A determination of the gradient flow scale t_0 on $N_f = 2 + 1$ CLS ensembles

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RQCD Collaboration

For details of this analysis and others see: **poster, Wolfgang Söldner**

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Precision determinations of the scale are needed: m_N, m_Ω, the leptonic decay constants f_π and f_K, ... have been used.

Intermediate or "theory" scales such as the gradient flow scales t_0 and w_0. are also employed.

Generally, these are observables that depend on the gauge fields, such that the dependence on m_q is mild and they can also be computed very precisely.

Gradient flow equation [Lüscher,1006.4518]:

$$a^2 \frac{d}{dt} V_t(x,\mu) = -g_0^2 \cdot \partial_{x,\mu} S_G(V_t) \cdot V_t(x,\mu) \qquad V_t(x,\mu)|_{t=0} = U(x,\mu)$$

where S_G is the lattice gauge action. **Flow time t has dimensions a^2.**

Define the gradient flow time t_0 in terms of the average action density $E(t)$

$$\boldsymbol{t^2\, E(t)\big|_{t=t_0} = c = 0.3}, \quad E(t) = \frac{1}{V_4} \int_{V_4} d^4x\, \frac{1}{4} G^a_{\mu\nu}(x,t) G^a_{\mu\nu}(x,t)$$

where $G^a_{\mu\nu}(x,t)$ is the field strength tensor evaluated using V_t.

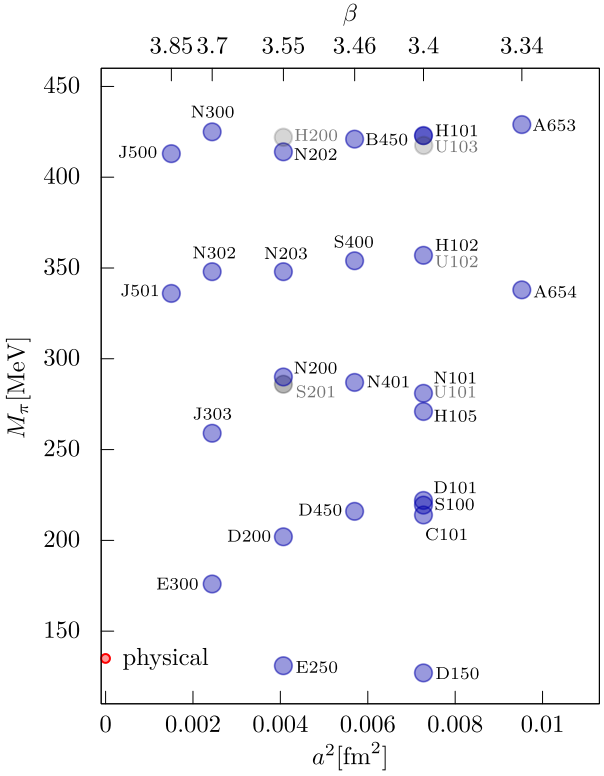
Determine $t_{0,ph}$ via $(\sqrt{t_0} M_\Xi)^{latt} = (\sqrt{t_0} M_\Xi)^{phys}$. Why m_Ξ? Statistically precise. Simultaneously fit baryon octet (N,Λ, Σ, Ξ) with SU(3) BChPT (a lot of data fitted with few parameters).

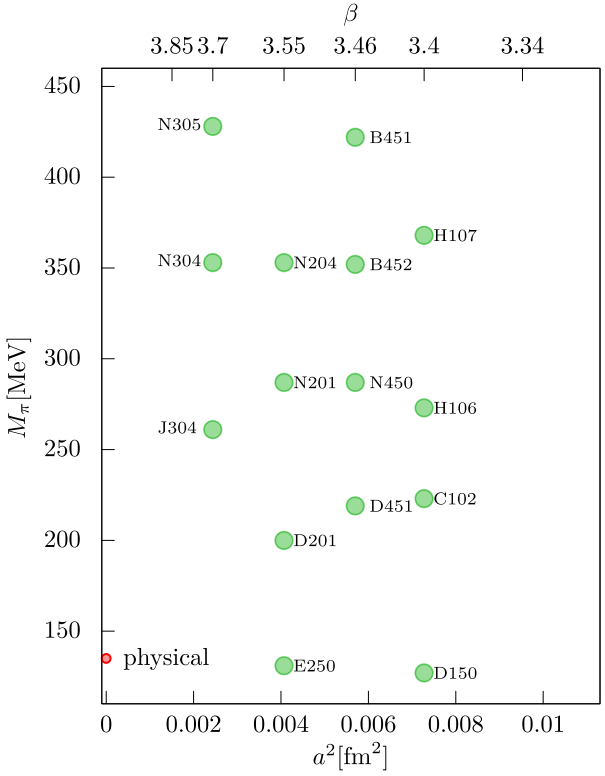
CLS ensembles: m_π vs a^2

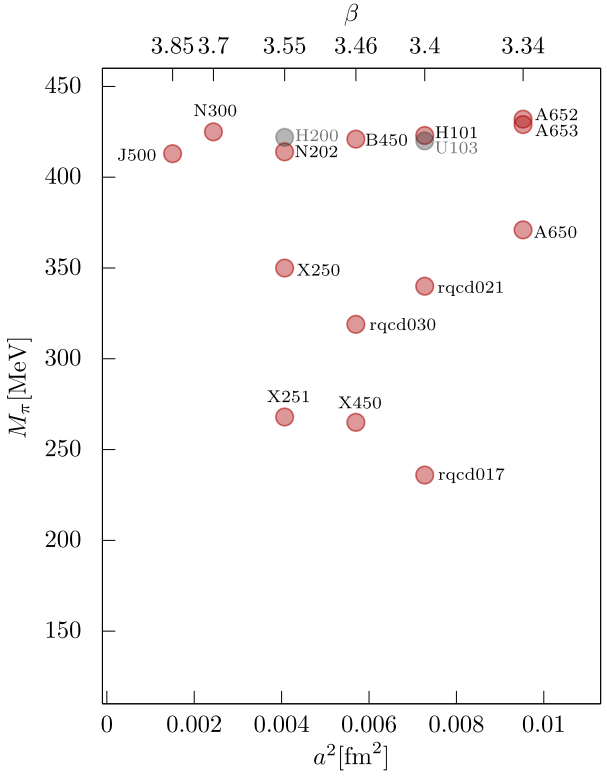
$N_f = 2 + 1$ flavours of non-perturbatively $O(a)$ improved Wilson fermions on tree level Symanzik improved glue.

High statistics: typically 6000-8000 MDUs, 1000-2000 configurations.

Aim to control all main sources of systematics (a, m_q and V): Six lattice spacings: $a = 0.1 - 0.04$ fm, $Lm_\pi \gtrsim 4$ with additional smaller volumes, $m_\pi = 410$ MeV down to m_π^{phys}.



