Scale Setting from a Mixed Action with Twisted Mass Quarks

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Setup: sea sector

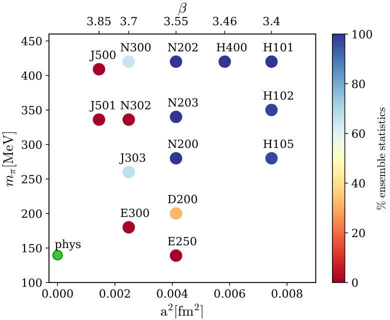
- CLS $N_f = 2 + 1$ ensembles [Lüscher and Schaefer, JHEP 1107 036; Bruno et al. JHEP 1502 043 - 1712.04884 - 2003.13359]
  - Lüscher-Weisz gauge action
  - Non-perturbatively $O(a)$-improved Wilson fermions
  - Open boundary conditions [Lüscher and Schaefer, 1206.2809]
  - Finite volume corrections: χPT LO, $m_\pi L$ & 4
- Chiral trajectory $trM_q = 2m_{0,ud} + m_{0,s} = cnst.$
  - Mass shift to renormalized chiral trajectory [Bruno, Korzec, Schaefer, 1608.08900], [Straßberger et al. 2112.06696]

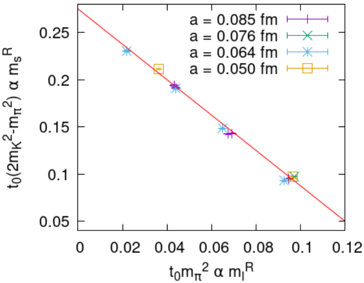
$$\phi_4^{guess} = 8t_0^{guess}\left(\tfrac{1}{2}m_\pi^2 + m_K^2\right)^{\text{phys}} \equiv 1.098(10).$$

$$\phi_4 \propto trM_q^R.$$

$$\langle O(m'_{0,q})\rangle = \langle O(m_{0,q})\rangle + \textstyle\sum_q (m'_{0,q} - m_{0,q})\dfrac{d\langle O(m_{0,q})\rangle}{dm_{0,q}}.$$

$$\frac{d\langle P_i\rangle}{dm} = \textstyle\sum_i \langle \frac{\partial P_i}{\partial m}\rangle - \left\langle (P_i - \langle P_i\rangle)\left(\frac{\partial S}{\partial m} - \langle \frac{\partial S}{\partial m}\rangle\right)\right\rangle.$$





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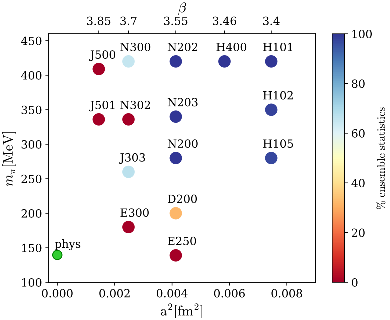
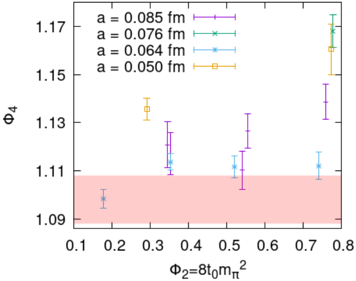




Setup: valence sector

- Wilson twisted mass valence quarks [ALPHA, hep-lat/0101001; Frezzotti and Rossi hep-lat/0306014, Pena et al., hep-lat/0405028]

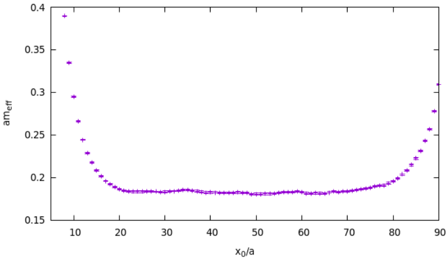
$$D_{tm} = \underbrace{\frac{1}{2}\sum_{\mu=0}^{3}\left[\gamma_\mu\left(r^*_\mu + r_\mu\right) \quad a\, r^*_\mu r_\mu\right] + \frac{i}{4} a c_{sw} \sum_{\mu,\nu=0}^{3} \sigma_{\mu\nu}\hat{F}_{\mu\nu}}_{D_W} + a m_0 + i\gamma_5\, a\mu_{0,q}.$$

  - Tuned to maximal twist: $m_0 = \tilde{m}_{cr} \leftrightarrow m_{ud} \equiv m_{12} \equiv \frac{m_u+m_d}{2} = 0$
  - Automatic $O(a)$-improvement $\rightarrow$ relevant for heavy quark physics
  - Residual cutoff effects $O(ag_0^4\, trM_q^{sea})$
  - Finite volume corrections: χPT LO, $m_\pi L$ & 4

- Motivation:
  - Alternative way to control cutoff effects $\rightarrow$ universality
  - Leptonic & semileptonic decays with heavy quarks
    [A. Conigli et al, 2112.00666]
    see talks by Alessandro Conigli & by Julien Frison
  - Light-quark sector (isospin limit): valence/sea matching, scale setting, light quark masses
    this talk & talk by Gregorio Herdoíza

- Lattice observable $O(x_0) = \{m_{\pi,K}, f_{\pi,K}, m_{12,13}\}$ & fit function(s) $f(x_0; p_1, p_2, ..., p_k)$

$$f(x_0) = p_1 + \sum_i p_i \exp\left(-q_i \frac{x_0}{a}\right) + \sum_i p_l \exp\left(-q_l \frac{T - x_0}{a}\right).$$



$m_\pi = m_K = 420$ MeV, $a = 0.085$ fm.

- Use different cuts of data for fit:

$$\chi_j^2 \to \chi_j^2 \frac{dof}{\chi^2_{j,exp}}, \quad \text{[ALPHA: JHEP 05 (2021) 288]}$$

$$IC_j = \chi_j^2 + 2n_{param} + 2n_{cuts},$$

$$W_j \propto \exp(-0.5\, IC_j).$$

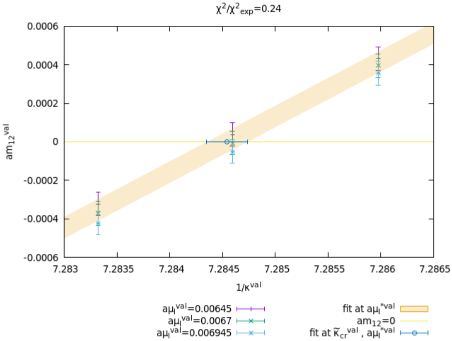
- Average fit parameters results & add systematic uncertainty:

