# Pseudoscalar Transition Form Factors and the Hadronic Light-by-Light Contribution to $a_\mu$

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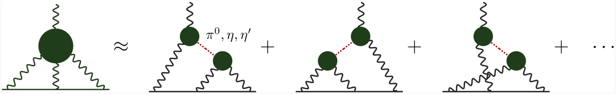
anr agence nationale de la recherche

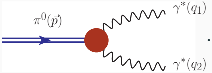
- Pseudoscalar Transition Form Factors (**TFF**s) play a crucial role in the determination of the Hadronic Light-by-Light contribution to the muon $g-2$.
- ‘Master equation’ relates the TFFs to pseudoscalar (p) pole contributions to $a_\mu$ (Knecht and Nyffeler, 2002)

$$a_\mu^{p-pole} = \left(\frac{\alpha_e}{\pi}\right)^3 \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^{1} d\tau \, \Big[ w_1(Q_1,Q_2,\tau) \mathscr{F}_{p\gamma^*\gamma^*}(-Q_1^2,-Q_3^2) \mathscr{F}_{p\gamma^*\gamma^*}(-Q_2^2,0) + w_2(Q_1,Q_2,\tau) \mathscr{F}_{p\gamma^*\gamma^*}(-Q_1^2,-Q_2^2) \mathscr{F}_{p\gamma^*\gamma^*}(-Q_3^2,0) \Big],$$

where $Q_3^2 = Q_1^2 + Q_2^2 + 2\tau Q_1 Q_2$ and $\tau = \cos\theta$ with $\theta$ the angle between $Q_1$ and $Q_2$ and $w_i(q_1,q_2,\tau)$ are known analytic weight functions.




- The TFF $\mathscr{F}_{p\gamma^*\gamma^*}$ encodes the interaction between a pseudoscalar and two photons. E.g. for the pion

$$\mathscr{F}_{\pi^0\gamma^*\gamma^*}(q_1^2, q_2^2) =$$


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# Motivation

1. $\pi^0$-pole
   - Contribution has been determined on the lattice by Mainz (Gérardin et al., 2016, 2019). Preliminary results by ETM (Burri et al., 2022) + (Kanwar, THU 10.40) .
   - Also computed in data-driven dispersive framework (Hoferichter et al., 2018).
2. $\eta, \eta'$-pole
   - No lattice nor dispersive results (Burri, THU 11:30).
   - TFF not well-known in relevant kinematical region.
   - Challenges for lattice QCD: mixing between $\eta, \eta'$ and sizable disconnected diagrams.

| **Contributions** | **Value** $\times 10^{11}$ |
|---|---|
| $\pi^0, \eta, \eta'$-poles | 93.8(4.0) |
| $\pi, K$-loops/boxes | -16.4(0.2) |
| $\pi\pi$ scattering | -8(1) |
| scalars + tensors | -1(3) |
| axial vectors | 6(6) |
| $u, d, s$-loops / short distance | 15(10) |
| $c$-loop | 3(1) |
| Total | 92(19) |

https://muon-gm2-theory.illinois.edu/white-paper/

# Phenomenology

- Normalization of TFF related to partial decay widths $\Gamma(p \to \gamma\gamma)$,

$$\Gamma(p \to \gamma\gamma) = \frac{\pi \alpha_e^2 m_p^3}{4} \mathscr{F}_{p\gamma^*\gamma^*}(0,0)$$

- Current values are:

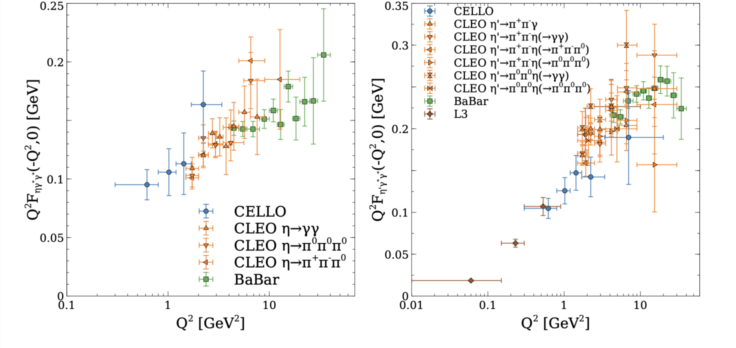
  1. $\Gamma(\pi^0 \to \gamma\gamma) = 7.802(0.117)$ eV (Larin et al., 2020).
  2. $\Gamma(\eta \to \gamma\gamma) = 0.516(0.18)$ keV (PDG, 2020).
  3. $\Gamma(\eta' \to \gamma\gamma) = 4.28(0.19)$ keV (PDG, 2020).