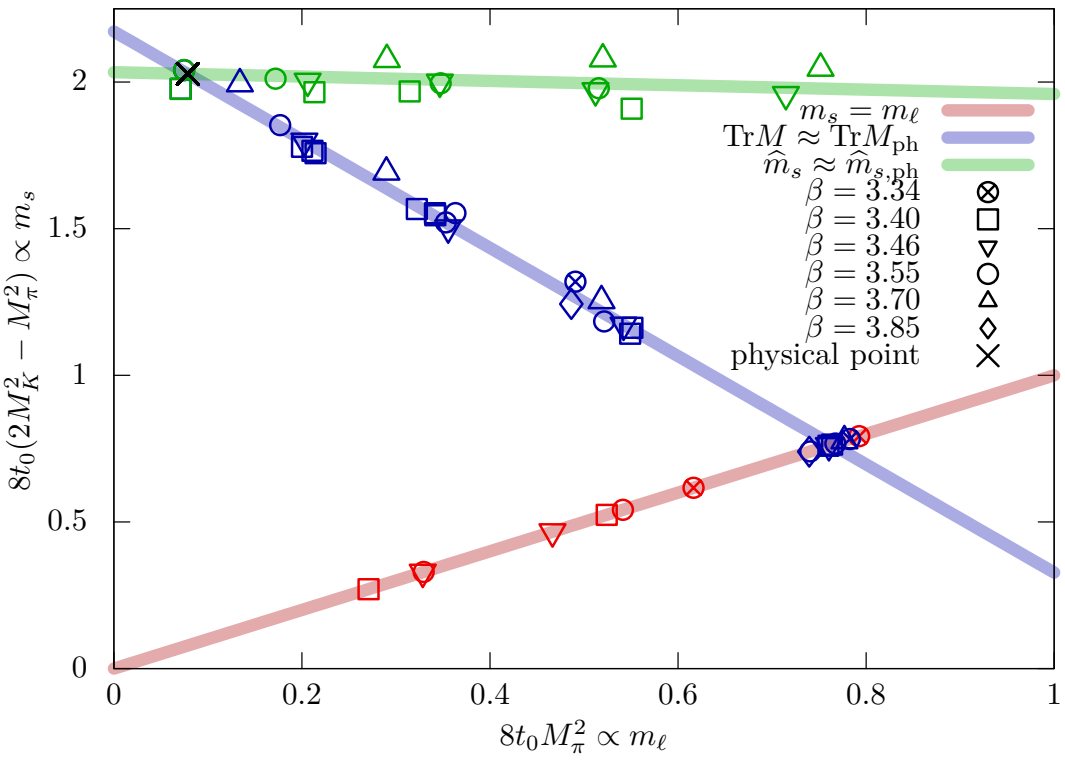




$2m_\ell + m_s = \text{const.}$ $\qquad$ $m_s = \text{const.}$ $\qquad$ $m_\ell = m_s$

$a < 0.06$ fm: open-boundary (ob) conditions in time. $a > 0.06$ fm: mixture of ensembles with periodic and ob conditions.

CLS ensembles: m_ℓ-m_s plane



Three trajectories: good control over the quark mass dependence. Can correct for mis-tuning of the trajectories. Observables sensitive to m_s, e.g. m_Ξ, are tightly constrained.

$2m_\ell + m_s = \mathrm{const.}$: investigate SU(3) flavour breaking (flavour average quantities roughly constant), approach to physical point involves $m_\pi \downarrow$ and $m_K \uparrow$.

$m_\ell = m_s$: important for determination of SU(3) ChPT low energy constants (and renormalisation factors).

$t_{0,ph}$, t_0^* and $t_{0,ch}$

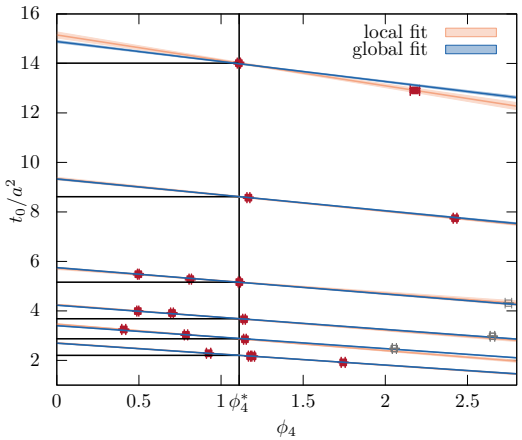
t_0 depends (mildly) on the quark mass: [Bär and Golterman,1312.4999]

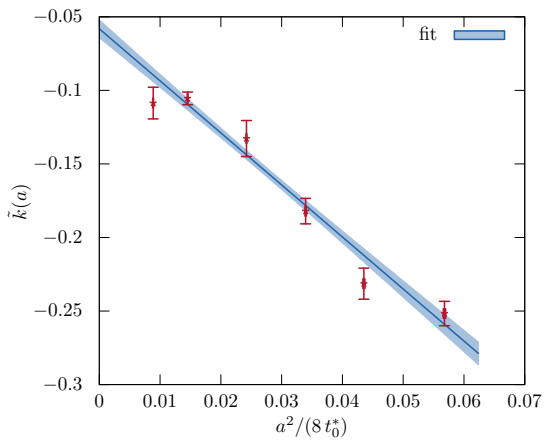
$$t_0(\overline{M}, \delta M) = t_{0,\mathrm{ch}} \left(1 + k_1 \frac{3\overline{M}^2}{(4\pi F_0)^2}\right) \approx t_{0,\mathrm{ch}} \left(1 + \tilde{k}_1 8 t_0 \overline{M}^2\right),$$

Define the scale t_0^* at the point along the symmetric line $m_s = m_\ell$ where

$$\phi_4^* = 8t_0^* \left(M_K^2 + \frac{M_\pi^2}{2}\right) = 12 t_0^* M_\pi^2 = 1.110.$$

Only requires an interpolation. Useful for representing relative change in the lattice spacing.





From a global fit to t_0:

$$\tilde{k}_1 = -0.0466(62), \qquad t_0^* = 0.9655(46) t_{0,\mathrm{ch}}, \qquad t_0^* = 0.99947(7) t_{0,\mathrm{ph}}.$$

Extrapolation of baryon multiplets

Extracting meson and baryon masses: **poster, Wolfgang Söldner**

Determine $t_{0,ph}$ using m_Ξ by performing a continuum, quark mass and finite volume extrapolation of the baryon octet (and decuplet) masses.

$$\overline{M}^2 = \frac{1}{3}(2M_K^2+M_\pi^2) \propto \overline{m} = \frac{1}{3}(2m_\ell+m_s), \quad \delta M^2 = 2(M_K^2-M_\pi^2) \propto \delta m = m_s-m_\ell$$

Rescale all masses by $\sqrt{8t_0}$: $B \in \{N, \Lambda, \Sigma, \Xi, \Delta, \Sigma^*, \Xi^*, \Omega\}$

$$\overline{\mathbb{M}} = \sqrt{8t_0}\,\overline{M}, \quad \delta\mathbb{M} = \sqrt{8t_0}\,\delta M, \quad \mathbb{m}_B = \sqrt{8t_0}\,m_B, \quad \mathbb{a} = \frac{a}{\sqrt{8t_0^*}},$$

Extrapolation performed using the fit form

$$\mathbb{m}_B(\mathbb{M}_\pi, \mathbb{M}_K, L, \mathbb{a}) = \left[\mathbb{m}_B(\mathbb{M}_\pi, \mathbb{M}_K, \infty, 0) + \delta m_B^{FV}(\mathbb{M}_\pi, \mathbb{M}_K, L)\right] \times \left[1 + \mathbb{a}^2\left(c + \bar{c}\,\overline{\mathbb{M}}^2 + \delta c_B\,\delta\mathbb{M}^2\right)\right].$$

Simultaneous fit to baryon multiplets with **all correlations taken into account.**

Discretisation coefficients: 6 parameters for the octet and decuplets baryons separately.

