Finite-volume effects in a calculation of the rare Hyperon decay $\Sigma^+ \to p\ell^+\ell^-$

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Motivation

- Rare $\Sigma^+ \to p\ell^+\ell^-$ motivated by Raoul Hodgson in previous talk:
  - LHCb measurement
  - baryonic analogue of rare $K \to \pi\ell^+\ell^-$
  - Flavour-changing neutral current $\Rightarrow$ potentially sensitive to new physics
- We must consider effects of simulating in a finite volume L
  - known formalism for rare kaon [Christ et al. (2015)]
    and for general case [Briceño et al. (2020)]
  - explicit formulation for $\Sigma^+ \to p\ell^+\ell^-$ requires careful considerations

Related RBC/UKQCD talks, both in this session:

- Ryan Hill: Rare $K^+ \to \pi^+\ell^+\ell^-$ decays with physical mass light-quarks
- Raoul Hodgson, Progress on the exploratory calculation of the rare Hyperon decay $\Sigma^+ \to p\ell^+\ell^-$

QUICK RECAP

$\Sigma^+ \to p\ell^+\ell^-$ amplitude

$$\mathcal{A}_\mu^{rs} = \int d^4x \, \langle p(\mathbf{p}), r | T[\mathcal{H}_W(x) J_\mu(0)] | \Sigma^+(\mathbf{k}), s \rangle \, , \quad (\mathcal{A}_\mu^{rs} = \bar{u}^r \tilde{\mathcal{A}}_\mu u^s)$$

with azimuthal *spin* components r, s.
$\tilde{\mathcal{A}}_\mu$ can be extracted from finite-volume estimator

$$F_\mu(\mathbf{k}, \mathbf{p})_L = i \int_0^\infty d\omega \, \frac{\rho_\mu(\omega)_L}{E_\Sigma(\mathbf{k}) - \omega} - i \int_0^\infty d\omega \, \frac{\sigma_\mu(\omega)_L}{\omega - E_\Sigma(\mathbf{k})}$$

where e.g.

$$\rho_\mu(\omega)_L = \sum_n \frac{C_{n,\mu}(\mathbf{k})}{2E_n(\mathbf{k})} \delta(E_n(\mathbf{k}) - \omega)$$

$$\bar{u}_p^r(\mathbf{p}) C_{n,\mu}(\mathbf{k}) u_\Sigma^s(\mathbf{k}) = \langle p(\mathbf{p}) | J_\mu(0) | E_n, \mathbf{k} \rangle_L \langle E_n, \mathbf{k} | \mathcal{H}_W(0) | \Sigma^+(\mathbf{k}) \rangle_L$$

with finite-volume energy states $|E_n, \mathbf{k}\rangle_L$.

$N\pi$ SYSTEM

- Finite-volume effects known for $K_L - K_S$ [Christ et al. (2015)]
- in there, driven by $\pi\pi$ intermediate state
- In our case, FV effects arise from $N\pi$ state
- formalism in principle known for general case (including this one) [Briceño et al. (2020)]
- practical challenges in the $N\pi$ system:
  - parity
  - non-degenerate masses m_π, m_N
  - spin

  $\Rightarrow$ index space:
    - J total angular momentum
    - $S = \frac{1}{2}$ spin (omitted in notation)
    - μ azimuthal component of total angular momentum
    - ℓ orbital angular momentum

PARITY

The weak Hamiltonian decomposes

$$\mathcal{H}_W(x) = \mathcal{H}_W^+(x) + \mathcal{H}_W^-(x)$$

into parity-positive and parity negative part. In a finite-volume matrix element like

$$\langle E_n, \mathbf{0} | \mathcal{H}_W(0) | \Sigma^+(\mathbf{0}) \rangle_L$$

this leads to

$$\begin{aligned} \mathcal{H}_W^+ | \Sigma^+ \rangle_L &: \quad \text{transforms in } G_1^+ \text{ irrep} \\ \mathcal{H}_W^- | \Sigma^+ \rangle_L &: \quad \text{transforms in } G_1^- \text{ irrep} \end{aligned}$$

$\Rightarrow$ necessitates scattering study of two separate sets of finite-volume states $|E_n, \mathbf{0}\rangle_L$