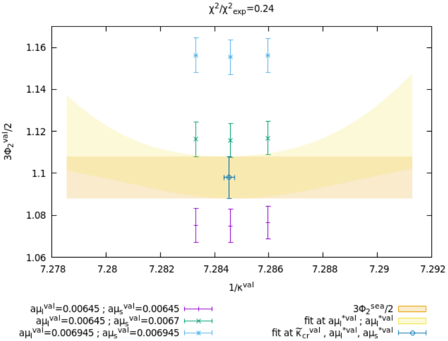
$$\langle p_1 \rangle_{model} = \sum_j p_1{}^{(j)} W_j,$$

$$\sigma^2_{1,syst} = \langle p_1{}^2 \rangle_{model} - \langle p_1 \rangle^2_{model}.$$

Matching valence & sea + maximal twist

- Matching & maximal twist condition $\phi_2^{val} \quad \phi_2^{sea}$, $\phi_4^{val} \quad \phi_4^{sea}$, $am_{12}^{val} \quad 0$
- Interpolate from valence tuning grid ($\kappa_l^{val} = \kappa_s^{val}, a\mu_l^{val}, a\mu_s^{val}$)





$$\phi_4 = 8t_0\left(\frac{1}{2}m_\pi^2 + m_K^2\right) = \frac{1}{2}\phi_2 + \phi_K.$$

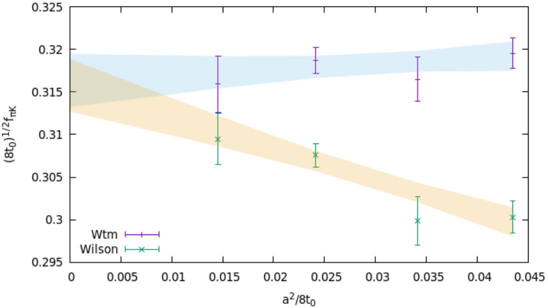
Continuum limit: universality checks $\quad f_{\pi K} = \frac{2}{3}\left(\frac{1}{2} f_\pi + f_K\right)$

- Cutoff effects:

$$\left(\sqrt{8t_0} f_{\pi K}\right)(\phi_2)\Big|_{latt} = \left(\sqrt{8t_0} f_{\pi K}\right)(\phi_2)\Big|_{cont} \quad \left(1 + p_3 \frac{a^2}{8t_0(\phi_2)}\right).$$

  - Finite volume corrections

- Symmetric point ensembles:
  - $m_{ud} = m_s \rightarrow m_\pi = m_K = 420$ MeV
  - $\phi_2^{sym} = \frac{2}{3}\phi_4^{phys} = 0.732(7) \rightarrow \left(\sqrt{8t_0} f_{\pi K}\right)(\phi_2^{sym})\Big|_{latt} = c_1 + c_2 \frac{a^2}{8t_0(\phi_2^{sym})}$.



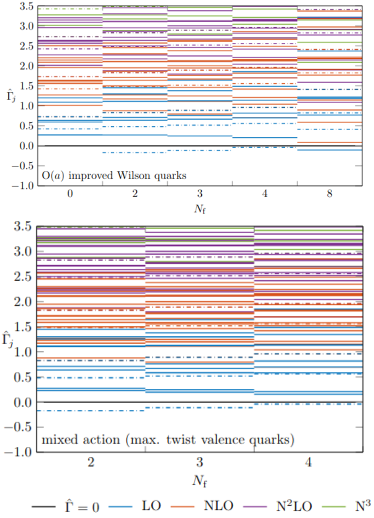
Wtm (MA) : $c_1 = 0.3163(31)$, $c_2 = +0.065(95)$, $\chi^2/\chi^2_{exp} = 0.89$.

Wilson (sea) : $c_1 = 0.3158(31)$, $c_2 = 0.37(10)$, $\chi^2/\chi^2_{exp} = 0.62$.

Continuum limit: universality checks

$$f_{\pi K} = \frac{2}{3}\left(\frac{1}{2} f_\pi + f_K\right)$$

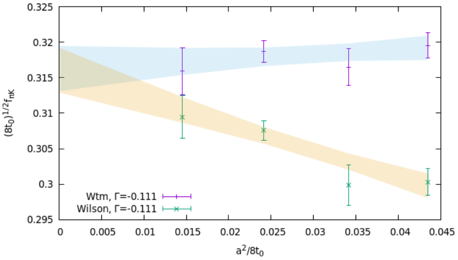
- Logarithmic corrections $a^2 \alpha_s^\Gamma(a^{-1}) \sim -a^2 \left(\left[ln(a\Lambda_{QCD})\right]^{-1}\right)^\Gamma$ [Husung, 2206.03536]



Plot from [Husung, 2206.03536]

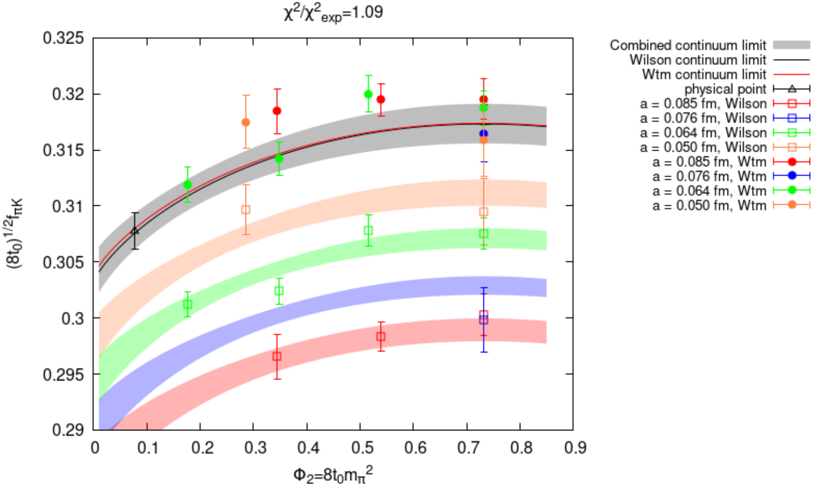
Take smallest value $\Gamma = -0.111$.

$\chi^2/\chi^2_{exp} \sim 0.6$ (Wilson), 0.9 (Wtm).