Natural choice for $\mathbb{m}_B(\mathbb{M}_\pi, \mathbb{M}_K, \infty, 0)$ is to use SU(3) baryon ChPT (and in a finite volume for δm_B^{FV}).

NNLO BChPT: octet baryons

BChPT: $O(p^3)$ baryon ChPT [Ellis et al.,nucl-th/9904017] with EOMS regularisation:

$$m_O(M_\pi, M_K, \infty, 0) = m_0 + \bar{b}\,\overline{M}^2 + \delta b_O\,\delta M^2 + \frac{m_0^3}{(4\pi F_0)^2}\left[g_{O,\pi} f_O\left(\frac{M_\pi}{m_0}\right) + g_{O,K} f_O\left(\frac{M_K}{m_0}\right) + g_{O,\eta_8} f_O\left(\frac{M_{\eta_8}}{m_0}\right)\right],$$

where

$$f_O(x) = -2x^3\left[\sqrt{1-\frac{x^2}{4}}\arccos\left(\frac{x}{2}\right) + \frac{x}{2}\ln(x)\right].$$

and $M_{\eta_8}^2 = (4M_K^2 - M_\pi^2)/3$.

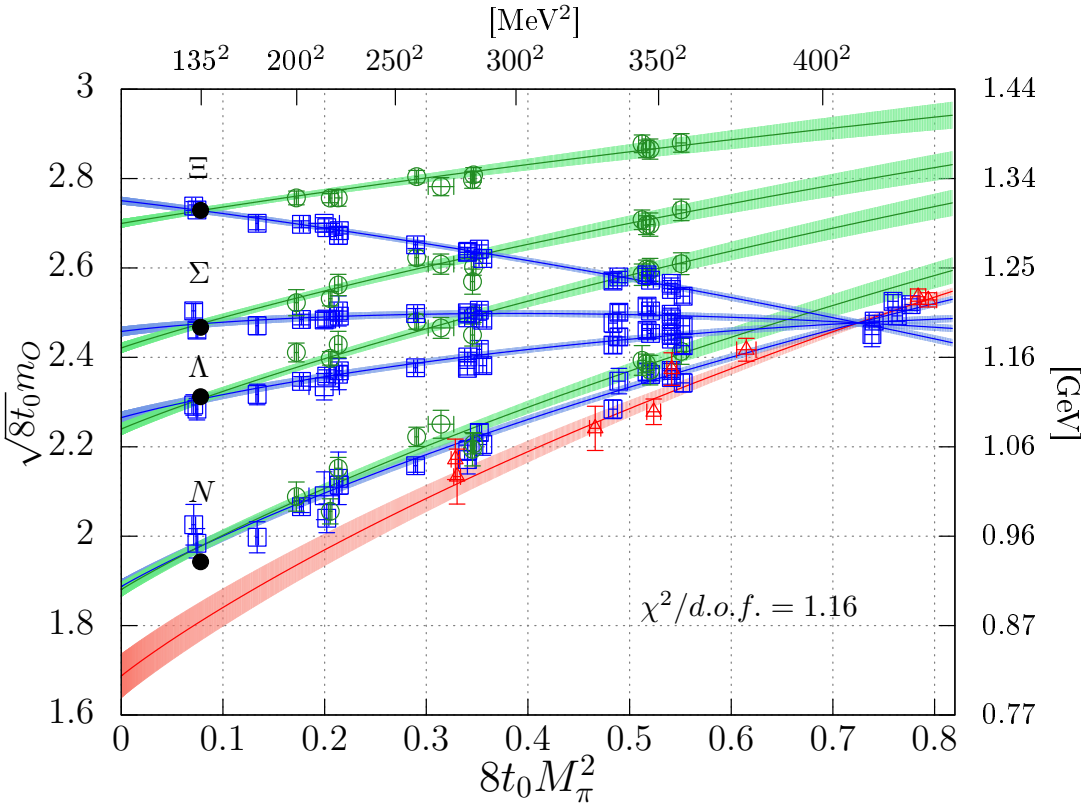
Depends on 6 low energy constants (LECs): m_0, $\bar{b}$ and δb_O (depending only on b_0, b_F **and** b_D due to SU(3) constraints), g_{O,π,K,η_8} (depending only on F **and** D, also appearing in ChPT expressions for g_A^O), appear in combination with F_0.

Heavy baryon limit (HBChPT) [Jenkins and Manohar,Phys. Lett. B 255 (1991) 558.]: $f(x) = -x^3 + \mathcal{O}(x^4)$.

BChPT in a finite volume (FV) gives the finite volume dependence of m_B (see e.g. [Ren et al.,1209.3641]). **No additional LECs!**

NNLO BChPT fit to the baryon octet: m_q dependence

Finite a and V terms included in the fit. 12 parameters to fit the 4 octet baryon masses.