  $\to$ Errors are relatively small (few %), so can be really useful to combine with lattice data, especially for $\eta, \eta'$.

- Such a constraint already tested for pion TFF in (Gérardin et al., 2019), reduced total error on $a_\mu^{\pi-\text{pole}}$ by more than 30%.

# Experimental Data TFF $\eta, \eta'$

(Aoyama et al., 2020)




- A lot of experimental data avalaible for TFF in singly virtual (SV) regime at large $Q^2$.
- No data in regime where both photons are virtual below 6 GeV$^2$.
- Absence of precise data at low $Q^2$, important region for $a_\mu^{\mathrm{p-pole}}$
  $\rightarrow$ can be provided by lattice QCD.
- Combination of lattice and experimental data can also be an interesting comparison to pure lattice result.

# Gauge Ensembles

$2+1+1$ dynamical staggered fermions with 4 steps of stout smearing
(same ensembles as for the LO HVP calculation (Borsanyi et al., 2021))

- Gauge ensembles at (nearly) physical pion & kaon mass.
- Exploit up to six different lattice spacings ranging between [0.0640 - 0.1315] fm.
- Consider boxes of $\sim 3, 4$ and 6 fm for finite-size effect studies.
- Ensembles in isosymmetric limit ($\rightarrow$ no mixing between $\pi^0$ and $\eta^{(\prime)}$).

# Transition Form Factor from the Lattice

The TFF for a pseudoscalar meson is extracted from matrix elements $M_{\mu\nu}$ (Ji and Jung, 2001) (Gérardin et al., 2016)

$$M_{\mu\nu}(p,q_1) = i\int d^4x\, e^{iq_1\cdot x}\,\langle\Omega|\,T\{J_\mu(x)J_\nu(0)\}\,|P(\vec{p})\rangle = \varepsilon_{\mu\nu\alpha\beta}\,q_1^\alpha q_2^\beta\,\mathscr{F}_{P\gamma^*\gamma^*}(q_1^2,q_2^2),$$

where $J_\mu$ is the EM current. (Euclidean) Matrix elements are related to 3-point correlation function $C^{(3)}_{\mu\nu}$ on lattice

$$C^{(3)}_{\mu\nu}(\tau,t_P) = a^6\sum_{\vec{x},\vec{z}}\langle J_\mu(\vec{z},\tau+t_P)J_\nu(\vec{0},t_P)P^\dagger(\vec{x},0)\rangle e^{i\vec{p}\cdot\vec{x}}e^{-i\vec{q}_1\cdot\vec{z}}.$$

where $\tau$ is the time-separation between the two EM currents and

1. $C^{(3)}_{\mu\nu}$ proportional to matrix elements $\tilde{A}_{\mu\nu}(\tau)$ that are related to $M^E_{\mu\nu}$ as

$$M^E_{\mu\nu} = \frac{2E_P}{Z_P}\int_{-\infty}^{\infty} d\tau\, e^{\omega_1\tau}\tilde{A}_{\mu\nu}(\tau),$$

2. $E_P, Z_P$ energy and overlap of the pseudoscalar that are extracted from two-point correlations functions.
3. $q_1 = (\omega_1,\vec{q}_1)$ and $q_2 = (E_P-\omega_1,\vec{p}-\vec{q}_1)$

# Correlation Function on the Lattice: Wick Contractions

$$C^{(3)}_{\mu\nu}(\tau, t_P) = a^6 \sum_{\vec{x},\vec{z}} \langle J_\mu(\vec{z}, \tau + t_P) J_\nu(\vec{0}, t_P) P^\dagger(\vec{x}, 0) \rangle e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}_1\cdot\vec{z}}.$$

The correlation function receives contributions from (potentially) four different Wick contractions

