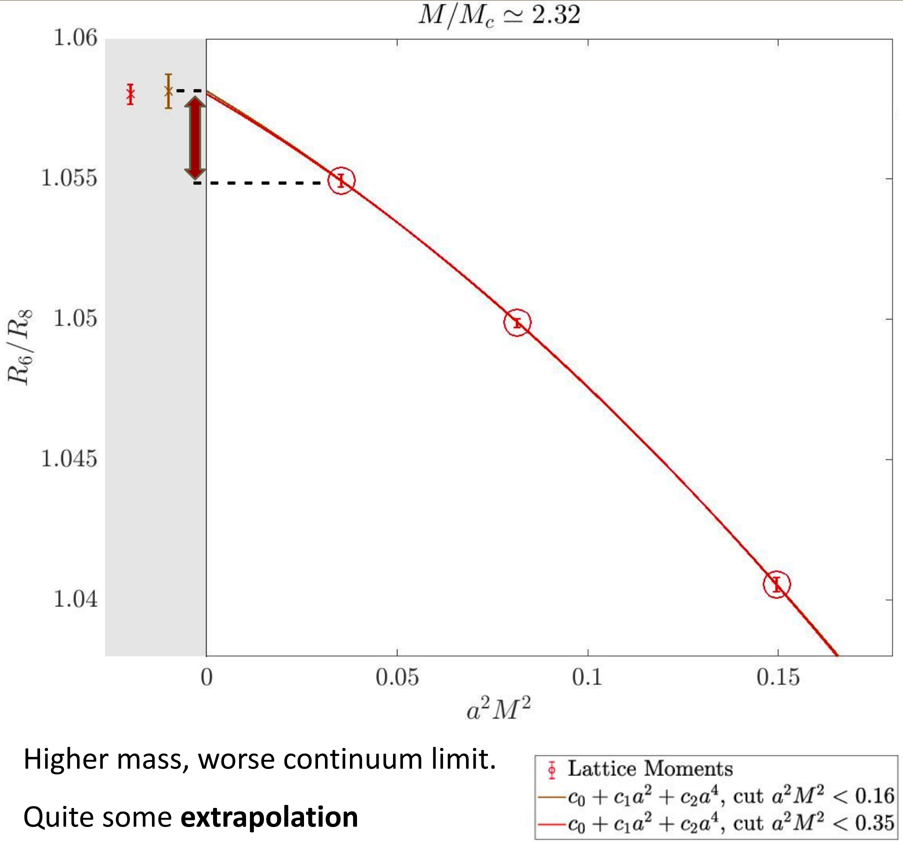

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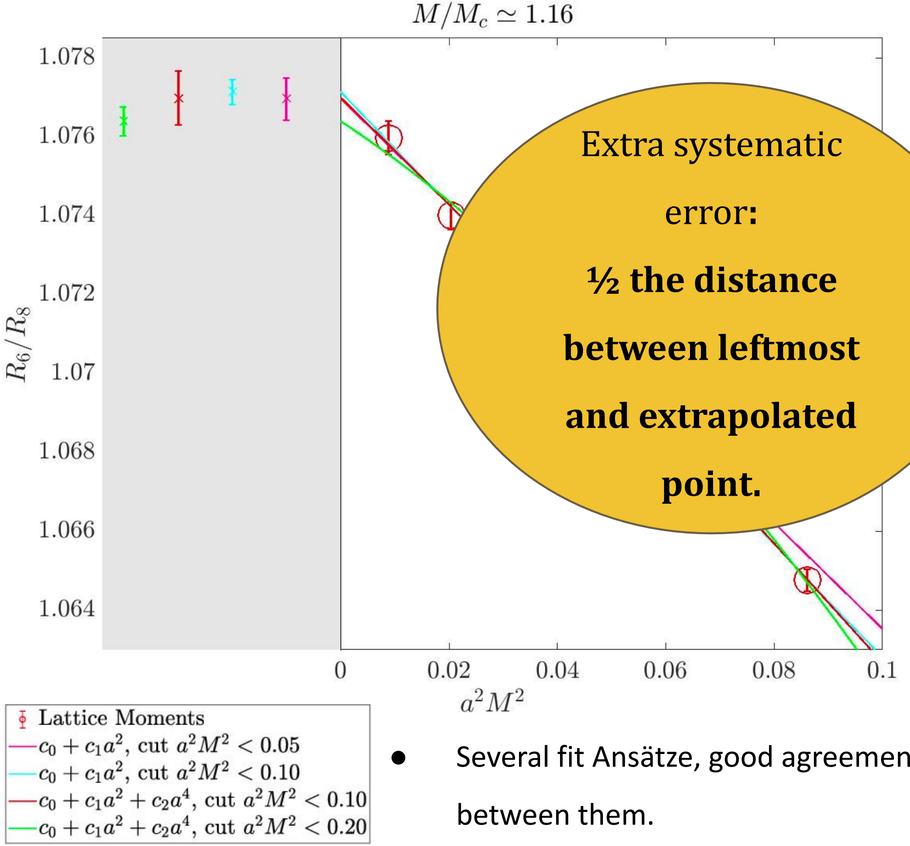
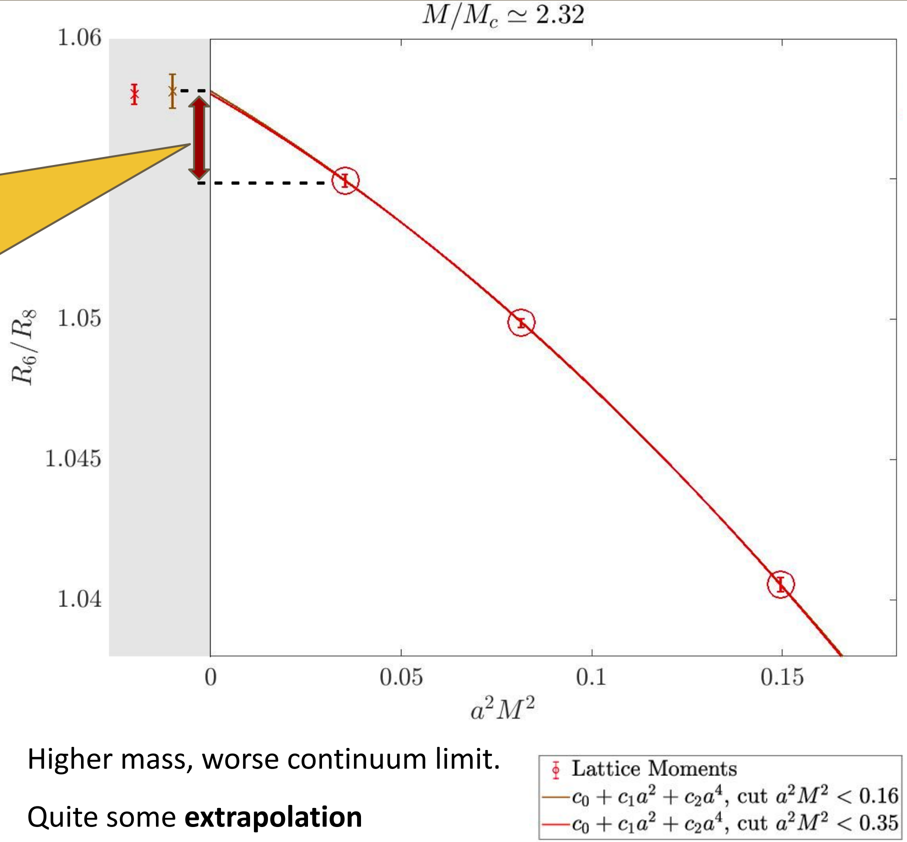


Extra systematic error:

½ the distance between leftmost and extrapolated point.



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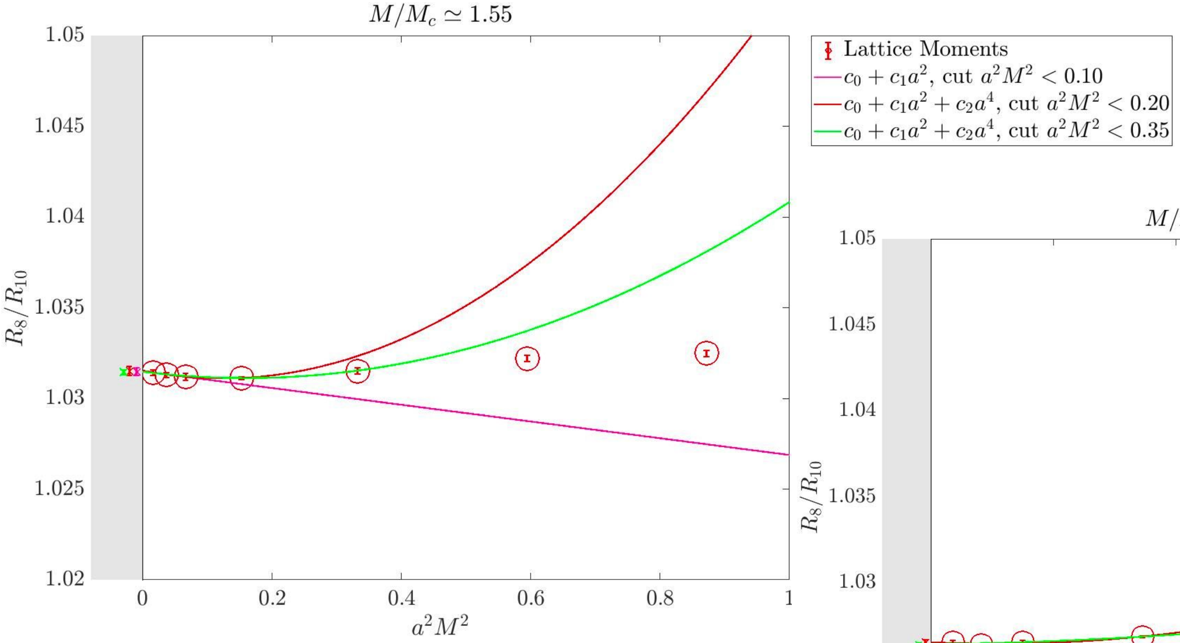