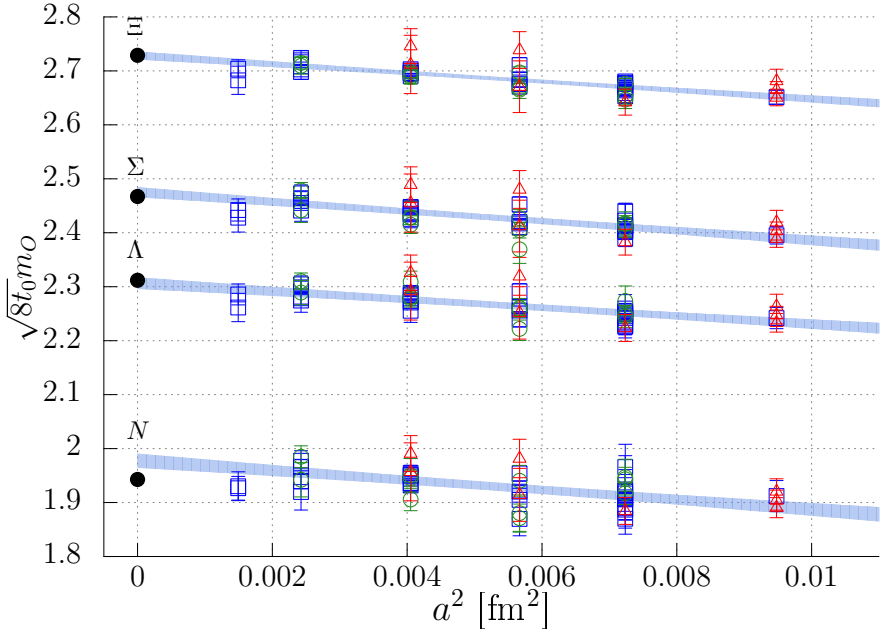


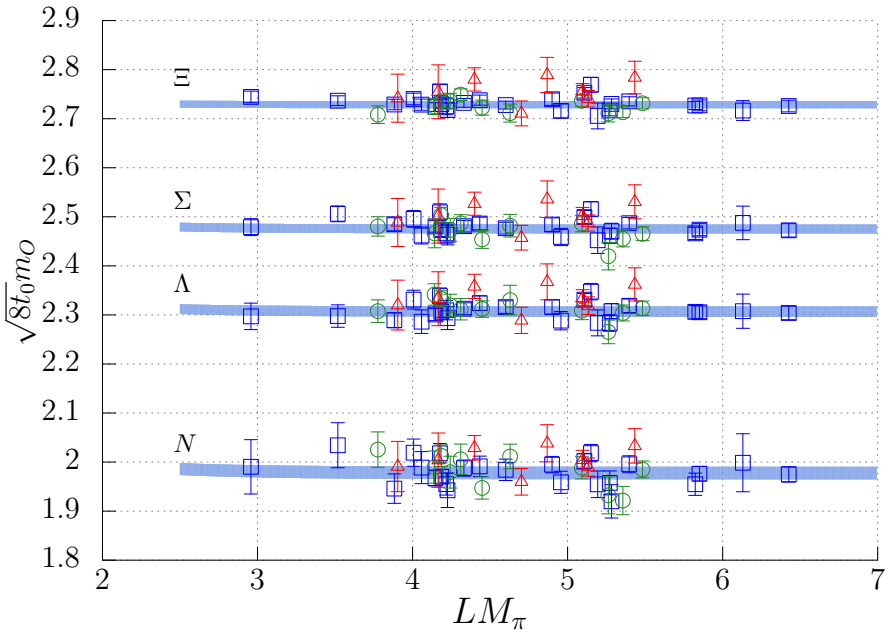
Curves show $\mathbb{m}_O(\mathbb{M}_\pi, \mathbb{M}_K, \infty, 0)$, while the data points are shifted to correct for finite a, finite V and mis-tuning of the trajectory.

$t_{0,ph}$ determined via an iterative procedure with QCD "expt." values for $M_{\pi,ph}$, $M_{K,ph}$ and $m_{\Xi,ph}$. $\sqrt{8t_{0,ph}} \sim 0.409$ fm.

Discretisation and finite volume effects

The data points are shifted to the physical point ($M_{\pi,ph}$, $M_{K,ph}$).





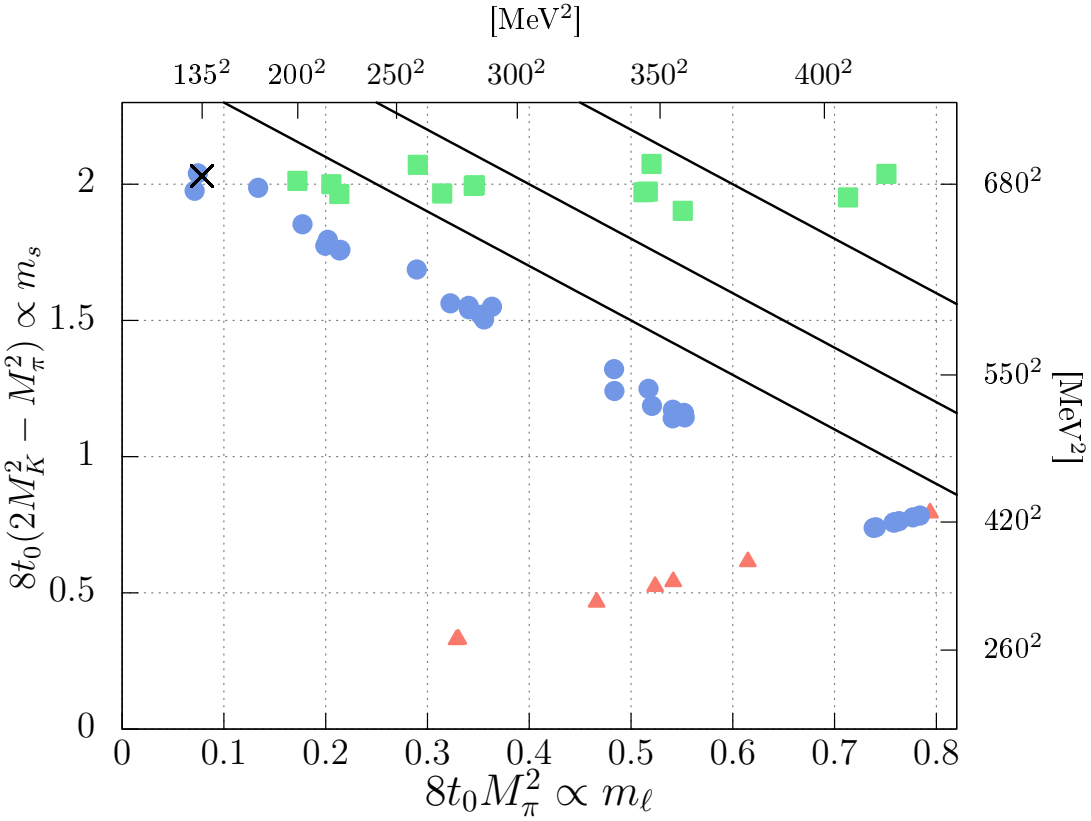
Discretisation effects are mild: $1 + \mathbb{a}^2 \left(c + \bar{c}\,\overline{\mathbb{M}}^2 + \delta c_O\,\delta\mathbb{M}^2\right)$. Around 3% from $a = 0.1$ fm to $a = 0$.

$\mathbb{a}^2 \delta c_O\,\delta\mathbb{M}^2$ terms are small. Higher order terms are not significant.

Finite volume effects are small, however, including FV terms in the fit (which involves no extra coefficients) improves the fit quality.

Assess the systematics of the fits: cuts on the data

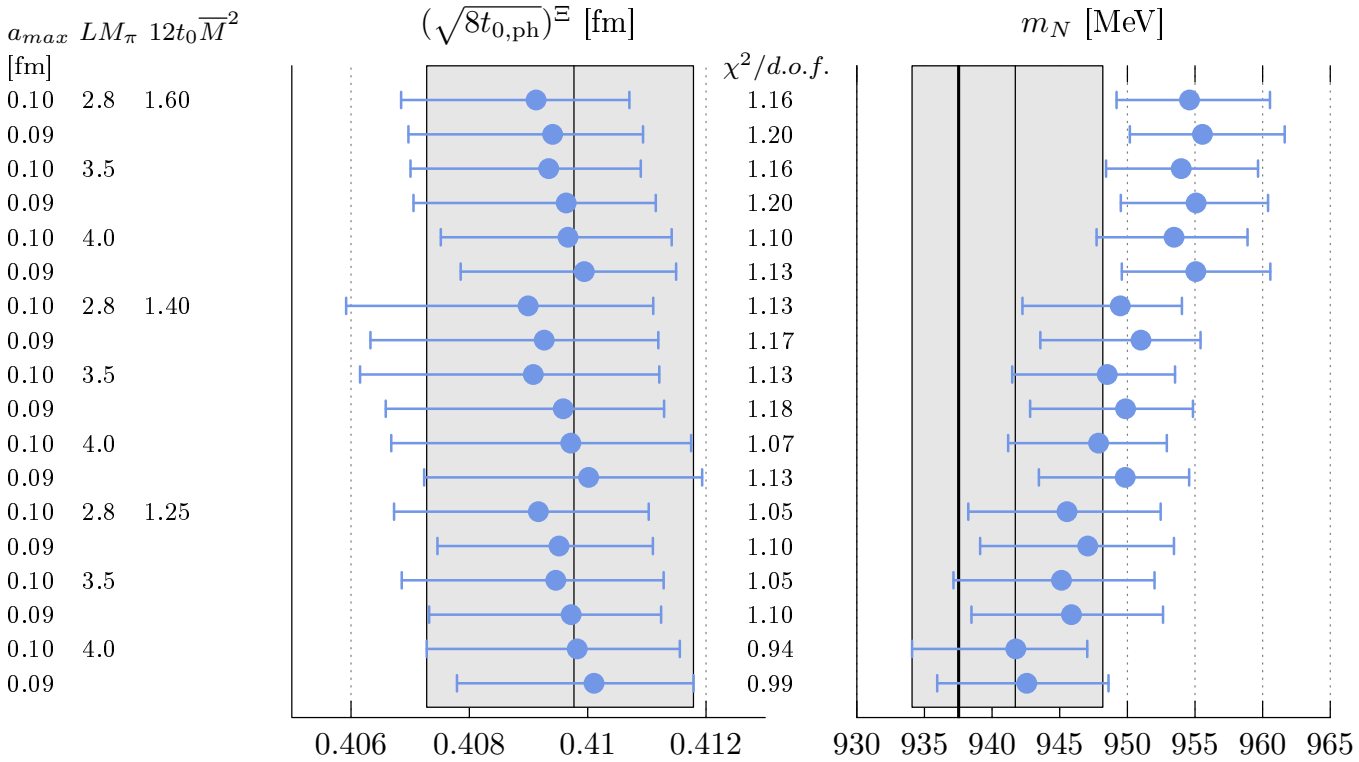
m_q dependence: (SU(3) BChPT) include $\overline{M}^2 < 498^2$ MeV^2 and impose further cuts $\overline{M}^2 < 466^2$ MeV^2, $\overline{M}^2 < 440^2$ MeV^2.



