# Hadronic contributions to the running of $\alpha_{\mathrm{QED}}$ from Lattice QCD

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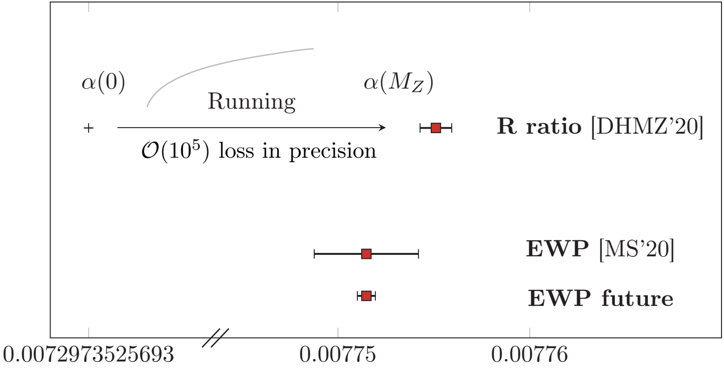
# Introduction

$\alpha(M_Z)$ important input parameter for electroweak precision tests




| | $\delta\alpha(M_Z)/\alpha(M_Z)$ |
|---|---|
| current precision | $\sim 10^{-4}$ |
| precision needed for future colliders | $\sim 5 \times 10^{-5}$ |

$\alpha(M_Z)$ one of the least well determined SM input parameters

# The running of $\alpha_{\mathrm{QED}}$

The running QED coupling $\alpha$ is given by

$$\alpha(s) = \frac{\alpha(0)}{1 - \Delta\alpha(s)} \,, \qquad \Delta\alpha(s) = \underbrace{\Delta\alpha_{\mathrm{lep}}(s)}_{\mathcal{O}(\alpha^4)} + \Delta\alpha^{(5)}_{\mathrm{had}}(s) + \underbrace{\Delta\alpha_{\mathrm{top}}(s)}_{\mathcal{O}(\alpha_s^3)}$$


$$\Delta\alpha(q^2)_{\mathrm{had}} = 4\,\pi\,\alpha\hat{\Pi}(q^2)_{\mathrm{had}} = 4\,\pi\,\alpha\left(\Pi(q^2) - \Pi(0)\right)_{\mathrm{had}}$$

$$-i\Pi_{\mu\nu}(q) = (ie)^2 \int d^4x e^{iq\cdot x} \langle 0|\,T\{J_\mu(x)J_\nu(0)\}|0\rangle = \underbrace{\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi\left(q^2\right)}_{\text{Lorentz inv. and current conservation}}$$

with $J_\mu(x) \equiv \sum_{f=u,d,s,c,\ldots} q_f \bar{\psi}_f(x)\gamma_\mu\psi_f(x)$

Largest uncertainty comes from non-perturbative effects in hadronic contribution in low energy region

# HVP: On the lattice

Study time evolution of Euclidean correlation function ($Q^2 = -q^2$) with
$J_\mu/e = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c$ electromagnetic current

$$C(t) = \frac{a^3}{3}\sum_{i=1}^{3}\sum_{\vec{x}}\langle J_i(x)J_i(0)\rangle = \underbrace{C^l(t) + C^s(t) + C^c(t) + C^{\text{disc}}(t)}_{\text{very different systematic errors}}$$

- Define $\forall Q\in\mathbb{R}$ [Bernecker et al '11]

$$\hat{\Pi}_{TL}(Q^2) \equiv \Pi_{TL}(Q^2) - \Pi^f_{TL}(0) = \frac{1}{3}\sum_{i=1}^{3}\frac{\Pi_{ii,TL}(0) - \Pi_{ii,TL}(Q)}{Q^2} - \Pi^f_{TL}(0)$$

$$= a\sum_{t=0}^{T-a}\text{Re}\left[\frac{e^{iQt} - 1}{Q^2} + \frac{t^2}{2}\right]\text{Re}\,C_{TL}(t)$$

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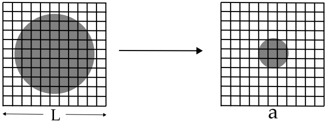
gives running: 