1. • For the $\pi^0$

$$P_{\pi^0}(x) = \frac{1}{\sqrt{2}} \left( \overline{u}\gamma_5 u(x) - \overline{d}\gamma_5 d(x) \right).$$

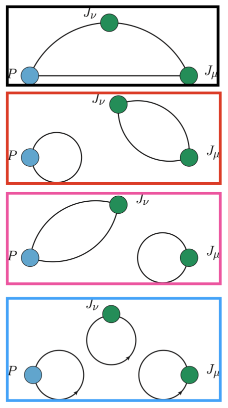
! We work in the isospin limit $\Rightarrow$ (2) and (4) do not contribute.
! Diagram (3) is small $\mathcal{O}(1-2\%)$ (Gérardin et al., 2019).

2. • For the $\eta, \eta'$

$$P_{\eta_8}(x) = \frac{1}{\sqrt{6}} \left( \overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) - 2\overline{s}\gamma_5 s(x) \right),$$

$$P_{\eta_0}(x) = \frac{1}{\sqrt{3}} \left( \overline{u}\gamma_5 u(x) + \overline{d}\gamma_5 d(x) + \overline{s}\gamma_5 s(x) \right).$$

! **All** four diagrams contribute.
! $\eta^8$ and $\eta^0$ mix to create physical $\eta, \eta'$.
! Diagram (2) is large!

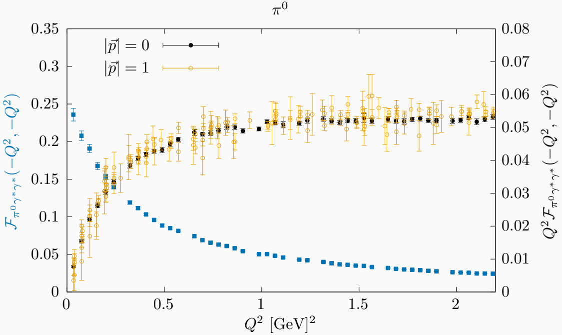



Some notation: Pseudoscalar is indicated by P and vector current by V, and a 'disconnection' by a hyphen. So (1) is PVV, (2) is P-VV (3) is PV-V and (4) P-V-V.

# $\pi^0$ TFF: Result on a Single Ensemble

First we check the pion TFF

- Simpler than $\eta, \eta'$
- Cross-check with previous lattice computations (Gérardin et al., 2016, 2019).




- $L/a = 96$, $a = 0.0640$ fm (6 fm box).
- Good agreement between $\vec{p} = \vec{0}$ & $\vec{p} = \frac{2\pi}{L}(0,0,1)$.
- Error on TFF grows with decreasing $Q^2$.

# Fitting the TFF

Continuous description of the TFF can be obtained using the (modified) $z-$**expansion**,

$$P(Q_1^2,Q_2^2)\mathscr{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)=\sum_{n,m=0}^{N}c_{nm}\left(z_1^n+(-1)^{N+n}\frac{n}{N+1}z_1^{N+1}\right)\times \left(z_2^m+(-1)^{N+m}\frac{m}{N+1}z_2^{N+1}\right),$$

where $z_k$ are conformal variables

$$z_k=\frac{\sqrt{t_c+Q_k^2}-\sqrt{t_c-t_0}}{\sqrt{t_c+Q_k^2}+\sqrt{t_c-t_0}},\quad k=1,2,$$

- $c_{nm}$ symmetric coefficients
- $t_0$ free parameter

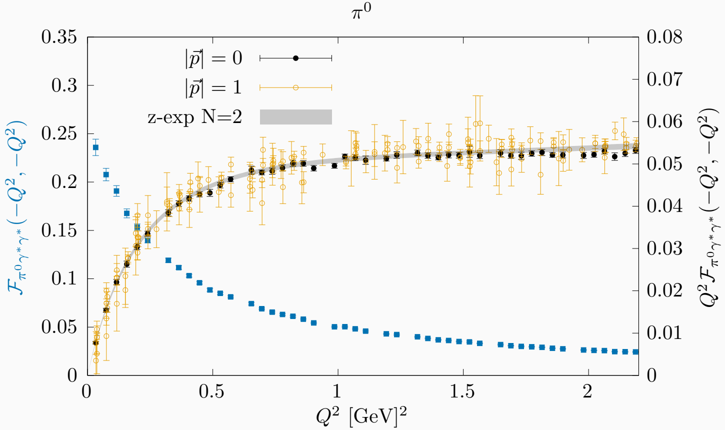
- ○ $t_c=4m_\pi^2$
- ○ $P(Q_1^2,Q_2^2)$ imposes short distance constraints

Advantages:

→ Fit is model-independent, only systematic is choice of $N$.

