Time windows of the muon HVP from twisted mass lattice QCD.

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(On behalf of the ETM Collaboration)

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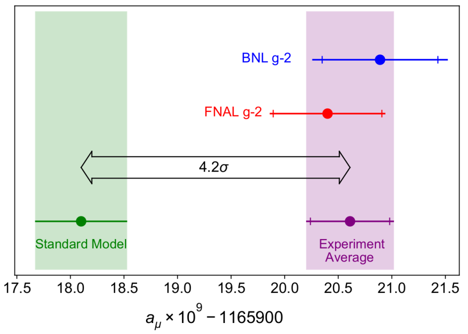
Lattice 2022, 9 August 2022, Bonn

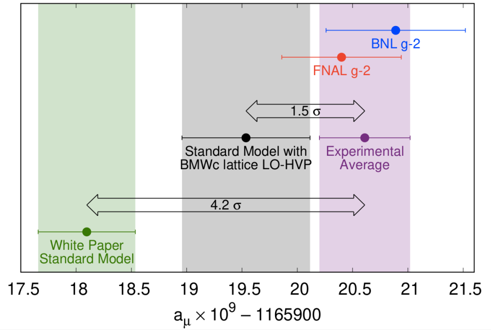
The beginning of the new $g-2$ puzzle

$$a_\mu \equiv \frac{g_\mu - 2}{2} = a_\mu^{QED} + a_\mu^{weak} + a_\mu^{\mathrm{HVP}} + a_\mu^{\mathrm{HLbL}}$$

[Fermilab plot, from PRL 126, 141801 (2021) Muon $g-2$ collaboration]



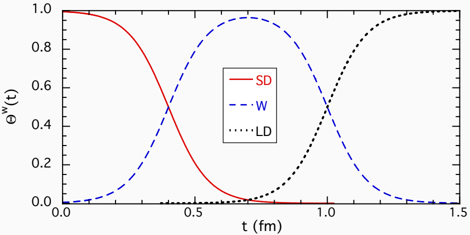
[BMWc version, from L. Lellouch slides at SchwingerFest, LA (June 2022)]

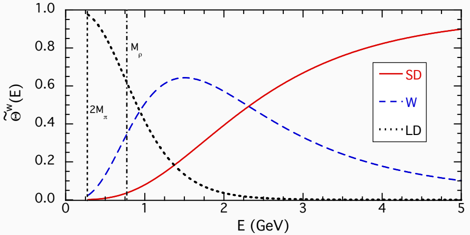


- In WP '20, SM prediction made use of a_μ^{HVP} from experimental $e^+e^- \to$ hadrons (R^{had}−ratio) using dispersion relations.
- Lattice calculation of $a_\mu^{\mathrm{LO-HVP}}$ by BMWc ('20) disagrees with R^{had} at $2.1\,\sigma$ level ($1.5\,\sigma$ with exp.).

Window observables of $a_\mu^{\mathrm{LO-HVP}}$ as a probe of $R^{had}(E)$

$$\underbrace{2\alpha_{em}^2 \int_0^\infty dt\; t^2 K(m_\mu t)\; V(t)}_{\text{lattice, SM}} \;=\; a_\mu^{\mathrm{LO-HVP}} \;=\; \underbrace{\frac{2\alpha_{em}^2 m_\mu^2}{9\pi^2} \int_{2M_\pi}^\infty \frac{dE}{E^2}\; \tilde{K}(E)\; R^{had}(E)}_{\text{dispersive, experimental}}$$





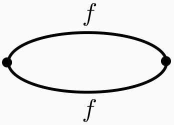
$$2\alpha_{em}^2 \int_0^\infty dt\; t^2 K(m_\mu t) V(t) \underline{\Theta^w(t)} = a_\mu^w = \frac{2\alpha_{em}^2 m_\mu^2}{9\pi^2} \int_{2M_\pi}^\infty \frac{dE}{E^2} \tilde{K}(E) R^{had}(E) \underline{\tilde{\Theta}^w(E)}$$

- a_μ^w, $w = \{SD, W, LD\}$, probe $R^{had}(E)$ more locally in energy E.
- Key test of SM (lattice QCD + QED) vs experimental $e^+e^- \to$ hadrons (independent of $g_\mu - 2$).
- Lattice data for $w = SD, W$ are very precise, no S/N problem.

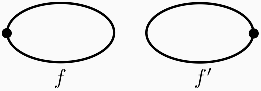
Outline of our twisted-mass lattice QCD calculation

We computed the u, d, s, c, quark-line connected and disconnected contributions to a_μ^{SD} and a_μ^{W} in the isospin symmetric limit $m_u = m_d$, neglecting α_{em}^3 QED effects.

$$V^{f}_{conn}(t) \equiv -\frac{1}{3} \sum_{i=1,2,3} \int d^3x \langle J_i^f(\vec{x},t) \; J_i^f(0) \rangle = q_f^2 \times$$



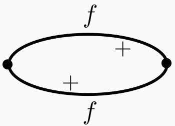
$$V^{ff'}_{disco}(t) \equiv -\frac{1}{3} \sum_{i=1,2,3} \int d^3x \langle J_i^f(\vec{x},t) \; J_i^{f'}(0) \rangle = -q_f q_{f'} \times$$



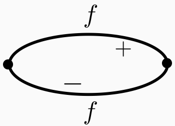
Dataset and disconnected (diagonal and off-diagonal) contributions already discussed in S. Bacchio's Talk.

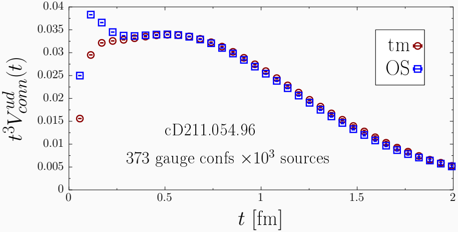
- For connected contributions, two different ways to approach the continuum limit.
- We computed $V^f_{conn}(t)$ employing two distinct lattice versions of the local e.m. current (peculiar to tm-LQCD):

$$J_\mu^{f,OS} \propto \bar{\psi}_f^{+} \gamma_\mu \psi_f^{+}$$



$$J_\mu^{f,tm} \propto \bar{\psi}_f^{+} \gamma_\mu \psi_f^{-}$$





- $\pm$ is the sign of the twisted Wilson parameter.
- Connected $V^{f,OS}_{conn}(t)$ and $V^{f,tm}_{conn}(t)$ differ by $\mathcal{O}(a^2)$ cut-off effects, including a^2-dependent FSEs.

General fit ansatz

For all connected contributions to a_μ^W and a_μ^{SD} we perform combined continuum fits employing both tm and OS lattice correlators.