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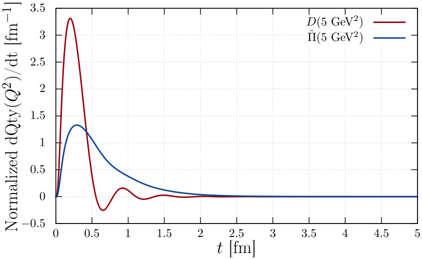
**Adler function**: single scale $Q^2$

$$D(Q^2) \equiv 12\pi^2 Q^2 \frac{\mathrm{d}\hat{\Pi}(Q^2)}{\mathrm{d}Q^2} = 24\pi^2 a \sum_t \underbrace{\left[-\frac{t\,\sin(Qt)}{2Q} - \frac{\cos(Qt)-1}{Q^2}\right]}_{k(t,Q^2)} C(t)$$



# Logarithmically enhanced discretization errors

Discretization errors for on-shell staggered quantities behave as

$$M(a) = M(0)\left\{1 + a^2 \sum_{n=0}^{\infty} c_n \alpha_s^n(1/a) + \mathcal{O}\left(a^4\right)\right\}$$

$D(Q^2)$ receives logarithmically enhanced $\mathcal{O}(a^2)$ artefacts [Cè et al '21]

$$D\left(Q^2, a\right) = D\left(Q^2, 0\right)\left\{1 + (aQ)^2 \sum_{n=0}^{\infty} c_n \alpha_s^n\left(\frac{1}{a}\right) + (aQ)^2 \ln(aQ)^2 \Gamma_0 + \mathcal{O}(a^4)\right\}$$

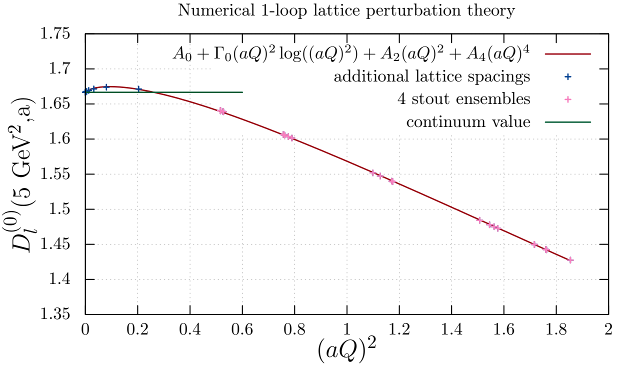
These arise from small separations between the currents

$$C(t) \overset{t\to 0}{\sim} \frac{1}{t^3}\left(\alpha_0 + \alpha_1 \frac{a^2}{t^2} + \alpha_2 \frac{a^4}{t^4} + \cdots\right)$$

$$D\left(Q^2, a\right)\Big|_{a^2} \sim \int_a \mathrm{d}t\ Q^2 t^4 C(t) \sim a^2 Q^2 \int_a \frac{\mathrm{d}t}{t} \sim \frac{1}{2} a^2 Q^2 \ln a^2$$

# Controlling the logarithmically enhanced term

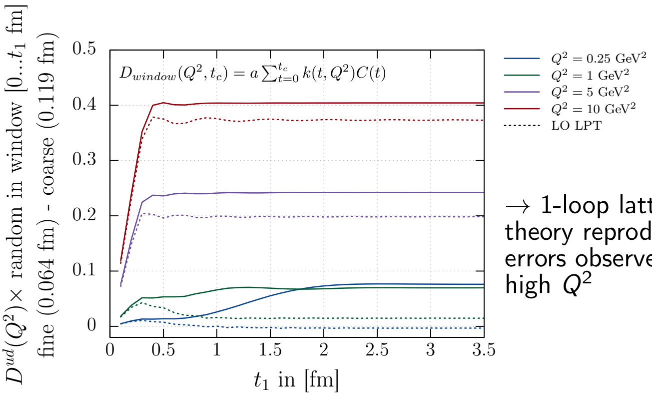
$$D\left(Q^2, a\right) = D\left(Q^2, 0\right)\left\{1 + (aQ)^2 \sum_{n=0}^{\infty} c_n \alpha_s^n \left(\frac{1}{a}\right) + (aQ)^2 \ln(aQ)^2 \Gamma_0 + \mathcal{O}(a^4)\right\}$$




Confirms leading coefficient of $\ln((aQ)^2)$ that we computed analytically in lattice perturbation theory: $\Gamma_0 = -1/30$

# I: Removal of LO discretization effects

$$\tilde{D}\left(Q^2,a\right) = D\left(Q^2,0\right)\left\{1 + (aQ)^2 \sum_{n=1}^{\infty} c_n \alpha_s^n \left(\frac{1}{a}\right) + (aQ)^2 \cancel{\ln(aQ)^2}\Gamma_0 + \mathcal{O}(a^4 \alpha_S(1/a))\right\}$$




→ 1-loop lattice perturbation theory reproduces well discretization errors observed in simulations at high $Q^2$