→ Obtain TFF in whole $(Q_1^2,Q_2^2)$-plane.

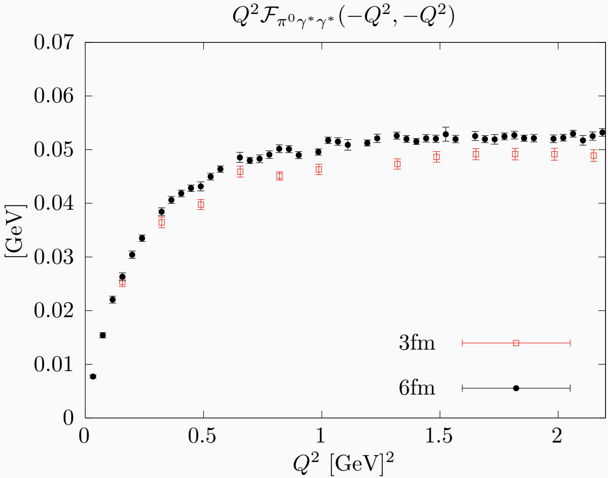
# $\pi^0$ TFF: Result on a Single Ensemble




- $L/a = 96$, $a = 0.0640$ fm (6 fm box).
- Uncorrelated z-expansion with N=2, gives $\chi^2/\mathrm{d.o.f.} = 1.06$.
- At this lattice spacing $a_\mu^{\pi-\mathrm{pole}}\Big|_{a=0.064\mathrm{fm}} = 63.3[2.9] \times 10^{-11}$
- Continuum value Mainz 2019: $a_\mu^{\pi-\mathrm{pole}} = 59.7[3.6] \times 10^{-11}$.

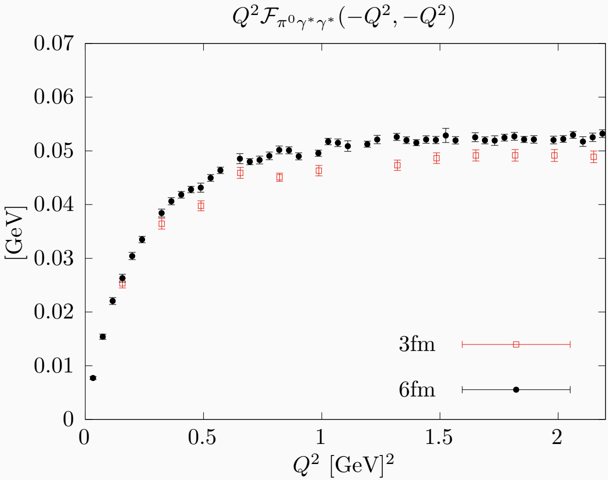
## Volume Effects $\pi^0$ TFF

- Smaller volumes reduce the cost of simulations drastically.
  $\rightarrow$ Could be useful for $\eta, \eta'$ TFF where the noise/signal ration increases rapidly.
- To test this possibility we study finite size effects (FSE) for the $\pi^0$ (precise data)
  $\rightarrow$ Compare signal at $a = 0.0640$ fm between 6fm and 3fm box



# Volume Effects $\pi^0$ TFF

- We see a discrepancy between the two box sizes.
- *Backward propagating pions* may contribute signifcantly to correlation function if time-exent is relatively small (for details see (Gérardin et al., 2016)).



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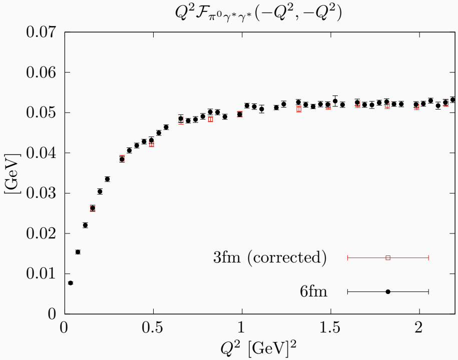


- Discrepancy can be satisfactorily explained by FTE correction
- We do not observe significant FSE for the $\pi^0$ and **thus compute the $\eta, \eta'$ TFFs mainly on small volumes (3fm and 4fm)**.