Continuum & chiral extrapolation: $\sqrt{8t_0} f_{\pi K}$

3 fits: Wilson, Wtm, Combined



- SU(3) NLO χPT:

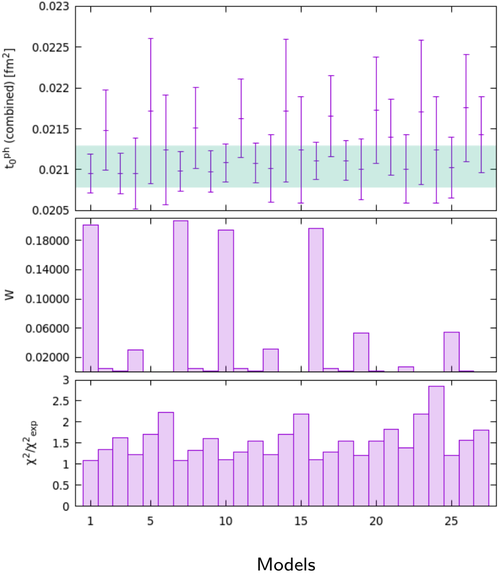
$$\left(\sqrt{8t_0} f_{\pi K}\right)(\phi_2)\Big|_{cont} = \frac{p_1}{8\pi\sqrt{2}}\left[1 - \frac{7}{6}\mathcal{L}\left(\frac{\phi_2}{p_1^2}\right) - \frac{4}{3}\mathcal{L}\left(\frac{\phi_4 - \phi_2/2}{p_1^2}\right) - \frac{1}{2}\mathcal{L}\left(\frac{4\phi_4/3 - \phi_2}{p_1^2}\right) + p_2\phi_4\right], \quad \mathcal{L}(x) = x\log(x)$$

- $O(a^2)$ cutoff effects: $\left(\sqrt{8t_0} f_{\pi K}\right)(\phi_2)\Big|_{latt} = \left(\sqrt{8t_0} f_{\pi K}\right)(\phi_2)\Big|_{cont} \times \left(1 + p_3\frac{a^2}{8t_0}\right)$

Systematic effects: model variations

Combined analysis Wilson & Wtm

$$W \sim \exp\left[-\frac{1}{2}\left(\chi^2\frac{dof}{\chi^2_{exp}} + 2n_{param} + 2n_{cut}\right)\right]$$



Model average:

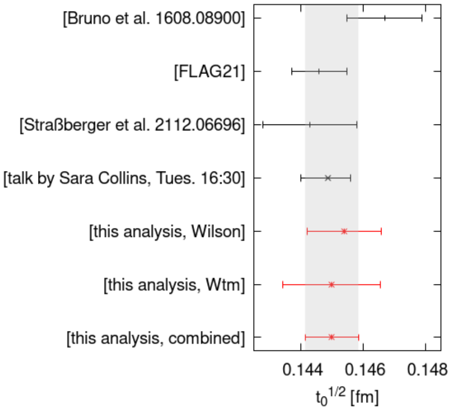
- Continuum ϕ_2 dependence:
  - SU(3) NLO χPT
  - Taylor in $(\phi_2 - \phi_2^{sym})$
- Cutoff dependence:
  - $O(a^2)$
  - $O(a^2 + \phi_2 a^2)$
  - $O(a^2 \alpha_s^{\hat{\Gamma}}(a^{-1}))$
    [Husung, 2206.03536]
- Cuts in data:
  - Remove $m_\pi = 420$ MeV
  - Remove $a = 0.085$ fm
- Physical inputs:

f_π^{isoQCD}, f_K^{isoQCD}, $m_{\pi^0}^{exp}$, $m_{K^0}^{exp}$ [FLAG]

t_0^{guess} ! ϕ_4^{phys} 1.098(10).

- 1: $SU(3)$ NLO χPT, $O(a^2)$
- 7: $SU(3)$ NLO χPT, $O(a^2\alpha_s^\Gamma)$
- 10: Taylor, $O(a^2)$
- 16: Taylor, $O(a^2\alpha_s^\Gamma)$

Preliminary physical results: t_0 CLS $N_f = 2 + 1$



$$\sqrt{t_0^{ph}} = 0.1467(10)(7) \text{ fm},$$

[Bruno et al. 1608.08900]

$$\sqrt{t_0^{ph}} = 0.1443(7)(13) \text{ fm},$$

[Straßberger et al. 2112.06696]

$$\sqrt{t_0^{ph}} = 0.1449^{(7)}_{(9)} \text{ fm},$$! m_Ξ

[talk by Sara Collins, Tues. 16:30]

$$\sqrt{t_0^{ph}} = 0.1454(10)(7) \text{ fm (Wilson)},$$

$$\sqrt{t_0^{ph}} = 0.1450(12)(10) \text{ fm (Wtm)},$$

$$\sqrt{t_0^{ph}} = 0.1450(8)(3) \text{ fm (combined)}.$$

0.6% relative error $\sqrt{t_0}$

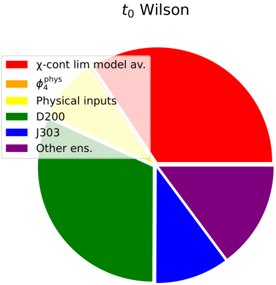
$$\phi_4^{ph} = 1.114(16)(11) \ (Wilson),$$

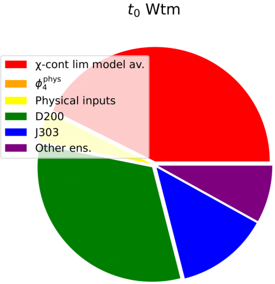
$$\phi_4^{ph} = 1.109(18)(16) \ (Wtm),$$

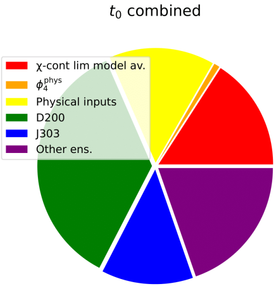
$$\phi_4^{ph} = 1.109(12)(5) \ (combined).$$

$$\phi_4^{guess} = 1.098(10)$$

Preliminary t_0 error budget







D200: $m_\pi = 200$ MeV, $a = 0.064$ fm

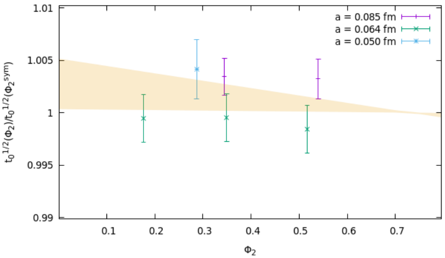
J303: $m_\pi = 260$ MeV, $a = 0.050$ fm

- Fit [Straßberger et al. 2112.06696]

$$\frac{\sqrt{t_0(\phi_2)/a^2}}{\sqrt{t_0(\phi_2^{sym})/a^2}} = \sqrt{1 + p_1(\phi_2 \quad \phi_2^{sym})} \ ! \ \sqrt{t_0^{sym}} = \frac{\sqrt{t_0^{ph}}}{\sqrt{1 + p_1(\phi_2^{ph} \quad \phi_2^{sym})}}.$$

- Lattice spacing

$$a = \sqrt{\frac{t_0^{sym}}{t_0(\phi_2^{sym})/a^2}} \frac{1}{\sqrt{1 + p_1(\phi_2^{ph} \quad \phi_2^{sym})}}.$$



β	Wilson a [fm]	Wtm a [fm]	Combined a [fm]
3.40	0.0855(6)(4)	0.0853(7)(6)	0.0853(5)(2)
3.46	0.0758(6)(4)	0.0756(6)(5)	0.0756(5)(2)
3.55	0.0637(4)(3)	0.0636(5)(5)	0.0636(4)(2)
3.70	0.0495(4)(2)	0.0493(4)(4)	0.0493(3)(1)

