HVP with C* boundary conditions

Part II

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Intro / Overview

- Part I
  - Strategy of calculation
  - Sources
  - Vector masses
  - Introduction to C* boundary conditions
- **Part II**
  - **Implications of C* boundary conditions**
  - **HVP in dynamical QCD+QED conditions**
  - **Outlook**
- Other related talks/posters
  - Sofie Martins: Finite-Size Effects of the Hadronic Vacuum Polarization Contribution to the Muon ($g-2$) with C* Boundary Conditions (*talk in this session at 09:40 AM*)
  - Jens Lücke: An update on QCD+QED simulations with C* boundary conditions (*talk in session "Hadron Spectroscopy and Interactions" on Fri 4:40 PM*)
  - Paola Tavella: Strange and charm contribution to the HVP from C* boundary conditions (*poster on Tue 8:00 PM*)
  - Alessandro Cotellucci: Tuning of QCD+QED simulations with C* boundary conditions (*poster on Tue 8:00 PM*)

Implications of C* boundary conditions

C* BOUNDARY CONDITIONS

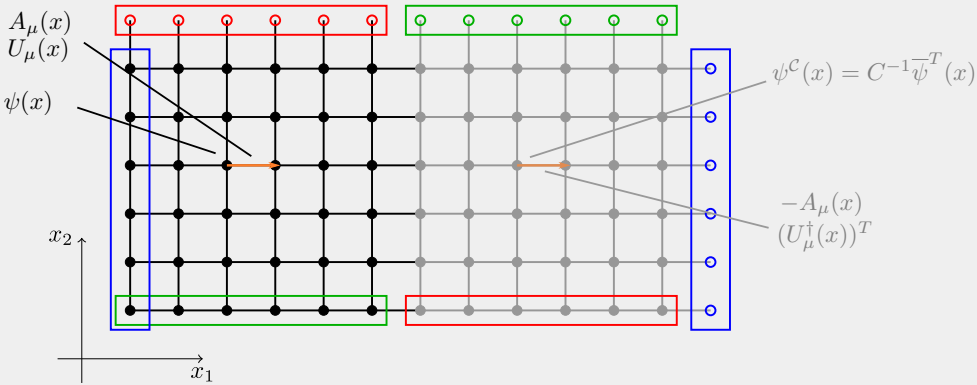

Figure: Fermions $\psi(x)$, QCD links $U_\mu(x)$, QED photon field $A_\mu(x)$.

- periodic boundaries on extended lattice, i.e. $\psi(x + 2L_1\hat{1}) = \psi(x)$

C* BOUNDARY CONDITIONS

Pros

+ simulation of dynamical QED from first principles.

Cons

- lattice volume doubled by introducing a mirror lattice.

C* BOUNDARY CONDITIONS

ψ and $\bar{\psi}$ not independent anymore[1]

	periodic boundary		C* boundary
Action:	$\sum_{x\in\Lambda_{\text{phys}}} \bar{\psi}(x)D\psi(x)$	$\longrightarrow$	$\sum_{x\in\Lambda_{\text{phys+mirror}}} -\frac{1}{2}\psi^T(x)CTD\psi(x)$
Integration measure:	$[\mathcal{D}\psi]_{\Lambda_{\text{phys}}}\left[\mathcal{D}\bar{\psi}\right]_{\Lambda_{\text{phys}}}$	$\longrightarrow$	$[\mathcal{D}\psi]_{\Lambda_{\text{phys+mirror}}}$
Determinant:[2]	$\det(D)$	$\longrightarrow$	$\text{Pf}(CTD)$
Wick-contraction :	$\overbracket{\psi(x)\bar{\psi}}(y) = D^{-1}(x\|y)$	$\longrightarrow$	$\overbracket{\psi(x)\psi^T}(y) = -D^{-1}(x\|y)TC^{-1}$

- C: charge-conjugation matrix
- T: translation operator flips physical $\leftrightarrow$ mirror lattice

$$T\psi(x_{\text{phys}}) = \psi(x_{\text{phys}} + L_1\hat{1})$$
$$T\psi(x_{\text{mirr}}) = \psi(x_{\text{mirr}} - L_1\hat{1})$$

[1] compare Lucini et al. (2016) [1], Patella (2017) [2]
[2] See poster by Alessandro Cotellucci (Tue 8:00 pm)

C* BOUNDARY CONDITIONS

Vector correlator turns into the usual expression (with modified Dirac operator)

$$\langle j_\mu(x) j_\nu(y) \rangle = \mathrm{tr}_{CD} \left[\gamma_\mu D^{-1}(x|y) \gamma_\nu D^{-1}(y|x) \right]$$

HVP in dynamical QCD+QED simulations

OPENQ*D CODE

Publicly available under `https://gitlab.com/rcstar/openQxD`.

