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Progress in calculation of the fourth Mellin moment of the pion LCDA using the HOPE method

THE HOPE COLLABORATION

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with
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Takeaways

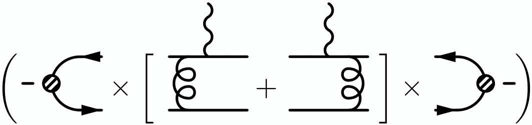
- Light cone distribution amplitude (LCDA) important in factorization theorem for pion electromagnetic form factor.
- Heavy-quark Operator Product Expansion (HOPE) $\rightarrow$ moments of LCDA.
- Preliminary results for $\langle \xi^4 \rangle$ at range of finite lattice spacings, twist-suppressions.

Factorization for $F_\pi(Q^2)$: 1979

- $\phi_M(x,\mu^2)$: Light Cone Distribution Amplitude (LCDA): Probability amplitude
- Factorization theorem:

$$F_\pi(Q^2) \underset{\text{large } Q^2}{=} \int_0^1 dxdy\phi_{\overline{M}}(y,Q^2)\,T_H(x,y,Q^2)\phi_M(x,Q^2)$$

$$\underset{\text{large } Q^2}{=} \int_0^1 dxdy$$



$$\underset{\text{large } Q^2}{=} \frac{16\pi\,\alpha_S(Q^2)}{Q^2} f_\pi^2 \omega_\phi^2(Q^2) \to \frac{16\pi\,\alpha_S(Q^2)}{Q^2} f_\pi^2$$

$$\omega_\phi(Q^2) = \frac{1}{3}\int_0^1 dx\,\frac{\phi(x,Q^2)}{x}$$

$F_\pi(Q^2)$

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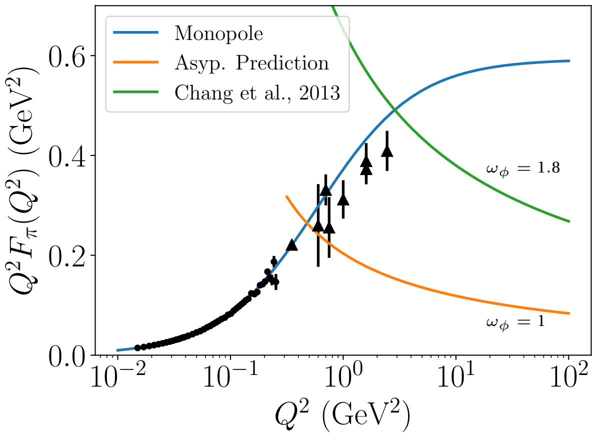
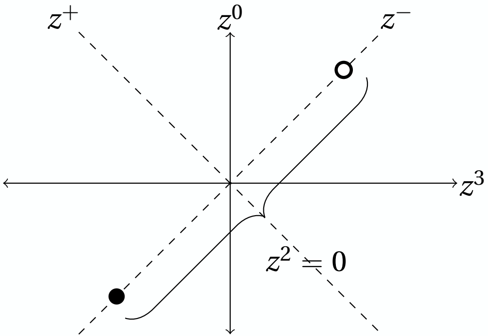


Figure 1: Data from Amendolia et al. (1986), Huber et al. (2008)

Calculating the Pion LCDA

- Only *ab-initio* method to calculate non-perturbative QCD: Lattice QCD.
- Problem: LCDA defined as

$$\left\langle \Omega \left| \bar{\psi}(z_-)\gamma_\mu\gamma_5 W[z_-, -z_-]\psi(-z_-) \right| \pi(\mathbf{p}) \right\rangle = i p_\mu f_\pi \int_{-1}^{1} d\xi \mathrm{e}^{i\xi p^+ \cdot z_-} \phi_\pi(\xi, \mu^2)$$



Heavy-quark OPE Detmold & Lin, 2006

THE HOPE COLLABORATION

- Consider matrix element

$$V^{\mu\nu}(p,q) = \int d^4 z e^{iq\cdot z} \langle\Omega| \, T\{J^{\mu}_{\Psi}(z/2) J^{\nu}_{\Psi}(-z/2)\} \, |\pi(\mathbf{p})\rangle$$

$$J^{\mu}_{\Psi} = \overline{\Psi}(x)\Gamma^{\mu}\psi(x) + \overline{\psi}(x)\Gamma^{\mu}\Psi(x)$$

- Perform operator product expansion:

$$\tilde{Q}^2 = -q^2 + m_{\Psi}^2 \qquad \text{large scale}$$

$$\tilde{\omega} = \frac{1}{\tilde{x}} = \frac{2p\cdot q}{\tilde{Q}^2} \qquad \text{expansion parameter}$$

$$V^{\mu\nu}_{\text{HOPE}}(p,q) = K[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \dots] + \underbrace{\mathcal{O}(\alpha_S)}_{\text{Perturbative corrections}} + \underbrace{\mathcal{O}(1/Q^3)}_{\text{Higher twist}}$$