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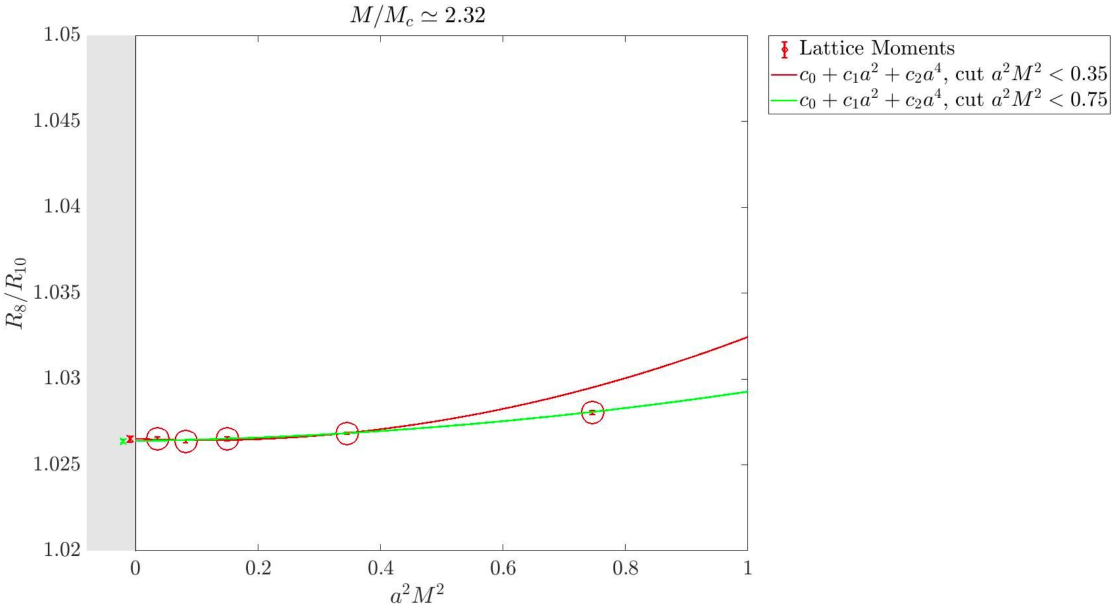
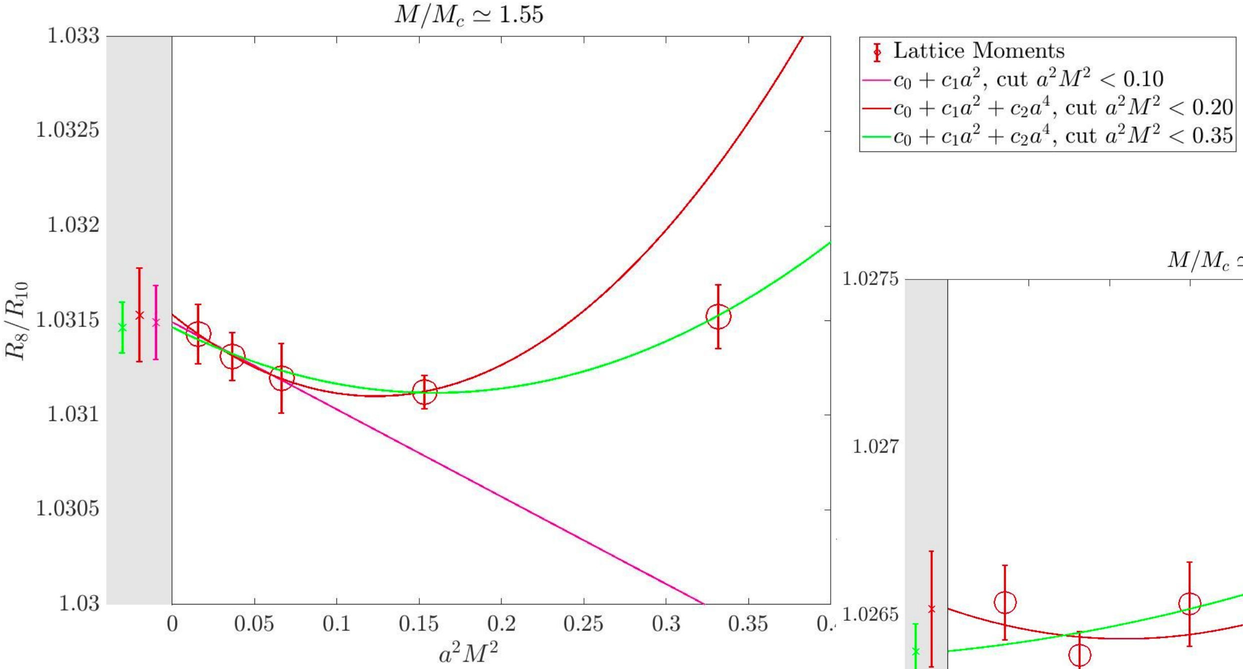
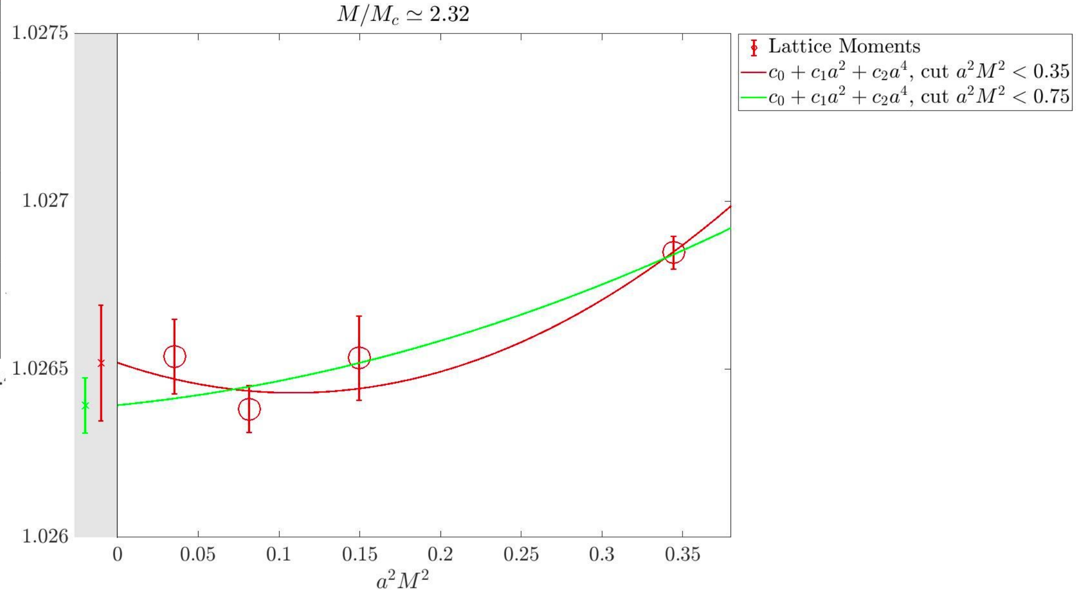


- Highest moments, **best continuum limit**.
- **Least perturbative**
- Coupling **very sensitive** to extrapolated value!

R_8/R_{10} zoom in:



- Coupling **very sensitive** to extrapolated value, signal for α is distance to 1!



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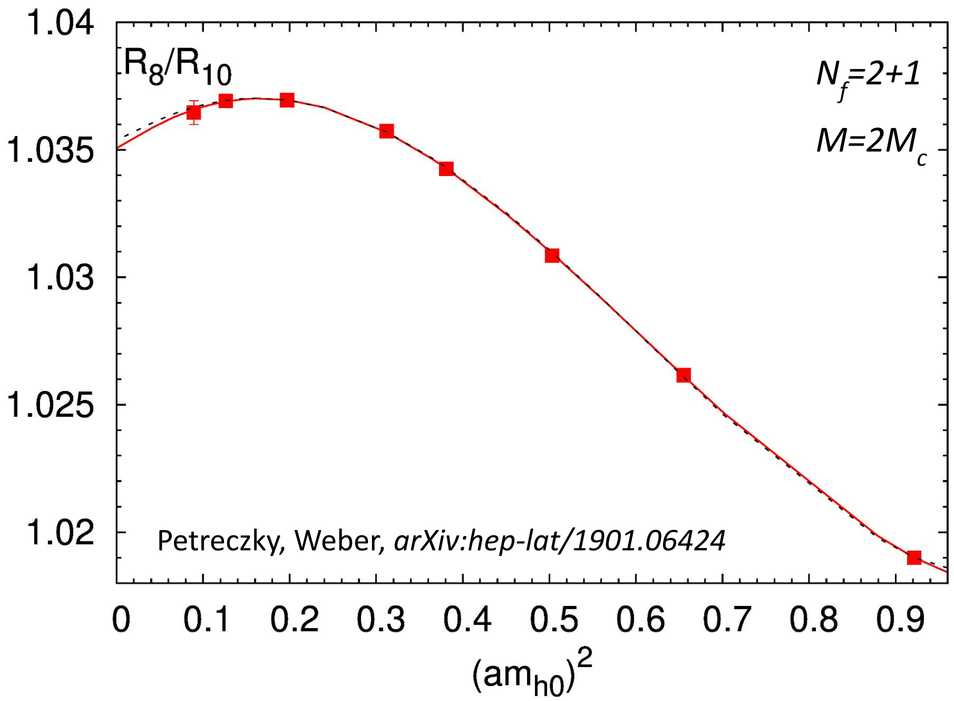
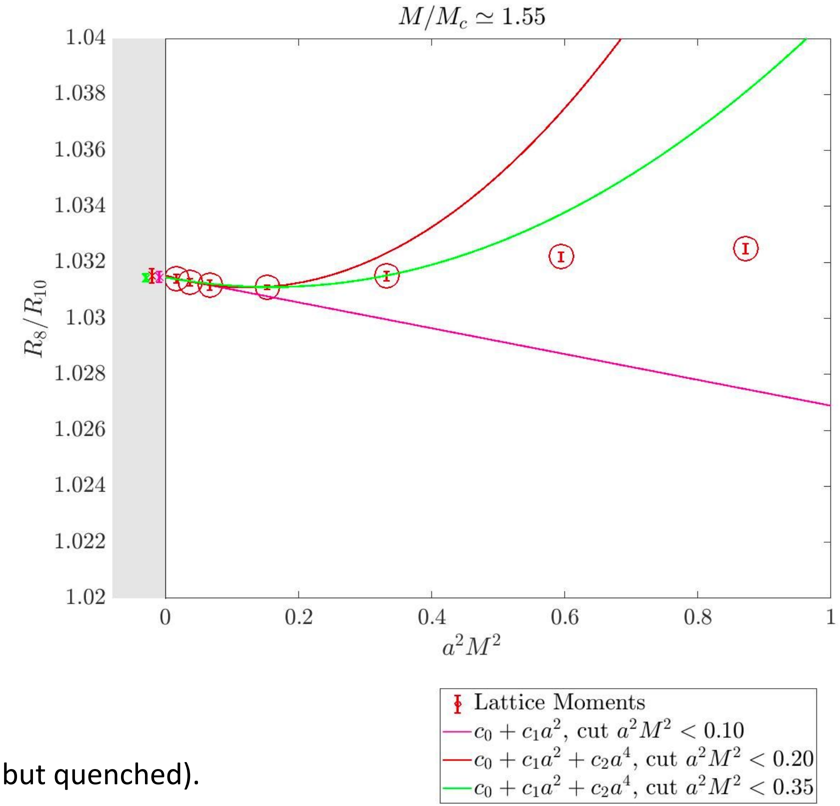
Comparison of Twisted Mass and Staggered





- Compared to other studies, less curvature and smaller lattice spacing (but quenched).

The Λ Parameter

- Observable does not depend on μ → in perturbative expansion there will leftover dependence due to truncation.

$$\mathcal{M}_n(m,\mu) \overset{\alpha\to 0}{\sim} \sum_{i=0}^{3} c_n^{(i)}(\mu/\overline{m}_{\overline{\rm MS}})\,\alpha^i_{\overline{\rm MS}}(\mu) + O(\alpha^4(\mu)),$$

Spurious μ dependence

Hard to estimate, systematic, scale dependent truncation error

$$\frac{\mathrm{d}\bar{g}}{\mathrm{d}\ln\mu} = \beta(\bar{g}(\mu)) \to \mathrm{d}\ln\mu = \int^{\bar{g}} \mathrm{d}x \frac{1}{\beta(x)} + C$$

$$\Lambda_{RGI} = \mu(b_0\bar{g}^2)^{-b_1/(2b_0^2)} \exp\left\{-\frac{1}{2b_0\bar{g}^2} - \int_0^{\bar{g}} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

- ❖ Contains info of coupling and running.
- ❖ Is an integration constant of RGEs.

$$\big(\alpha(\mu),\, m(\mu)\big) \longleftrightarrow \big(\Lambda_{RGI},\, M_{RGI}\big)$$

- Run to infinite energy via 5L β-function and 4L τ (mass anomalous dimension)
- Given $\alpha(\mu_s)$ and $z = \sqrt{8t_0}M_{RGI}$, we can compute Λ.