⇒ reduce discretization errors of simulation results via

$$D(Q^2,a) \to \tilde{D}(Q^2,a) = D(Q^2,a) + D^{(0)}(Q^2,0) - D^{(0)}(Q^2,a)$$

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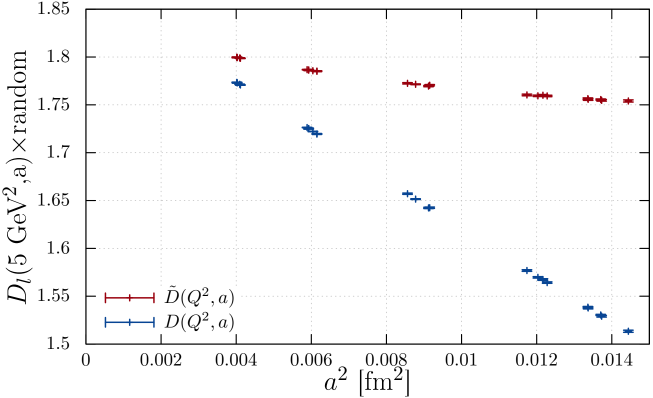



$\Rightarrow$ reduce discretization errors of simulation results via

$$D(Q^2, a) \to \tilde{D}(Q^2, a) = D(Q^2, a) + D^{(0)}(Q^2, 0) - D^{(0)}(Q^2, a)$$

# II: Removal of a higher order discretization effect

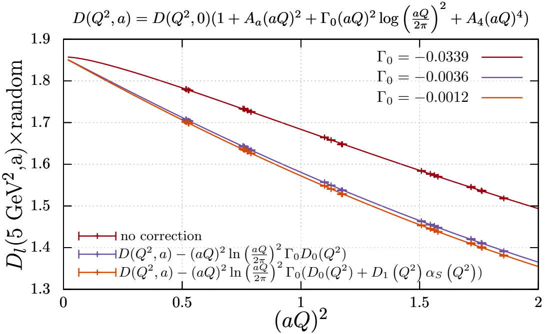
$$D\left(Q^2,a\right)=\left\{D_0\left(Q^2\right)+D_1\left(Q^2\right)\alpha_S\left(Q^2\right)+\mathcal{O}\left(\alpha_S^2\right)\right\}\cdot\left\{1+\Gamma_0(aQ)^2\ln\left(\frac{aQ}{2\pi}\right)^2+\mathcal{O}\left((aQ)^2\right)\right\}$$

$\Gamma_0$ and $D_1\left(Q^2\right)$ [A.O G. Kallen, A. Sabry '55] are known

Define additionally subtracted $\bar{D}\left(Q^2,a\right)$

$$\bar{D}\left(Q^2,a\right)=\tilde{D}\left(Q^2,a\right)-(aQ)^2\ln\left(\frac{aQ}{2\pi}\right)^2\Gamma_0 D_1\left(Q^2\right)\alpha_S\left(Q^2\right)$$

which has no longer any log enhanced discretization errors




From 1-loop lattice perturbation theory: $\Gamma_{0,\mathrm{ana}}=-1/30$

# III: Taking the continuum limit using $\hat{Q}$

A bosonic propagator on the lattice takes momentum $\hat{Q} = 2/a \sin(aQ/2)$.
Equally well justified to replace

$$D\left(\hat{Q}^2\right) = \hat{Q}^2 \frac{\partial \hat{\Pi}(\hat{Q}^2)}{\partial \hat{Q}^2} = 2\int_0^\infty \mathrm{d}t \; k(\hat{Q},t) C(t)$$

Now, also $k(\hat{Q},t)$ receives $\mathcal{O}(a^2)$ corrections:

$$D_f^{(0)}\left(\hat{Q}^2, a\right)\Big|_{a^2} = 2\int \mathrm{d}t \left( C_f(t)^{(0)}\Big|_{a^2} k(t,\hat{Q})\Big|_{a^0} + C_f(t)^{(0)}\Big|_{a^0} k(t,\hat{Q})\Big|_{a^2} \right)$$

Computing the relevant integral analytically gives

$$D_f^{(0)}\left(\hat{Q}^2, a\right)\Big|_{a^2} \xrightarrow[m\to 0]{a\to 0} N_c q_f^2 \hat{Q}^2 \ln\left(\frac{a\hat{Q}}{2\pi}\right)^2 \left(-\frac{1}{12} + \frac{1}{20} + \frac{1}{12}\right)$$