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Overview of Calculation

- Pion LCDA $\phi_\pi(\xi, \mu^2)$ important in description of $F_\pi(Q^2)$
- Long-range sensitive: non-perturbative.
- HOPE method allows for determination of moments:

$$\langle \xi^n \rangle (\mu^2) = \int_{-1}^{1} d\xi\, \xi^n \phi(\xi, \mu^2)$$

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$$V^{\mu\nu}_{\mathsf{LQCD}}(p, q; a) = \int d^4 z e^{iq\cdot z} \langle \Omega | \, T\{J^\mu_\Psi(z/2) J^\nu_\Psi(-z/2)\} \, |\pi(\mathbf{p})\rangle$$

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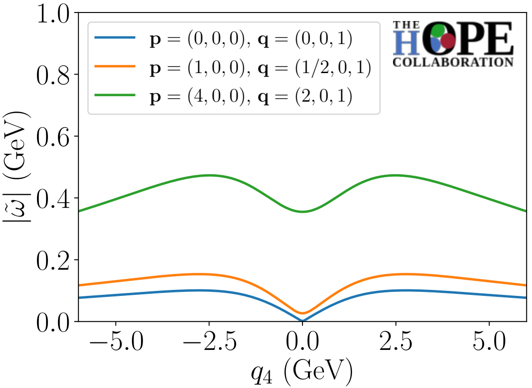
$\rightarrow$ $\langle \xi^2 \rangle, \ \langle \xi^4 \rangle$

Optimizing Kinematics

$$V^{\mu\nu}(p,q) = K[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \dots], \qquad \tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2}$$

- Choose $\mathbf{p} = (2,0,0) \times 2\pi/L$



Lattice Details

$L^3 \times T$	a (fm)	N_{cfg}	N_Ψ
$24^3 \times 48$	0.0813	6500	2
$32^3 \times 64$	0.0600	4500	3
$40^3 \times 80$	0.0502	$\mathcal{O}(5000)$	4
$48^3 \times 96$	0.0407	$\mathcal{O}(5000)$	5

- Quenched approximation with $m_\pi = 550$ MeV
- Wilson-clover fermions with non-perturbatively tuned c_{SW}
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Still to come...

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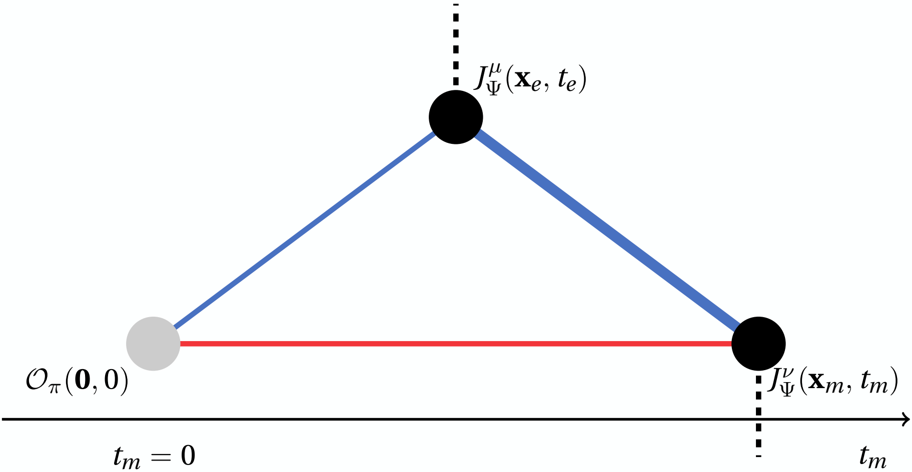
Matrix Element Calculated

$$C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) = \int d^3x_e d^3x_m e^{i\mathbf{p}_e\cdot\mathbf{x}_e + i\mathbf{p}_m\cdot\mathbf{x}_m} \langle\Omega|\, T\{J_\Psi^\mu(x_e) J_\Psi^\nu(x_m)\mathcal{O}_\pi^\dagger(0)\}\,|\Omega\rangle$$



Ratio Method

- Excited state dependent only on sum $t_e + t_m$.

$$C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) = R^{\mu\nu}(t_e - t_m; \mathbf{p}, \mathbf{q}) \frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2} + \dots,$$

- Define $t_+ = t_e + t_m$, $t_- = t_e - t_m$.

$$C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) = R^{\mu\nu}(t_-; \mathbf{p}, \mathbf{q}) \frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})t_+/2} + \dots,$$

- Consider two sets of time; (t_e, t_m) and $(t'_e, t'_m) = (t_e + \delta, t_m - \delta)$

$$t'_+ = (t_e + \delta) + (t_m - \delta) = t_e + t_m = t_+$$
$$t'_- = (t_e + \delta) - (t_m - \delta) = t_e - t_m + 2\delta \neq t_-$$

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Ratio Method

- Construct ratio with fixed t_+, varying t_- $(\delta = -1)$

$$\mathcal{R} = \frac{C_3^{\mu\nu}(t_e - 1, t_m + 1; \mathbf{p}_e, \mathbf{p}_m)}{C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m)}$$
$$= \frac{R^{\mu\nu}(t_e - t_m - 2; \mathbf{p}, \mathbf{q})}{R^{\mu\nu}(t_e - t_m; \mathbf{p}, \mathbf{q})} \frac{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}}{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}} \left[1 + \dots\right]$$