- General structure of the fit ansatz ($w = \{SD, W\}$):

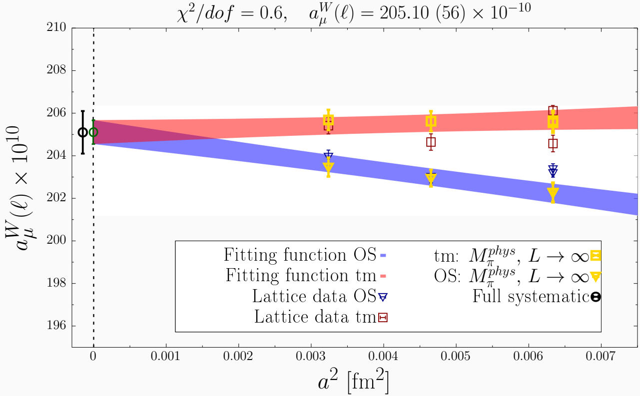
$$a_\mu^w(\ell) = \left[a_\mu^{w,cont}(\ell) + \Delta a_\mu^w(L) + F_1^r a^2 \frac{\partial}{\partial M_\pi} \Delta a_\mu^w(L)\right]$$

$$\times \left[1 + A\,(M_\pi - M_\pi^{phys}) + D_1^r \,\frac{a^2}{[\log(a^2/w_0^2)]^{n_r}} + D_2^r\, a^4\right]$$

- $a_\mu^{w,cont}$,F_1^r,A,D_1^r and D_2^r are free fitting parameters ($n_r = \{0,3\}$).
- $a_\mu^{w,cont}$,A do not depend upon the regularization $r = \{tm, OS\}$.
- $\Delta a_\mu^w(L)$: based on MLLGS modelization of Finite Size Effect, tuned to reproduce known lattice results (only for $u+d$ contribution).
- Term proportional to F_1^r describes a^2-dependent FSEs due to $\mathcal{O}(a^2)$ distortions of the pion spectrum.

Connected-light contribution to the intermediate window: $a_\mu^W(\ell)$

Light $(u+d)$ connected contribution to a_μ^W



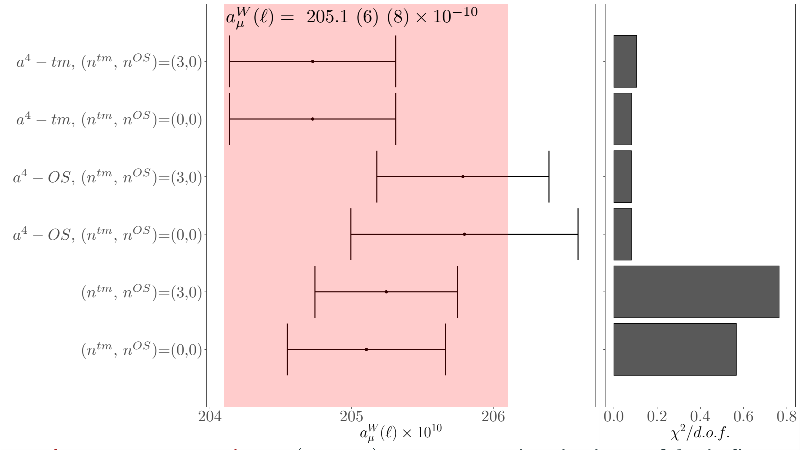
- Typical accuracy of $0.1 - 0.2\%$ for all ensembles and regularizations.
- At $a \sim 0.08$ fm evidence for a^2-dependent FSEs ($F_1^{tm} \neq F_1^{OS}$).
- Negligible for OS and:

$$\Delta a_\mu^{W;tm}(\ell, L \sim 7.6 \text{ fm} \ \ vs. \ \ L \sim 5.1 \text{ fm}) \sim 1.5\ (5) \times 10^{-10}$$

- We show continuum/infinite-volume extrapolation performed using MLLGS-like ansatz (slide 5) w.o. a^4 terms or Logs.

Analysis of the systematics

We selected all fits ($N \simeq \mathcal{O}(10)$) leading to $\chi^2/d.o.f. < 1.8$.

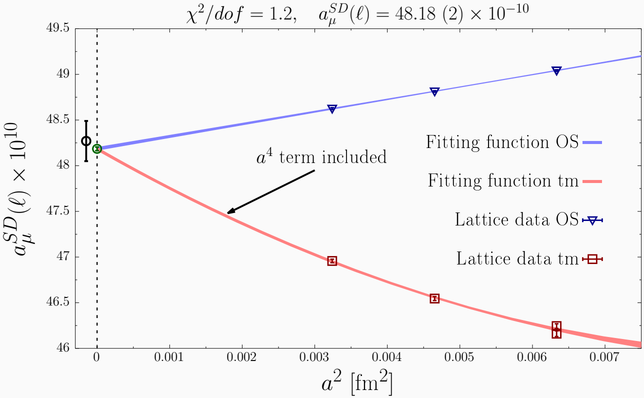


Averaging procedure: (x_k, σ_k) is mean and std. dev. of k-th fit

$$\bar{x} = \sum_{k=1}^{N} \omega_k x_k, \quad \sigma_{\bar{x}}^2 = \sum_{k=1}^{N} \omega_k \sigma_k^2 + \sum_{k=1}^{N} \omega_k (x_k - \bar{x})^2, \quad \omega_k = 1/N$$

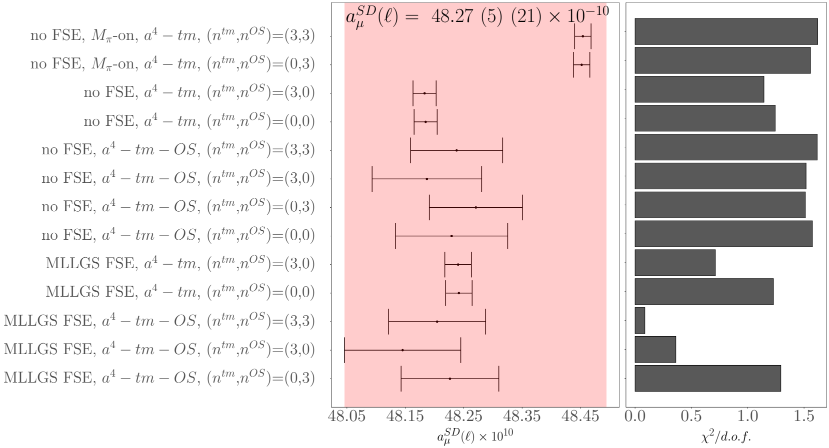
Connected-light contribution to the short-distance: $a_\mu^{SD}(\ell)$

$a_\mu^{SD}(\ell)$



- Achieved a remarkable precision better than 0.1%.
- We subtracted from the raw lattice data the free-theory cut-off effects for both regularizations to remove $a^2 \log(a)$ lattice artifacts [Cè, Harris, Meyer et al. (2021)].
- a^4 term on tm regularization necessary to have a good χ^2/dof.

Analysis of the systematics for $a_\mu^{SD}(\ell)$



- Final error entirely due to systematics in continuum extrapolation.
- $A\left(M_\pi - M_\pi^{phys}\right)$ term not visible.
- Alternative continuum limit extrapolation with ultra-short distance regulator (Backup).