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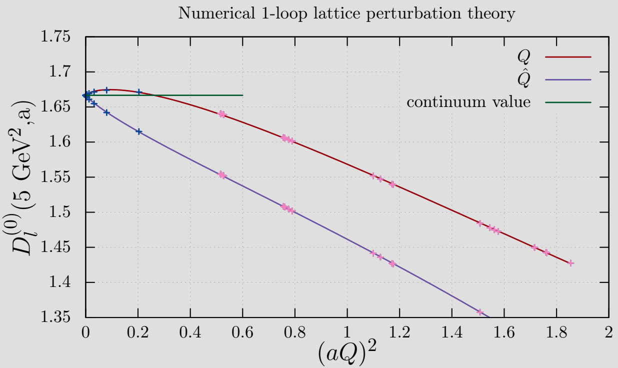
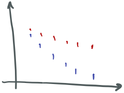



Log coefficients for $Q$ and $\hat{Q}$ have different signs!

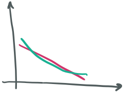
# Systematic error of continuum extrapolation

Set the scale using the Ω-mass: $a^2 = ((aM_{\Omega^-})/[M_{\Omega^-}]_*)^2$

Cuts in the lattice spacing (no cut, 0.111 fm, 0.095 fm)



Different choices of corrections (always correct with $D^{(0)}(Q^2) - D^{(0)}(Q^2, a)$, subtract/do not subtract additional discretization effect and twice this correction), using $\hat{Q}$ and $Q$ etc.



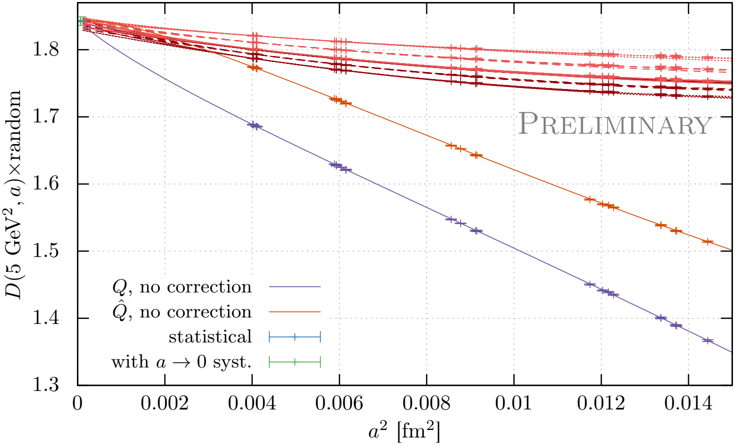
Different choices of fit functions for continuum extrapolation (fix/do not fix $\Gamma_0 = 0$, switch between $n \in [0, 3]$ for $\alpha_s^n(1/a)$ etc.)

Estimate uncertainty using AIC weights

# $D^l(5\ \mathbf{GeV}^2)$: Preliminary global fit + $a \to 0$ syst.

Strong IB and QED corrections included to $\mathcal{O}(\delta m, e^2)$ as in [BMWc '20]

$$D(Q^2, a) = D(Q^2, 0) + \underbrace{A_2[a\alpha_s^n(1/a)]^2 + A_{2l}a^2\log(a^2)(+A_4[a^2\alpha_s^n(1/a)]^2)}_{\text{continuum extrapolation}}$$



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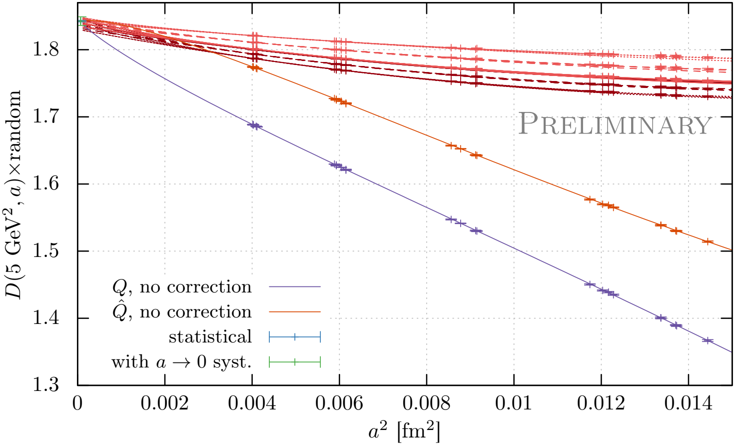

