Hadronic vacuum polarization from step scaling

Lattice 2022

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Contents

Adler function from low to high energies

Hadronic vacuum polarization

- The hadronic contributions to $\alpha_{em}(M_Z^2)$ are given by the HVP $\Pi(M_Z^2)$

$$\Pi(Q^2)(Q^2\delta_{\mu\nu} - Q_\mu Q_\nu) = \Pi_{\mu\nu}(Q^2) = \int d^4x e^{iQ\cdot x}\langle j_\mu(x) j_\nu(0)\rangle$$

- Interested in the difference

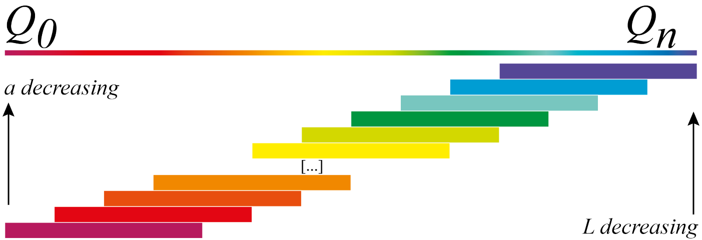
$$\Pi(Q_n^2) - \Pi(Q_0^2)$$
$$Q_0 \sim 1\,\text{GeV},\ \ Q_n \sim 100\,\text{GeV}$$

  - Computationally unfeasible
  - Use a strategy similar to step scaling [Lüscher, Weisz, Wolff, Sommer ...]
  - Divide the range into many steps $Q_0, Q_1 = 2Q_0, ..., Q_n = 2^n Q_0$

Strategy

- Discrete Adler function:

$$\Delta(Q^2) = \Pi(4Q^2) - \Pi(Q^2)$$
$$\Pi(Q_n^2) - \Pi(Q_0^2) = \Delta(Q_0^2) + \Delta(Q_1^2) + \ldots + \Delta(Q_n^2)$$



- Compute each step in decreasing volumes
- Higher energies are less sensitive to FV effects

Test the strategy in QED_2

- Partition function:

$$Z = \int [dU] \exp(-S_g[U]) \det M(m, U)$$

$$S_g = \beta \sum_{x \in \Lambda} \mathrm{Re}\, U_{x,0} U_{x+\hat{0},1} U^*_{x+\hat{1},0} U^*_{x,1}$$

$$M = \sum_{\mu=0,1} \eta_{x,\mu} \nabla_\mu + m$$

- Line of constant physics (LCP) defined via

$$\beta m^2 = 0.8$$

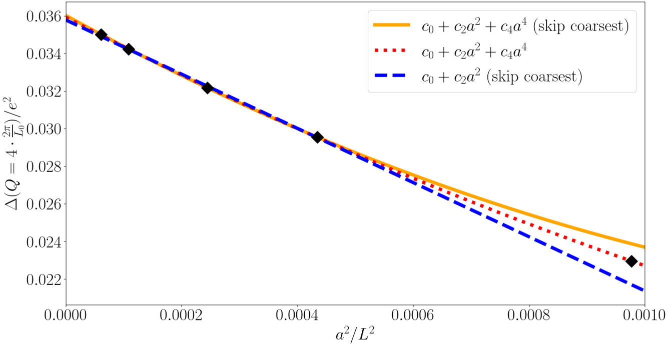
- Volumes given by

$$m L_0 = 16, \;\; L_0 / L_n = 2^n$$

- Metropolis + top. update, reweighting

Continuum Extraplation

- Cont. extrapolation from $L/a = 32, 48, 64, 96, 128$ ($\beta \propto a^{-2}$, $m \propto a$)
- Sweet spot for lattice artifacts and FV effects, $Q = 4 \cdot \frac{2\pi}{L}$
- No logarithmically enhanced lattice artefacts $a^2 \log(a^2)$
- Systematic error from different cont. extrapolations

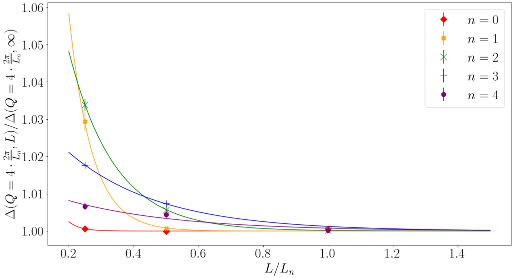


Finite volume effects

- Same momentum in three different volumes

$$Q_n = 4 \cdot \frac{2\pi}{L_n} = 2 \cdot \frac{2\pi}{L_{n+1}} = 1 \cdot \frac{2\pi}{L_{n+2}}$$