Strange and charm connected contributions to a_μ^W and a_μ^{SD}

Calculation details

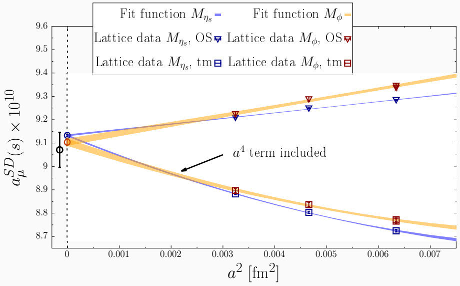
Strange contributions

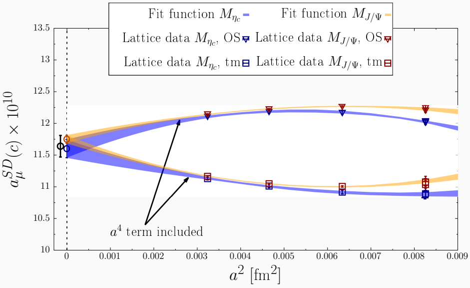
- Valence s quark mass tuned alternatively using M_{η_s} or M_ϕ as input.
- Both determinations included in final analysis of systematics.
- Subtraction of perturbative $\mathcal{O}(\alpha_s^0)$ cut-off effects in $a_\mu^{SD}(s)$.
- Finite size effects and M_π^{sea} mistuning effects not visible within accuracy.

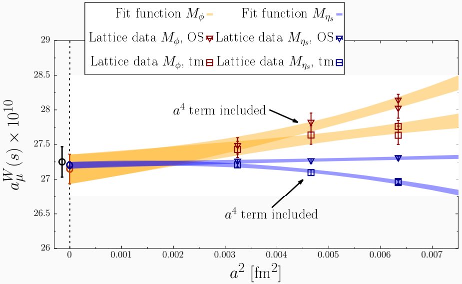
Charm contributions

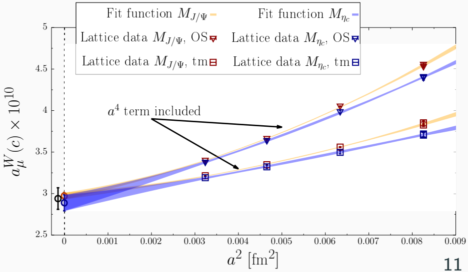
- Valence c quark mass tuned alternatively using M_{η_c} or $M_{J/\Psi}$ as input.
- Both determinations included in final analysis of systematics.
- Added a fourth (coarser) lattice spacing $a \sim 0.09$ fm with pion masses $M_\pi^{sea} \in [250 - 350]$ MeV to improve continuum limit extrapolation.
- No M_π^{sea} dependence observed, negligible finite size effects.
- Subtraction of perturbative $\mathcal{O}(\alpha_s^0)$ cut-off effects in $a_\mu^{SD}(c)$. More effective if evaluated with $m_q = m_c^{bare}$.

Strange and charm connected contributions









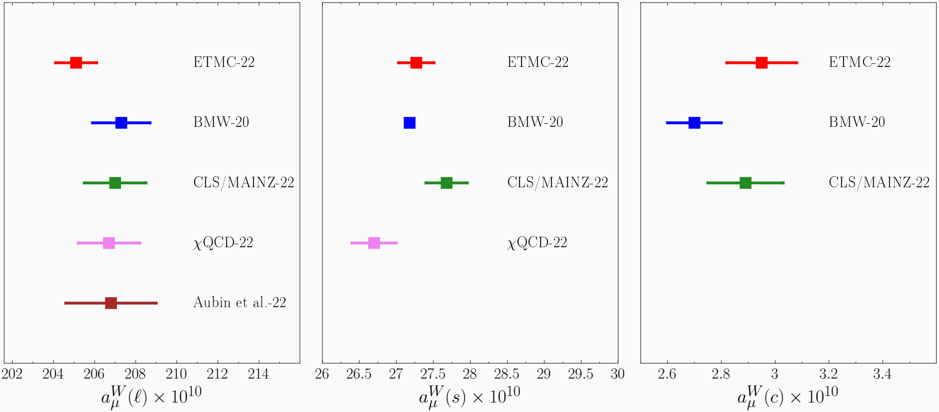
Summary

	$a_\mu^{SD} \times 10^{10}$	$a_\mu^{W} \times 10^{10}$
ℓ	48.27 (22)	205.1 (1.0)
s	9.071 (75)	27.27 (24)
c	11.64 (16)	2.95 (13)
disco	−0.006 (5)	−0.77 (17)
IB	0.03*	0.43 (4)**
b	0.32 (2)***	–
total	69.33 (29)	235.0 (1.1)

- * *rhad software package.* 0.04% *of the total* a_μ^{SD} *(or* 0.1σ*).*
- ** *From Borsanyi et al. (Nature, 2021).* 0.18% *of the total* a_μ^{W} *(or* 0.4σ*).*
- *** *rhad & lattice.* 0.46% *of total* a_μ^{SD} *(or* 1.1σ*).*

Precision achieved on a_μ^{W} and a_μ^{SD} is $\sim 0.5\%$.

Per-flavour lattice comparisons

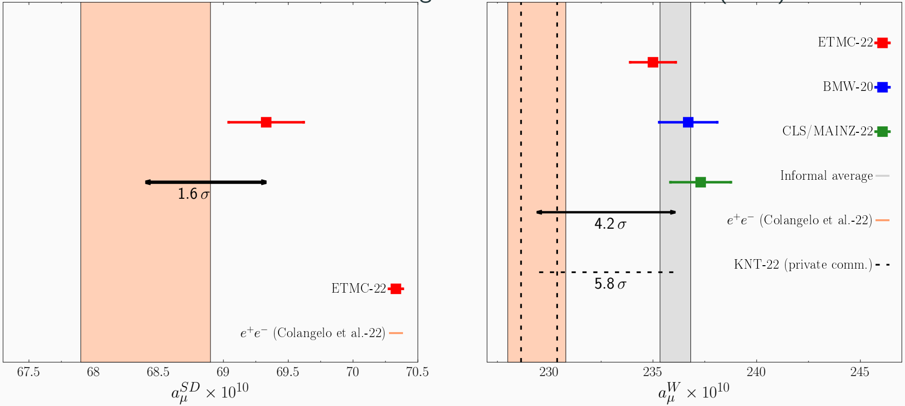


$[a_\mu^{SD} + a_\mu^W] \times 10^{10}$	ℓ	s	c	total, incl. disc., IB, b
ETMC-22*	253.4 (1.2)	36.3 (0.3)	14.6 (3)	304.3 (1.4)
Fermilab/HPQCD/MILC-22	253.5 (0.9)	36.3 (0.2)	14.63 (5)	303.8 (1.1)

*Preliminary: conservative error (Assuming 100% correlation between a_μ^{SD} and a_μ^W).

Comparison with $e^+e^- \to$ hadrons results

$e^+e^- \to$ hadrons from Colangelo et al. arXiv:2205.12963 (2022).