Conclusions

- **Mixed action**: Wilson twisted mass quarks on Wilson fermions

- **Finite volume effects** corrections for $m_{\pi,K}$ and $f_{\pi,K}$

- **Scale setting**: lattice spacing a and t_0 combining Wilson & mixed action results

$$\sqrt{t_0^{ph}} = 0.1450(8)(3), \;\; 0.6\ \%\ \text{relative error.}$$

- **Systematic effects**: cuts in data & fit functions

- Adding **lighter ensembles** and **finer lattice spacings**

- Light **quark masses**: m_{ud}, m_s
  see talk by Gregorio Herdoíza

- **Heavy-quark physics:** leptonic & semileptonic decays
  see talk by Alessandro Conigli & Julien Frison

The End

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Backup

Twisted mass fermions: maximal twist

- Twisted mass Dirac operator:

$$D_{tm} = \frac{1}{2}\sum_{\mu=0}^{3}\left[\gamma_\mu\left(r_\mu^{*} + r_\mu\right) \quad a\, r_\mu^{*}\, r_\mu\right] + \frac{i}{4} a c_{sw} \sum_{\mu,\nu=0}^{3} \sigma_{\mu\nu}\hat{F}_{\mu\nu} + a m_0 + i\gamma_5\; a\mu_{0,q}.$$

- 2-point correlation functions:

$$P^{rs}(x) = \bar{\psi}^r(x)\gamma_5\psi^s(x),\;\; A_0^{rs}(x) = \bar{\psi}^r(x)\gamma_0\gamma_5\psi^s(x),$$

$$C_A(x_0, y_0) = \quad \frac{a^6}{L^3}\sum_{\vec{x},\vec{y}} h P^{rs}(x)P^{sr}(y) i\,,\;\; C_P(x_0, y_0) = \quad \frac{a^6}{L^3}\sum_{\vec{x},\vec{y}} h A_0^{rs}(x)P^{sr}(y) i\,.$$

- PCAC Ward identity:

$$\frac{\left(\partial_0 C_A^{rs} + a c_A \partial_0^2 C_P^{rs}\right)(x_0, y_0)}{C_P^{rs}(x_0, y_0)} = m_{rs}(x_0, y_0) \quad \frac{m_r + m_s}{2}.$$

- Renormalized standard quark mass:

$$m_{rs}^R = Z_P^{-1} Z_A\; m_{rs}.$$

- Maximal twist condition:

$$\omega_l = atan\,\frac{\mu_l^R}{m_{12}^R} = \frac{\pi}{2}\;\;)\;\; m_0 = \tilde{m}_{cr}\;\;,\;\; m_{12} = 0.$$

Observables improvement

- Wilson PCAC quark masses

$$m_{rs}^R = \frac{Z_A}{Z_P}\left(1 + a\,(\bar{b}_A \quad \bar{b}_P)\mathrm{tr}M_q + a\left(\tilde{b}_A \quad \tilde{b}_P\right) m_{rs}\right)\, m_{rs} + O(a^2).$$

- Wtm PCAC quark masses

$$m_{rs}^R = \frac{1}{Z_P}\left(1 + a\,(\bar{b}_A \quad \bar{b}_P)\mathrm{tr}M_q + a\left(\tilde{b}_A \quad \tilde{b}_P\right) m_{rs}\right)\, m_{rs} + O(a^2) + O(a\mu_{rs}^2).$$

- Wilson pseudoscalar decay constants

$$f_{\pi,K}^R = Z_A\left(1 + a\bar{b}_A\mathrm{tr}M_q + a\tilde{b}_A m_{rs}\right) f_{\pi,K}^{bare} + O(a^2).$$

- Wtm pseudoscalar decay constants

$$f_{\pi,K}^R = \left(1 + a\,(\bar{b}_\mu \quad \bar{b}_P)\mathrm{tr}M_q\right) f_{\pi,K}^{bare} + O(a^2).$$

$$\bar{b}_{A,P,\mu} = O(g_0^4)$$

Finite volume effects

- Finite Volume χPT LO corrections (w/ pions & kaons):

$$\frac{m_\pi(L) \quad m_\pi(\,1\,)}{m_\pi(\,1\,)} = \frac{1}{2}\xi_\pi \tilde{g}_1(\lambda_\pi), \qquad \frac{f_\pi(L) \quad f_\pi(\,1\,)}{f_\pi(\,1\,)} = \quad 2\xi_\pi \tilde{g}_1(\lambda_\pi) \quad \xi_K \tilde{g}_1(\lambda_K),$$

$$\frac{m_K(L) \quad m_K(\,1\,)}{m_K(\,1\,)} = 0. \qquad \frac{f_K(L) \quad f_K(\,1\,)}{f_K(\,1\,)} = \quad \frac{3}{4}{}_\pi \tilde{g}_1(\lambda_\pi) \quad \frac{3}{2}\xi_K \tilde{g}_1(\lambda_K).$$

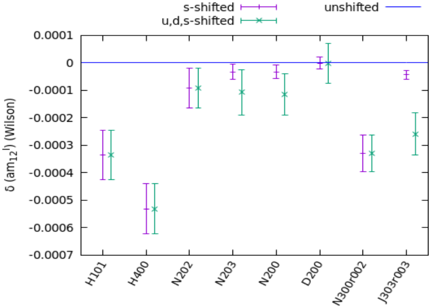
$$\lambda_{\pi,K} = m_{\pi,K} L, \qquad \tilde{g}_1(x) = \sum_{n=1}^{\infty} \frac{4m(n)}{p\,\bar{n}x} K_1(^{p}\,\bar{n}x), \qquad \xi_{\pi,K} = \left(\frac{m_{\pi,K}}{4\pi f_{\pi,K}}\right)^2.$$

- Smallest $m_\pi L$:

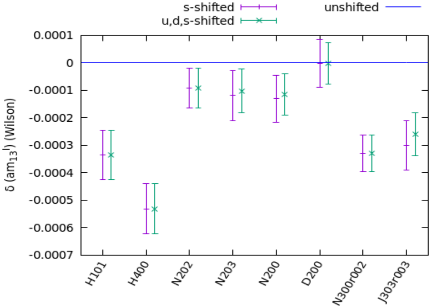
ensemble	m_π [MeV]	$m_\pi L$	af_π	af_π^∞
H105	320	3.9	0.05743(99)	0.05764(99)
D200	200	4.1	0.04225(15)	0.04231(15)

Mass shifting q={u,d,s} vs. q={s} Wilson m_{PCAC}

$$\langle O(m'_{0,q})\rangle = \langle O(m_{0,q})\rangle + \sum_q (m'_{0,q} - m_{0,q}) \frac{d\langle O(m_{0,q})\rangle}{dm_{0,q}}.$$