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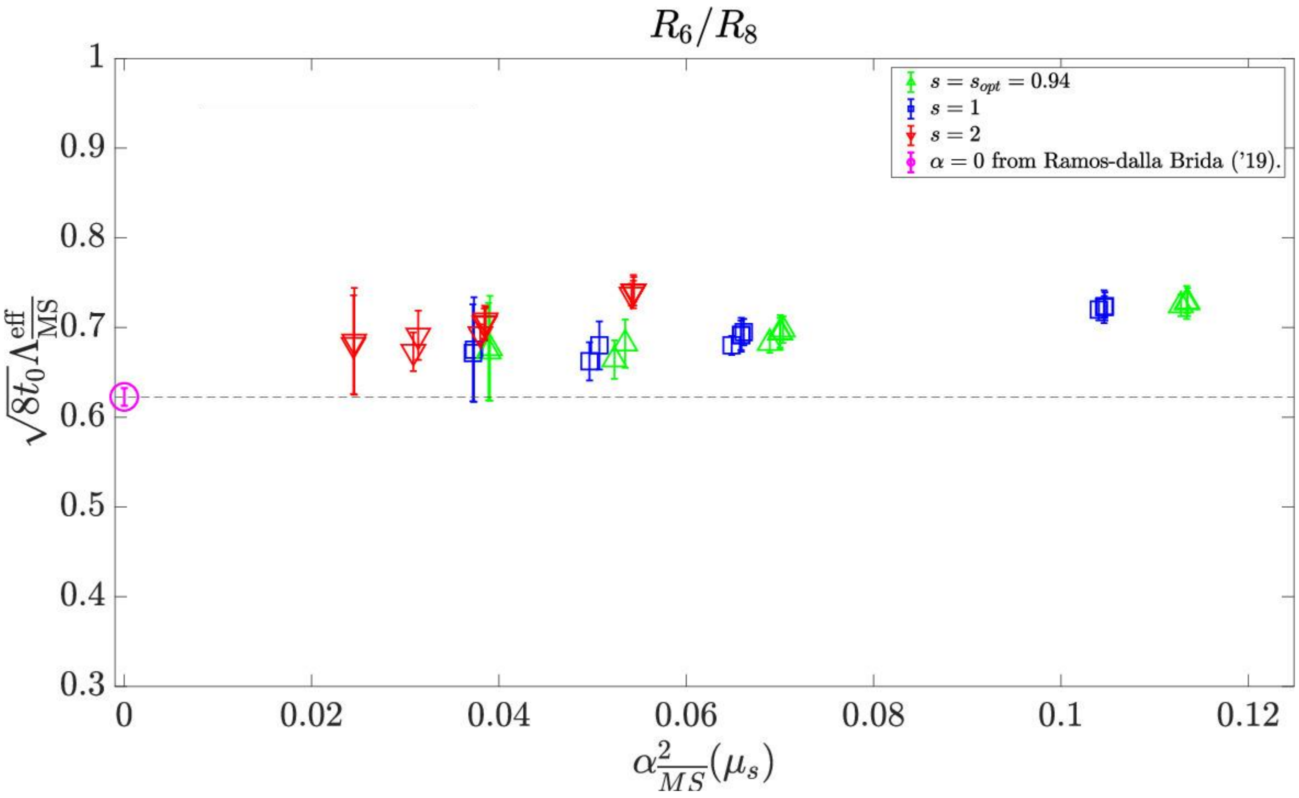
Λ Plot from R_6/R_8

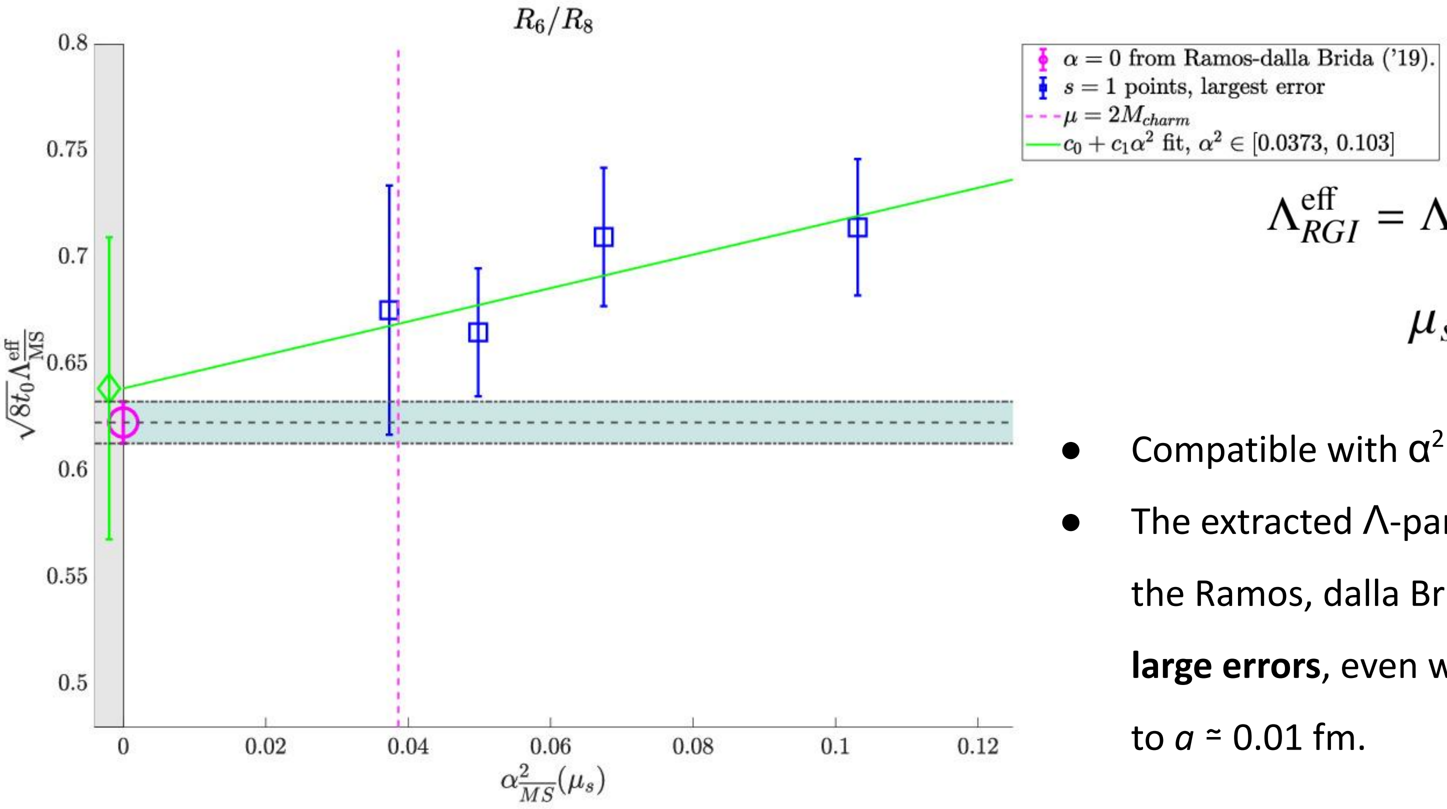


$$\Lambda^{\text{eff}}_{RGI} = \Lambda_{RGI} + \mathcal{O}\left(\alpha^2_{\overline{\text{MS}}}(\mu_s)\right)$$

$$\mu_s = s\bar{m}_{\overline{\text{MS}}}(\mu_s)$$

- **Largest mass dropped** here, not possible to take a continuum limit.

Λ Fit from R_6/R_8

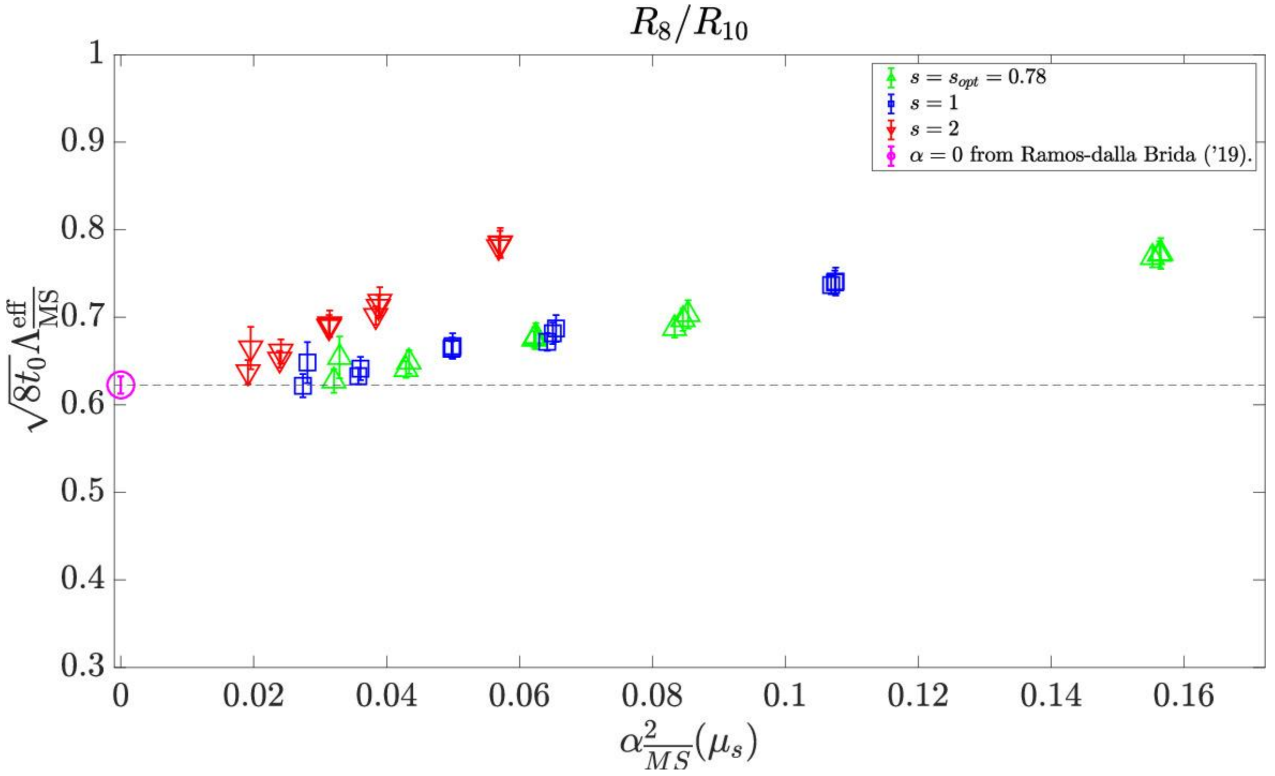


$$\Lambda^{\mathrm{eff}}_{RGI} = \Lambda_{RGI} + O\left(\alpha^2_{\overline{\mathrm{MS}}}(\mu_s)\right)$$

$$\mu_s = s\overline{m}_{\overline{\mathrm{MS}}}(\mu_s)$$

- Compatible with α^2 behavior.
- The extracted Λ-parameter does agree with the Ramos, dalla Brida result, but has very **large errors**, even with lattice spacing down to $a \simeq 0.01$ fm.