- Need two t_e, ie t_e and $t_e - 1$
- No need for 2-point data!
- No renormalization required.
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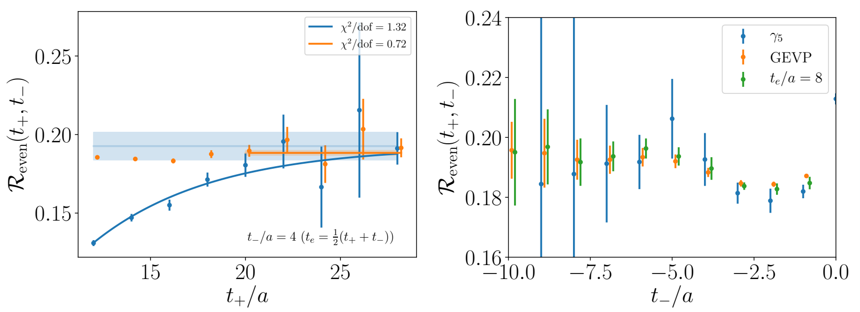
Operator Optimization

- Momentum smearing (Bali et al)
- Variational analysis:

$$\mathcal{O}_\pi(x) = c_1\mathcal{O}_1(x) + c_2\mathcal{O}_2(x), \quad \mathcal{O}_1(x) = \overline{\psi}\gamma_5\psi, \;\; \mathcal{O}_2(x) = \overline{\psi}\gamma_4\gamma_5\psi$$

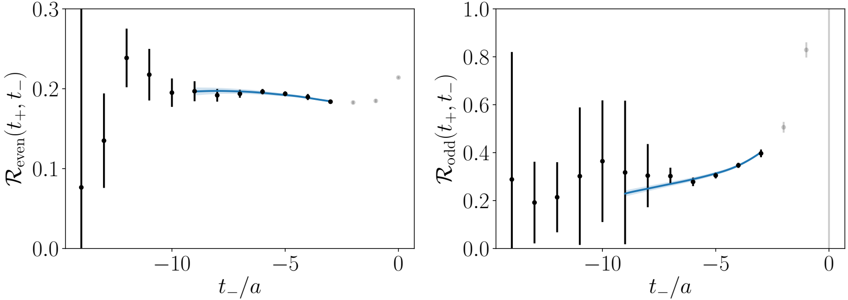
$$\begin{aligned} C^{\mu\nu}_{3,\mathsf{GEVP}}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) &= c_1 C^{\mu\nu}_{3,\gamma_5}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) \\ &+ c_2 C^{\mu\nu}_{3,\gamma_4\gamma_5}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) \end{aligned}$$

$$\mathcal{R} == \frac{R^{\mu\nu}(t_e - t_m - 2; \mathbf{p}, \mathbf{q})}{R^{\mu\nu}(t_e - t_m; \mathbf{p}, \mathbf{q})} \frac{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}}{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}} \left[1 + \ldots\right]$$



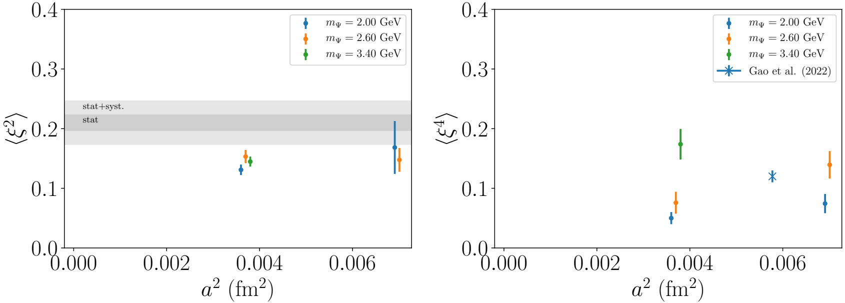
Example at fixed lattice spacing

$$\mathcal{R}(t_-, \mathbf{p}, \mathbf{q}, m_\psi, \langle \xi^2 \rangle, \langle \xi^2 \rangle; a) = \mathcal{R}_{\mathsf{HOPE}}(t_-, \mathbf{p}, \mathbf{q}, m_\psi, \langle \xi^2 \rangle, \langle \xi^2 \rangle) + \mathcal{O}(a^2)$$



$L/a = 24$, $m_\Psi = 2.0$ GeV, $\langle \xi^2 \rangle = 0.17 \pm 0.04$, $\langle \xi^4 \rangle = 0.07 \pm 0.02$

Status of Calculation



$$\langle \xi^n \rangle (a, m_\Psi) = \langle \xi^n \rangle + \frac{A}{m_\Psi} + Ba^2 + Ca^2 m_\Psi + Da^2 m_\Psi^2$$

Further Work & Conclusions

- HOPE can be used to extract Mellin moments.
- Introduced ratio method: no renormalzation.
- 2 lattice spacings, 3 heavy quark masses.
- Aim for 2 more lattice spacings + more heavy quark masses to enable continuum, twist-2 extrapolation.

HIGHER-TWIST/LATTICE ARTIFACTS TRADEOFF