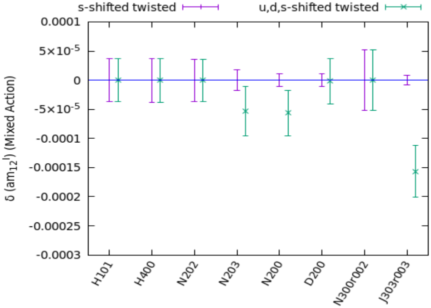


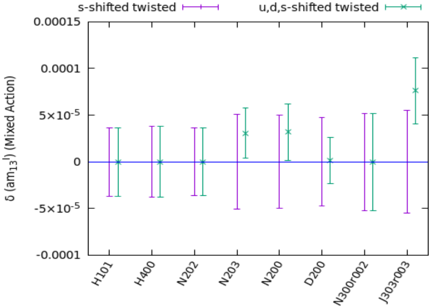
- Wilson data
- Improved light-light PCAC quark masses
- Difference between observables shifted with $q = \{u, d, s\}$ or $q = \{s\}$ (unshifted result as offset)



- Wilson data
- Improved light-strange PCAC quark masses
- Difference between observables shifted with $q = \{u, d, s\}$ or $q = \{s\}$ (unshifted result as offset)

$$hO(m'_{0,q})i = hO(m_{0,q})i + \sum_q (m'_{0,q} \quad m_{0,q}) \frac{d\, hO(m_{0,q})i}{dm_{0,q}}.$$

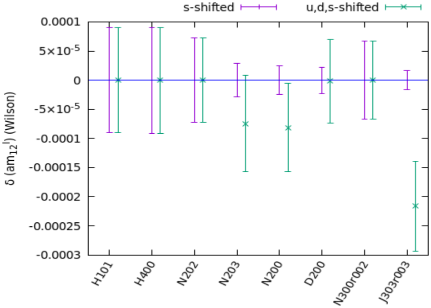


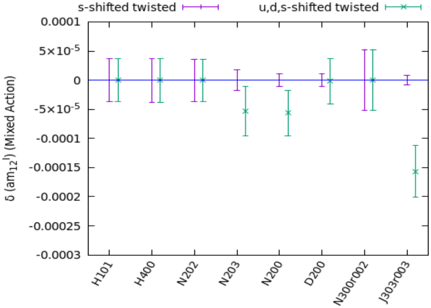


- Wtm data
- Twisted light masses after matching
- Difference between observables shifted with $q = fu, d, sg$ or $q = fsg$ ($q = fsg$ result as offset)

- Wtm data
- Twisted light-strange masses after matching
- Difference between observables shifted with $q = fu, d, sg$ or $q = fsg$ ($q = fsg$ result as offset)

$$hO(m'_{0,q})i = hO(m_{0,q})i + \sum_q (m'_{0,q} \quad m_{0,q}) \frac{d\, hO(m_{0,q})i}{dm_{0,q}}.$$

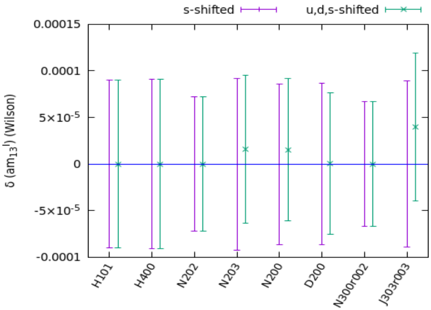


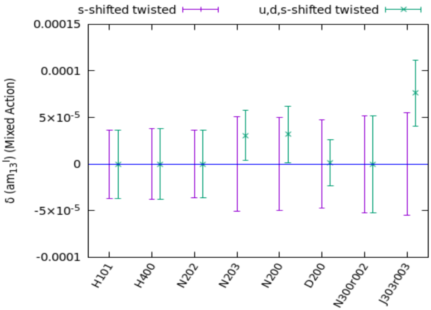


- Wilson data
- Improved light-light PCAC quark masses
- Difference between observables shifted with $q = fu, d, sg$ or $q = fsg$ ($q = fsg$ result as offset)

- Wtm data
- Twisted light masses after matching
- Difference between observables shifted with $q = fu, d, sg$ or $q = fsg$ ($q = fsg$ result as offset)

$$\langle O(m'_{0,q})\rangle = \langle O(m_{0,q})\rangle + \sum_q (m'_{0,q} - m_{0,q}) \frac{d\langle O(m_{0,q})\rangle}{dm_{0,q}}.$$



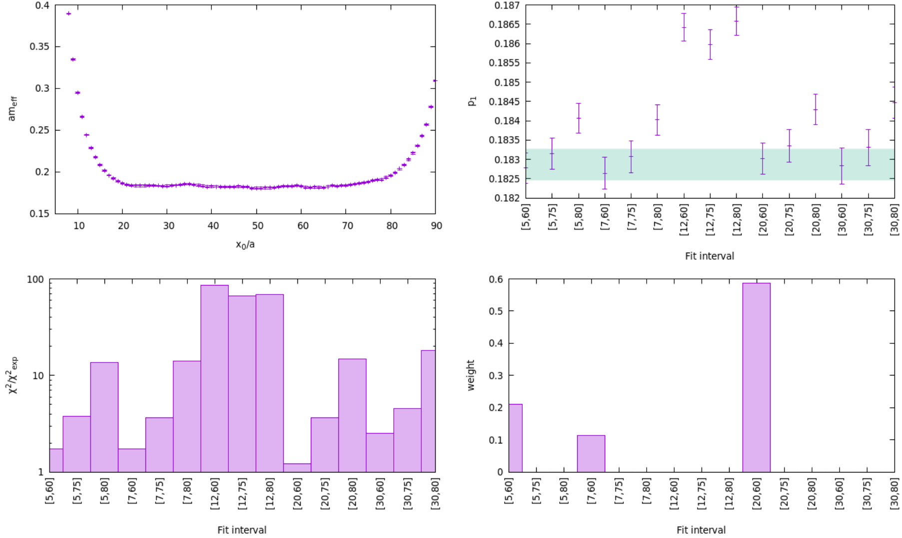


- Wilson data
- Improved light-strange PCAC quark masses
- Difference between observables shifted with $q=\{u,d,s\}$ or $q=\{s\}$ ($q=\{s\}$ result as offset)

- Wtm data
- Twisted light-strange masses after matching
- Difference between observables shifted with $q=\{u,d,s\}$ or $q=\{s\}$ ($q=\{s\}$ result as offset)

Bayesian averages: ground state [Jay, Neil, 2008.01069]

$$\chi_j^2 \; ! \; \chi_j^2 \frac{dof}{\chi^2_{j,exp}}, \qquad IC_j = \chi_j^2 + 2n_{param} + 2n_{cut} \;, \qquad W_j \quad \exp(\quad 0.5 \; IC_j).$$



Tuning grid interpolations I

- Need to match valence and sea

$$\phi_2^{val} \quad \phi_2^{sea},\ \phi_4^{val} \quad \phi_4^{sea}.$$

- Tune to maximal twist

$$am_{12}^{val} \quad 0.$$

- Interpolate from valence tuning grid $(\kappa^{val}, a\mu_l^{val}, a\mu_s^{val})$.