Finite V: start with $L > 2.3$ fm and impose further cuts $LM_\pi > 3.5$ and $LM_\pi > 4.0$.

Finite a: include six lattice spacings ($0.04 \leq a \leq 0.1$ fm) and only consider cut $a < 0.09$ fm.

Variation with cuts on the data



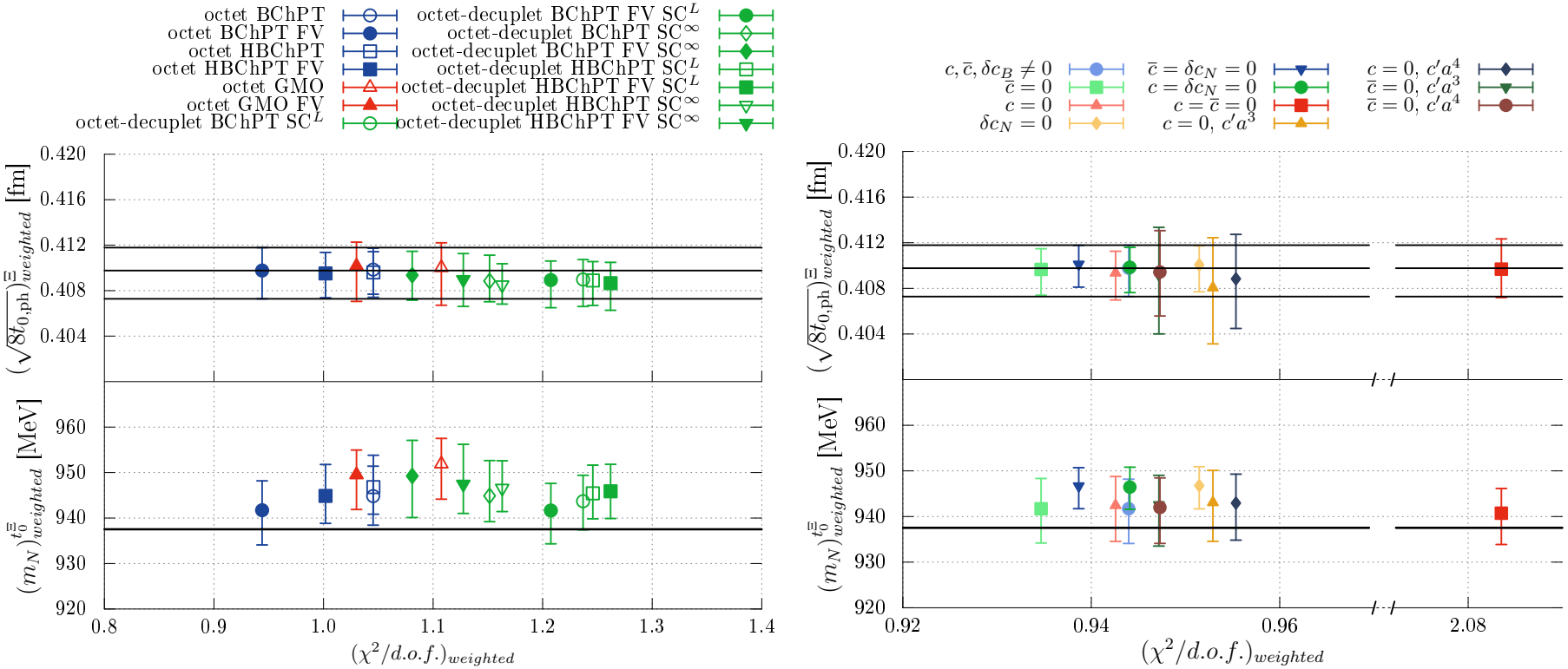
$\chi^2/d.o.f$ improves with cuts on $\overline{M}^2$. Note, $73 \leq N_{\mathrm{DF},j} \leq 125$

Values of $\sqrt{8t_{0,ph}}$ obtained are consistent. Agreement of m_N with QCD "expt." value improves with cuts on $\overline{M}^2$.

Grey bands indicate the weighted average of the results using a modified AIC (following [BMWc,2002.12347]) $w_j = A \exp\left[-\frac{1}{2}\left(\chi_j^2 - N_{\mathrm{DF},j})\right)\right]$