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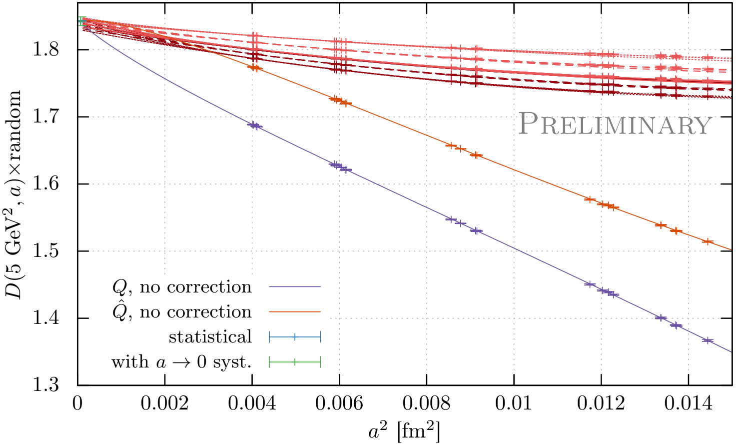

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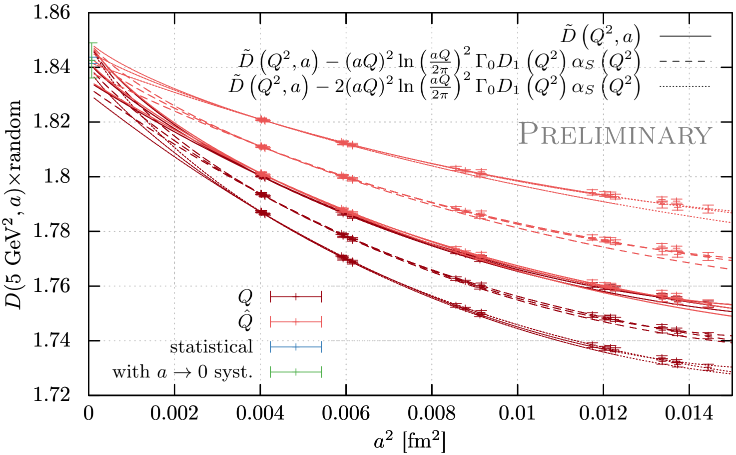
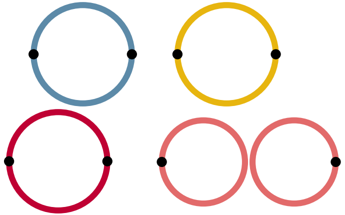


# Conclusion

Aim: compute Adler function on the lattice to yield $\Delta\alpha^{(5)}_{\mathrm{had}}(M_Z^2)$ with uncertainty $\lesssim 1.5 \times 10^{-4}$



- Complete analysis for all flavours (light, strange, charm, disconnected) with QED and strong isospin breaking corrections and various values of $Q^2$
- Remove finite volume effects
- Perform analytic taste improvement (important for small values of $Q^2$)
- Assess systematic error by: performing cuts in the lattice spacing, varying choice of fit function, varying choice of improvements, variation of fit ranges for hadron masses, varying experimental values of hadron masses etc.

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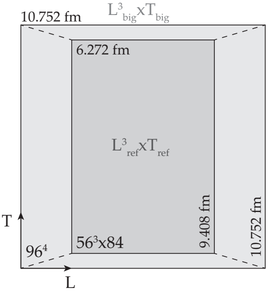


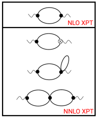


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$$D(Q^2)(L_{\text{big}}, T_{\text{big}}) - D(Q^2)(L_{\text{ref}}, T_{\text{ref}})]_{\text{4HEX}}$$
$$= [D(Q^2)(L_{\text{big}}, T_{\text{big}}) - D(Q^2)(L_{\text{ref}}, T_{\text{ref}})]_{\text{4HEX}}$$
$$+ [D(Q^2)(\infty, \infty) - D(Q^2)(L_{\text{big}}, T_{\text{big}})]_{\chi\text{PT}}$$

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- Remove finite volume effects

$$[D(Q^2)^{\mathrm{l}}]_0(L,a) \to [D(Q^2)^{\mathrm{l}}]_0(L,a) + \frac{10}{9}\left[\Delta_{\mathrm{RHO-SRHO}} D(Q^2,L)\right] + \frac{10}{9}\left[\Delta_{L_{\mathrm{ref}}-L} D(Q^2)^{\mathrm{RHO}}\right]$$

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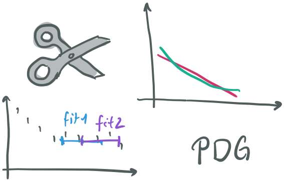