$$\phi_2^{val} = \frac{p_1}{a\mu_l^{val}} \left(\frac{1}{\kappa^{val}} \quad \frac{1}{\tilde{\kappa}_{cr}^{val}} \right)^2 + p_2(a\mu_l^{val}),$$

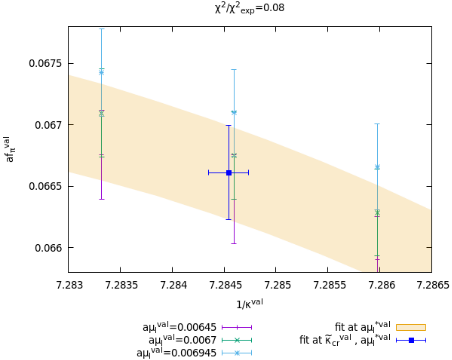
$$\phi_4^{val} = \frac{p_3}{a\mu_l^{val}} \left(\frac{1}{\kappa^{val}} \quad \frac{1}{\tilde{\kappa}_{cr}^{val}} \right)^2 + \frac{p_4}{a\mu_s^{val}} \left(\frac{1}{\kappa^{val}} \quad \frac{1}{\tilde{\kappa}_{cr}^{val}} \right)^2 + p_5(a\mu_l^{val}) + p_6(a\mu_s^{val}),$$

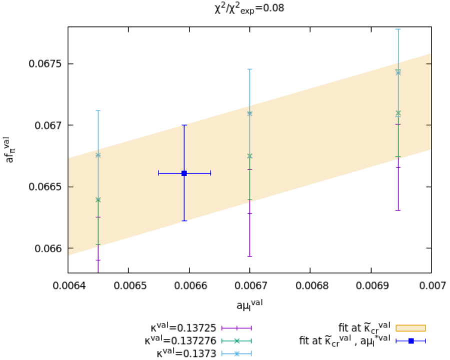
$$am_{12}^{val} = p_7 \left(\frac{1}{\kappa^{val}} \quad \frac{1}{\tilde{\kappa}_{cr}^{val}} \right) + p_8(a\mu_l^{val}).$$

- Interpolate pseudoscalar decay constants to matching point $(\kappa_c^{val}, a\mu_l^{*val}, a\mu_s^{*val})$

$$af_\pi^{val} = r_1 \left(\frac{1}{\kappa^{val}} \quad \frac{1}{\tilde{\kappa}_{cr}^{val}} \right)^2 + r_2 \left(\frac{1}{\kappa^{val}} \quad \frac{1}{\tilde{\kappa}_{cr}^{val}} \right) + r_3 (a\mu_l^{val}) + r_4,$$

$$af_K^{val} = r_1' \left(\frac{1}{\kappa^{val}} \quad \frac{1}{\tilde{\kappa}_{cr}^{val}} \right)^2 + r_2' \left(\frac{1}{\kappa^{val}} \quad \frac{1}{\tilde{\kappa}_{cr}^{val}} \right) + r_3'(a\mu_l^{val}) + r_4'(a\mu_s^{val}) + r_5'.$$



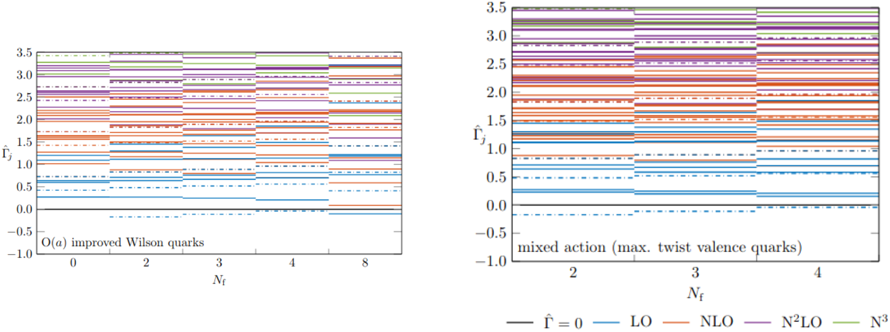


- Cutoff effetcs: [Husung, 2206.03536]

$$a^2\alpha_s^\Gamma(a^{-1}) \qquad a^2\left(\frac{1}{ln(a\Lambda_{QCD})}\right)^\Gamma \quad ! \quad ln\left(a^{guess}\Lambda_{QCD}\right),\quad a^{guess} = \frac{a}{\sqrt{8t_0(\phi_2)}}\sqrt{8t_0^{guess}}.$$

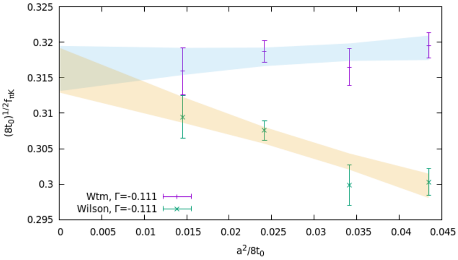
- On symmetric ensembles:

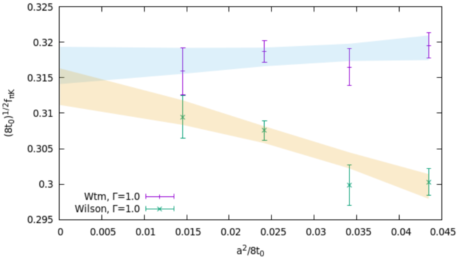
$$\left(\sqrt{8t_0}f_{\pi K}\right)(\phi_2^{sym}) = c_1 + c_2\frac{a^2}{8t_0(\phi_2^{sym})}\left(\quad \frac{1}{ln(a\Lambda_{QCD})}\right)^\Gamma .$$

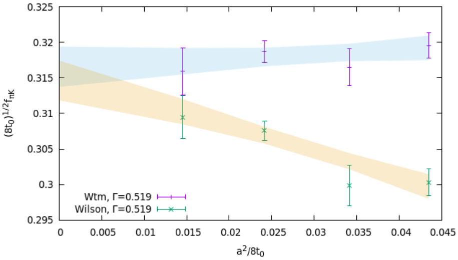


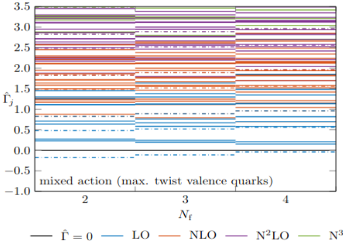
Plot from [Husung, 2206.03536]

- Cutoff effetcs $a^2\alpha_s^{\Gamma}(a^{-1}) \sim -a^2\left[ln(a\Lambda_{QCD})\right]^{-1}$ [Husung, 2206.03536], [FLAG] $\chi^2/\chi^2_{exp} \sim 0.6 - 0.9$.