Final result: $\sqrt{8t_{0,ph}} = 0.4098^{(20)}_{(25)}$ **fm**

Variation with the continuum fit form and discretisation terms



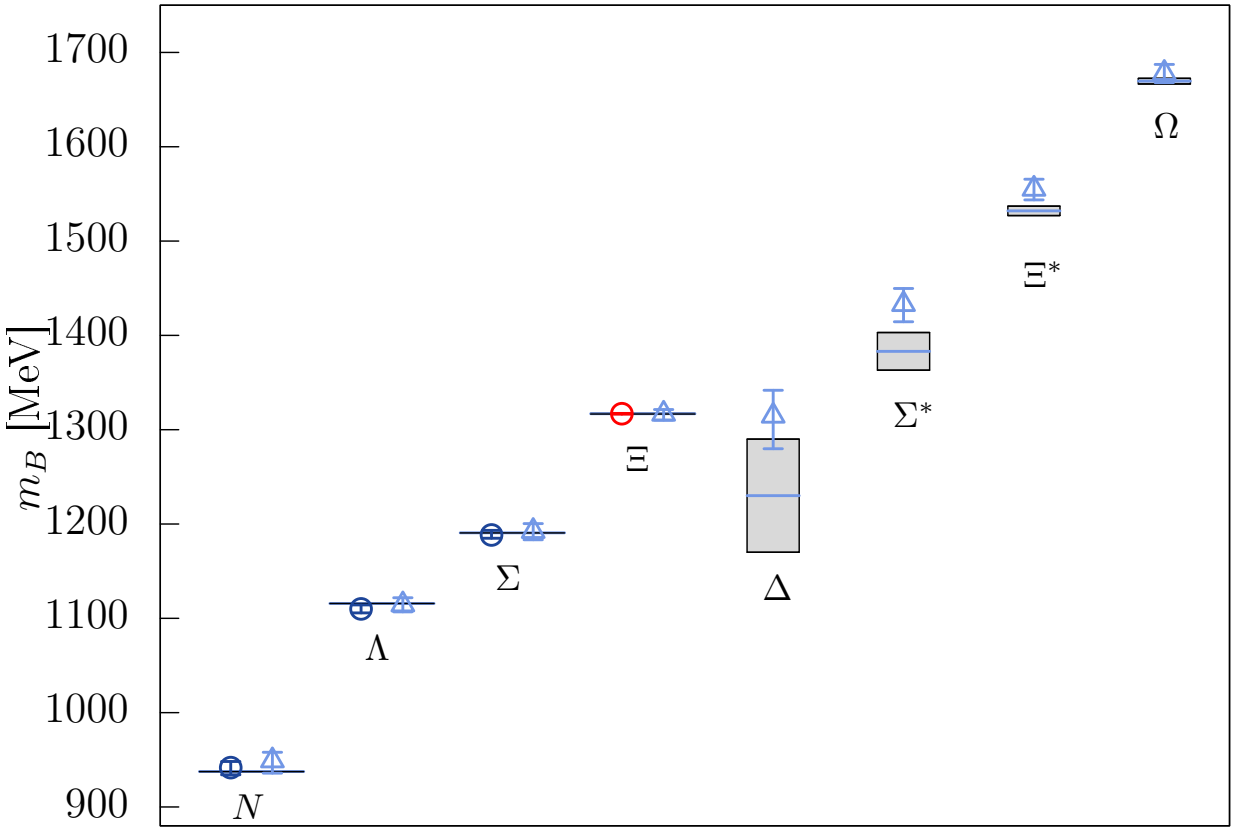
Quality of the fit varies depending on the fit form, however,
the results for $\sqrt{8t_{0,ph}}$ are very stable (m_Ξ is tightly constrained).

The variations are well within overall error for final value of $\sqrt{8t_{0,ph}}$.

Including FV terms in the fit improves the fit quality.

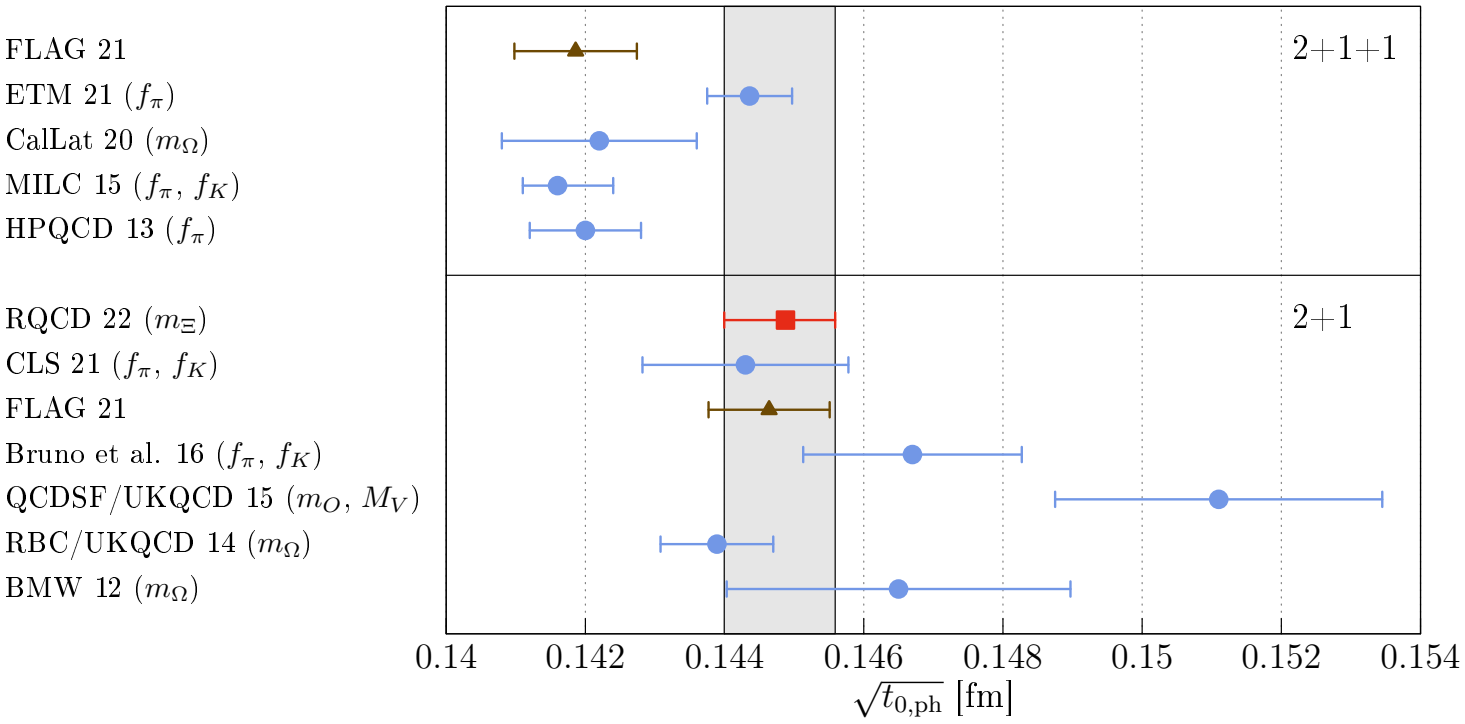
Low lying baryon spectrum

- ○ Octet baryon spectrum from BChPT, finite a and FV fits (determine $\sqrt{8t_{0,ph}}$) Agreement with QCD "expt." masses within 1% overall uncertainty.
- △ Octet and decuplet masses from SSE BChPT, finite a and FV fits.



Unstable decuplet baryons: grey bands indicate the expt. Breit-Wigner width. Proper treatment via the Lüscher formalism required.

Comparison with other determinations of $\sqrt{t_{0,ph}}$



Previous determinations on the $N_f = 2 + 1$ CLS ensembles, [Bruno et al.,1608.08900] and [CLS 21,2112.06696] (preliminary).

Baryon sigma terms: $\sigma_{\pi B}$ and σ_{sB}

The sigma terms, can be obtained via the Feynman-Hellmann theorem.

$$\sigma_{qB} = m_q \left[\frac{\langle B \,|\bar{q}\mathbb{1}q|\, B\rangle}{\langle B|B\rangle} - \langle \Omega \,|\bar{q}\mathbb{1}q|\, \Omega\rangle \right] = m_q \frac{\partial m_B}{\partial m_q},$$

where $|\Omega\rangle$ denotes the vacuum. Consider $\sigma_{\pi B} = \sigma_{uB} + \sigma_{dB}$ and σ_{sB}.