- FV effects $\propto e^{-m_\pi L}$
- $Q = 4 \cdot \frac{2\pi}{L} \Rightarrow$ FV effects below 0.05%

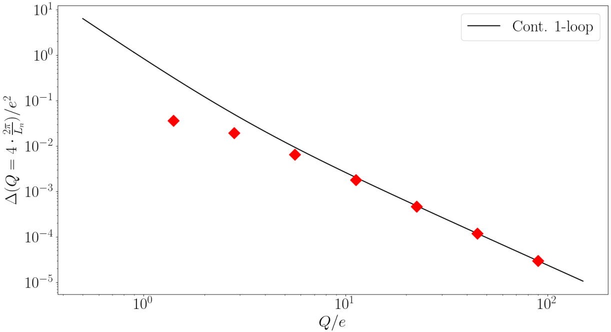


Discrete Adler function QED_2

- 1-loop PT [Adams'98]:

$$\Pi(q^2)_{PT}/e^2 = \frac{1}{\pi}\frac{1}{q^2}\left(1 + \frac{2m^2}{q^2}\frac{1}{R}\log\frac{1+R}{1-R}\right) \quad R = \sqrt{1-\frac{4m^2}{Q^2}}$$

- For large Q^2 $\Delta(Q^2) \propto \frac{1}{Q^2}$
- Deviation from higher order effects (checked by perfroming simulations with $e \to 0$)



Discrete Adler function QED_2

Q/e	$\Delta(Q)/e^2$	stat.$[10^{-6}]$	cont. extrap.$[10^{-6}]$	FV $[10^{-6}]$	total$[10^{-6}]$
1.405	0.036009	31	16	18	37
2.810	0.019362	17	7	10	13
5.620	0.006515	4	< 1	10	10
11.240	0.001807	< 1	1	4	4
22.480	0.000467	< 1	< 1	< 1	< 1
44.960	0.000118	< 1	< 1	< 1	< 1
89.920	0.000029	< 1	< 1	< 1	< 1

- Adding up gives final result:

$$\Pi(2^{14}Q_0^2) - \Pi(Q_0^2) = 0.064308[41, 0.06\%]$$

- Here we used analytic LCP (known in QED_2)

Lines of constant physics $(\beta(a),\ m(\beta))$

Determination of the LCP

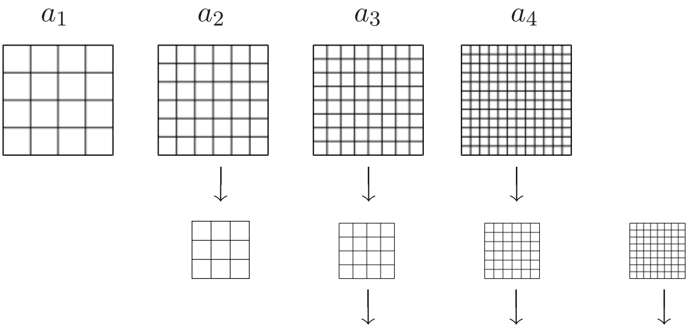
- General strategy to determine $m(\beta)$ and $\beta(a)$ twofold:

(A) Determine $\beta(a)$ to ensure that we will have in all of our steps the same physical volume

(B) Use Δ (Adler function) to fix m as a function of β

(C) Please note that this should be done simultaneously

General strategy



1. Assume we know $\beta(a_1) = \beta_1,\ \beta(a_2) = \beta_2,\ \beta(a_3) = \beta_3$
2. Determine $\beta(a_4)$ with cont. extrapolation of an Observable $\langle O \rangle$ sensitive to β
3. Halve phys. size $L_{2,3,4}/2$ and switch to $\langle O \rangle$ with smaller FV effects
4. Shift $a_2 \to a_1,\ a_3 \to a_2,\ a_4 \to a_3$ and go to 1.