- open source (GNU GPLv2),
- see Campos et al., (2020) [3] for an introduction.
- available solvers:
  - conjugate gradient on the normal equations (CGNE)
  - generalized conjugate residual using Schwarz alternating procedure (SAP+GCR) [3]
  - (inexact) deflation-accelerated solver (DFL+SAP+GCR) [4]

[3] Lüscher (2003) [4]
[4] Lüscher (2007) [5]

Used ensembles / parameters

Ensemble	A400	A360	A380
flavors	3(u/d/s) + 1(c)	1(u) + 2(d/s) + 1(c)	1(u) + 2(d/s) +1(c)
α	0.0	0.04	1/137
$m_\pi [MeV]$	400	360	380

QCD $\leftrightarrow$ QCD+QED

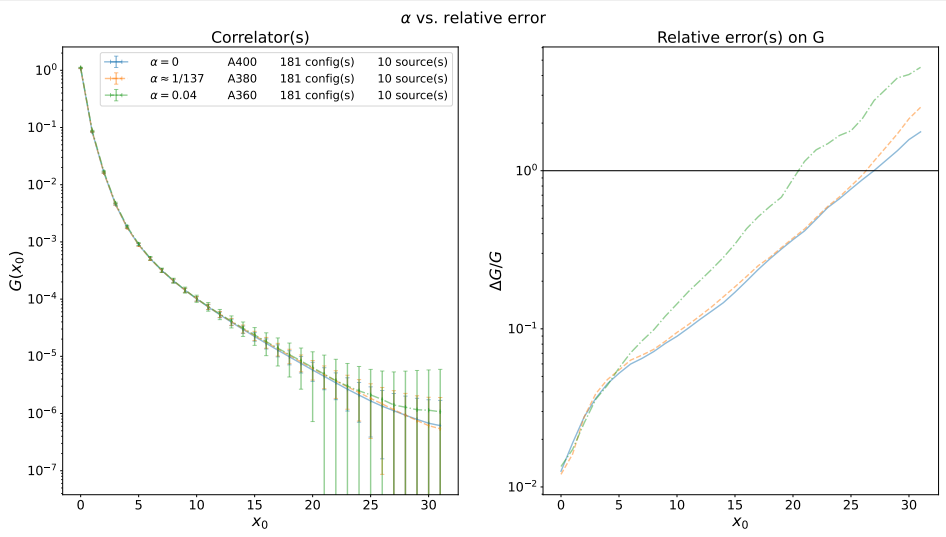

Figure: Relative error comparison. $G(x_0) \sim \sum_{k=1}^{3} \sum_{\vec{x}} \langle j_k(x_0, \vec{x}) j_k(0, \vec{0}) \rangle$

HVP with QCD+QED

- At large times, signal/noise dominates $\implies$ cutoff at x_{cut}, remaining part is modeled using a model function.
- single-exponential, m_0 taken from mass spectroscopy, amplitude A taken from 1-parameter fit to correlator.

$$G(x_0) = \begin{cases} G^{\text{lattice}}(x_0) & x_0 < x_{\text{cut}} \\ Ae^{-m_0 x_0} & x_0 \geq x_{\text{cut}} \end{cases}$$

HVP with QCD+QED

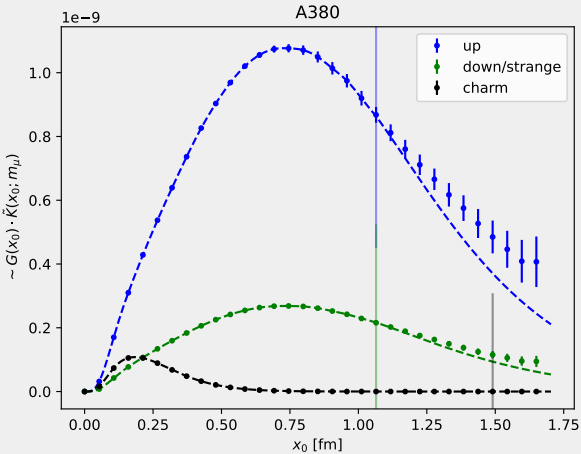

Figure: Integrand for $g-2$ for $\alpha \approx 1/137$, conserved-local current: regions left of vertical line use lattice data, right of vertical line use single-exponential fit.