# Correlation Function on the Lattice: Wick Contractions

$$C^{(3)}_{\mu\nu}(\tau, t_P) = a^6 \sum_{\vec{x},\vec{z}} \langle J_\mu(\vec{z}, \tau + t_P) J_\nu(\vec{0}, t_P) P^\dagger(\vec{x}, 0)\rangle e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{q}_1\cdot\vec{z}}.$$

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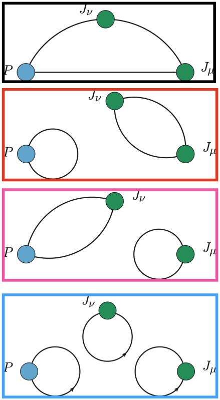
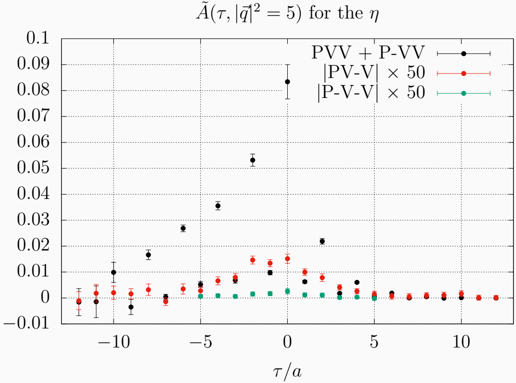


# $\eta$ Transition Form Factor Integrand

- PVV and P-VV together form the bulk of the signal.
- PV-V and P-V-V contributions are significantly smaller.

→ When computing the TFF we currently ignore the PV-V and P-V-V.




- Ensemble with $L/a = 32$, $a = 0.1315$ fm (4fm box).

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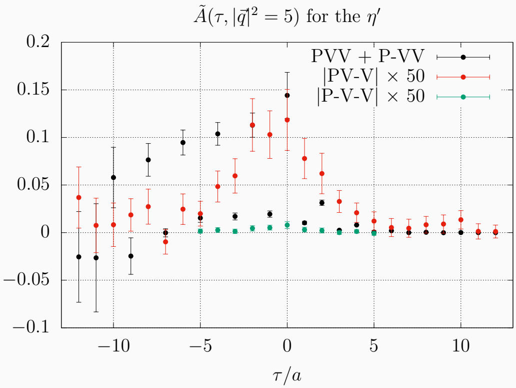
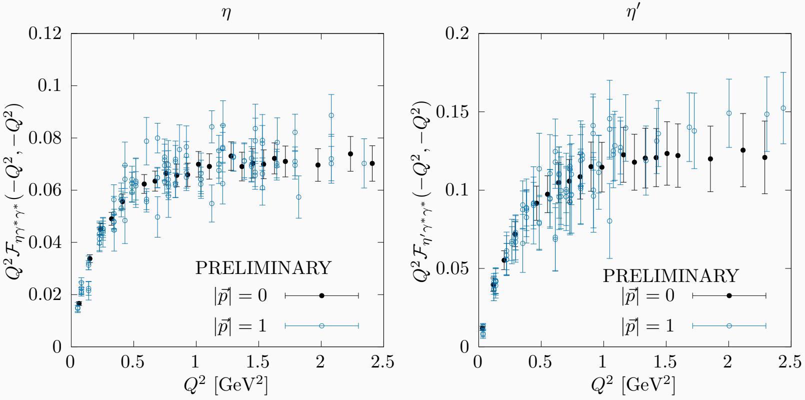