FINITE-VOLUME $\rightarrow$ INFINITE-VOLUME AMPLITUDE

- $F_\mu(\mathbf{k}, \mathbf{p})_L$ contains poles in volume L, when $E_n(L) \rightarrow E_\Sigma$

$$\int_0^\infty d\omega \, \frac{\rho_\mu(\omega)_L}{E_\Sigma - \omega} = \sum_n \frac{C_{n,\mu}}{2E_n(L)\big(E_\Sigma - E_n(L)\big)} \, .$$

- infinite-volume amplitude can be obtained via

$$\tilde{\mathcal{A}}_\mu(k, p) = F_\mu(\mathbf{k}, \mathbf{p})_L + \Delta F_\mu(\mathbf{k}, \mathbf{p})_L$$

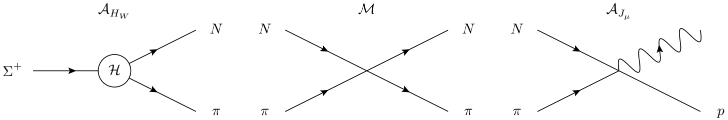
- $\tilde{\mathcal{A}}_\mu(k, p)$ does not contain poles

$\Rightarrow$ the poles in $F_\mu(\mathbf{k}, \mathbf{p})_L, \Delta F_\mu(\mathbf{k}, \mathbf{p})_L$ must cancel exactly

FINITE-VOLUME CORRECTION

$$\Delta F_{\mu}(\mathbf{k},\mathbf{p})_L = i\mathcal{A}_{J_\mu}(E_\Sigma,\mathbf{k},\mathbf{p})\mathcal{F}(E_\Sigma,\mathbf{k},L)\mathcal{A}_{H_W}(E_\Sigma,\mathbf{k})$$

$$\mathcal{F}(E,\mathbf{P},L) = \frac{1}{F(E,\mathbf{P},L)^{-1} + \mathcal{M}(E_{\mathrm{cm}})}$$



$\Rightarrow$ Knowledge of these 3 amplitudes and mathematical function F^{-1} allows to determine $\tilde{\mathcal{A}}_{\mu}(k,p)$ from finite-volume result $F_{\mu}(\mathbf{k},\mathbf{p})_L$

$F(E, \mathbf{P}, L)^{-1}$ FOR SPIN $1/2$

For two non-identical **scalar** particles with masses m_π, m_N

$$F_{\ell' m', \ell m}(E, \mathbf{P}, L) = \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \right] f_{\ell' m', \ell m}(E, \mathbf{P}, \mathbf{k}, m_\pi, m_N)$$

which can be promoted to the case of particles with spin

$$F_{J'\ell'\mu', J\ell\mu}(E, \mathbf{P}, L) = \sum_{m,\sigma,m'} \langle \ell m; \frac{1}{2}\sigma | J\mu \rangle \langle \ell' m'; \frac{1}{2}\sigma | J'\mu' \rangle F_{\ell' m', \ell m}(E, \mathbf{P}, L)$$

$N\pi$ scattering amplitude

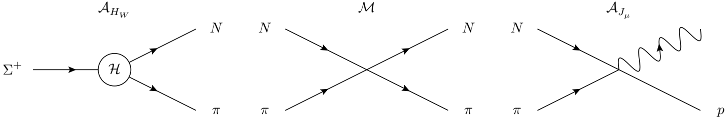
$$\mathcal{M}_{J'\ell'\mu',J\ell\mu}(E_{\mathrm{cm}}) = \delta_{J'J}\delta_{\ell'\ell}\delta_{\mu'\mu}\frac{8\pi E_{\mathrm{cm}}}{p\cot\delta_{J,\ell}(p) - ip}$$

transition amplitudes from Lellouch-Lüscher formalism near finite-volume energies E_n

$$\lim_{E\to E_n(L)}\left(E - E_n(L)\right)\frac{1}{\mathcal{M}(E_n^{\mathrm{cm}}) + F^{-1}(E_n, \boldsymbol{P}, L)} = \mathcal{E}^{(n),\mathrm{in}} \otimes \mathcal{E}^{(n),\mathrm{out}}$$

$$\left\langle E_n, \boldsymbol{k}\middle|\mathcal{H}_W(0)\middle|\Sigma^+(\boldsymbol{k}), s\right\rangle_L = \mathcal{E}^{(n),\mathrm{out}} \cdot \mathcal{A}_{H_W}(E, \boldsymbol{k})\; u_\Sigma^s(\boldsymbol{k})$$

$$\left\langle p(\boldsymbol{p}), r\middle|J_\mu(0)\middle|E_n, \boldsymbol{k}\right\rangle_L = \bar{u}_p^r(\boldsymbol{p})\;\mathcal{A}_{J_\mu}(E, \boldsymbol{k}, \boldsymbol{p}) \cdot \mathcal{E}^{(n),\mathrm{in}}$$