- Tension in a_μ^W rises to 4.2σ if we combine ETMC '22, BMW '20 and CLS/Mainz '22 (informal average $\to$ next WP).
- Deviation of $e^+e^- \to$ hadrons data w.r.t. the SM in the low and (possibly) intermediate energy regions, but not in the high energy region.

Conclusions

- We performed a first-principle evaluation of the isosymmetric QCD contribution to a_μ^{SD} and a_μ^{W}.
- Thanks to our dedicated simulations at the $(\simeq)$ physical point, and to a high-statistics computation of the VV correlator, we achieved a relative precision smaller than 0.5% on both a_μ^{SD} and a_μ^{W}.

- For a_μ^{SD} our determination agrees with $R(E)$ results at $1.6\,\sigma$ level (No NP at high E, consistent with EW precision tests).

- Our result for a_μ^{W} confirms the tension between lattice (SM) and dispersive (data-driven) $e^+e^- \to$ hadrons results. Current tension between lattice (informal average) and $R(E)$-driven results is $4.2\,\sigma$ (problems in dispersive analysis? NP at intermediate E?).

Work in progress

- Inclusion of $\mathcal{O}(\alpha_{em}^3)$ QED-effects and strong IB contribution.
- New dedicated ensembles to study FSEs, analysis of a_μ^{LD}.

Extended Twisted Mass Collaboration

Thanks for your attention!

Backup slides

Renormalization constants

Hadronic determination of Z_V

We define:

$$R_V(t) = 2\mu_q \frac{C_{PP}^{tm}(t)}{\partial_t C_{AP}^{tm}(t)}$$

P and A are pseudoscalar and axial local bare currents in C_{AP} and C_{PP}.

Owing to the exact flavor symmetry of massless Wilson fermions ($\mu_q = 0$):

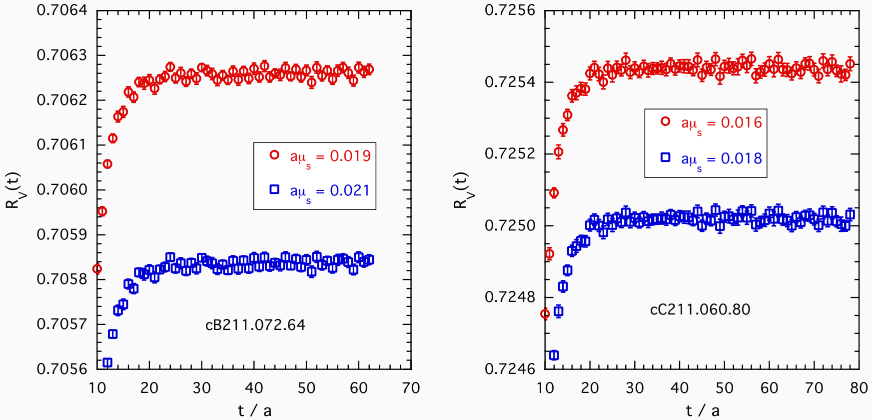
$$R_V(t) \stackrel{t/a \gg 1}{=} Z_V$$

- Freedom in the choice of μ_q:

$$Z_V(\mu_q) = Z_V(0) + \mathcal{O}(a^2 \mu_q \Lambda_{QCD}, a^2 \mu_q^2)$$

- We choose μ_q such that $M_P^{tm} = M_{\eta_s}^{isoQCD}$.
- $M_{\eta_s}^{isoQCD} = 689.89\ (50)\ \mathrm{MeV}$, [*Borsanyi et al., Nature (2021)*].

R_V around the physical strange quark mass



- Remarkable precision better than 0.01%.

Hadronic determination of Z_A

We define: $$R_A(t) = 2\mu_q \frac{C_{PP}^{OS}(t)}{\partial_t C_{AP}^{OS}(t)}$$

P and A are pseudoscalar and axial local bare currents in C_{AP} and C_{PP}. In the large time limit $t/a \gg 1$ ($X = tm, OS$):

$$C_{PP}^{X}(t) \to |G_P^X|^2 \frac{e^{-M_P^X t} + e^{-M_P^X (T-t)}}{2M_P^X}, \; R_A(t) \to 2a\mu_\ell \frac{Z_A}{f_P^{OS}} \frac{G_P^{OS}}{M_P^{OS} \sinh\left(aM_P^{OS}\right)}$$

We impose (true up to $\mathcal{O}(a^2)$ lattice artifacts):

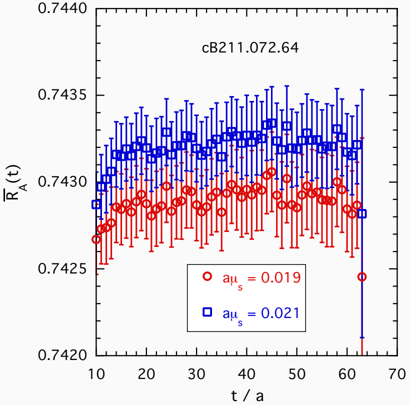
$$\boxed{f_P^{OS} = f_P^{tm} = 2a\mu_\ell \frac{G_P^{tm}}{M_P^{tm} \sinh\left(aM_P^{tm}\right)}}$$

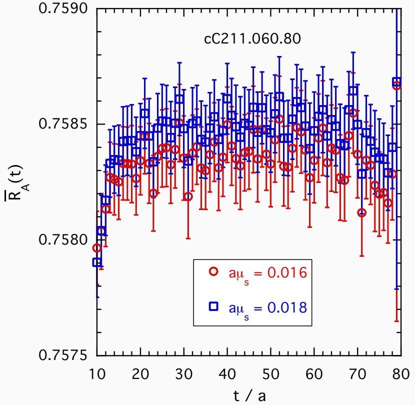
and using $\frac{Z_P}{Z_S} = \frac{G_P^{OS}}{G_P^{tm}}$, we can extract Z_A through

$$\boxed{\bar{R}_A(t) \equiv R_A(t) \frac{M_P^{OS} \sinh\left(aM_P^{OS}\right)}{M_P^{tm} \sinh\left(aM_P^{tm}\right)} \frac{Z_S}{Z_P} \to Z_A}$$

$Z_A(\mu_q) = Z_A(0) + \mathcal{O}(a^2 \mu_q \Lambda_{QCD}, a^2 \mu_q^2) \implies$ any choice of μ_q is legitimate.

$\bar{R}_A$ around the physical strange quark mass.





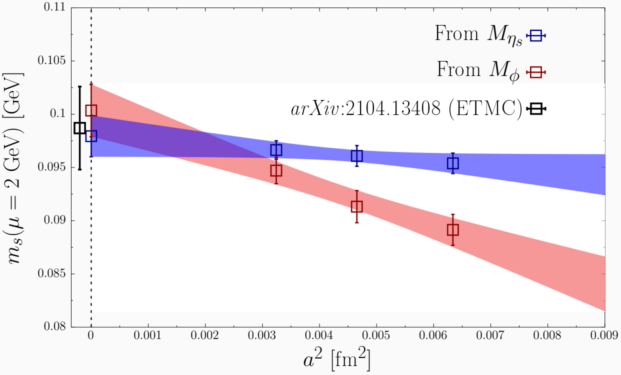
We interpolate $Z_A(\mu_q)$ at $Z_A(\mu_q = \mu_s^{phys})$ with $M_{\eta_s}(\mu_s^{phys}) = 689.89\ (50)\ \mathrm{MeV}$.