Using the Gell-Mann-Oakes-Renner (GMOR) relation: $M_{PS}^2 \approx B_0(m_{q1} + m_{q2})$

$$\sigma_{\pi B} \approx \tilde{\sigma}_{\pi B} = M_\pi^2 \frac{\partial m_B}{\partial M_\pi^2} \qquad \sigma_{sB} \approx \tilde{\sigma}_{sB} = M_{s\bar{s}}^2 \frac{\partial m_B}{\partial M_{s\bar{s}}^2}$$

where $M_{s\bar{s}}^2 = 2M_K^2 - M_\pi^2$.

Difficult to determine σ_{sN} via indirect (FH) approach (in particular if m_s is kept roughly constant).

Baryon octet sigma terms

Weighted avg.	$\tilde{\sigma}_{\pi N}$	$\tilde{\sigma}_{\pi \Lambda}$	$\tilde{\sigma}_{\pi \Sigma}$	$\tilde{\sigma}_{\pi \Xi}$
MeV	$44.0^{(4.4)}_{(4.7)}$	$27.6^{(4.3)}_{(4.9)}$	$24.9^{(4.6)}_{(5.0)}$	$10.1^{(4.4)}_{(5.4)}$
	$\tilde{\sigma}_{sN}$	$\tilde{\sigma}_{s\Lambda}$	$\tilde{\sigma}_{s\Sigma}$	$\tilde{\sigma}_{s\Xi}$
MeV	$3.7^{(59.3)}_{(60.8)}$	$112.7^{(63.3)}_{(59.9)}$	$194.1^{(67.8)}_{(60.7)}$	$266.69^{(70.3)}_{(68.4)}$

cf. [BMWc,1109.4265] (more precise results for the nucleon in [BMWc,1510.08013] and [BMWc,2007.03319])

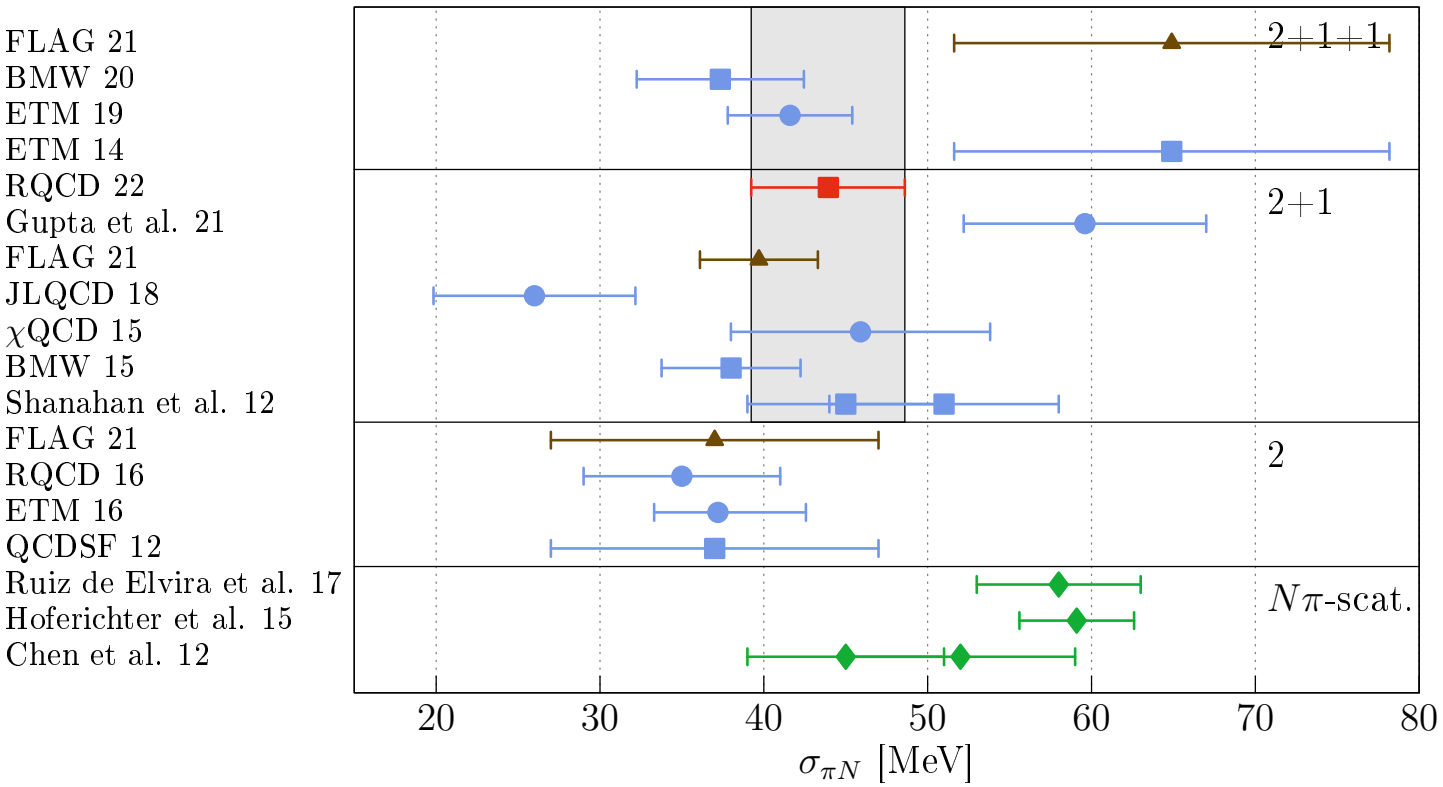
	$\sigma_{\pi N}$	$\sigma_{\pi \Lambda}$	$\sigma_{\pi \Sigma}$	$\sigma_{\pi \Xi}$
MeV	$39(4)^{(18)}_{(7)}$	$29(3)^{(11)}_{(5)}$	$23(3)^{(19)}_{(3)}$	$15(2)^{(8)}_{(3)}$
	σ_{sN}	$\sigma_{s\Lambda}$	$\sigma_{s\Sigma}$	$\sigma_{s\Xi}$
MeV	$67(27)^{(55)}_{(47)}$	$180(26)^{(48)}_{(77)}$	$245(29)^{(50)}_{(72)}$	$312(32)^{(72)}_{(77)}$

and [Shanahan et al.,1205.5365] (single lattice spacing)

	$\sigma_{\pi N}$	$\sigma_{\pi \Lambda}$	$\sigma_{\pi \Sigma}$	$\sigma_{\pi \Xi}$
MeV	47(6)(5)	26(3)(2)	20(2)(2)	8.9(7)(4)
	σ_{sN}	$\sigma_{s\Lambda}$	$\sigma_{s\Sigma}$	$\sigma_{s\Xi}$
MeV	22(6)(0)	141(8)(1)	172(8)(1)	239(8)(1)

Comparison with other determinations of $\sigma_{\pi N}$

$\tilde{\sigma}_{\pi B} \longrightarrow \sigma_{\pi B}$: $\partial M_\pi/\partial m_\ell$, $\partial M_K/\partial m_\ell$, $\partial M_K/\partial m_s$ obtained from fit of $m_{\ell,s}$ w.r.t. M_π and M_K using $O(p^4)$ SU(3) ChPT, see **talk by G. Bali, Wed. 14:00**.