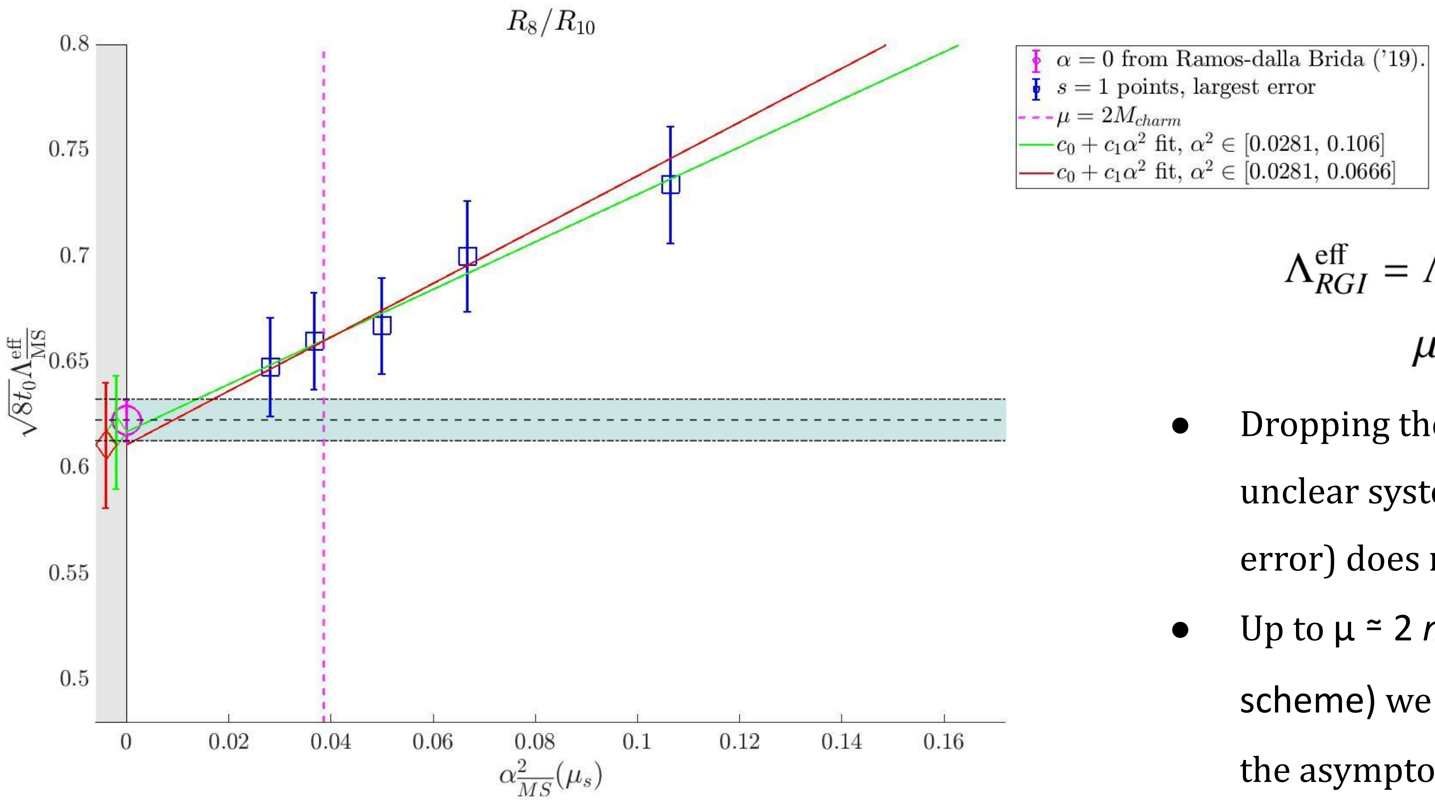
Λ Plot from R_8/R_{10}



$$\Lambda^{\text{eff}}_{RGI} = \Lambda_{RGI} + O\left(\alpha^2_{\overline{\text{MS}}}(\mu_s)\right)$$

$$\mu_s = s\overline{m}_{\overline{\text{MS}}}(\mu_s)$$

Λ Fit from R_8/R_{10}



$$\Lambda^{\text{eff}}_{RGI} = \Lambda_{RGI} + O\left(\alpha^2_{\overline{\text{MS}}}(\mu_s)\right)$$

$$\mu_s = s\overline{m}_{\overline{\text{MS}}}(\mu_s)$$

- Dropping the largest mass (with its unclear systematic continuum limit error) does not change much the result.
- Up to μ ≃ 2 m_{charm} ≃ 2.5 GeV (MS-bar scheme) we see 10% deviation from the asymptotic value

Summary and Outlook

- In improved theory with lattice spacings down to ***a* ≃ 0.01 fm**, the **continuum limit** is very **challenging**.
- **Higher moments** are dominated by longer distances**.**
- We see compatibility with $\Lambda^{eff} = \Lambda + O(\alpha^2)$, where the **slope increases with n.**
- The extracted quenched **Λ-parameters** agree with the Ramos, dalla Brida result.
- R_8/R_{10} has **10% deviations in** Λ w.r.t. the $\alpha \to 0$ result **up to** $\mu \simeq 2\, m_{charm} \simeq$ **2.5 GeV (MS-bar scheme).**
- How **reliable continuum limits** with $N_f > 0$ and the extracted α are is unclear.
- **Precise** lattice results of the coupling are crucial for the **world average**, with **influence** on many observables such as $H \to gg$, BSM searches, EW vacuum stability and many more.

Thank You!

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 813942.

Questions Please!

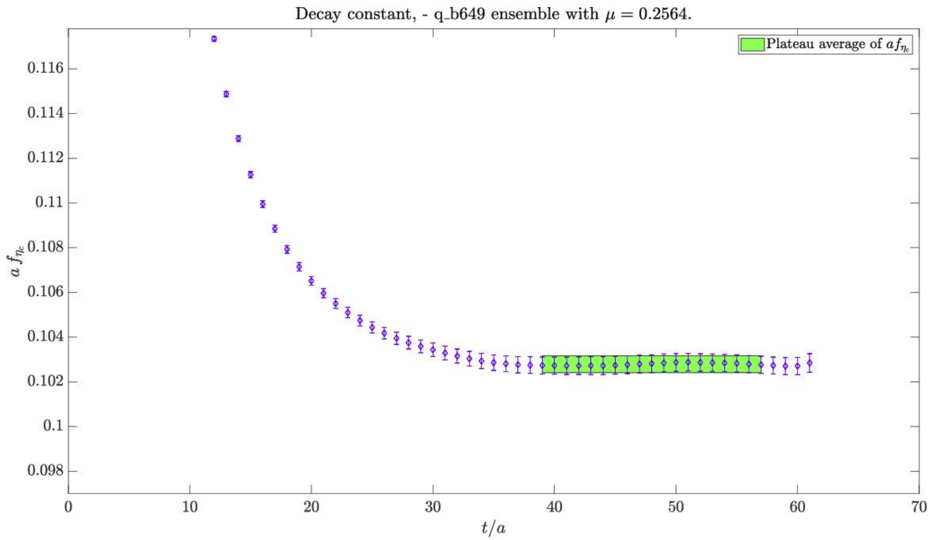
Cross Checks - I

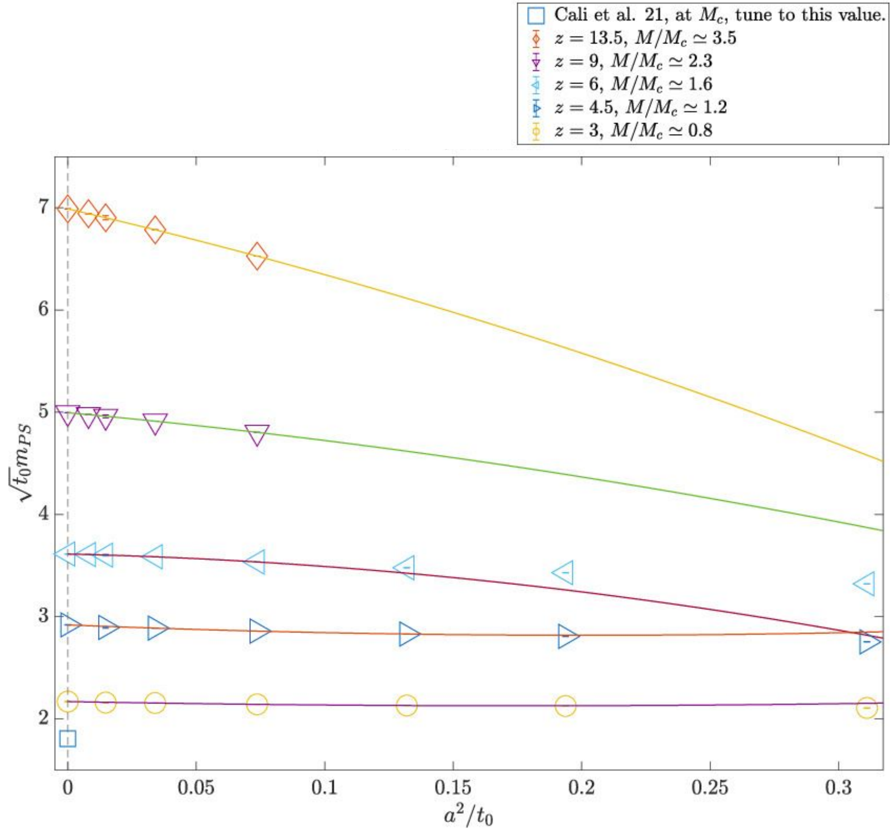
We cross checked several things:

A. variation of results when varying cutoff of sum

B. double check of perturbative inversion and running

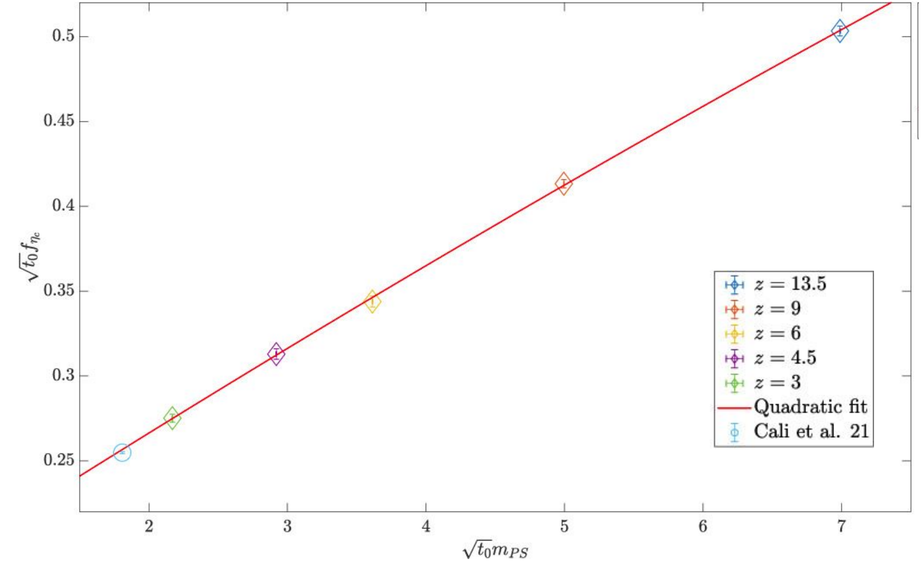
C. measuring of decay constant and pseudoscalar mass:





Cross Checks - II

No hints of issues, no mistunings, all consistent.



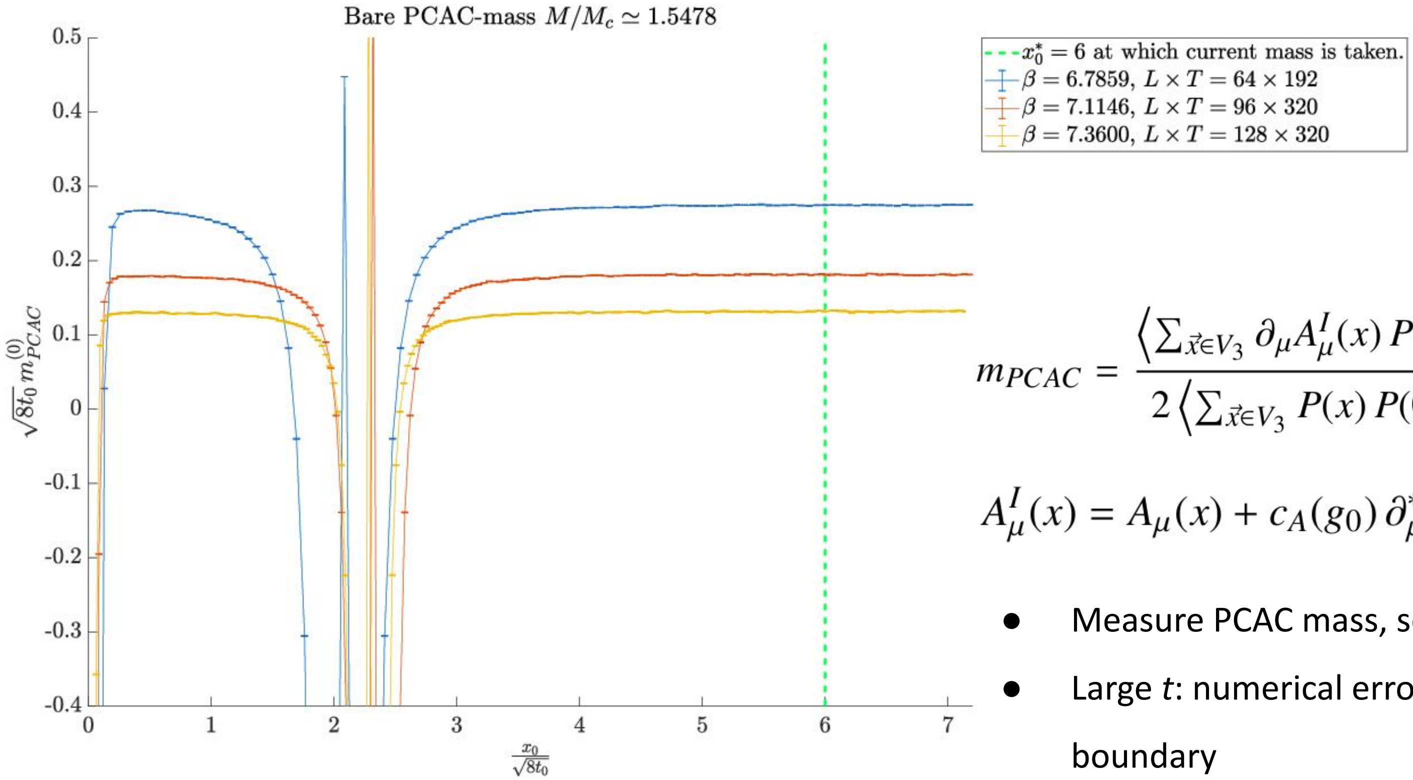
D. Finite Volume Effects? We computed analytically the continuum TL (where FV effects are expected to be larger):