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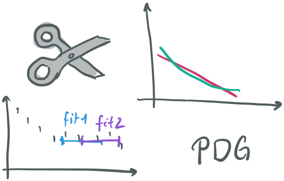


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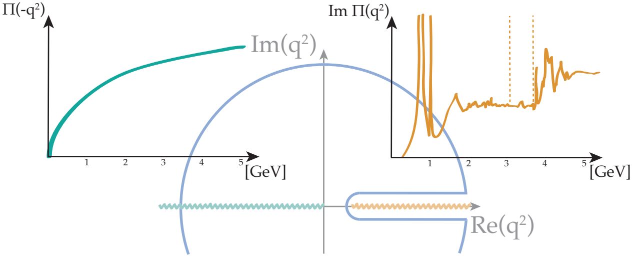
Particularly interesting in light of $a_\mu$ [BMWc '20], low energy running of $\alpha$ might be enhanced as seen in [Mainz '22]
**Stay tuned for results on $\alpha_{\text{QED}}(M_Z)$!**

# Backup

# $\hat{\Pi}(-q^2)$ accessible in Lattice QCD

Complementary to data driven approach




Run to $M_Z$ using **Euclidean split method**

$$\Delta\alpha_{\text{had}}^{(5)}\left(M_z^2\right) = \Delta\alpha_{\text{had}}^{(5)}\left(-Q_0^2\right)_{\text{LQCD}} + \left[\Delta\alpha_{\text{had}}^{(5)}\left(-M_z^2\right) - \Delta\alpha_{\text{had}}^{(5)}\left(-Q_0^2\right)\right]_{\text{pQCD/ R-ratio}}$$

$$+ \left[\Delta\alpha_{\text{had}}^{(5)}\left(M_z^2\right) - \Delta\alpha_{\text{had}}^{(5)}\left(-M_z^2\right)\right]_{\text{pQCD}}$$

# Removal of yet another discretization effect

$$D\left(Q^2, a\right) = \left\{D_0\left(Q^2\right) + D_1\left(Q^2\right)\alpha_S\left(Q^2\right) + \mathcal{O}\left(\alpha_S^2\right)\right\} \cdot \left\{1 + \Gamma_0 (aQ)^2 \ln\left(\frac{aQ}{2\pi}\right)^2 + \mathcal{O}\left((aQ)^2\right)\right\}$$

which can be rewritten as

$$D\left(Q^2, a\right) = D_0(Q^2, a^2) \times \left\{1 + \frac{D_1}{D_0}\left(Q^2\right)\alpha_s\left(Q^2\right) + \cdots\right\}\left\{1 + \Gamma_1 \alpha_s\left(\frac{1}{a}\right)(aQ)^2 \ln\left(\frac{aQ}{2\pi}\right)^2 + \cdots\right.$$

with $D_0(Q^2, a^2)$ our result from LO LPT $\rightarrow$ define additionally subtracted $\bar{D}\left(Q^2, a\right)$

$$\begin{aligned}
\bar{D}\left(Q^2, a\right) &= \tilde{D}\left(Q^2, a\right) - (aQ)^2 \ln\left(\frac{aQ}{2\pi}\right)^2 \Gamma_0 D_1\left(Q^2\right)\alpha_S\left(Q^2\right) \\
&= D\left(Q^2\right) + \Gamma_1 D_0\left(Q^2\right)\alpha_S\left(\frac{1}{a}\right)(aQ)^2 \ln\left(\frac{aQ}{2\pi}\right)^2 + \cdots \\
&= D\left(Q^2\right)\left\{1 + A\left(Q^2\right)\alpha_S\left(\frac{1}{a}\right)(aQ)^2 \ln\left(\frac{aQ}{2\pi}\right)^2 + \cdots\right\}
\end{aligned}$$

with

$$A\left(Q^2\right) = \Gamma_1 \frac{D_0\left(Q^2\right)}{D\left(Q^2\right)} = \Gamma_1\left\{1 - \alpha_S\left(Q^2\right)\frac{D_S}{D_0}\left(Q^2\right) + \mathcal{O}(\alpha_S^2\left(Q^2\right))\right\}$$

so that the $Q^2$-dependence of $A\left(Q^2\right)$ only appears as a higher order in $\alpha_S$ in the expansion of $\bar{D}\left(Q^2, a\right)$.