HVP with QCD+QED

Preliminary results with conserved-local current, blinded ($Z_V = 1$)

ensemble	flavor	$a_\mu^{HVP} \times 10^{10}$
A400 $\alpha = 0$	up/down/strange	319(8)
	charm	10.0(1)
A360 $\alpha = 0.04$	up	309(11)
	down/strange	77(2)
	charm	10.6(1)
A380, $\alpha \approx 1/137$	up	331(7)
	down/strange	83(2)
	charm	9.8(1)

Error contributions

for $\alpha \approx 1/137$, $m_\pi = 380\ \mathrm{MeV}$, up flavor, conserved-local current

	variation w.r.t.	relative error
statistical	jackknife	1.21%
	vector mass[5]	1.36%
	relative scale setting[5,6]	0.92%
systematic	fit range	0.14%
	cutoff	0.03%
	excited states	1.20%
total		2.37%

unaccounted errors: physical pion mass, scale setting, continuum extrapolation.

[5] by error propagation

[6] input $\Delta a/a = 0.53\%$, sensitivity to lattice scaling agrees with Della Morte et al. (2017) [6]

Isospin corrections

QCD

$$S = S_F[U] + S_G[U],$$
$$U \in \mathrm{SU}(3).$$

- effects due to $\alpha, m_u - m_d \neq 0$ need to be included $\implies$ needs 1+16 more inversions per source and flavor.

QCD + QED

$$S = S_F[U, A_\mu] + S_G[U] + S_\gamma[A_\mu],$$
$$U \in \mathrm{SU}(3), e^{-iA_\mu} \in U(1).$$

- only effects due to $m_u \neq m_d$ need to be included $\implies$ needs 1 more inversion per source and flavor.

[6] see Duncan et al. (1996) [7], De Divitiis et al. (2012) [8], Boyle et al. (2017) [9].

Finite size effects

Hansen-Patella method[7]:

- expansion in $e^{-|\mathbf{n}|m_\pi L}$, with $\mathbf{n} \in \mathbb{Z}^4$ due to 2-pion states.
- periodic boundary conditions: $\mathcal{O}(e^{-m_\pi L})$.
- Leading order vanishes for C* boundary conditions .[8]

finite size effects due to QED

- expansion in $1/L$ due to 1-photon states.
- For hadron masses effects are smaller than with QED_L[9].

[7] See Hansen and Patella (2020) [10]
[8] See talk by Sofie Martins (this session at 09:40) for details.
[9] see Lucini et al., JHEP (2016) [1].

Outlook

- model part: include excited states.
- Variance reduction
  - Low mode averaging
  - Extended/stochastic sources (for vector masses)
- Chiral extrapolation
- Continuum extrapolation
- Disconnected contributions

We acknowledge access to Piz Daint at the Swiss National Supercomputing Centre, Switzerland under the ETHZ's share with the project IDs s1101, eth8 and go22.

References I

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[2] A. Patella, *C* boundary conditions in openQ*D code*, URL: gitlab.com/rcstar/openQxD/-/blob/master/doc/cstar.pdf.

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[10] M. T. Hansen and A. Patella, "Finite-volume and thermal effects in the leading-HVP contribution to muonic $g-2$", Journal of High Energy Physics **2020**, 29 (2020), arXiv:2004.03935.

[11] J. C. Collins, A. V. Manohar, and M. B. Wise, "Renormalization of the vector current in QED", Physical Review D **73**, 105019 (2006),

Used ensembles / parameters

Ensemble	A400	A360	A380
flavors	3(u/d/s) + 1(c)	1(u) + 2(d/s) + 1(c)	1(u) + 2(d/s) +1(c)
β	3.24	3.24	3.24
α	0.0	0.04063(6)	0.00708(2) $\approx 1/137$
m_π [MeV]	399(3)	359(3)	380
a [fm]	0.05393(24)	0.05054(27)	0.05323(28)
size	32x32x32x64	32x32x32x64	32x32x32x64
#configs	200	181	200

Renormalization

- For QCD: conserved-local current requires renormalization.

$$j_\mu^{\text{ren}}(x) = Z_V j_\mu^{\text{local}}(x) + \mathcal{O}(a).$$

Use ratio between conserved-local and local-local correlator:

$$Z_V = \left\langle \frac{\langle j_\mu^{\text{ps}}(x) j_\mu^{\text{loc}}(0)\rangle}{\langle j_\mu^{\text{loc}}(x) j_\mu^{\text{loc}}(0)\rangle} \right\rangle.$$

- For QCD+QED: even conserved current is subject to renormalization.[10]

$$j_\mu^{\text{ren}}(x) = j_\mu^{\text{ps}}(x) + \frac{1 - Z_3^{-1}}{e_0} \partial_\nu F^{\nu\mu}$$

[10] See Collins et al. PRD (2006) [11], e_0 is bare electric charge, Z_3 renormalization constant of $A_\mu(x)$.