$$\frac{\Delta G_n^L}{G_n(\infty)} \overset{mL\to\infty}{\sim} \frac{\pi}{2}\Gamma(3/2)\,\Gamma\left(\frac{n-2}{2}\right)\left(\frac{2}{mL}\right)^{\frac{3-n}{2}} \mathrm{e}^{-mL}\left(1+O\left(\frac{1}{mL}\right)\right).$$

Table 4: Relative TL-FV effects, normalized by $L=\infty$ value, as function of $y = Lm_*$. For $M/M_c > 1.1$, we have $y > 14$.

y	5	6	7	8	9	10	11	12	13	14	15
$n=4$	0.015	0.0060	0.0024	0.00093	0.00036	1.4e-04	5.5e-05	2.1e-05	8.0e-06	3.1e-06	1.2e-06
$n=6$	0.037	0.018	0.0083	0.0037	0.0016	7.1e-04	3.0e-04	1.3e-04	5.2e-05	2.1e-05	8.7e-06
$n=8$	0.19	0.11	0.058	0.030	0.015	0.0071	0.0033	1.5e-03	6.8e-04	3.0e-04	1.3e-04
$n=10$	1.39	0.97	0.61	0.36	0.20	0.11	0.054	0.027	0.013	0.0063	3.0e-03

PCAC Data I

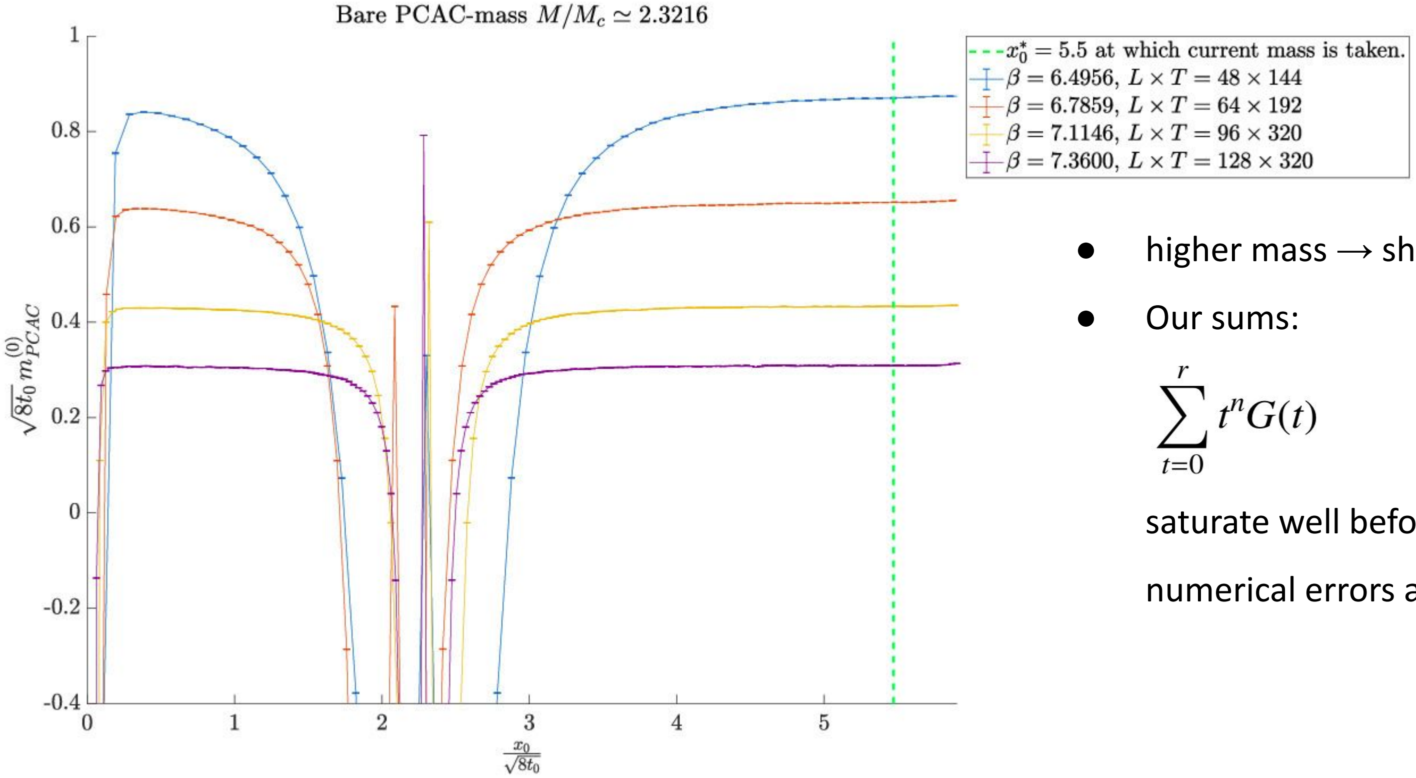


$$m_{PCAC} = \frac{\left\langle \sum_{\vec{x}\in V_3} \partial_\mu A^I_\mu(x)\, P(0) \right\rangle}{2\left\langle \sum_{\vec{x}\in V_3} P(x)\, P(0) \right\rangle} = \frac{\left\langle \sum_{\vec{x}\in V_3} \partial_0 A^I_0(x)\, P(0) \right\rangle}{2\left\langle \sum_{\vec{x}\in V_3} P(x)\, P(0) \right\rangle},$$

$$A^I_\mu(x) = A_\mu(x) + c_A(g_0)\, \partial^*_\mu P(x)$$

- Measure PCAC mass, select one value in plateau.
- Large t: numerical errors grow + states from $t = T$ boundary

PCAC Data II



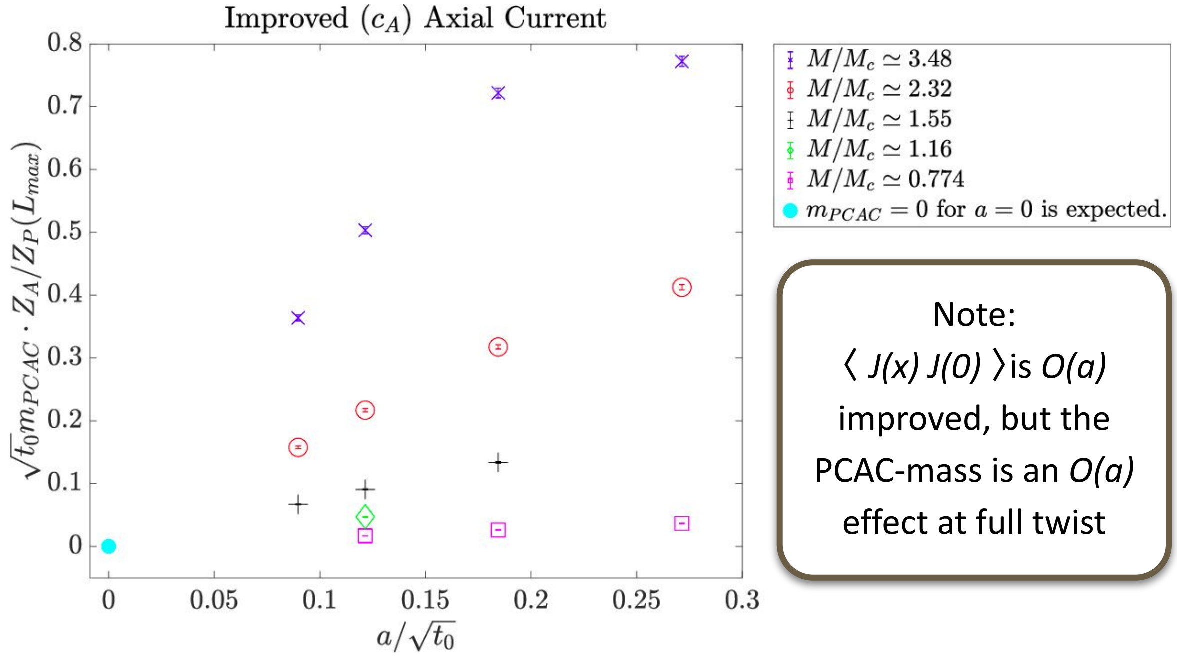
- higher mass $\longrightarrow$ shorter plateau
- Our sums:

$$\sum_{t=0}^{r} t^n G(t)$$

saturate well before the point where

numerical errors appear.