SINGLE-CHANNEL CASE, $\mathbf{P} = \mathbf{0}$

$$\mathcal{M}(E) = \frac{8\pi E}{p}\frac{1}{\cot\delta(E) - i}\,, \qquad F(E,L) = \frac{p}{8\pi E}\left[\cot\phi(E,L) + i\right]$$

- pseudophase $\phi(E,L)$
- scattering phase shift $\delta^{\ell=1}(E)$

We can then relate finite- and infinite-volume matrix elements

$$\bar{u}_p^r(\boldsymbol{p})\,\mathcal{A}_\mu(E_n,\mathbf{0},\boldsymbol{p})\mathcal{A}_{H_W}(E_n,\mathbf{0})\,u_\Sigma^s(\mathbf{0}) = \left\langle p(\boldsymbol{p}),r\middle|J_\mu(0)\middle|E_n,\mathbf{0}\right\rangle_L$$

$$\times\frac{4\pi}{p}e^{+2i\delta(E)}\left[\frac{\partial\phi(E,L)}{\partial E} + \frac{\partial\delta(E)}{\partial E}\right]_{E=E_n}\left\langle E_n,\mathbf{0}\middle|\mathcal{H}_W(0)\middle|\Sigma^+(\mathbf{0}),s\right\rangle_L$$

COMPARISON TO $K_L - K_S$ EXAMPLE

In the single-channel case, $\mathbf{P} = \mathbf{0}$, the result is equivalent to the one from [Christ et al. (2015)] [1]

$$\Delta F_\mu(\mathbf{0}, \boldsymbol{p})_L = -i \frac{p_\Sigma}{8\pi M_\Sigma} \mathcal{A}_\mu(M_\Sigma, \mathbf{0}, \boldsymbol{p}) e^{-2i\delta(M_\Sigma)} \mathcal{A}_{H_W}(M_\Sigma, \mathbf{0})$$
$$\times \left(\cot\left[\delta(M_\Sigma) + \phi(M_\Sigma, L)\right] + i \right)$$

[1] here, the imaginary part is kept as part of physical amplitude, but not included in $K_L - K_S$.

VOLUME TUNING

For an ensemble with a carefully tuned volume $\bar{L}$, like the one considered in [Christ et al. (2015)], which satisfies

$$E_{\bar{n}}(\bar{L}) = M_{\Sigma}$$

for the $\bar{n}^{th}$ finite-volume energy level, we can show that

$$\begin{aligned} F_{\mu}(\mathbf{0}, \boldsymbol{p})_L &= \mathcal{O}(\delta L^{-1}) \\ \Delta F_{\mu}(\mathbf{0}, \boldsymbol{p})_L &= \mathcal{O}(\delta L^{-1}) \\ F_{\mu}(\mathbf{0}, \boldsymbol{p})_L + \Delta F_{\mu}(\mathbf{0}, \boldsymbol{p})_L &= \mathcal{O}(\delta L^{0}) \end{aligned}$$

i.e. the leading singularity cancels around $\delta L = L - \bar{L}$.
Furthermore, we computed the $\mathcal{O}(\delta L^1)$ correction term (given in arXiv: 2208.XXXXX, Mathematica version available on request)

CONCLUSIONS

- finite-volume correction to rare-hyperon decay $\Sigma^+ \to p\ell^+\ell^-$
  - non-degenerate masses m_N, m_π
  - spin $\langle \ell m; \frac{1}{2}\sigma | J\mu \rangle$
  - parity $\mathcal{H}_W(x) = \mathcal{H}_W^+(x) + \mathcal{H}_W^-(x)$
- single-channel case, $\mathbf{P} = \mathbf{0}$ case reproduces known result from $K_L - K_S$ [Christ et al. (2015)]
- leading-order cancellation, $\mathcal{O}(\delta L)$ correction term available
- exploratory $\Sigma^+ \to p\ell^+\ell^-$ lattice computation underway, see previous talk by Raoul Hodgson
  - FV effects relevant at the physical point

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