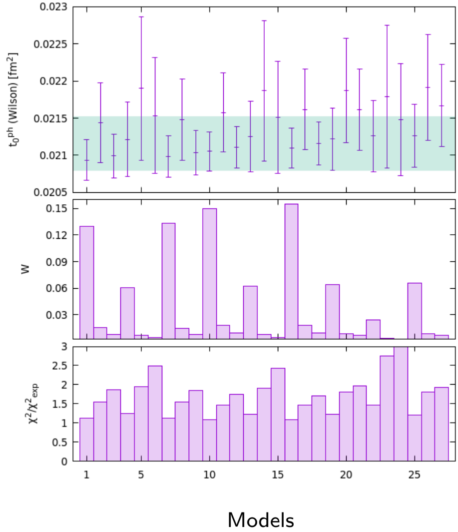








Plot from [Husung, 2206.03536]

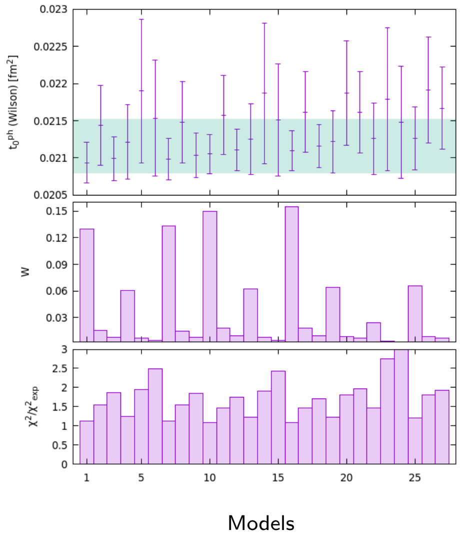


Bayesian model average:

- Continuum ϕ_2 dependence:
  - SU(3) NLO χPT
  - Taylor in $(\phi_2 - \phi_2^{sym})$
- Cutoff dependence:
  - $O(a^2)$
  - $O(a^2 + \phi_2 a^2)$
  - $O(a^2 \alpha_s^{\hat{\Gamma}}(a^{-1}))$
- Cuts in data:
  - Remove $m_\pi = 420$ MeV
  - Remove $a = 0.085$ fm
- Physical inputs:

  f_π^{isoQCD}, f_K^{isoQCD}, $m_{\pi^0}^{exp}$, $m_{K^0}^{exp}$ [FLAG]

  t_0^{guess} ! ϕ_4^{phys} 1.098(10).



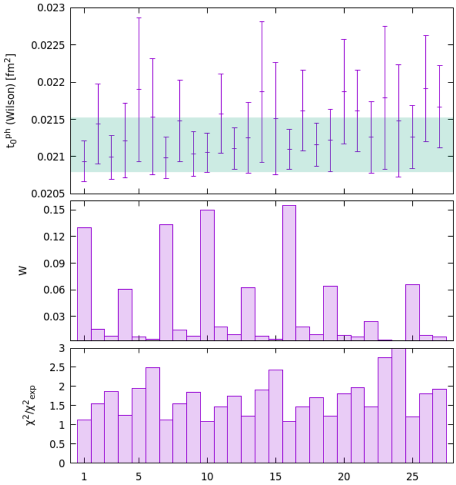
Comparisson to [Straßberger et al. 2112.06696]:

- Continuum ϕ_2 dependence:
  - SU(3) NLO χPT
  - Taylor in $(\phi_2 - \phi_2^{sym})$
- Cutoff dependence:
  - $O(a^2)$
  - $O(a^2 + \phi_2 a^2)$
  - $O(a^2 \alpha_s^{\hat{\Gamma}}(a^{-1}))$
- Cuts in data:
  - Remove $m_\pi = 420$ MeV
  - Remove $a = 0.085$ fm
- Physical inputs:

$$f_\pi^{isoQCD},\ f_K^{isoQCD},\ m_{\pi^0}^{exp},\ m_{K^0}^{exp} \quad \text{[FLAG]}$$

$$t_0^{guess}\ !\ \phi_4^{phys} \quad 1.098(10).$$

Systematic effects: Wilson



Comparisson to [Straßberger et al. 2112.06696]:

- Continuum ϕ_2 dependence:
  - SU(3) NLO χPT
  - Taylor in $(\phi_2 - \phi_2^{sym})$
- Cutoff dependence:
  - $O(a^2)$
  - $O(a^2 + \phi_2 a^2)$
  - $O(a^2 \alpha_s^{\hat{\Gamma}}(a^{-1}))$
- Cuts in data:
  - Remove $m_\pi = 420$ MeV
  - Remove $a = 0.085$ fm
- Physical inputs:

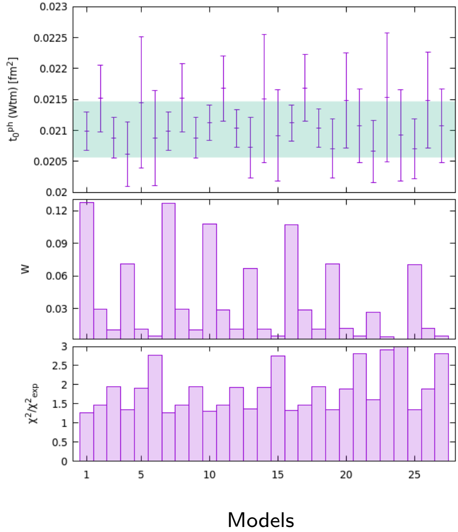
  $f_\pi^{isoQCD}, \ f_K^{isoQCD}, \ m_{\pi^0}^{exp}, \ m_{K^0}^{exp}$ [FLAG]

  t_0^{guess} ! ϕ_4^{phys} 1.098(10).

$$\sqrt{t_0^{ph}} = 0.1447(10)(14) \text{ fm (this work, Wilson)},$$

$$\sqrt{t_0^{ph}} = 0.1443(7)(13) \text{ fm}$$ [Straßberger et al. 2112.06696]

Systematic effects: Wtm



Bayesian model average:

- Continuum ϕ_2 dependence:
  - SU(3) NLO χPT
  - Taylor in $(\phi_2 - \phi_2^{sym})$
- Cutoff dependence:
  - $O(a^2)$
  - $O(a^2 + \phi_2 a^2)$
  - $O(a^2 \alpha_s^{\hat{\Gamma}}(a^{-1}))$
- Cuts in data:
  - Remove $m_\pi = 420$ MeV
  - Remove $a = 0.085$ fm
- Physical inputs:

  f_π^{isoQCD}, f_K^{isoQCD}, $m_{\pi^0}^{exp}$, $m_{K^0}^{exp}$ [FLAG]

  t_0^{guess} ! ϕ_4^{phys} 1.098(10).