Choosing the observables

- Observable sensitive to β: $t_0^{(n)}$-scale from gradient flow

$$\frac{d}{dt}U_t(x,\mu) = -\left[\partial_{x,\mu}\frac{1}{\beta}S_g(U_t)\right]U_t(x,\mu)$$

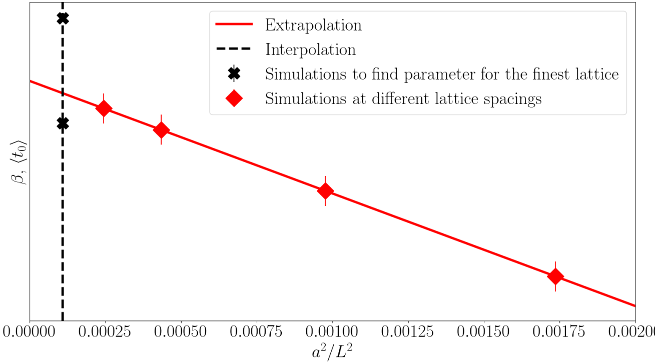
$$t_0^{(n)} \cdot E(U_{t_0^{(n)}})/(L/a)^2 = 0.375 \cdot \left(\frac{1}{2}\right)^{2n}$$

- Observable sensitive to m:

$$\Delta^{(n)} = \Delta(Q_n^2, 2^n m)$$

Mass doubling favors mass dependence of the discrete Adler function

Determination of the LCP



- Intersection point gives the LCP
- Generalization to D parameters possible (D observables needed)

Estimation of uncertainties

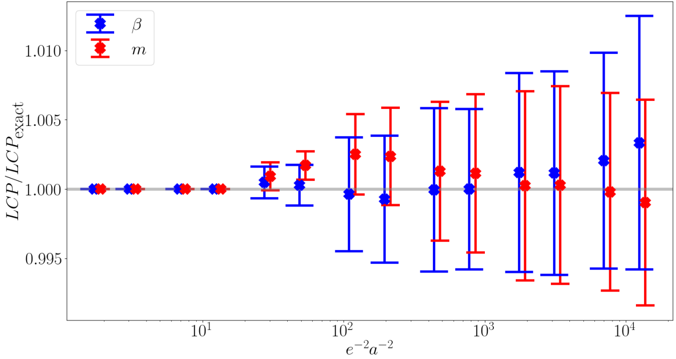
- Systematic uncertainties from continuum extrapolation
- Many extrapolations weighted with AIC:

$$\exp\left\{-\frac{1}{2}(\chi^2 + 2n_f - n_p)\right\}$$

- Recursive structure $\Rightarrow$ Dependence on the previous estimations
  - 36 different fits $\Rightarrow 36^6 \approx 10^{10}$ fits after 6 steps
  - We reduce this number by choosing a handful of random representants after every step
- Statistical uncertainties from a Jackknife analysis
- Whole procedure repeated on the $N_J = 48$ Jackknife bins

Results

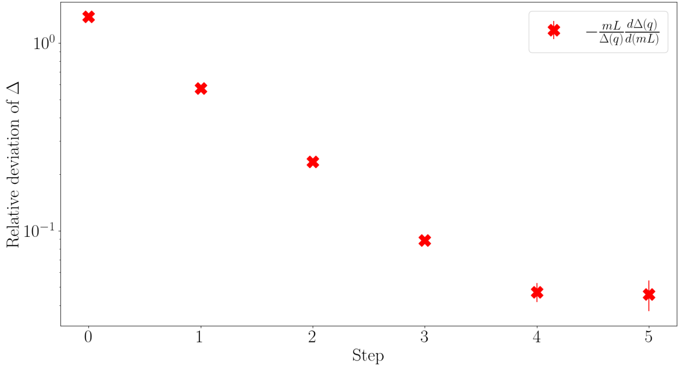
Results for the LCP in QED_2



- Good agreement with exact results
- Uncertainties $\lesssim 1\%$

How LCP uncertainty effects Adler function?

- Uncertainty from $\delta\beta(a)$ estimated from $\frac{1}{Q^2}$ dependence
- Uncertainty from $\delta m(a)$ estimated from measuring mass dependence of $\Delta(Q^2)$



Results and Conclusion

- The final results of the total study:

$$\Pi\left(2^{14}Q_0^2\right) - \Pi(Q_0^2) = 0.064308[186, 0.3\%]$$

- Error budget:

Stat. error:	$32 \cdot 10^{-6}$	0.05%
Finite volume error:	$14 \cdot 10^{-6}$	0.02%
Cont. Extrap. error:	$19 \cdot 10^{-6}$	0.03%
β estimation error:	$155 \cdot 10^{-6}$	0.24%
m estimation error:	$95 \cdot 10^{-6}$	0.14%

- Seems to work in QED_2
- Does the method work in QCD to calculate physical observables?

Thank you for your attention!

I am happy to answer any questions you have!

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