Strange and charm quark masses

Strange quark mass in the $\overline{\mathrm{MS}}$ scheme at $\mu = 2$ GeV.

We determine m_s^{phys} using two different hadronic inputs (both included in the a_μ^W and a_μ^{SD} analyses).

- From the mass of the *fictious* η_s meson: $M_{\eta_s}^{\mathrm{isoQCD}} = 689.9(5)$ MeV.
- From the mass of the vector ϕ meson: $M_\phi^{PDG} = 1.01946(2)$ GeV.

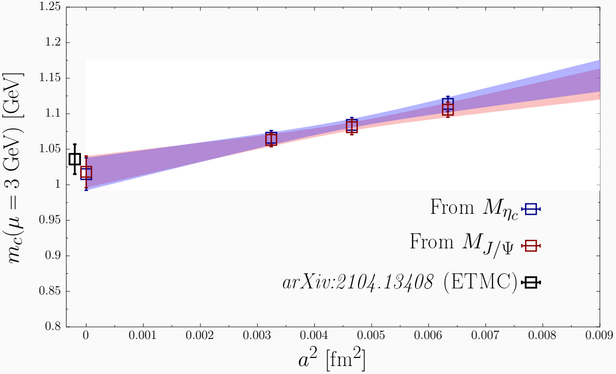


Good agreement with previous ETMC determination from m_K.

Charm quark mass in the $\overline{\mathrm{MS}}$ scheme at $\mu = 3$ GeV.

We determine m_c^{phys} using two different hadronic inputs (both included in the a_μ^W and a_μ^{SD} analyses).

- From the mass of the η_c meson: $M_{\eta_c} = 2.984\ (4)_{disco}$ GeV.
- From the mass of the J/Ψ resonance: $M_{J/\Psi} = 3.097\ (1)_{disco}$ GeV.

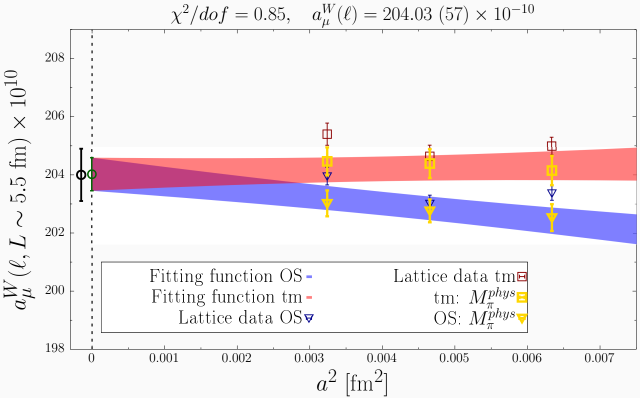


Good agreement with previous ETMC determination from m_D.

Cross-check for $a_\mu^W(\ell)$

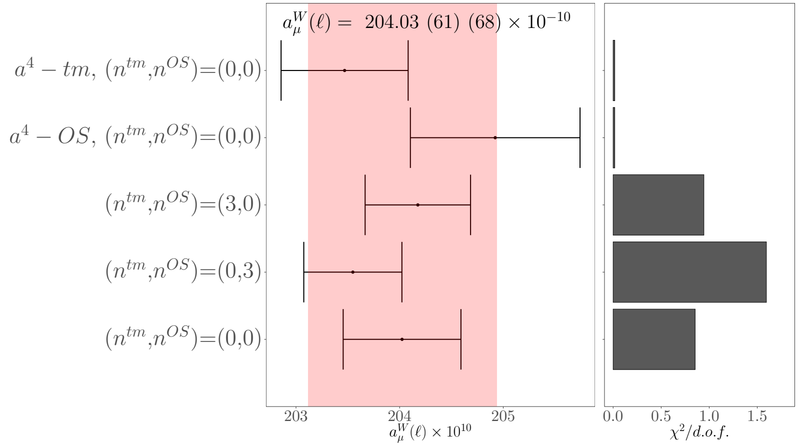
Cross-check: $a_\mu^W(\ell)$ at fixed physical volume

We performed the continuum extrapolation at fixed $L \sim 5.46$ fm, which corresponds to the cC.060.80 and cD.054.96 ensemble volumes.



- At $a \sim 0.08$ fm we interpolated $a_\mu^W(\ell)$ at $L \sim 5.46$ fm, linearly in $e^{-M_\pi L}$, using our results at $L \sim 5.10$ fm and $L \sim 7.64$ fm.

Analysis of the systematics at $L \sim 5.46$ fm



We add to $a_\mu^W(\ell, L \sim 5.46 \text{ fm})$, the FSE estimated from the MLLGS-model multiplied by the correcting factor $\kappa = 1.25\ (25)$:

$$\Delta a_\mu^W(\ell, L \sim 5.46 \text{ fm}) = 1.00(20) \times 10^{-10} \implies a_\mu^W(\ell) = 205.03(93) \times 10^{-10}$$

Perturbative subtraction at tree level in $a_\mu^{SD}(\ell)$

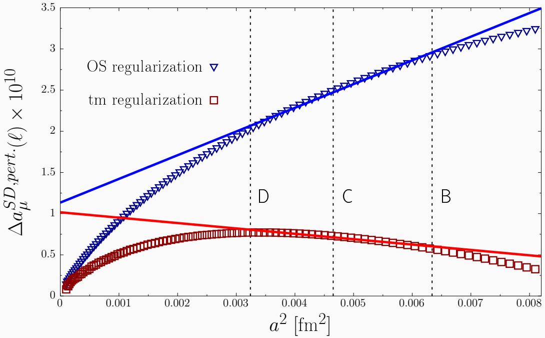
The dangerous $a^2 \log(a)$ cut-off effects

Lattice evaluation of a_μ^{SD} suffers from dangerous $a^2 \log(a^2)$ artifacts generated by the short-times integration [Cè, Harris, Meyer et al. (2021)]

$$V(t \ll m^{-1}, a) \propto \frac{1}{t^3}\left[1 + \sum_{n=1}^{\infty} c_n \cdot \left(\frac{a}{t}\right)^{2n}\right], \qquad K(m_\mu t \ll 1) \propto t^2$$

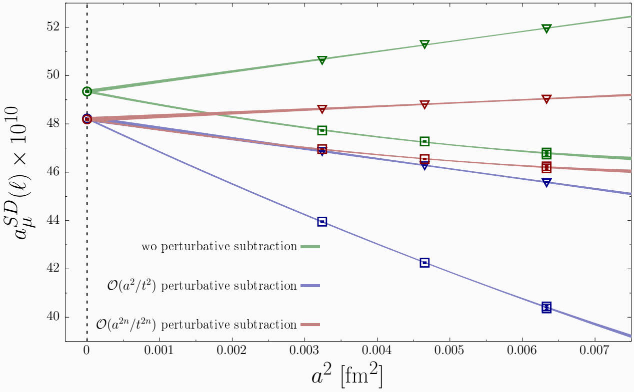
$$\implies a_\mu^{SD} \simeq \int_a^{t_0} dt\, V(t,a)\, t^2 K(m_\mu t) = A + D\, a^2 \log(a^2) + \mathcal{O}(a^2)$$

Naive continuum extrapolation of free theory cut-off effects



- $a^2 \log(a^2)$ cut-off effects already present in the free-theory correlator.
- $\Delta a_\mu^{SD,pert.}(\ell)$ are cut-off effects of the tm (OS) $\mathcal{O}(\alpha_s^0)$ massless correlator.