Model list I

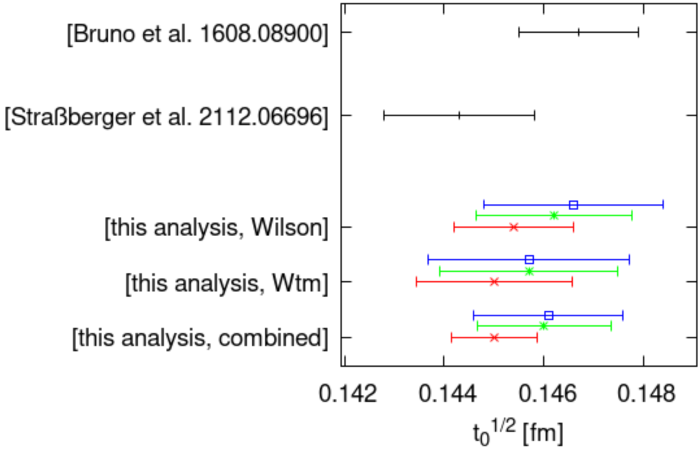
model	$(8t_0 f_{\pi K})^{cont}$	cutoff	cuts
1	$SU(3)$ NLO	$O(a^2)$	–
2	$SU(3)$ NLO	$O(a^2)$	$\beta = 3.40$
3	$SU(3)$ NLO	$O(a^2)$	$m_\pi = 420$ MeV
4	$SU(3)$ NLO	$O(a^2 + \phi_2 a^2)$	–
5	$SU(3)$ NLO	$O(a^2 + \phi_2 a^2)$	$\beta = 3.40$
6	$SU(3)$ NLO	$O(a^2 + \phi_2 a^2)$	$m_\pi = 420$ MeV
7	$SU(3)$ NLO	$O(a^2 \alpha_s^\Gamma)$	–
8	$SU(3)$ NLO	$O(a^2 \alpha_s^\Gamma)$	$\beta = 3.40$
9	$SU(3)$ NLO	$O(a^2 \alpha_s^\Gamma)$	$m_\pi = 420$ MeV
10	$(\phi_2 \quad \phi_2^{sym})^2$	$O(a^2)$	–
11	$(\phi_2 \quad \phi_2^{sym})^2$	$O(a^2)$	$\beta = 3.40$
12	$(\phi_2 \quad \phi_2^{sym})^2$	$O(a^2)$	$m_\pi = 420$ MeV
13	$(\phi_2 \quad \phi_2^{sym})^2$	$O(a^2 + \phi_2 a^2)$	–

Model list II

model	$(8t_0 f_{\pi K})^{cont}$	cutoff	cuts
14	$(\phi_2 \quad \phi_2^{sym})^2$	$O(a^2 + \phi_2 a^2)$	$\beta = 3.40$
15	$(\phi_2 \quad \phi_2^{sym})^2$	$O(a^2 + \phi_2 a^2)$	$m_\pi = 420$ MeV
16	$(\phi_2 \quad \phi_2^{sym})^2$	$O(a^2 \alpha_s^\Gamma)$	–
17	$(\phi_2 \quad \phi_2^{sym})^2$	$O(a^2 \alpha_s^\Gamma)$	$\beta = 3.40$
18	$(\phi_2 \quad \phi_2^{sym})^2$	$O(a^2 \alpha_s^\Gamma)$	$m_\pi = 420$ MeV
19	$(\phi_2 \quad \phi_2^{sym})^2, (\phi_2 \quad \phi_2^{sym})^3$	$O(a^2)$	–
20	$(\phi_2 \quad \phi_2^{sym})^2, (\phi_2 \quad \phi_2^{sym})^3$	$O(a^2)$	$\beta = 3.40$
21	$(\phi_2 \quad \phi_2^{sym})^2, (\phi_2 \quad \phi_2^{sym})^3$	$O(a^2)$	$m_\pi = 420$ MeV
22	$(\phi_2 \quad \phi_2^{sym})^2, (\phi_2 \quad \phi_2^{sym})^3$	$O(a^2 + \phi_2 a^2)$	–
23	$(\phi_2 \quad \phi_2^{sym})^2, (\phi_2 \quad \phi_2^{sym})^3$	$O(a^2 + \phi_2 a^2)$	$\beta = 3.40$
24	$(\phi_2 \quad \phi_2^{sym})^2, (\phi_2 \quad \phi_2^{sym})^3$	$O(a^2 + \phi_2 a^2)$	$m_\pi = 420$ MeV
25	$(\phi_2 \quad \phi_2^{sym})^2, (\phi_2 \quad \phi_2^{sym})^3$	$O(a^2 \alpha_s^\Gamma)$	–
26	$(\phi_2 \quad \phi_2^{sym})^2, (\phi_2 \quad \phi_2^{sym})^3$	$O(a^2 \alpha_s^\Gamma)$	$\beta = 3.40$
27	$(\phi_2 \quad \phi_2^{sym})^2, (\phi_2 \quad \phi_2^{sym})^3$	$O(a^2 \alpha_s^\Gamma)$	$m_\pi = 420$ MeV

Model averaging: effect of modifying the model weights

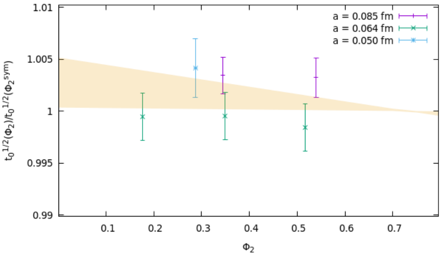
- $IC = \chi^2 + 2n_{param} + 2n_{cut}$,
- $IC = \chi^2 + 2n_{param}$,
- $IC = \chi^2$,



Scale setting: symmetric point

- Fit [Straßberger et al. 2112.06696]

$$\frac{\sqrt{t_0(\phi_2)/a^2}}{\sqrt{t_0(\phi_2^{sym})/a^2}} = \sqrt{1 + p_1(\phi_2 \quad \phi_2^{sym})} \;\; ! \;\; \sqrt{t_0^{sym}} = \frac{\sqrt{t_0^{ph}}}{\sqrt{1 + p_1(\phi_2^{ph} \quad \phi_2^{sym})}}.$$



$$\sqrt{t_0^{sym}} = 0.1450(11)(7) \text{ fm (Wilson)},$$
$$\sqrt{t_0^{sym}} = 0.1446(12)(10) \text{ fm (Wtm)},$$
$$\sqrt{t_0^{sym}} = 0.1446(9)(3) \text{ fm (combined)}.$$

Twist angles

- Light twist angle $\omega_l = \pi/2$ by construction (maximal twist)
- Interpolate am_{34}^{val} to matching & maximal twist point $(\tilde{\kappa}_{cr}^{val}, a\mu_l^{*val}, a\mu_s^{*val})$
- Complementary strange twist angle $\theta = \pi/2 \quad \omega_{strange}$

$$tan\ \theta_{strange} = \frac{Z_A}{Z_P} \frac{1}{Z_P^{-1}} \frac{am_{34}^{val}(\tilde{\kappa}_{cr}^{val}, a\mu_l^{*val}, a\mu_s^{*val})}{a\mu_s^{*val}}.$$