# Removal of yet another discretization effect

Now

$$\alpha_s\left(\frac{1}{a}\right)=\frac{4\pi}{\beta_0\ln(1/a\Lambda)^2}$$

with $\beta_0=11-2/3n_f$ at 1-loop. Hence,

$$\alpha_S\left(\frac{1}{a}\right)\ln(aQ)^2\simeq\frac{4\pi\left\{\ln Q^2/\Lambda^2-\ln\left(\frac{1}{a\Lambda}\right)^2\right\}}{\beta_0\ln(1/a\Lambda)^2}=\frac{\alpha_s\left(\frac{1}{a}\right)}{\alpha_s\left(Q^2\right)}-\frac{4\pi}{\beta_0}$$

and there are no longer any log enhanced discretization errors, except for $\alpha_S$ suppressed terms, such as

$$\sim(aQ)^2\ln(aQ)^2\left\{\Gamma_0 D_2\left(Q^2\right)\alpha_S^2\left(Q^2\right)\right\}$$

and higher orders, that should be small.

# Isospin breaking effects

- Include $e^2$ effects and $\delta m = m_d - m_u$ difference by expanding on isospin symmetric configurations [DeDivitiis et al. '13], distinguish between sea and valence charges

$$\langle O \rangle \simeq \frac{\int [dU][dA] e^{-S[U,A]}\, \mathrm{dets}_0 \left(1 + e_s \frac{\mathrm{dets}_1'}{\mathrm{dets}_0} + e_s^2 \frac{\mathrm{dets}_2''}{\mathrm{dets}_0}\right) \left(O_0 + \frac{\delta m}{m_l} O_m' + e_v O_1' + e_v^2 O_2''\right)}{\int [dU] e^{-S_g[U]} \int [dA] e^{-S_\gamma[A]}\, \mathrm{dets}_0 \left(1 + e_s \frac{\mathrm{dets}_1'}{\mathrm{dets}_0} + e_s^2 \frac{\mathrm{dets}_2''}{\mathrm{dets}_0}\right)}$$

$$= \langle O_0 \rangle_0$$

$$+ \langle O_m' \rangle_0$$

$$+ \left\langle O_2'' \right\rangle_0$$

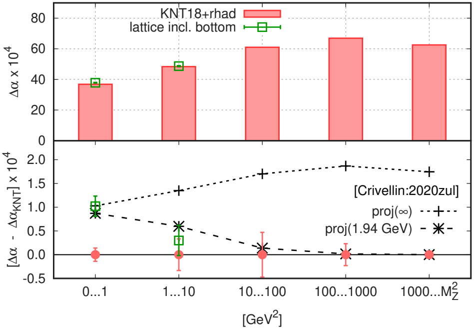
$$+ \left\langle O_1' \frac{\mathrm{dets}_1'}{\mathrm{dets}_0} \right\rangle_0$$

$$+ \left\langle [O_0 - \langle O_0 \rangle_0] \frac{\mathrm{dets}_2''}{\mathrm{dets}_0} \right\rangle_0$$

where $\mathrm{dets}[U, A; \{m_f\}, \{q_f\}, e] = \prod_f \det M_f \left[V_U e^{ieq_f A}, m_f\right]^{1/4}$ (M fermionic matrix),
$O_0 = O|_{(0,0)}$, $O_m' = m_l \partial_{\delta m} O|_{(0,0)}$, $O_1' = \partial_{e_v} O|_{(0,0)}$ and $O_2'' = \frac{1}{2} \partial_{e_v}^2 O|_{(0,0)}$

# Connection to $a_\mu$?

- first exploration of connection $a_\mu^{\text{LO-HVP}} \leftrightarrow \Delta^{(5)}_{\text{had}}\alpha(M_Z^2)$ [Passera et al '08]
- In [Crivellin et al '20]: BMW results suggest a $4.2\sigma$ overshoot in $\Delta^{(5)}_{\text{had}}\alpha(M_Z^2)$ compared to result of fit to EWPO
- They assume 2.8% relative deviation in R-ratio for all $s$ ($\sim$ excess BMW found in $a_\mu^{\text{LO-HVP}}$)
- Hypothesis is not consistent with [BMWc '17] nor new result




- values of $a_\mu^{\text{LO-HVP}}$ even as large as needed to explain $a_\mu^{\text{exp}}$ do not necessarily imply $\Delta^{(5)}_{\text{had}}\alpha(M_Z^2)$ in conflict withs EWPO [Malaescu et al '20, de Rafael '20 & Colangelo et al '20]