Perturbative $\mathcal{O}(\alpha_s^0)$ subtraction of cut-off effects



- No perturbative subtraction: continuum limit missed by $\simeq 1 \times 10^{-10}$ (effect larger than any other source of systematics).
- LO $\mathcal{O}(\frac{a^2}{t^2})$ subtraction: sufficient to get correct continuum limit.
- Full $\mathcal{O}(a^{2n}/t^{2n})$ subtraction: makes lattice data flatter.

Cross-check for $a_\mu^{SD}(\ell)$

Cross-check for $a_\mu^{SD}(\ell)$ (I)

Introduce an UV regulator t_{min} and define:

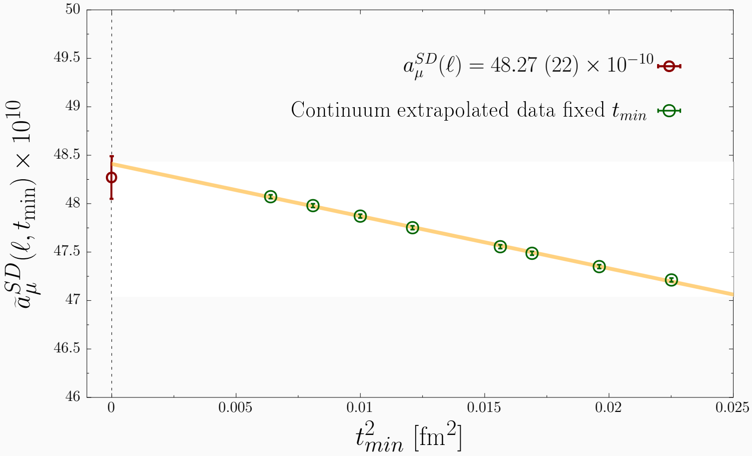
$$a_\mu^{SD}(\ell, t_{min}) \equiv 4\alpha_{em}^2 \int_{t_{min}}^{\infty} dt\ t^2 K(m_\mu t)\, V_\ell(t) \cdot [1 - \Theta(t, t_0, \Delta)]$$

- Switch the order in which the limit $a \to 0$ and $t_{\rm min} \to 0$ are performed (perform $a \to 0$ first).
- Residual Logs $a^2/[\log(a^2/w_0^2)]^n$ generated by integration at short-times become $a^2/[\log(t_{min}^2/w_0^2)]^n$.
- After performing cont. extr. at fixed t_{min}, take $t_{min} \to 0$ limit.
- To speed-up convergence we add continuum perturbative $\mathcal{O}(\alpha_s^0)$ contribution in $t \in [0, t_{min}]$:

$$\widetilde{a}_\mu^{SD}(\ell, t_{min}) = a_\mu^{SD}(\ell, t_{min}) + \int_0^{t_{min}} dt\ t^2 K(m_\mu t)\, V_\ell^{pert.}(t) \cdot [1 - \Theta(t, t_0, \Delta)]$$

- We use $t_{min} \in [0.08, 0.15]$ fm, kept fixed for all ensembles.

Cross-check for $a_\mu^{SD}(\ell)$ (II)



- For small t_{min}: $\tilde{a}_\mu^{SD}(\ell, t_{\min}) \propto t_{min}^2$. $\tilde{a}_\mu^{SD}(\ell, 0) = 48.41\ (3) \times 10^{-10}$
- Good agreement with our previous determination:

$$a_\mu^{SD}(\ell) = 48.27\ (22) \times 10^{-10}$$

Finite size effects

Finite Size Effects in the MLLGS-model (I)

Assuming the dominance of $\pi\pi$ states contributions in $V_\ell(t)$:

$$V_\ell(t,L) \sim V_{\pi\pi}(t,L) = \sum_n \nu_n |A_n|^2 e^{-\omega_n t}, \qquad \underbrace{\delta_{11}(k_n) + \phi\left(\frac{k_n L}{2\pi}\right) = n\pi}_{\text{Lellouch-Luscher FV quantisation}}$$

- A_n and δ_{11} depend on the time-like pion form factor $F_\pi(\omega)$.
- In th MLLGS model:

$$F_\pi^{(GS)}(\omega) = \frac{M_\rho^2 - A_{\pi\pi}(0)}{M_\rho^2 - \omega^2 - A_{\pi\pi}(\omega)}$$

- The (twice-subtracted) $\pi\pi$ amplitude $A_{\pi\pi}$ depends on three parameters M_ρ, M_π and $g_{\rho\pi\pi}$:

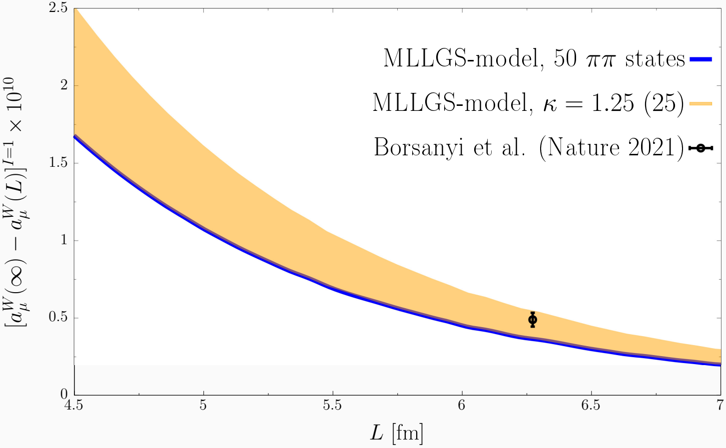
$$\Gamma_{\rho\pi\pi}(\omega) = \frac{g_{\rho\pi\pi}^2}{6\pi}\frac{k^3}{\omega^2}, \qquad k \equiv \sqrt{\omega^2/4 - M_\pi^2}$$

- FSE determined from $V_{\pi\pi}^{\infty}(t) - V_{\pi\pi}(t,L)$ with

$$V_{\pi\pi}^{\infty}(t) = \frac{1}{48\pi^2}\int_{2M_\pi}^{\infty} d\omega\ \omega^2 \left[1 - \frac{4M_\pi^2}{\omega^2}\right]^{3/2} |F_\pi(\omega)|^2 e^{-\omega t}$$

Finite Size Effects in the MLLGS-model (II)

$$M_\pi \sim 135 \text{ MeV}\ , \qquad M_\rho \sim 775 \text{ MeV}\ , \qquad \Gamma_{\rho\pi\pi} \sim 149 \text{ MeV}$$



- Correcting factor $\kappa = 1.25\ (25)$ to take into account deviation between the FSEs from the MLGGS-model and the BMW result.