Some tension between the lattice results and the phenomenological results of [Hoferichter et al.,1506.04142] obtained using $N\pi$ scattering data.

Direct determination by [Gupta et al.,2105.12095] is also larger.

Consistency with direct determinations: talk by Pia Petrak, Tues. 18:10

Summary and outlook

Determination of Wilson flow scale at the physical point with 0.5% uncertainty via m_Ξ:

$$\sqrt{8t_{0,ph}} = 0.4098^{(20)}_{(25)} \text{ fm}$$

- ⋆ Utilising data along three trajectories in the quark mass plane means mistuning of the trajectories can easily be corrected. Provided tight constraints on m_Ξ at the physical point.
- ⋆ Discretisation and finite volume effects are mild for our ensembles.
- ⋆ Covariant BChPT provided a reasonable description of the data for our range of M_π and M_K ($\overline{M}^2 < 440^2$ [MeV^2]). (Large N_{DF} for the fits).
- ⋆ QCD "expt." octet baryon spectrum reproduced to within 1% uncertainties.
- ⋆ As a by product extract the sigma terms $\sigma_{\pi B}$ and σ_{sB}.
- ⋆ Also extracted the LECs which are not well known for SU(3) BChPT.

In the future:

- ⋆ Improve the $t_{0,ph}$ determination: e.g. already have additional measurements on many ensembles, which are not included in the current analysis.
- ⋆ Test ChPT further through determination of LECs via simultaneous fits to related observables, e.g. $M_{\pi,K}$, $m_{\ell,s}$ and $F_{\pi,K}$ and m_B and g_A^B.

Correcting expt. masses for electrical and quark mass isospin breaking effects

QCD values for M_π and M_K from [FLAG 16,1607.00299].

mass	value/MeV	Expt./MeV	mass	value/MeV	Expt./MeV
M_π	**134.8(3)**	134.98 (π^0)			
M_K	**494.2(3)**	497.61 (K^0)			
m_N	937.53(6)	938.27 (p)	m_Δ	1231(60)	1231(56) (Δ^0)
m_Λ	1115.68(1)	1115.68	m_{Σ^*}	1383(20)	1383(18) (Σ^{*+})
m_Σ	1190.67(12)	1192.64 (Σ^0)	m_{Ξ^*}	1532(5)	1532(5) (Ξ^{*0})
m_Ξ	**1316.9(3)**	1314.86 (Ξ^0)	m_Ω	1669.5(3.0)	1672.45(29)

For the octet: convention that neutral particles do not receive QED corrections.
Assume same charge-isospin breaking effects for the whole octet $\Delta m_N - \Delta m_\Sigma - \Delta m_\Xi = 0$,
Coleman-Glashow theorem (verified to within 0.13 MeV by [BMWc,1406.4088]),

$$\Delta m_N = m_p - m_n \approx -\delta m^{\mathrm{QCD}} + \delta m^{\mathrm{QED}}, \qquad \Delta m_\Sigma = m_{\Sigma^+} - m_{\Sigma^-} \approx -2\delta m^{\mathrm{QCD}},$$

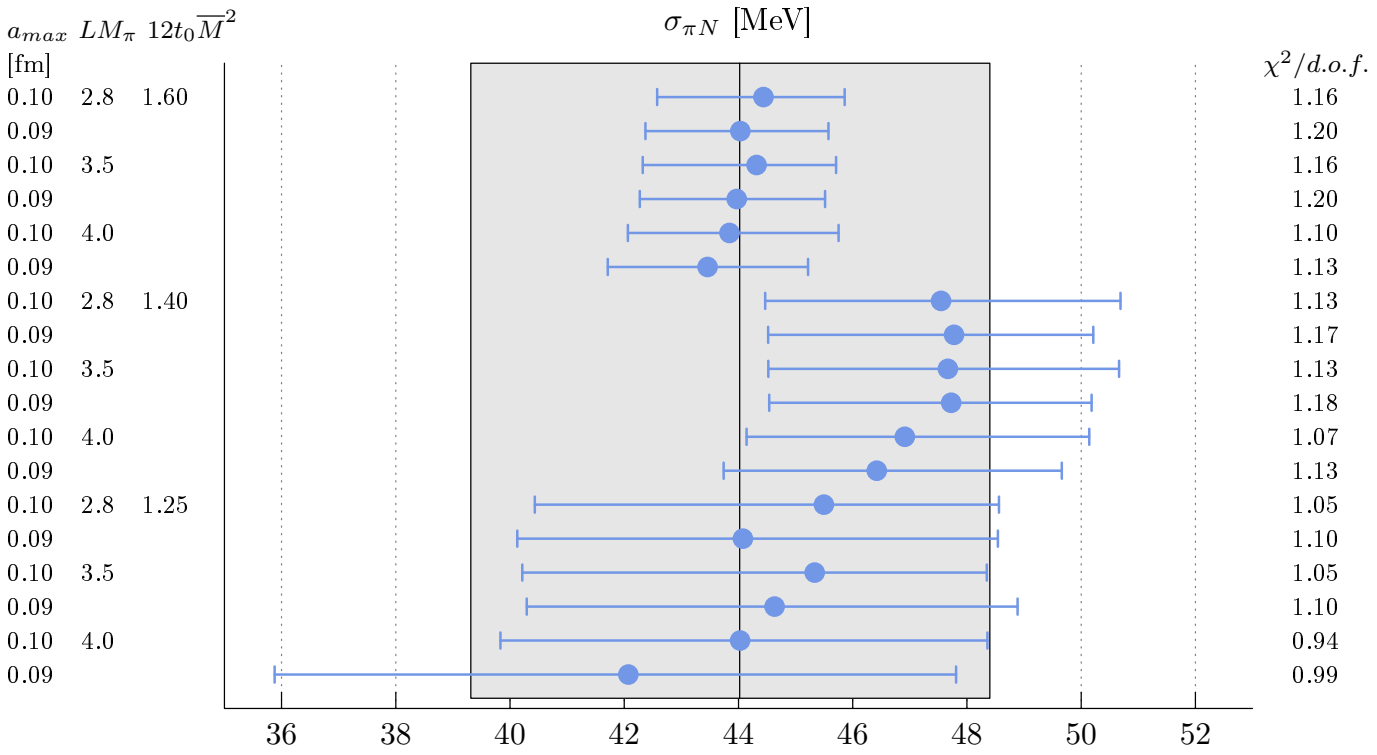
$$\Delta m_\Xi = m_{\Xi^0} - m_{\Xi^-} \approx -\delta m^{\mathrm{QCD}} - \delta m^{\mathrm{QED}}, \qquad m_\Xi = \frac{1}{2}\left(m_{\Xi^0} + m_{\Xi^-} - \delta m^{\mathrm{QED}}\right)$$

$\delta m^{\mathrm{QCD}} \approx 4$ MeV and $\delta m^{\mathrm{QED}} \approx 3$ MeV. Assume same δm^{QED} for m_Ω.

For the unstable decuplet baryons, the Breit-Wigner mass is given with an error of $\Gamma/2$ (with rounding).

Variation of $\tilde{\sigma}_{\pi N}$ with cuts on the data

NNLO BChPT, continuum limit, finite volume fits to the baryon octet.



Variation of $\tilde{\sigma}_{\pi N}$ with the continuum fit form