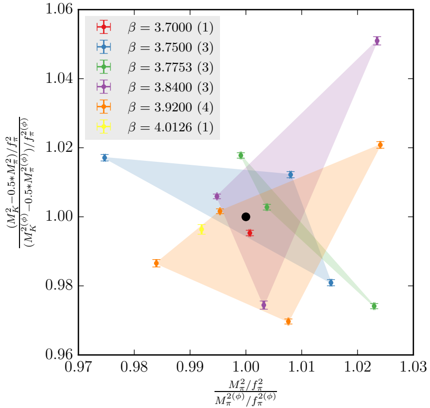
# Simulations and global fit

31 high-statistics simulations, $N_f$=2+1+1 flavors of 4-stout staggered quarks

$$D(Q^2) = D(Q^2,0) + \underbrace{A_2[a\alpha_s^n(1/a)]^2 + A_{2l}a^2\log(a^2) + A_4[a^2\alpha_s^n(1/a)]^2}_{\text{continuum extrapolation}}$$

6 $a$'s: $0.134 \to 0.064\,\mathrm{fm}$

| $\beta$ | $a$ [fm] | $T \times L$ | #conf |
|---|---|---|---|
| 3.7000 | 0.1315 | 64 × 48 | 904 |
| 3.7500 | 0.1191 | 96 × 56 | 2072 |
| 3.7753 | 0.1116 | 84 × 56 | 1907 |
| 3.8400 | 0.0952 | 96 × 64 | 3139 |
| 3.9200 | 0.0787 | 128 × 80 | 4296 |
| 4.0126 | 0.0640 | 144 × 96 | 6980 |




Over 25,000 gauge configurations, 10's of millions of measurements

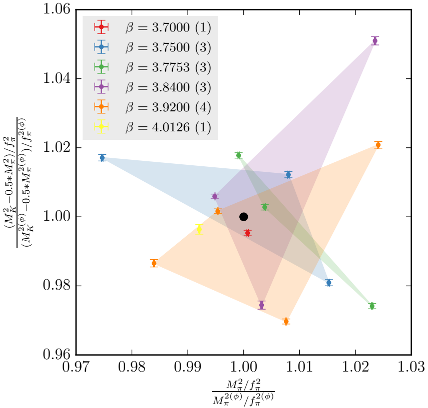
# Simulations and global fit

31 high-statistics simulations, $N_f$=2+1+1 flavors of 4-stout staggered quarks

$$D(Q^2) = \underbrace{A}_{\text{continuum extrapolation}} + \underbrace{(B_0 + B_2 a^2)X_l + (C_0 + C_2 a^2)X_s}_{\text{interpolation to physical point}}$$

$$X_l = \frac{M_{\pi^0}^2}{M_\Omega^2} - \left[\frac{M_{\pi^0}^2}{M_\Omega^2}\right]_*$$

$$X_s = \frac{M_{K_\chi}^2}{M_\Omega^2} - \left[\frac{M_{K_\chi}^2}{M_\Omega^2}\right]_*$$

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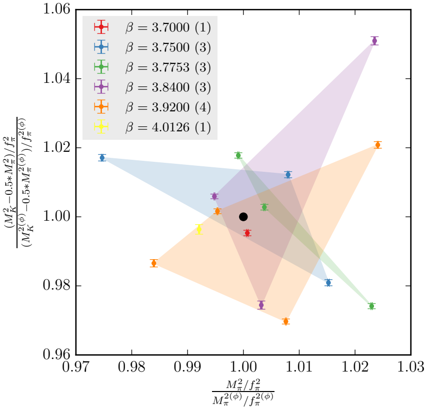
# Simulations and global fit

31 high-statistics simulations, $N_f$=2+1+1 flavors of 4-stout staggered quarks

$$D(Q^2) = \underbrace{A}_{\text{continuum extrapolation}} + \underbrace{BX_l + CX_s}_{\text{interpolation to physical point}} + \underbrace{DX_{\delta m} + EX_{vv} + FX_{vs} + GX_{ss}}_{\text{extrapolation to physical } \delta m, e^2}$$

$$X_{vv} = e_v^2$$
$$X_{vs} = e_v e_s$$
$$X_{ss} = e_s^2$$
$$X_{\delta m} = \frac{M_{K^0}^2 - M_{K^+}^2}{M_\Omega^2}$$




For sea-quark QED corrections

| $\beta$ | $a$ [fm] | $T \times L$ | #conf |
|---|---|---|---|
| 3.7000 | 0.1315 | 48 × 24 | 716 |
| | | 64 × 48 | 300 |
| 3.7753 | 0.1116 | 56 × 28 | 887 |
| 3.8400 | 0.0952 | 64 × 32 | 4253 |

Over 25,000 gauge configurations, 10's of millions of measurements