Disconnected contributions

Disconnected contributions per flavor

ℓ, s, c are flavour-diagonal light, strange and charm contributions.

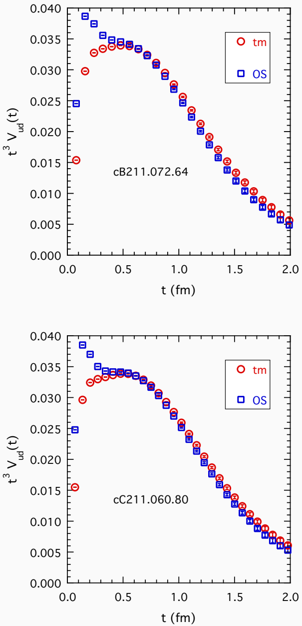
us, uc and sc are flavour off-diagonal light-strange, light-charm and strange-charm contributions.

$a_\mu^{SD}(disco) \times 10^{12}$						
Ensemble	ℓ	s	c	us	uc	sc
cB211.072.64	$-3.36\ (11)$	$-2.090\ (58)$	$-1.18\ (18)$	$+5.29\ (12)$	$-1.52\ (23)$	$+1.67\ (17)$
cC211.060.80	$-3.36\ (14)$	$-2.090\ (69)$	$-0.78\ (13)$	$+5.53\ (19)$	$-1.48\ (24)$	$+1.37\ (18)$
cD211.054.96	$-3.54\ (16)$	$-2.084\ (71)$	$-0.71\ (14)$	$+5.60\ (19)$	$-1.50\ (22)$	$+1.27\ (17)$

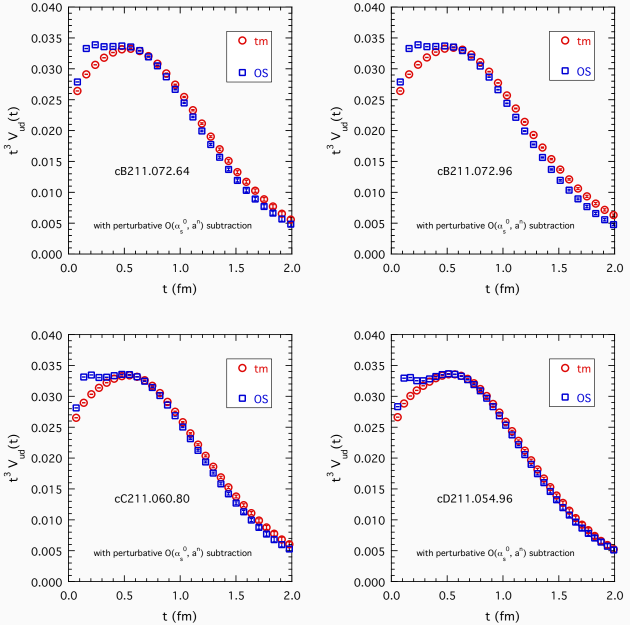
$a_\mu^{W}(disco) \times 10^{10}$						
Ensemble	ℓ	s	c	us	uc	sc
cB211.072.64	$-1.086\ (48)$	$-0.149\ (20)$	$-0.030\ (57)$	$+0.635\ (48)$	$+0.00\ (7)$	$-0.02\ (5)$
cC211.060.80	$-1.300\ (63)$	$-0.159\ (27)$	$-0.033\ (46)$	$+0.726\ (73)$	$-0.03\ (9)$	$+0.04\ (6)$
cD211.054.96	$-1.201\ (52)$	$-0.149\ (28)$	$+0.018\ (56)$	$+0.627\ (70)$	$+0.02\ (9)$	$-0.02\ (7)$

Effect of perturbative subtraction in vector correlator

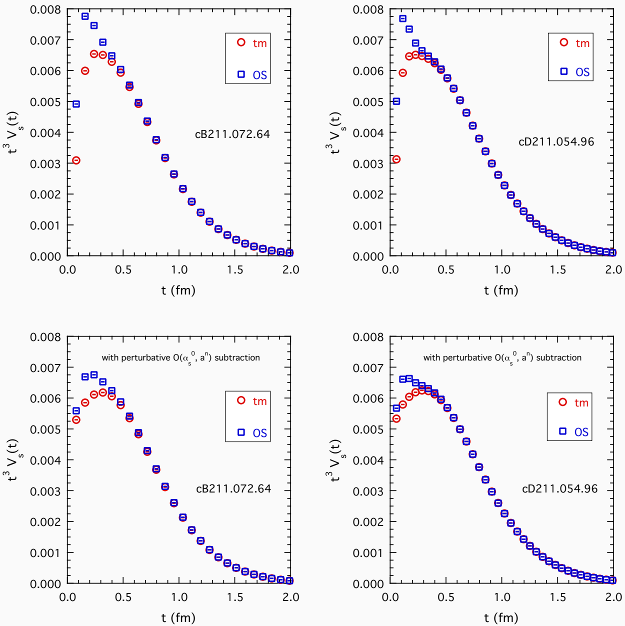
Light-connected (Unsubtracted)



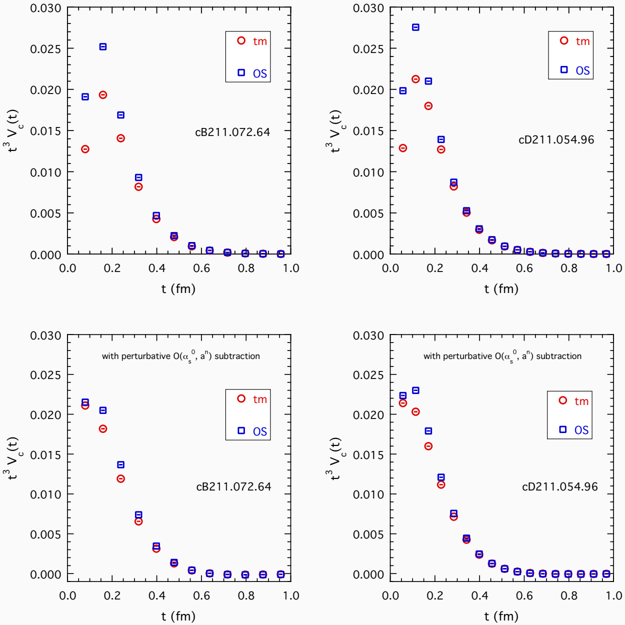
Light-connected (subtracted)



Strange-connected



Charm-connected



Charged, neutral and OS pion masses

Charged, neutral and OS pion masses

- In our twisted-mass mixed action setup, three different type of pions are present (neutral, charged and OS), with masses differing from each other by $\mathcal{O}(a^2)$ cut-off effects.
- OS pion is the heaviest, neutral pion the lightest.
- Charged (or tm) pion is the only Goldstone at finite a ($M_{\pi^+}(m_q = 0) = 0$).