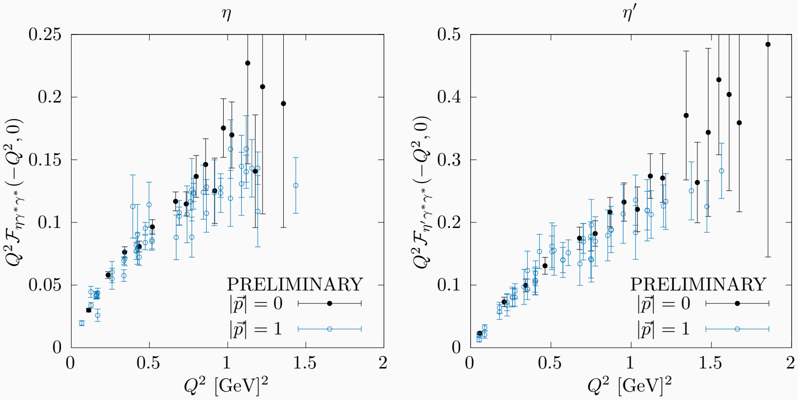

- Ensemble with $L/a = 32$, $a = 0.1315$ fm (4fm box).

# $\eta, \eta'$ TFF: Result on a Single Ensemble




- Ensemble with $L/a = 32$, $a = 0.1315$ fm.
- Good agreement between the two $\eta^{(\prime)}(\vec{p})$ frames with $\vec{p} = \vec{0}$ & $\vec{p} = \frac{2\pi}{L}(0,0,1)$.
- Errors larger than for $\pi^0$ because of difficulties mentioned before.
- Statistical error only.

# $\eta, \eta'$ TFF: Result on a Single Ensemble




- Ensemble with $L/a = 32$, $a = 0.1315$ fm.
- Good agreement between $\vec{p} = \vec{0}$ & $\vec{p} = \frac{2\pi}{L}(0,0,1)$.
- Preliminary z-expansion fits with N=2 at this lattice spacing give
  $a_\mu^{\eta\text{-pole}}\Big|_{a=0.1315\text{fm}} = 28[5] \times 10^{-11}$,
  $a_\mu^{\eta'\text{-pole}}\Big|_{a=0.1315\text{fm}} = 30[10] \times 10^{-11}$ (stat error only).

## Summary

- $\pi^0$ TFF
  - Level of desired precision already reached.
  - Need to estimate different systematics & extrapolate to physical point.
  - Estimate size of the PV-V contribution.
- $\eta, \eta'$ TFF
  - Preliminary data looks good in different kinematical regimes.
  - Data on several lattice spacings has been generated.
  - Add at least one big volume (6fm) $\rightarrow$ FSE small but better resolution.
- $\pi^0, \eta, \eta'$

  Goal of $\lesssim 10\%$ error on $a_\mu^{(\pi^0+\eta+\eta')\text{-poles}}$. Combining lattice results with experimental data can help achieving this goal.

$\rightarrow$ Updates on the analysis for the $\eta, \eta'$ will be presented at the muon $g-2$ workshop in Edinburgh.

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# Staggered Mesonic Operators

1. Two (taste-singlet) operators couple to the $\eta^{(\prime)}$ mesons (Golterman, 1986):
   - **3-link operator** $\mathscr{O}_3$ (couples to spin $\otimes$ taste $= \gamma_4\gamma_5 \otimes 1$ and $1 \otimes \gamma_4\gamma_5$), defined as (Altmeyer et al., 1993)

$$\mathscr{O}_3(x) = \frac{1}{6}\sum_{ijk} \varepsilon_{ijk}\overline{\chi}(x)[\eta_i\Delta_i[\eta_j\Delta_j[\eta_k\Delta_k]]]\chi(x) \equiv \overline{\chi}(x)\hat{O}_3\chi(x),$$

$$\text{Symmetric shift} \quad \Delta_\mu\chi(x) = \frac{1}{2}\left[U_\mu(x)\chi(x+\hat{\mu}) + U_\mu^\dagger(x-\hat{\mu})\chi(x-\hat{\mu})\right].$$

     - Con: Oscillating parity partner state (scalar).
   - **4-link operator** $\mathscr{O}_4$ (couples to $\gamma_5 \otimes 1$), defined as

$$\xrightarrow{\text{Used in analysis}} \boxed{\mathscr{O}_4(x) = \frac{1}{2}\eta_4(x)\left[\overline{\chi}(x)\hat{O}_3\chi_+(x) + \overline{\chi}_+(x)\hat{O}_3\chi(x)\right],}$$

$$\chi_+(x) = U_0(x)\chi(x+\hat{0}).$$

     - Con: Non-local in time.
     - Pro: Parity partner state with exotic quantum number (no contribution).
2. We use the conserved vector current.