Monitoring Full Twist

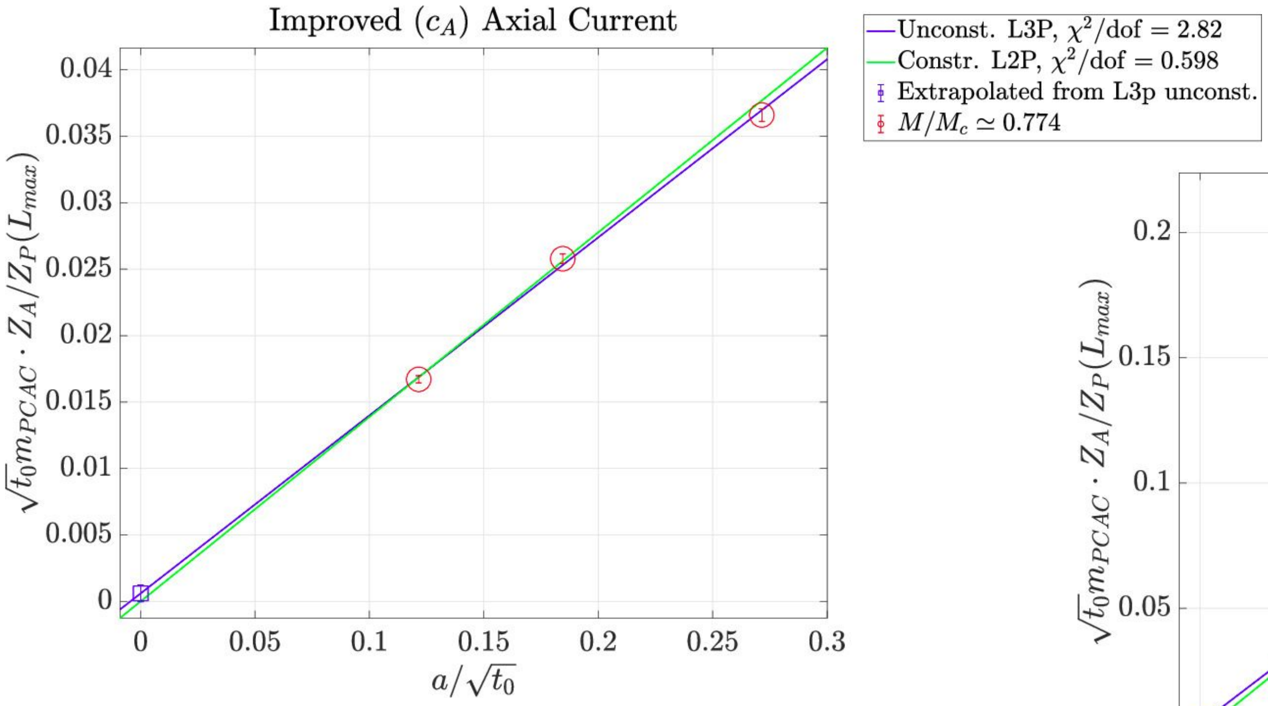


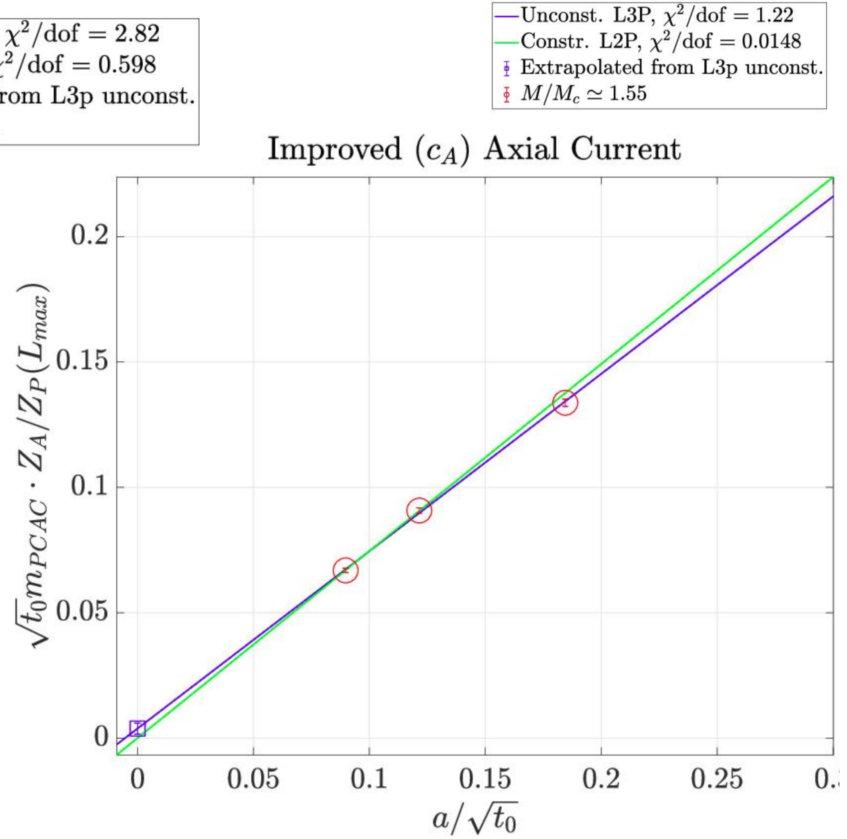
Note:
⟨ *J(x) J(0)* ⟩is *O(a)* improved, but the PCAC-mass is an *O(a)* effect at full twist

- Renormalized PCAC mass vs *a*
- All errors included: statistic, Z_A, Z_P
- also systematic error on κ, since there it was determined at fixed *L/a* = 16. Possible NP effects $O\left((a/L)^3 f(L/r)\right)$, but they are under control:
- *L/a*=8 ⟶ *L/a*=16 gives 2.0e-05 effect, propagate into m_{PCAC} ⟶ 5.5e-04 effect

$$m_{PCAC} = \frac{\left\langle \sum_{\vec{x}\in V_3} \partial_\mu A^I_\mu(x)\, P(0) \right\rangle}{2\left\langle \sum_{\vec{x}\in V_3} P(x)\, P(0) \right\rangle} = \frac{\left\langle \sum_{\vec{x}\in V_3} \partial_0 A^I_0(x)\, P(0) \right\rangle}{2\left\langle \sum_{\vec{x}\in V_3} P(x)\, P(0) \right\rangle}, \quad A^I_\mu(x) = A_\mu(x) + c_A(g_0)\, \partial^*_\mu P(x)$$

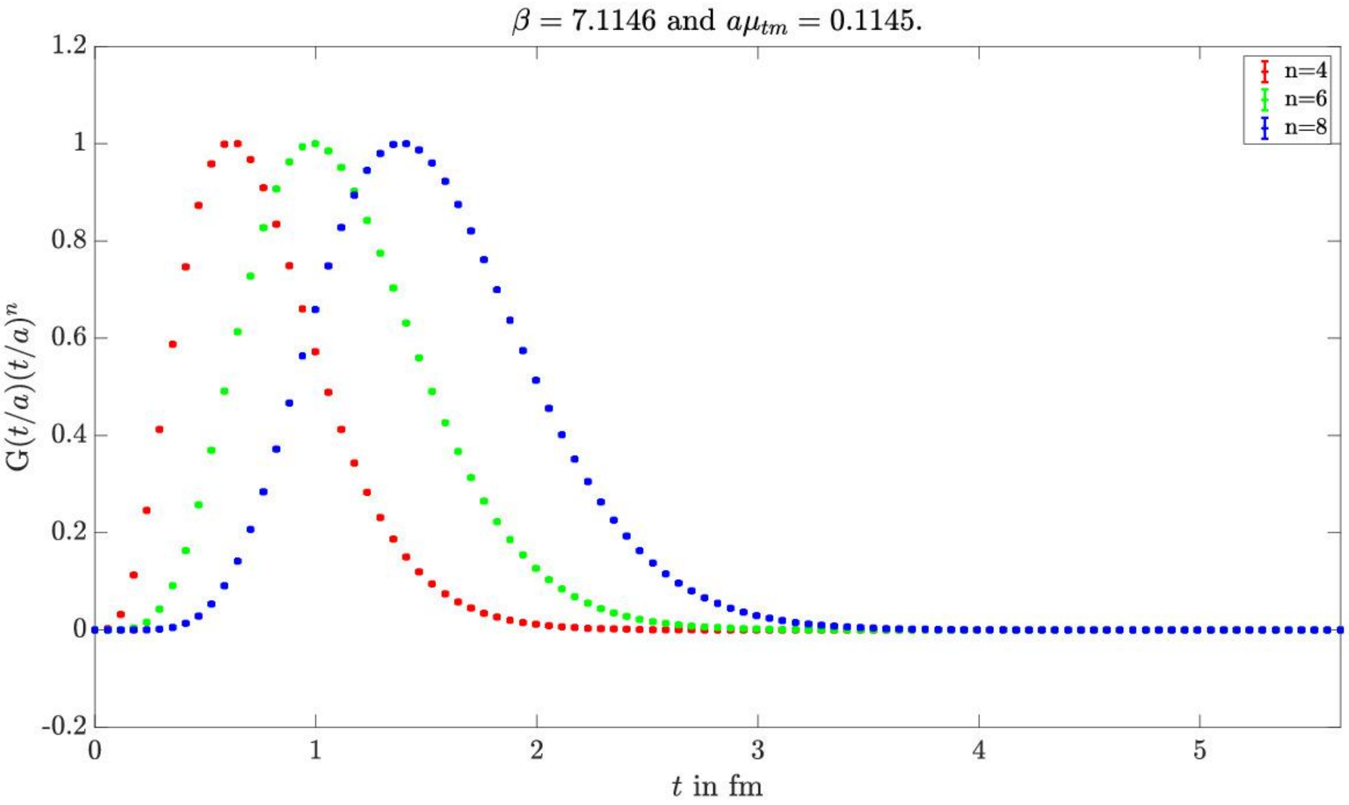
PCAC vs a - I





- Fits constrained through 0 show <u>linear behavior</u>

Scale of Moments



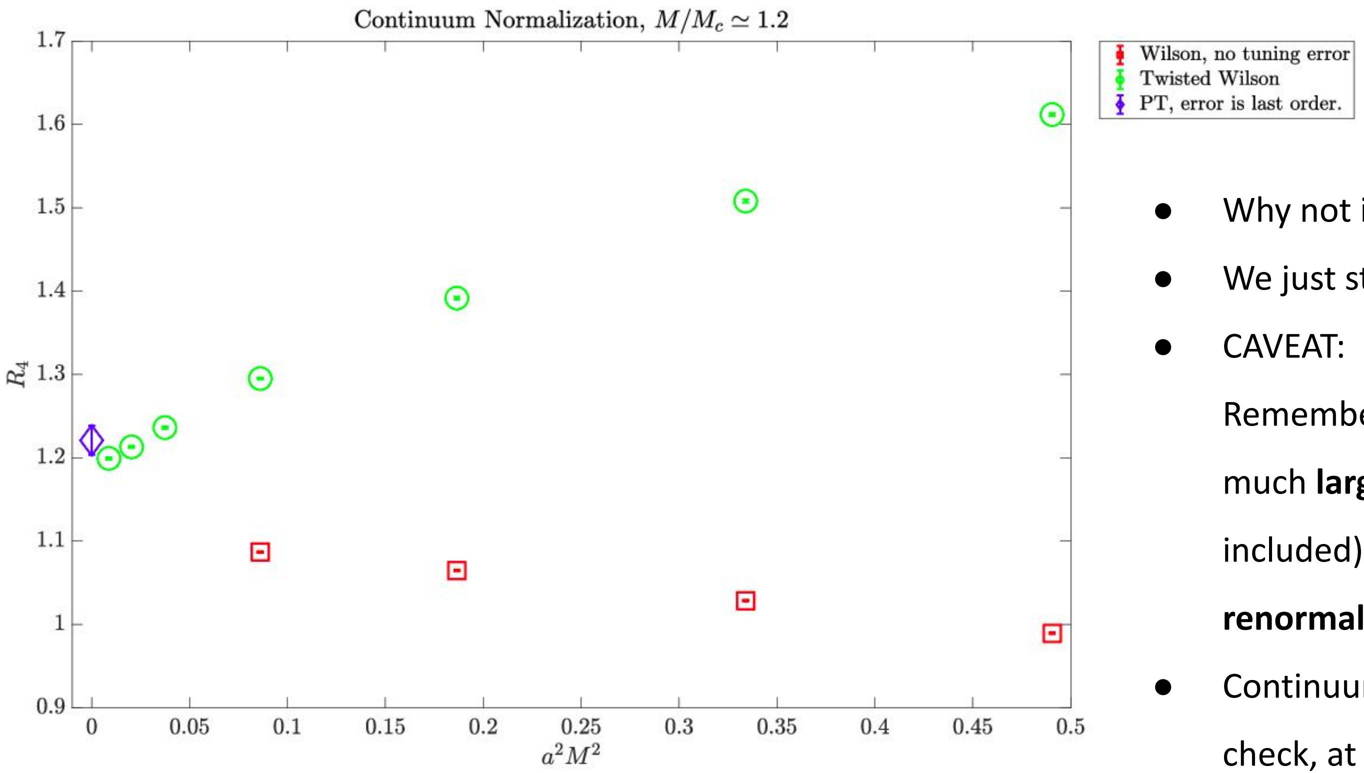
$$\mathcal{M}_n(m) = \int_{-\infty}^{\infty} \mathrm{d}t\, t^n G(t,m)$$

★ Integrands normalized so their height is 1

★ Peak moves to the right (long distance) with n and with inverse mass → **high *n* is less perturbative**

★ Energy **scale** of moments is somewhat **worrying**

"Untwisted" Wilson Fermions



- Why not improved Wilson Fermions?
- We just started looking into this.
- CAVEAT:

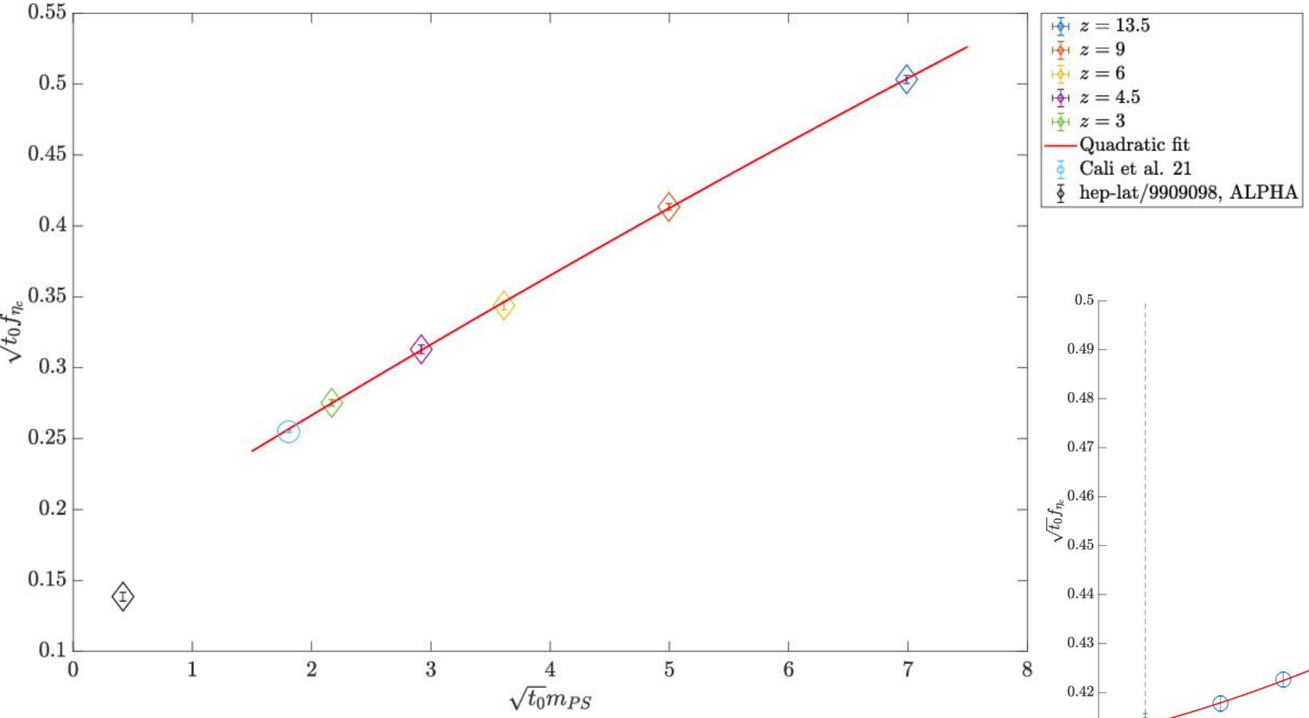
  Remember, Wilson fermions will have much **larger error** bars (here not yet included) because of the **explicit renormalization factors**.
- Continuum PT just as a reference/sanity check, at this stage.

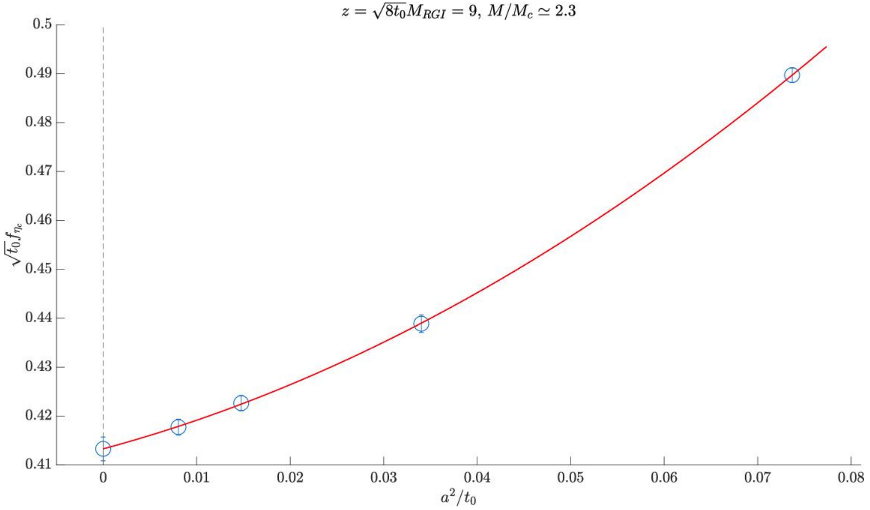
Expansion of Ratios of Moments

$$\lim_{a\to 0} \frac{R_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{R_{n+2}(\sqrt{8t_0}M_{RGI}, aM_{RGI})} = \frac{\overline{m}_{\overline{\mathrm{MS}}}(\mu)}{\overline{m}_{\overline{\mathrm{MS}}}(\mu)} \frac{\left(\sum_{i\geq 0}^{L} c_n^{(i)}(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu))\, \alpha^i_{\overline{\mathrm{MS}}}(\mu) + \mathcal{O}(\alpha^{L+1})\right)^{\frac{1}{n-4}}}{\left(\sum_{i\geq 0}^{L} c_{n+2}^{(i)}(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu))\, \alpha^i_{\overline{\mathrm{MS}}}(\mu) + \mathcal{O}(\alpha^{L+1})\right)^{\frac{1}{n-2}}}$$

$$= \sum_{i\geq 0}^{L} \tilde{c}_n^{(i)}(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu))\, \alpha^i_{\overline{\mathrm{MS}}}(\mu) + \mathcal{O}(\alpha^{L+1})$$

Extended m_{PS} and f_η





Twisted Mass Renormalization Factors

- For some scheme S, renormalization scale μ, we have for a doublet of mass -degenerate Wilson, twisted mass fermions out of full twist:

$$\overline{m}_S(\mu) = \lim_{a\to 0}\left[Z_P^S(a\mu, g_0)\right]^{-1}\sqrt{\mu_{tm}^2 + Z_A^2(g_0)m_{PCAC}^2}$$

$$= \lim_{a\to 0}\left[Z_P^S(a\mu, g_0)\right]^{-1}\sqrt{\mu_{tm}^2 + Z_A^2(g_0)Z^2(g_0)m_q^2}$$

$$m_q = m_0 - m_{cr.}$$

$$Z(g_0) = \frac{Z_m^S(a\mu, g_0)Z_P^S(a\mu, g_0)}{Z_A(g_0)}$$

Twisted PS and Disconnected Diagrams

$$P^i(x) = im_0\overline{\psi}(x)\gamma_5\frac{\tau^i}{2}\psi(x)$$

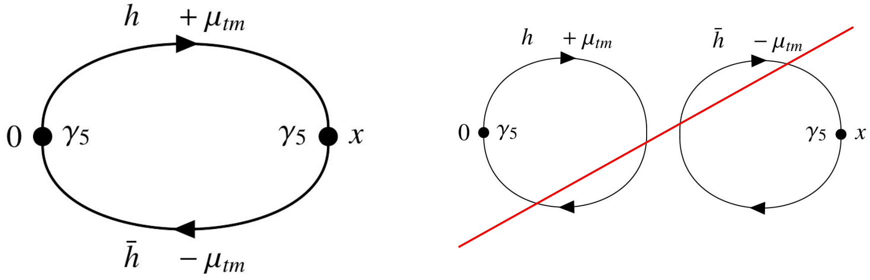
$$P^{\pm}(x) = P^1(x) \pm iP^2(x)$$

$$P^{1,2}(x) \xrightarrow{\text{chiral rotation}} P^{1,2}(x)$$

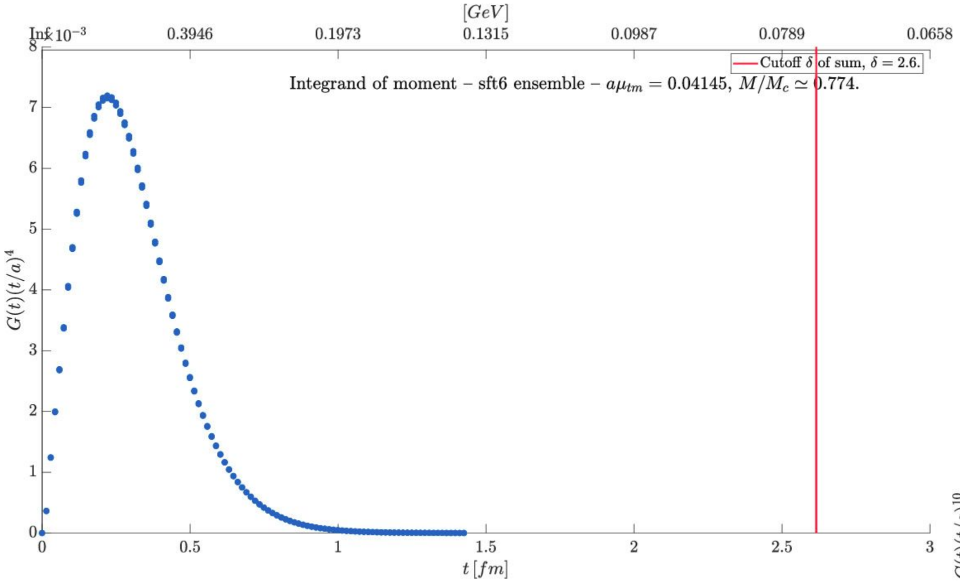
$$P^{\pm}(x) \xrightarrow{\text{chiral rotation}} P^{\pm}(x)$$

$$G(t) = -\mu_{tm}^2 \sum_{\vec{x}} a^3 \left\langle \mathrm{tr}\left[S_h(x,0)\gamma_5 S_{h'}(0,x)\gamma_5\right]\right\rangle$$

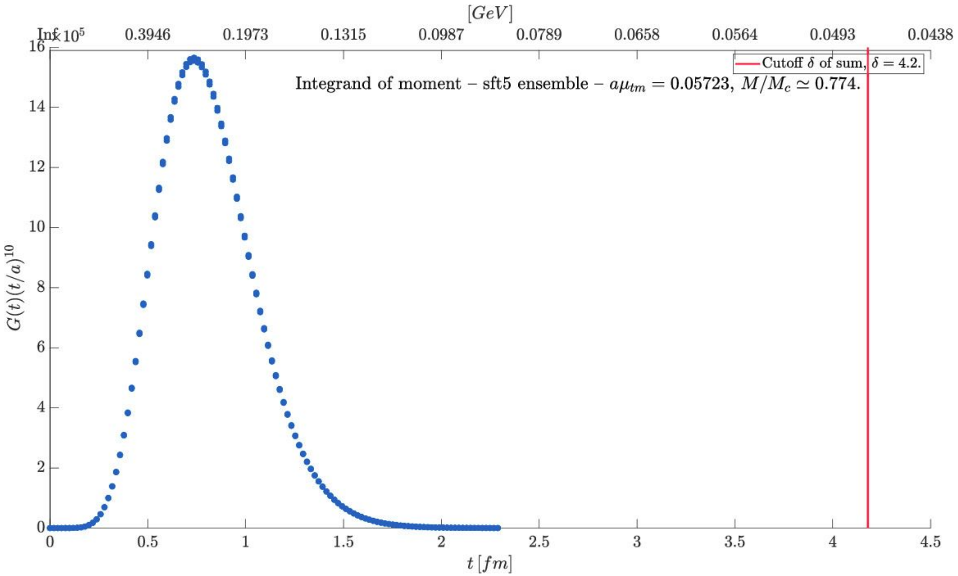
The 2 different flavors mean no disconnected diagrams are present:



Saturation of Sum



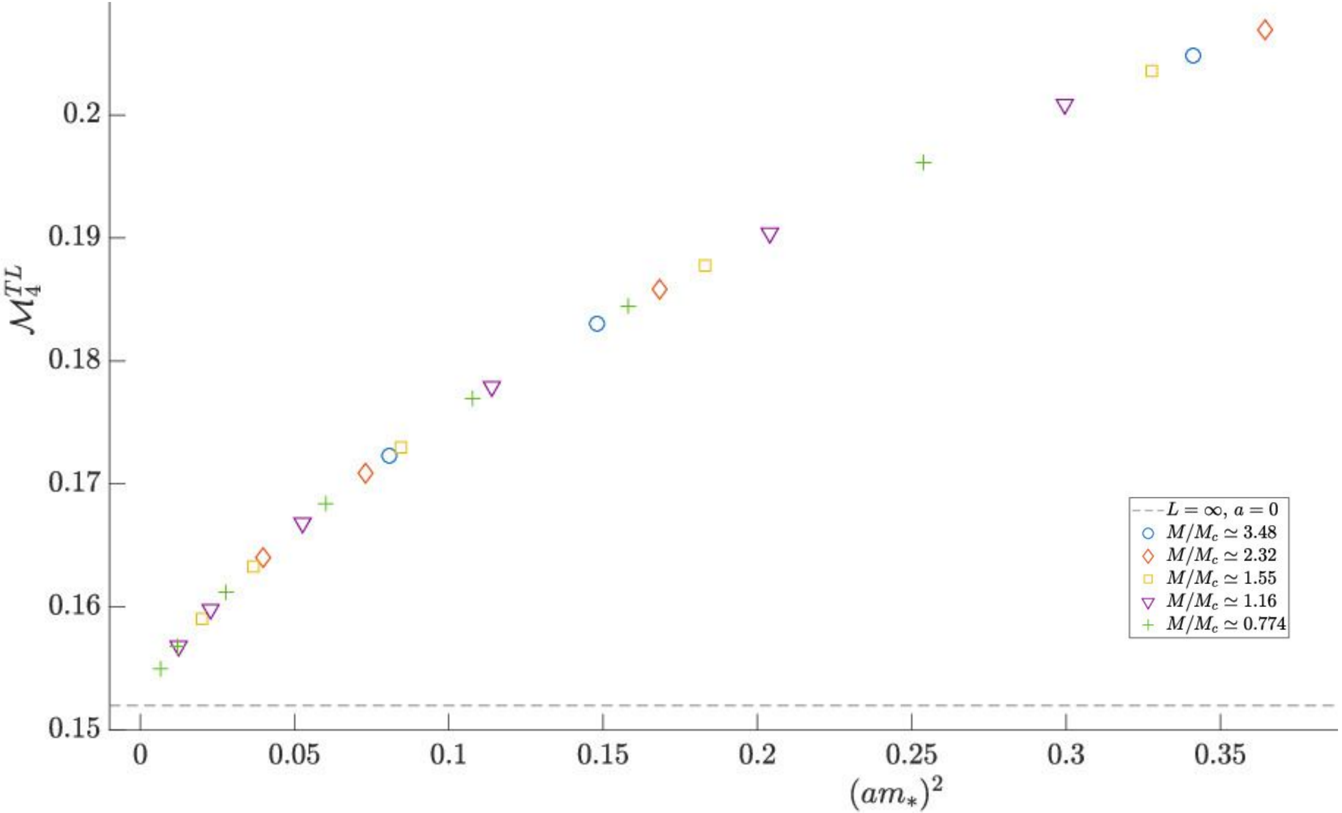
- The moments saturate after a certain number of time slices have been summed up.
- The sum is cut off when the plateau has been reached.
- This is a lot before the PCAC mass signal is lost due to numerical errors.



More (TL) Finite Volume

Finite lattice spacing: TL $\rightarrow \beta = \infty$, inputs are L/a and $a\mu_{\text{tm}}$. Changing $L\mu_{\text{tm}}$, points remain on same curve. Figure:

only lattice artefacts can be seen, no FV corrections.



Constant Mass Trajectory

- Line of constant "physics": at every a we tune the bare mass in order to keep some renormalized mass fixed.
- We keep the renormalization group invariant mass fixed (scheme independent!):

$$M_{RGI} = \lim_{\mu\to\infty} \overline{m}_X(\mu) \left[2b_0\, \bar{g}_X^2(\mu)\right]^{-d_0/(2b_0)}$$

RGI-mass (parameters) are like running to infinite energy.

$$z := \sqrt{8t_0} M_{RGI} = \frac{\sqrt{8t_0}}{a} a M_{RGI} = \frac{\sqrt{8t_0}}{a} \underbrace{\frac{M_{RGI}}{\overline{m}_{SF}(\mu)\, Z_P^{SF}(a\mu,\beta)}}_{\text{Literature, from [1]}} a\mu_{tm} + O((a\mu_{tm})^2)$$

Choose

Measure

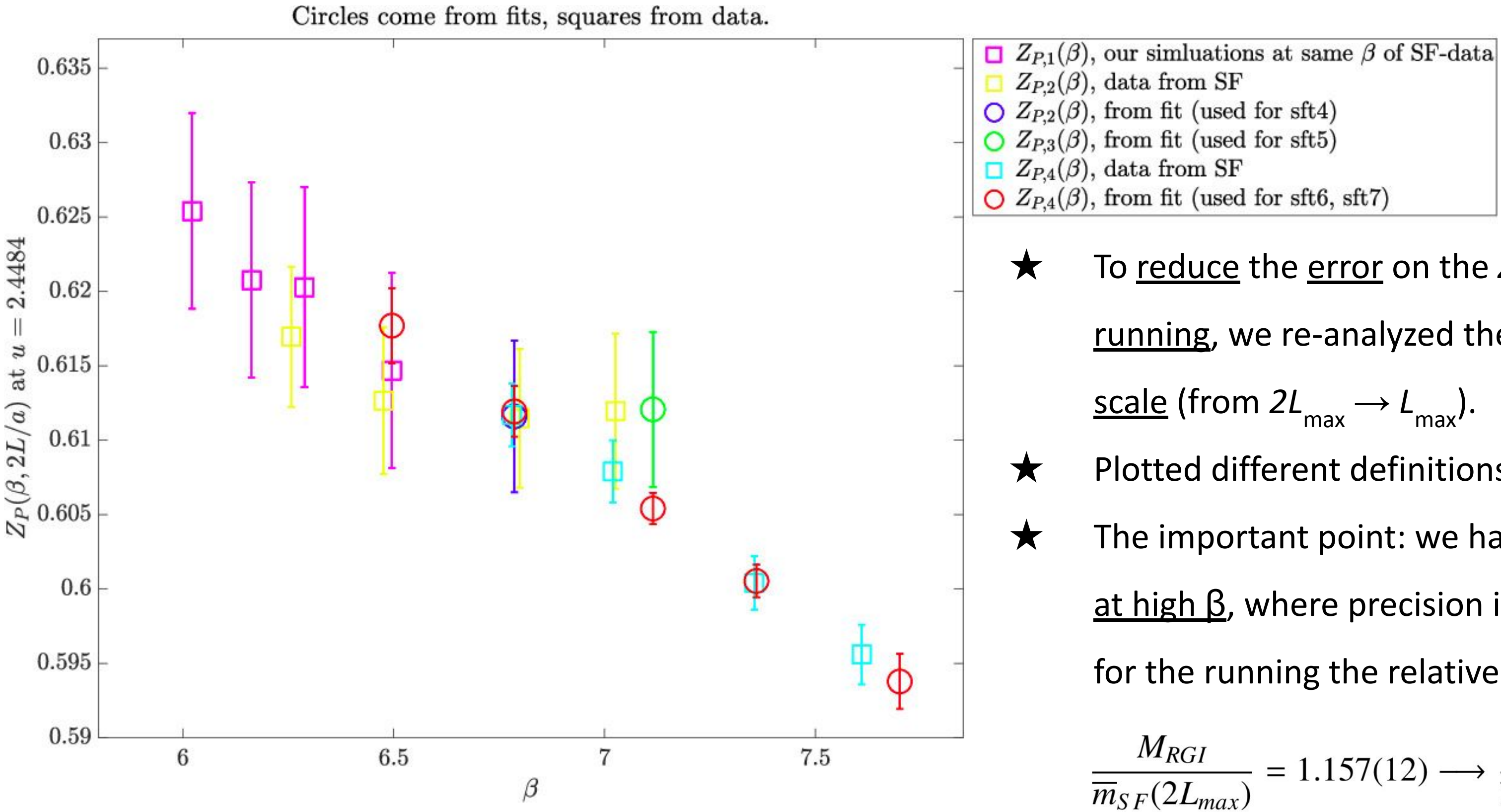
Tune

Some numbers: $$M_{RGI} = (\hbar c)\frac{z}{\sqrt{8t_0}} \simeq 5.75\,, 3.83\,, 2.56\,, 1.92\,, 1.28\,\text{GeV}$$

$$M_c^{RGI}\Big|_{N_f=0} = 1.654(45)\,\text{GeV}$$ [Rolf and Sint, *hep-ph/0110139*]

[1] Capitani, Lüscher, Sommer, Wittig, [*hep-lat/9810063*]

Reanalysis of Quenched SF Data



★ To reduce the error on the Z_P-factor and the running, we re-analyzed the data of [1] at a lower scale (from $2L_{max} \to L_{max}$).

★ Plotted different definitions of Z_P, labeled $Z_{P,i}$.

★ The important point: we have rather small errors at high β, where precision is most important. Also for the running the relative error decreases:

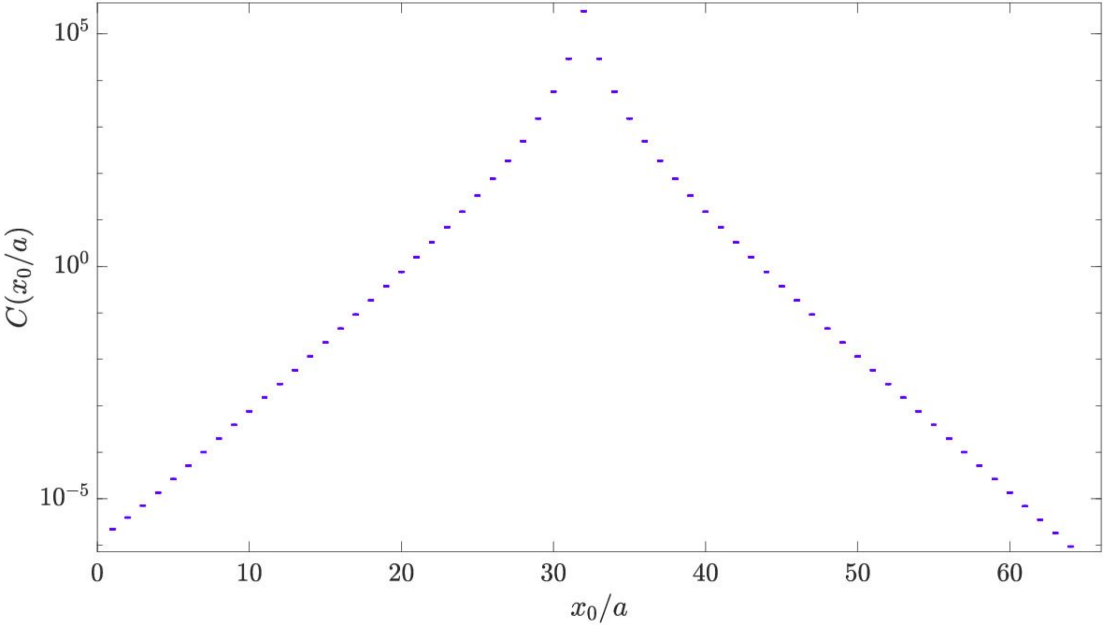
$$\frac{M_{RGI}}{\overline{m}_{SF}(2L_{max})} = 1.157(12) \longrightarrow \frac{M_{RGI}}{\overline{m}_{SF}(L_{max})} = 1.379(11)$$

[1] Capitani, Lüscher, Sommer, Wittig, [*hep-lat/9810063*]

Absence of Boundary Effects

- **Source** placed at **1 fm from boundary**, checked absence of boundary effects

Example correlator: no asymmetry around source (no <u>boundary effects</u>) can be seen within precision; β = 6.7859, $M/M_c \simeq 1.6$



[1] Lüscher, Sint, Sommer, Weisz, Wolff. [*arXiv:hep-lat/9609035*]
[2] Lüscher, *arXiv:hep-lat/1006.4518*