ensemble	M_{π^+} (MeV)	M_{π}^{OS} (MeV)
cB211.072.64	$140.2\ (0.2)$	$297.5\ (0.7)$
cB211.072.96	$140.1\ (0.2)$	$298.4\ (0.5)$
cC211.060.80	$136.6\ (0.2)$	$248.9\ (0.5)$
cD211.054.96	$140.8\ (0.3)$	$210.0\ (0.4)$

- Charged-neutral pion mass splitting estimated on the cB211.072.64 ensemble (*ETMC (2018) arXiv:1807.00495*)

$$M_{\pi^+} - M_{\pi^0} = 31\ (22)\ \mathrm{MeV}$$

- Estimated to be $\sim 16\ (11)\ \mathrm{MeV}$ on our finest lattice spacing.

tm mixed-action setup

Mixed action

Local and renormalizable mixed action employed
[Frezzotti and Rossi (2004)] :

$$S = S_{YM}[U] + S_{q,sea}[\Psi_\ell, \Psi_h, U] + S_{q,val}[q_f^\eta, U] + S_{q,ghost}[\phi_f^\eta, U]$$

- Gluonic sector: improved Iwasaki action $S_{YM}[U]$ (not detailed here).
- Fermionic sector: sea quark action $S_{g,sea}$ written in terms of degenerate quark doublet $\Psi_\ell^t = \{u_{sea}, d_{sea}\}$, and heavy non-degenerate doublet $\Psi_h^t = \{c_{sea}, s_{sea}\}$.
- Fermionic sector: valence quark action $S_{g,val}$ written in terms of quark fields q_f^η, where $f = u, d, s, c$, and the replica index η runs from 1 to 3 with darkred$r_f^\eta = (-1)^{\eta+1}$.
- Ghost sector $S_{q,ghost}$ introduced to cancel contribution of $S_{q,val}$ to fermionic determinant.

$S_{q,sea}$ and $S_{q,val}$

Sea quark action

$$S_{q,sea} = a^4 \sum_x \left\{ \bar{\Psi}_\ell(x) \left[\gamma \cdot \tilde{\nabla} + \mu_\ell - i\gamma_5 \tau^3 W_{cr}^{clov.} \right] \Psi_\ell \right.$$

$$\left. + \bar{\Psi}_h(x) \left[\gamma \cdot \tilde{\nabla} + \mu_\sigma + \mu_\delta \tau^3 - i\gamma_5 \tau^1 W_{cr}^{clov.} \right] \Psi_h \right\}$$

Valence quark action

$$S_{q,val} = a^4 \sum_x \bar{q}_f^\eta(x) \left[\gamma \cdot \tilde{\nabla} + m_f - \underbrace{r_{f,\eta}}_{(-1)^{\eta+1}} i\gamma_5 W_{cr}^{clov.} \right] q_f^\eta(x)$$

Critical Wilson-clover operator

$$W_{cr}^{clov.} = -\frac{a}{2} \nabla^* \cdot \nabla + m_{cr} + \frac{c_{SW}}{32} \gamma_\mu \gamma_\nu \left[Q_{\mu\nu} - Q_{\nu\mu} \right]$$

Role of the replica fields (I)

- We want to make sure that any connected or disconnected single-flavour contribution ($G =$ connected ℓ, s or c ; sum of disconnected) we considered, admits (separately) a well defined continuum limit.
- For connected contributions this must be true for both tm and OS regularizations.
- Given the mixed action S, for each contribution $a_\mu^{w,G}$ to a_μ^w considered separately, suitable renormalized currents $J^{G,ren}, J'^{G,ren}$, exist such that:

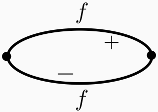
$$a_\mu^{w,G} = 4\alpha_{em}^2 \int dt\ K_\mu(t)\ M_w(t)\ \frac{1}{3} \int d^3x\ \langle J_i^{G,ren}(\vec{x},t) J_i'^{G,ren}(0) \rangle$$

$$M_w(t) = \begin{cases} 1 - \Theta(t, t_0, \Delta) & w = \mathrm{SD} \\ \Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta) & w = \mathrm{W} \end{cases}$$

Role of the replica fields (II)

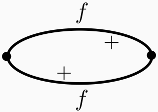
The mixed action S serves to this goal, and the QCD correlators of interest can be easily reconstructed as $a \to 0$.

- Connected, flavour f, contrib. in tm reg. (RC $= Z_A$):



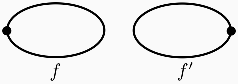
$$= \langle J_\mu^{f,tm}(x) J_\mu^{f,tm}(0) \rangle, \text{ with } J_\mu^{f,tm}(x) = \bar{q}_f^1(x)\gamma_\mu q_f^2(x)$$

- Connected, flavour f, contrib. in OS reg. (RC $= Z_V$):



$$= \langle J_\mu^{f,OS}(x) J_\mu^{f,OS}(0) \rangle, \text{ with } J_\mu^{f,OS}(x) = \bar{q}_f^1(x)\gamma_\mu q_f^3(x)$$

- Disconnected, flavours f, f', contrib. in OS reg. (RC $= Z_V$) :



$$= -\langle J_\mu^{f}(x) J_\mu^{f'}(0) \rangle, \text{ with } J_\mu^{f(f')}(x) = \bar{q}_{f(f')}^{1(3)}(x)\gamma_\mu q_{f(f')}^{1(3)}(x)$$

- For Wilson fermions $Z_V^0 = Z_V$ (known to 2-loops order, proof to appear in paper).

Comparison with e^+e^- and other lattice determinations

Comparison with e^+e^-

window (w)	$a_\mu^w(\mathrm{LQCD})$	$a_\mu^w(e^+e^-)$ [1]	Δa_μ^w	$a_\mu^w(2\pi)$ [1]	$\Delta a_\mu^w / a_\mu^w(2\pi)$
SD	$69.3\ (0.3)$ [*]	$68.4\ (0.5)$	$0.9\ (0.6)$	$13.7\ (0.1)$	$0.066\ (43)$
W	$235.0\ (1.1)$ [*]	$229.4\ (1.4)$	$5.6\ (1.9)$	$138.3\ (1.2)$	$0.040\ (14)$
HVP	$707.5\ (5.5)$ [2]	$693.0\ (3.9)$	$14.5\ (6.7)$	$494.3\ (3.6)$	$0.029\ (14)$

[*] = this work.

[1] = *Colangelo et al. arXiv:2205.12963 (2022).*

[2] = *Borsanyi et al. (Nature, 2021).*

- $a_\mu^W(2\pi)$ is $\pi - \pi$ states contribution below $1\ \mathrm{GeV}$ from e^+e^- [1].
- Almost constant rescaling of $a_\mu^w(2\pi)$ ($\sim 4\%$) sufficient to reconcile all window contributions.